Relationship between Two FCA Approaches on Heterogeneous Formal Contexts*

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Abstract. We show a relationship between two theoretical approaches of Formal Concept Analysis working with so-called heterogeneous formal context i.e. such context in which each object and attribute can have own data-type. One of them is presented in [19]; each value in a formal context is some Galois connection between the lattices corresponding to the appropriate object and attribute. Another approach is presented in our paper [1] and it is a unifying platform of approaches from [14] and [11], [12]. In this paper, we prove that each of them can be derived from another.

Keywords: Formal Concept Analysis, Galois connection, G-ideal

1 Introduction

The Formal Concept Analysis is a well-known data-mining method on a rectangle matrix of data where each row corresponds to some object, each column corresponds to some attribute and a matrix field value expresses the presence of the column attribute to the row object. One of the goals of this method is to find so-called concepts – the stable (in some sense) pairs of subsets of objects and attributes. This method can be considered as a nice application of the algebraic notion of a Galois connection. The Formal Concept Analysis is based and deeply described in the classical Ganter & Wille's book [9] where authors concentrate mainly to the so-called crisp case with binary data in the matrix. The natural question arose: What if the matrix data have a non-binary character?

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Beside the conceptual scaling from [9] which returns concepts with crisp subsets in both coordinates, some other answers arose which return concepts with fuzzy subsets at least in one coordinate: The first one was done by Burusco & Fuentes-Gonzalez [8] and it was improved (independently) by Bělohlávek [2], [3] and Pollandt [21], [22] which use values from the same residual lattice for values of the matrix and for the fuzziness of subsets of the objects and the attributes. Another approach independently (and with slight differences) given by Ben Yahia & Jaoua [7], Bělohlávek, Sklenář, & Zacpal [4], and Krajči [10] was not so symmetric – it considers fuzzy subsets in one coordinate and crisp (binary) subsets in another one. All these approaches where covered by a common platform – so-called generalized concept lattices [12], [13] which diversifies fuzziness of subsets of the attributes, fuzziness of subsets of the objects and moreover fuzziness of the matrix values.

Then Medina and Ojeda-Aciego brought the idea of multi-adjointness used in logic-programming [16], [17], [18] to the Formal Concept Analysis too [14], [15]. Because of this novelty and originality, this approach is not (at least immediately) covered by the above-mentioned generalized concept lattices.

This fact has inspired us to modify our old approach in such a way that this will work with different mutual relationships between the objects and the attributes. Moreover we work with different lattices for different rows and columns and for the matrix data. To compare with the till known approaches which works with attributes and objects of the same type, an important advantage of this new, totally diversifying, approach is the possibility to apply the Formal Concept Analysis to heterogeneous data too. This is the reason why we will call this new approach heterogeneous. We have described this approach in [1] and recall it in Section 2.

Another answer to the problem of data heterogeneity was given by [19] and [20]. In this approach, each datum in a formal context are not a simple number or other singular value but (sic!) a Galois connection which describes in a some way the behavior between the corresponding object and attribute. We recall this approach in Section 3.

2 Heterogeneous formal context

In this section we recall the basic definitions and results from [1].

Let A and B be non-empty sets. Let $\mathcal{P} = ((P_{a,b}, \leq_{P_{a,b}}) : a \in A, b \in B)$ be a system of posets and let R be a function from $A \times B$ such that $R(a,b) \in P_{a,b}$, for all $a \in A$ and $b \in B$. Let $\mathcal{C} = ((C_a, \leq_{C_a}) : a \in A)$ and $\mathcal{D} = ((D_b, \leq_{D_b}) : b \in B)$ be systems of complete lattices. (For simplicity, we will omit the indices of all noticed $\leq_?$, it will be always clear which of one is used.)

Let $\odot = ((\bullet_{a,b}) : a \in A, b \in B)$ be a system of operations such that $\bullet_{a,b}$ is from $C_a \times D_b$ to $P_{a,b}$ and it is isotone and left-continuous in both arguments, i. e.

