Shape is physical

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Abstract. In a simplified vision the universe is made by two main ingredients: atoms and bits. Within this paradigm the shape of things is just the content of information associated with each thing. In this paper we briefly discuss the role of information in defining shapes and specifically address the connection between information, entropy and energy, with the aim of showing that changing shape of a physical system is necessarily connected with a change in entropy and thus involves energy.

Keywords: shape, entropy, energy, information, Shannon, Boltzmann, reversibility, dissipation, statistics.

1 Introduction

One of the most ambitious tasks of science is the explanation of the entire universe using few fundamental concepts. In an extremely simplified vision the universe is composed by two main ingredients: atoms and bits. In this perspective the atoms (or some more elementary finite set of components[1]) are similar to LEGO bricks whose dispositions are encoded in bits of information. The laws of physics are wise algorithms that allow the reduction of the amount of information necessary to describe its entire functioning. Within this paradigm the shape of things is just a physical representation of the content of information associated with each thing. The word "information" itself comes form the Latin "in forma" meaning "in shape" and implies that "information" is what you need to know in order to put things into a proper shape. In fact, this activity in modern Italian (close relative of ancient Latin) is called "formare" equivalent of "forming", "shaping".

In the following we will focus on the role of information in defining shapes and specifically address the connection between information, entropy and energy, when the shape of a physical object changes. The paper is organized as follows: in the second section we introduce the concept of shape. In the third section we discuss the dynamical evolution of shapes and show how we can attribute entropy to different shapes. In the fourth section we discuss the relation between shapes, entropy and information. Finally, in the fifth section we discuss the relation between shape change and energy and draw some conclusions in view of the second principle of Thermodynamics.

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[1] Particle Physics has been digging into this problem for the last 50 years at least.

2 Shapes

In order to fix our ideas about shapes, we start with discussing a simple example. Let's consider an "object" made by 4 pink[2] squares that can be positioned in a 2x2 = 4 sites flat space box (see Fig. 1, left).

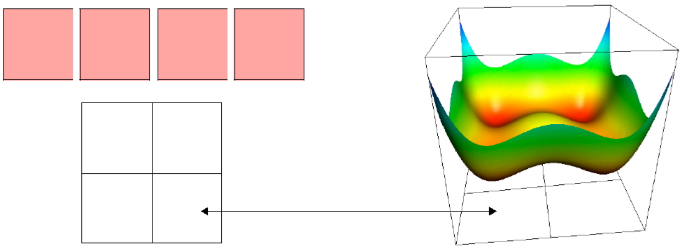

Fig. 1. Left: four squares can occupy up to four spots in a 2x2 box. Right: confining potential energy landscape with four minima sites each representing a dynamically stable equilibrium point for a material particle.

This "object" can be thought as a simple schematization of a physical system composed by four material particles that can be hosted in a confining potential energy landscape with four minima sites each representing a dynamically stable equilibrium point for the particle (Fig. 1, right).

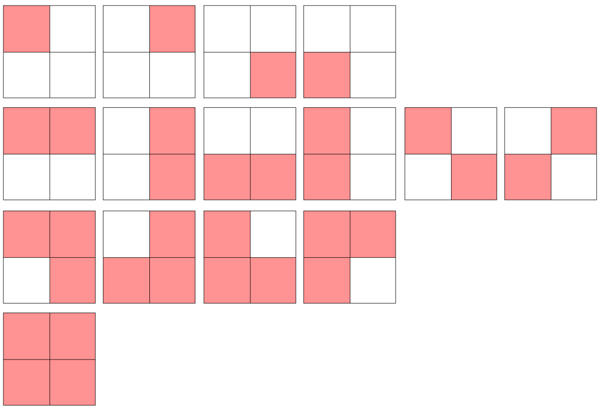

Fig. 2. Four squares can occupy up to four spots in a 2x2 box. In the first row from above we show the case in which the four squares occupy the same spot. The second row shows the possible shapes with the four squares occupying two spots. The third and fourth rows show respectively the case with three and four spots occupied.

Each particle can occupy equally well every potential minimum site and each potential minimum site can host up to four particles at once. The occupancy of a minimum site is signaled by a pink square on the corresponding 2x2 box (Fig. 1, left).

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[2] Grey in B/W version of the figure.