1a) $c_1 \leq c_2$ implies $c_1 \bullet_{a,b} d \leq c_2 \bullet_{a,b} d$ for all $c_1, c_2 \in C_a$ and $d \in D_b$,
1b) $d_1 \leq d_2$ implies $c \bullet_{a,b} d_1 \leq c \bullet_{a,b} d_2$ for all $c \in C_a$ and $d_1, d_2 \in D_b$,

2a) if $c \bullet_{a,b} d \leq p$ for some $d \in D_b$, $p \in P_{a,b}$ and for all $c \in X \subseteq C_a$ then $\sup X \bullet_{a,b} d \leq p$,
2b) if $c \bullet_{a,b} d \leq p$ for some $c \in C_a$, $p \in P_{a,b}$ and for all $d \in Y \subseteq D_b$ then $c \bullet_{a,b} \sup Y \leq p$.

Then the tuple $\langle A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot \rangle$ will be called a *heterogeneous formal context.*

Notice that if $C_a = D_b$ and $\bullet_{a,b}$ is commutative these conditions can be reduced to these two:

1) $c_1 \leq c_2$ implies $c_1 \bullet_{a,b} d \leq c_2 \bullet_{a,b} d$ for all $c_1, c_2, d \in C_a = D_b$,
2) if $c \bullet_{a,b} d \leq p$ for some $d \in C_a = D_b$, $p \in P$ and for all $c \in X \subseteq C_a = D_b$ then $\sup X \bullet_{a,b} d \leq p$.

Let F be the set of all functions f with the domain A such that $f(a) \in C_a$, for all $a \in A$ (i. e., more formally, $F = \prod_{a \in A} C_a$) and G be the set of all functions g with the domain B such that $g(b) \in D_b$, for all $b \in B$. (i. e. $G = \prod_{b \in B} D_b$).

Define the following mapping $\nearrow : G \to F$: If $g \in G$ then $\nearrow(g) \in F$ is defined by

$$(\nearrow(g))(a) = \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a,b)\}.$$

Symmetrically define the mapping $\swarrow : F \to G$: If $f \in F$ then $\swarrow(f) \in G$ is defined as following:

$$(\swarrow(f))(b) = \sup\{d \in D_b : (\forall a \in A) f(a) \bullet_{a,b} d \leq R(a,b)\}.$$

Theorem 1. *Let $f \in F$ and $g \in G$. Then the following conditions are equivalent:*

1) $f \leq \nearrow(g)$.
2) $g \leq \swarrow(f)$.
3) $f(a) \bullet_{a,b} g(b) \leq R(a,b)$ for all $a \in A$ and $b \in B$.

Corollary 1. *Mappings $\nearrow$ and $\swarrow$ form a Galois connection.*

Corollary 2.

1a) $g_1 \leq g_2$ implies $\nearrow(g_1) \geq \nearrow(g_2)$.
1b) $f_1 \leq f_2$ implies $\swarrow(f_1) \geq \swarrow(_2)$.
2a) $g \leq \swarrow(\nearrow(g))$.
2b) $f \leq \nearrow(\swarrow(f))$.
3a) $\nearrow(g) = \nearrow(\swarrow(\nearrow(g)))$.
3b) $\swarrow(f) = \swarrow(\nearrow(\swarrow(f)))$.

We use a Galois connection $(\nearrow, \swarrow)$ for the concept lattice construction via classical Ganter-Wille's approach from [9].

Lemma 1. *1) Let $\{g_i : i \in I\} \subseteq G$. Then*

$$\nearrow\left(\bigvee_{i \in I} g_i\right) = \bigwedge_{i \in I} \nearrow(g_i).$$

2) Let $\{f_i : i \in I\} \subseteq F$. *Then*

$$\swarrow\left(\bigvee_{i\in I} f_i\right) = \bigwedge_{i\in I} \swarrow(f_i).$$

By a *concept* we will understand a pair $\langle g, f\rangle$ from $G \times F$ such that $\nearrow(g) = f$ and $\swarrow(f) = g$.

Lemma 2. *If* $\langle g_1, f_1\rangle$ *and* $\langle g_2, f_2\rangle$ *are concepts then* $g_1 \leq g_2$ *iff* $f_1 \geq f_2$.

This lemma allows to define the following ordering of concepts: $\langle g_1, f_1\rangle \leq \langle g_2, f_2\rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$).

The poset of all such concepts ordered by $\leq$ will be called a *heterogeneous concept lattice* and denoted by $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$.

The following theorem shows that the word *lattice* in its name corresponds with reality.

Theorem 2. *(The Basic Theorem on Heterogeneous Concept Lattices)*

1) A heterogeneous concept lattice $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$ *is a complete lattice in which*

$$\bigwedge_{i\in I}\langle g_i, f_i\rangle = \left\langle \bigwedge_{i\in I} g_i, \nearrow\left(\swarrow\left(\bigvee_{i\in I} f_i\right)\right)\right\rangle$$

and

$$\bigvee_{i\in I}\langle g_i, f_i\rangle = \left\langle \swarrow\left(\nearrow\left(\bigvee_{i\in I} g_i\right)\right), \bigwedge_{i\in I} f_i\right\rangle.$$

2) For each $a \in A$, $b \in B$, *let* $P_{a,b}$ *have the least element* $0_{P_{a,b}}$ *such that* $0_{C_a} \bullet_{a,b} d = c \bullet_{a,b} 0_{D_b} = 0_{P_{a,b}}$, *for all* $c \in C_a$, $d \in D_b$. *Then a complete lattice* L *is isomorphic to* $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$ *if and only if there are mappings* $\alpha : \bigcup_{a\in A}(\{a\} \times C_a) \to L$ *and* $\beta : \bigcup_{b\in B}(\{b\} \times D_b) \to L$ *such that:*
   a) α *does not increase in the second argument (for the fixed first one).*
   b) β *does not decrease in the second argument (for the fixed first one).*
   c) $\mathrm{Rng}(\alpha)$ *is* inf*-dense in* L.
   d) $\mathrm{Rng}(\beta)$ *is* sup*-dense in* L.
   e) For every $a \in A$, $b \in B$ *and* $c \in C_a$, $d \in D_b$

$$\alpha(a, c) \geq \beta(b, d) \quad \textit{if and only if} \quad c \bullet_{a,b} d \leq R(a, b).$$

3 Galois connectional approach

In this section, we recall the basic definitions and results of approach from [19], [20] which is inspired by the (homogeneous) approach from [23].

Let A and B be non-empty sets. Let $\mathcal{C} = ((C_a, \leq_{C_a}) : a \in A)$ and $\mathcal{D} = ((D_b, \leq_{D_b}) : b \in B)$ be systems of complete lattices. Let $\mathcal{G} = ((\phi_{a,b}, \psi_{a,b}) : a \in A, b \in B)$ be a system of (antitone) Galois connection s.t. $(\phi_{a,b}, \psi_{a,b})$ is a Galois connection from $(C_a, \leq_{C_a})$ to $(D_b, \leq_{D_b})$. (Again we will omit the indices of all noticed $\leq_?$.)

Define the following mapping $\uparrow : G \to F$: If $g \in G$ then $\uparrow(g) \in F$ is defined by

$$(\uparrow(g))(a) = \bigwedge_{b \in B} \psi_{a,b}(g(b)).$$

Symmetrically define the mapping $\downarrow : F \to G$: If $f \in F$ then $\downarrow(f) \in G$ is defined as following:

$$(\downarrow(f))(b) = \bigwedge_{a \in A} \phi_{a,b}(f(a)).$$

Theorem 3. $(\uparrow, \downarrow)$ *is a Galois connection.*

Hence the classical Ganter-Wille's process can be used for the concept lattice construction, so it can be obtained the following.

By a *concept* in this approach it will be understand a pair $\langle g, f\rangle$ from $G \times F$ such that $\uparrow(g) = f$ and $\downarrow(f) = g$.

Lemma 3. *If* $\langle g_1, f_1\rangle$ *and* $\langle g_2, f_2\rangle$ *are concepts then* $g_1 \leq g_2$ *iff* $f_1 \geq f_2$.

This lemma allows to define the following ordering of concepts: $\langle g_1, f_1\rangle \leq \langle g_2, f_2\rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$).

The poset of all such concepts ordered by $\leq$ will be called a *connectional concept lattice* and denoted by $\mathrm{CCL}(A, B, \mathcal{C}, \mathcal{D}, \mathcal{G}, \downarrow, \uparrow, \leq)$.