An object like this can show a number of different shapes represented by the different dispositions of 4 pink squares. Specifically, in Fig. 2 we display all possible shapes of our object starting from the first row with the simplest shape: four squares occupy the same spot. The second row presents all the possible shapes where two spots are occupied at the same time, while the third and fourth rows present respectively the case with three and four spots occupied. As is apparent each row comes with a different number of shapes. To summarize our example in Fig. 2: here each box presents a different shape. Each shape is realized by a specific disposition of the original squares. Clearly if we have four squares and throw all of them at random in the box we will end up with one of the 15 different shapes presented in Fig. 2. In this sense the 15 shapes cover all the possible shapes that we can realize with the four squares according to the given rule: *one or more squares can occupy the same site at the same time*.

As we have seen each shape is realized by a specific disposition of the four squares. It is interesting to count how many different ways we have to realize a given shape, simply by changing the disposition of the squares in each site.

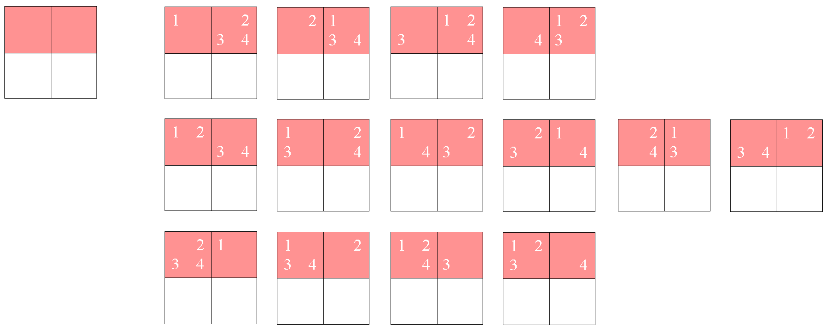
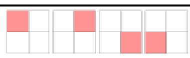
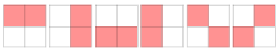
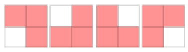
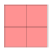


Fig. 3. The dispositions of four squares in two sites give place to 14 different configurations. All the configurations realize the same shape.

This problem is easily solved by combinatorial analysis, however for the sake of simplicity we present the explicit calculation for the case represented in Fig. 3. Here we have one of the 15 shapes and we explicitly show all the possible realization of this shape by distributing the four squares (numbered here form 1 to 4). These are 14 different configurations. It is important to note that, due to the fact that the squares are indistinguishable, each configuration realizes the very same shape. In Fig. 2 the shape in the first row represents an especially simple case. In this case there is only one configuration per shape because there is only one way to assemble the four squares into one site. In the table below we list the number of possible configurations associated with each shape, computed with the method that we just discussed.

As is apparent the shapes characterized by the same number of occupied sites have the same number of configurations. For sake of simplicity we call these group of shapes a "class". Thus, as an example, the different shapes s_{2a}, s_{2b}, s_{2c}, s_{2d} belong to class s_2 and all of them have the same number of configurations. Table 1 shows also the total number of configurations for each class. From the table is clear that class s_3 has the highest number of configurations.

Table 1. The table shows the different shapes and for each shape the number of configurations and the associated entropy. As is apparent the shapes characterized by the same number of occupied sites have the same number of configurations and thus the same entropy.

Shape number	Shape Picture	Num config. N_i	Num of total config. for the class $N_i\ n_i$	Shape Entropy log (N_i) in bit
S_{1a}, S_{1b}, S_{1c}, S_{1d}		1	4	log(1) = 0
S_{2a}, S_{2b}, S_{2c}, S_{2d}, S_{2e}, S_{2f}		14	84	log(14) = 3.81
S_{3a}, S_{3b}, S_{3c}, S_{3d}		36	144	log(36) = 5.17
S_{4a}		24	24	log(24) = 4.58

3 Shapes in motion

In a world where things are made by tiny particles at a certain temperature the shape of things changes spontaneously with time according to a diffusion process that is a manifestation of the second principle of Thermodynamics. A typical example would be the shape of airplane contrails (like the one in Fig. 4). In this case the initial positions of the particles of condensed water vapor generated by the exhaust of aircraft engines fit in a straight narrow line. As time passes however, the line gets smeared and eventually disappears.