Theorem 4. *(The Basic Theorem on Connectional Concept Lattices)*

1) *A connectional concept lattice* $\mathrm{CCL}(A, B, \mathcal{C}, \mathcal{D}, \mathcal{G}, \downarrow, \uparrow, \leq)$ *is a complete lattice in which*

$$\bigwedge_{i \in I} \langle g_i, f_i\rangle = \left\langle \bigwedge_{i \in I} g_i, \uparrow\left(\downarrow\left(\bigvee_{i \in I} f_i\right)\right)\right\rangle$$

and

$$\bigvee_{i \in I} \langle g_i, f_i\rangle = \left\langle \downarrow\left(\uparrow\left(\bigvee_{i \in I} g_i\right)\right), \bigwedge_{i \in I} f_i\right\rangle.$$

2) *A complete lattice* L *is isomorphic to* $\mathrm{CCL}(A, B, \mathcal{C}, \mathcal{D}, \mathcal{G}, \downarrow, \uparrow, \leq)$ *if and only if there are mappings* $\alpha : \bigcup_{a \in A}(\{a\} \times C_a) \to L$ *and* $\beta : \bigcup_{b \in B}(\{b\} \times D_b) \to L$ *such that for every* $a \in A$, $b \in B$ *and* $c \in C_a$, $d \in D_b$

$$\alpha(a, c) \geq \beta(b, d) \qquad \textit{iff} \qquad d \leq \phi_{a,b}(c) \qquad \textit{iff} \qquad c \leq \psi_{a,b}(d).$$

4 Heterogeneous approach can be expressed by connectional one

In this section, we modify method from [19] which was used for the proof that connectional approach covers a generalized approach from [11] and [12]. It used a notion of G-ideals defined in [24].

Let $(L, \leq_L)$, $(M, \leq_M)$ be complete lattices. Then $J \subseteq L \times M$ is called a *G-ideal* of $L \times M$ when the following conditions hold:

1) If $(\ell, m) \in J$ and $(\ell', m') \leq (\ell, m)$ (coordinate-wise, i.e. $\ell' \leq \ell$ and $m' \leq m$) then $(\ell', m') \in J$.
2) If $\{(\ell_i, m_i) : i \in I\} \subseteq J$ then $(\bigvee_{i\in I} \ell_i, \bigwedge_{i\in I} m_i), (\bigwedge_{i\in I} \ell_i, \bigvee_{i\in I} m_i) \in J$. If $I = \emptyset$ then $(0_L, 1_M), (1_L, 0_M) \in J$.

Theorem 5. *[24] Let $(L, \leq_L)$, $(M, \leq_M)$ be complete lattices.*

1) If (ϕ, ψ) is an (antitone) Galois connection from $(L, \leq_L)$ to $(M, \leq_M)$ then

$$\{(\ell, m) : \phi(\ell) \geq_M m\} = \{(\ell, m) : \psi(m) \geq_L \ell\}$$

is a G-ideal on $L \times M$.

2) If J is a G-ideal on $L \times M$ then the mappings $\phi : L \to M$ and $\psi : M \to L$ defined by

$$\phi(\ell) = \bigvee\{m \in M : (\ell, m) \in J\}$$

and

$$\psi(m) = \bigvee\{\ell \in L : (\ell, m) \in J\}$$

form a Galois connection from $(L, \leq_L)$ to $(M, \leq_M)$.

Moreover, this correspondences between Galois connections and G-ideals are each other inverse.

The paper [19] uses these facts in the following way:

Lemma 4. *Let $(L, \leq_L)$, $(M, \leq_M)$ be complete lattices, $(P, \leq_P)$ be poset and $\bullet : L \times M \to P$ is isotone and left-continuous in both arguments. Then*

$$\{(\ell, m) : \ell \bullet m \leq p\}$$

is a G-ideal.

Assume that we have a heterogeneous concept lattice HCL$(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$. For each $a \in A$ and $b \in B$ define

$$J_{a,b} = \{(c, d) \in C_a \times D_b : c \bullet_{a,b} d \leq R(a, b)\},$$

by the previous Lemma 4 we know that $J_{a,b}$ is a G-ideal on $C_a \times D_b$. Then again for each $a \in A$ and $b \in B$ define the mappings $\phi_{a,b} : C_a \to D_b$ and $\psi_{a,b} : D_b \to C_a$ defined by

$$\phi_{a,b}(c) = \bigvee\{d \in D_b : (c, d) \in J_{a,b}\}$$

and

$$\psi_{a,b}(d) = \bigvee\{c \in C_a : (c, d) \in J_{a,b}\}$$

and we know by Theorem 5 that $(\phi_{a,b}, \psi_{a,b})$ is a Galois connection from C_a to D_b. Finally, we define mappings $\downarrow$ and $\uparrow$ as before:

$$(\uparrow(g))(a) = \bigwedge_{b \in B} \psi_{a,b}(g(b)), \qquad (\downarrow(f))(b) = \bigwedge_{a \in A} \phi_{a,b}(f(a)).$$

Theorem 6. $(\uparrow, \downarrow) = (\nearrow, \swarrow)$.