Fig. 4. Airplane contrail. Note that the shape of the trail changes from left to right. In fact the rightmost part in the picture is older than the leftmost and time has operated in changing the shape. [3]

Another example is represented by the change in shape of ink drops in a water bowl. The drops initially confined within a small volume tend very soon to change shape and form filaments that expand until they fade away. Still another example is represented by the dramatic change of shape of buildings made of concrete during an earthquake. Initially the building shakes, oscillates and bends. Eventually it may crash by dividing into pieces of different sizes, spread in a volume much larger than the initial volume occupied by the building. All these cases are examples of physical phenomena whose main aspect, the change of shape, can be described by using the simple model introduced in Fig. 1. Here the shape change is produced by one or more

[3] Picture by Fir0002/Flagstaffotos distributed on Wikimedia Commons via GNU license (http://commons.wikimedia.org/wiki/Commons:GNU_Free_Documentation_License_1.2).

material particles that change site by crossing the potential barrier, under the action of some external force.

In order to gain some insight on how the shapes change one into another we can study the following problem. Let's suppose that we apply a random shaking to our system of particles. To visualize the phenomenon we can think of marbles in an egg carton that sits on a table during an earthquake. We are interested to learn how one specific shape changes into another specific shape. In practice, the shaking being random, the changes will also show some randomness. A proper treatment of this kind of problems requires a stochastic dynamic approach. This can be done following Langevin[1] approach, where a proper equation of motion for each marble is written and statistically solved in the presence of a stochastic force with known statistical properties or, equivalently, with a Fokker-Plank[1] approach where the probability density function of the outcomes is directly addressed via one or more partial differential equations. However, even without embarking on such a detailed analysis we show that some important results can be established by introducing a physical quantity called "shape entropy". In the next section we will show that, based on the sole knowledge of this quantity, we can make predictions on the shape change attitudes of our system. For the moment let's do some experiments with our shaking system by selecting a specific shape to start with and waiting some defined amount of time before checking what has become of it.

After some experiments we learned the following things: 1) the outcome of the experiment shows a random character and we can extract useful knowledge by taking average and using statistics. Specifically as outcome of the experiments we computed the statistical distribution of different shapes obtained after waiting some time. The distribution counts how many times we end up with s_1, s_2, s_3 and s_4. 2) No matter what is the initial shape that we select, after some time we end up always with the same kind of distribution. 3) The time that we need to wait is a function of how strongly we shake: the strongest the shorter. However, in the long run we always end up with the same distribution.

In Fig. 5 we plot the result of our "experiment": the distribution of shapes expressed as frequency (number of times that specific shape appears divided by the number of attempts) for 1000 attempts. We did our "experiment" by digitally simulating the dynamics of the marbles, so technically this is not an experiment with real marbles.

By looking at the frequency distribution we can promptly deduce that if we start from s_1 very easily we will end up changing our shape class. In fact the probability (frequency) that we remain with the same class is below 1.5%, while if we start from s_3 we have a fairly large probability (approx 56%) of remaining in the same class. If we focus our attention on the single shape instead, we could observe that the probability of ending up in a shape s_{ij} is

$$P_{ij} = P_i / n_i \qquad (\mathbf{1})$$

With i=1…4 being the class index and j=a,b,c,.. being the shape index (as in Table 1) and where n_i is the integer that represent the number of different shapes in that class: n_1=4, n_2=6, n_3=4, n_4=1. P_i is the probability of ending up in the class i:

$$P_i = N_i\, n_i / N \qquad (\mathbf{2})$$

Where N_i (see Table 1) is the number of different configuration in each shape belonging to the class i (all the shapes in a class have the same number of possible configurations) and N is the total number of possible configurations:

$$N = N_1 n_1 + N_2 n_2 + N_3 n_3 + N_4 n_4 \qquad \textbf{(3)}$$

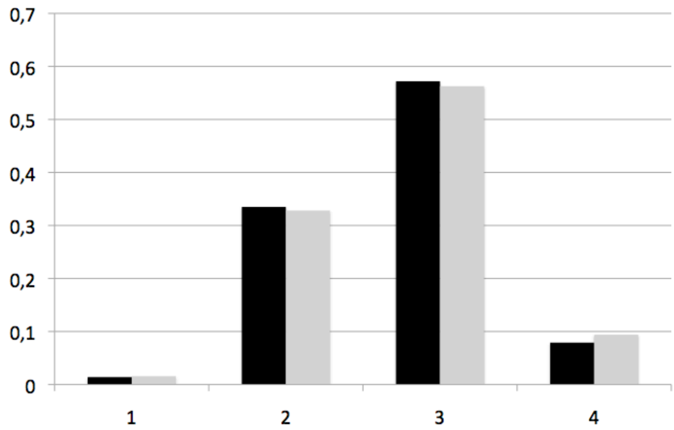


Fig. 5. Frequency of a class (number of times a shape class appeared divided by the number of attempts) versus the class number. The back bars represent the result of the digital simulation of shape changes (over 1000 attempts) while the grey bars represent the theoretical prediction P_i in (2).