Proof. We prove $\uparrow = \nearrow$ only, the second equality can be proved dually. Let $g \in G$ and $a \in A$, we are going to prove $(\uparrow(g))(a) = (\nearrow(g))(a)$.

By the definition we have

$$(\uparrow(g))(a) = \bigwedge_{b \in B} \psi_{a,b}(g(b)) = \bigwedge_{b \in B} \bigvee\{c \in C_a : (c, g(b)) \in J_{a,b}\}$$

$$= \bigwedge_{b \in B} \bigvee\{c \in C_a : c \bullet_{a,b} g(b) \leq R(a, b)\}$$

and

$$(\nearrow(g))(a) = \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a, b)\}.$$

Denote

$$X = \{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a, b)\}$$

and, for each $b \in B$,

$$X_b = \{c \in C_a : c \bullet_{a,b} g(b) \leq R(a, b)\},$$

then we want to prove $\bigwedge_{b \in B} \sup X_b = \sup X$.

- $\geq$ For each $b \in B$ we have $X_b \supseteq X$ hence $\sup X_b \geq \sup X$. It follows that $\bigwedge_{b \in B} \sup X_b \geq \sup X$.
- $\leq$ Let $b \in B$. Then for each $c \in X_b$ we have $c \bullet_{a,b} g(b) \leq R(a, b)$. By the left-continuity of $\bullet_{a,b}$ in the first argument we have $\sup X_b \bullet_{a,b} g(b) \leq R(a, b)$. Because clearly $\bigwedge_{b' \in B} \sup X_{b'} \leq \sup X_b$, by the isotonity of $\bullet_{a,b}$ in the first argument $\bigwedge_{b' \in B} \sup X_{b'} \bullet_{a,b} g(b) \leq R(a, b)$. This holds for each $b \in B$, which means that $\bigwedge_{b' \in B} \sup X_{b'} \in X$, hence $\bigwedge_{b' \in B} \sup X_{b'} \leq \sup X$.

□

5 Connectional approach can be expressed by heterogeneous one

In this section we show opposite direction to the previous one, namely that the heterogeneous approach covers the connectional one, moreover by the surprisingly simply way.

Firstly, one fact from [24] analogous to Lemma 1:

Lemma 5. *Let $(L, \leq_L)$, $(M, \leq_M)$ be complete lattices and (ϕ, ψ) be a Galois connection from $(L, \leq_L)$ to $(M, \leq_M)$.*

1) For arbitrary subset $\{\ell_i : i \in I\}$ of L

$$\phi\left(\bigvee_{i\in I} \ell_i\right) = \bigwedge_{i\in I} \phi(\ell_i).$$

2) For arbitrary subset $\{m_i : i \in I\}$ of M

$$\psi\left(\bigvee_{i\in I} m_i\right) = \bigwedge_{i\in I} \psi(m_i).$$

We use it in the following way:

Theorem 7. *Let $(L, \leq_L)$, $(M, \leq_M)$ be complete lattices and (ϕ, ψ) be a Galois connection from $(L, \leq_L)$ to $(M, \leq_M)$. Let $\bullet : L \times M \to (\{0, 1\}, \leq)$ be defined in the following way:*

$$\ell \bullet m = \begin{cases} 0 & \textit{if } \phi(\ell) \geq m \textit{ (iff } \psi(m) \geq \ell\textit{)}, \\ 1 & \textit{elsewhere.} \end{cases}$$

Then $\bullet$ is isotone and left-continuous in both arguments.

Proof. Because of duality, it is enough to prove isotonity and left-continuity in the first argument.

- Let $\ell_1, \ell_2 \in L$ where $\ell_1 \leq \ell_2$ and $m \in M$. We want to prove that $\ell_1 \bullet m \leq \ell_2 \bullet m$.
  - If $\ell_2 \bullet m = 1$, the inequality is trivial.
  - If $\ell_2 \bullet m = 0$, then by the definition $\phi(\ell_2) \geq m$. Because (ϕ, ψ) be a Galois connection and $\ell_1 \leq \ell_2$, we have $\phi(\ell_1) \geq \phi(\ell_2)$ which by transitivity implies $\phi(\ell_1) \geq m$. So, by the definition $\ell_1 \bullet m = 0$ hence $\ell_1 \bullet m \leq \ell_2 \bullet m$.
- Let $m \in M$, $X \subseteq L$ and $\ell \bullet m \leq p$ for all $\ell \in X$. We want to prove that $\sup X \bullet m \leq p$.
  - If $p = 1$, the inequality is trivial.
  - If $p = 0$, then by the definition $\phi(\ell) \geq m$ for all $\ell \in X$ which means $\bigwedge_{\ell\in X} \phi(\ell) \geq m$. Because (ϕ, ψ) is a Galois connection, by Lemma 5 we have $\bigwedge_{\ell\in X} \phi(\ell) = \phi(\sup_{\ell\in X} \ell)$. This implies $\phi(\sup_{\ell\in X} \ell) \geq m$, so, by the definition $\sup_{\ell\in X} \ell \bullet m = 0$.

Assume that we have a connectional concept lattice $\mathrm{CCL}(A, B, \mathcal{C}, \mathcal{D}, \mathcal{G}, \downarrow, \uparrow, \leq)$. For each $a \in A$ and $b \in B$ take the same $P_{a,b} = (\{0, 1\}, \leq)$, $R(a, b) = 0$ (sic!) and $\bullet_{a,b} : C_a \times D_b \to P_{a,b}$ such that for all $c \in C_a$ and $d \in D_b$,

$$c \bullet_{a,b} d = \begin{cases} 0 & \text{if } \phi_{a,b}(c) \geq d \text{ (iff } \psi_{a,b}(d) \geq c\text{)}, \\ 1 & \text{elsewhere.} \end{cases}$$

By Theorem 7 $\bullet_{a,b}$ is isotone and left-continuous in both arguments, so we have a frame for heterogeneous approach a we can define the mappings $\nearrow$ and $\swarrow$ as before.

Theorem 8. $(\nearrow,\swarrow) = (\uparrow,\downarrow)$.

Proof. We prove $\nearrow = \uparrow$ only, the second equality can be proved dually. Let $g \in G$ and $a \in A$. Then by the definitions we have

$$(\nearrow(g))(a) = \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a,b)\} =$$
$$= \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq 0\} =$$
$$= \sup\{c \in C_a : (\forall b \in B) \psi_{a,b}(g(b)) \geq c\} =$$
$$= \sup\{c \in C_a : \bigwedge_{b \in B} \psi_{a,b}(g(b)) \geq c\} = \bigwedge_{b \in B} \psi_{a,b}(g(b)) = (\uparrow(g))(a).$$

□

6 Conclusions

In this paper we recall two rather new common platform for till-known fuzzifications of the Formal Concept Analysis which work on the context without limitation of the same data-types of objects and/or attributes. The first one arises, defined in [1], as rather straightforward extension of the previous so-called generalized approach from [11] and [12] to such heterogeneous context. The second one, from [19] and [20] is based on interesting idea to put some Galois connection to each field of the table. We show that each of these two approaches covers and is covered by the other one (in some canonical way).

In the end, let us say one "philosophical" aspect about our approach (that from Section 2). In this case, a pair consisting of some $\bullet$ and some value is put into each field of the table. The part $\bullet$ can be understood as behavior of the corresponding object with respect to the corresponding attribute. This behavior can be known long before than data come to the table, hence it can be thought as *metadata*. Data can change through the time but this metadata are fixed. In other words, we divide information on relationship of an object and an attribute to the stable and dynamic part. (Of course, this division has meaning only in the case that we consider possible changing of the data in the table.) In our opinion, the connectional approach has not this advantage, because it mixes metadata and data parts.

Then we can formulate this problem: In Section 5 we can see a surprising (and maybe suspicious) transformation of connectional approach to heterogeneous with the data part constantly equal to 0, i.e. all this is transformed to metadata part. The question is: Is there some other (natural, canonical) transformation which is not constant?

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