P_i is plotted in Fig. 5 (grey bars) in good agreement with the digital simulations (black bars). As anticipated, we are now in position to introduce a quantitative measure of the tendency of one shape to change into another. We call "shape entropy"[4] the quantity

$$S_{ij} = k \log(N_i), \qquad \textbf{(4)}$$

Where the base of the logarithmic function is immaterial being k an arbitrary multiplicative constant. By using the shape entropy just introduced we can enunciate the following prediction for the shape dynamics: *the larger the shape entropy variation between the final and the initial shape is, the larger is the probability for the change to occur spontaneously*.

[4] This is not the first attempt to use the notion of entropy associated with shapes. In [2] "shape entropy" is based on the random variable "curvature" of 2D images and in [3] includes 3D meshes. In [4] a kind of entropy is computed using the random features of ellipsoidal grains. In [5] a cumulative distribution of a random variable is used to define the information content and applied to image alignment and shape classification[6]. The role of thermodynamics entropy is stabilizing the shape of Mithocondria is briefly mentioned in [7].

4 Shapes, Entropy and Information

In the previous section we have seen that if we let the shape associated with our marbles system evolve under a random force, it will most likely evolve toward certain shapes compared to others. This tendency is quantified by the entropy difference.

If our system can be considered at thermal equilibrium at a certain temperature T[5] then the random shaking will be provided for free by its thermal energy and its dynamical evolution will be subjected to the second law of thermodynamics that requires that the system evolves toward the maximization of its thermodynamical entropy. This last quantity, initially introduced by Clausius, has received a microscopic explanation by Boltzmann and Gibbs in terms of the number of configurations accessible to a physical state under certain constraints. Specifically in the Boltzmann formulation the entropy is expressed by:

$$S_B = k_B \log(\Omega), \tag{5}$$

Where Ω is the number of the equally probable microstates accessible by an isolated system (microcanonical ensamble) corresponding to a macrostate with given energy and K_B is the Boltzmann constant. As it is apparent, apart from scale factors associated with the value of the constant pre-factor, the shape entropy that we have introduced in (4) coincides with the Boltzmann entropy provided we interpret the number of configurations N_i for a given shape as the number of accessible microstates for a given macrostate of the thermodynamical system.

It is interesting to note that there does exist a close connection with information theory and specifically with the quantity of information that we can associate with a shape. Let's consider for a moment the expression for the microscopic entropy of a physical system in the form introduced by Gibbs. This form is required when the probabilities of occurrence for the microstates are not necessarily the same for each microstate:

$$S_G = -K_B \sum_l p_l \log(p_l) \tag{6}$$

Where p_l is the probability of the microstate of index l and the sum is taken over all the microstates. This expression reduces easily to (5) when all the p_l are equal. Equation (6) is formally equal to the expression introduced in 1948 by Shannon to quantify the information content of a given message chosen from a set of all possible messages. In this case p_l represents the probability of receiving the message with index l and K_B is a constant that fixes the units.

Based on this analogy we now compute the quantity of information a la Shannon[6] for the different shapes in our experiment. In order to do that. we associate an information content to a shape by introducing a binary representation for each shape. By the moment that each shape is characterized by the disposition of four squares (marbles) in four sites, it is clear that we need 2 bits per marble according to the coding system in Fig. 6.

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[5] In practice it has to be in thermal contact with a large heat reservoir at temperature T.

[6] Generalizations of the Shannon formulation are represented by the Rényi and Hartley entropy.

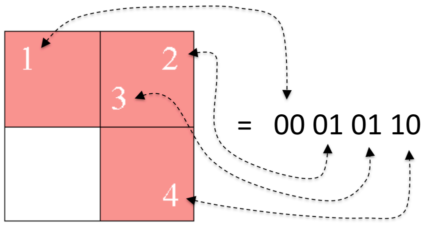

Fig. 6. Information coding for shape s_{3a}

Now, the question we want to answer is the following: how much information is contained in a shape represented by a certain sequence of binary digits? The answer is promptly obtained by applying (6) in the Shannon interpretation. As we have seen the probability of the sequence representing shapes belonging to the same class is the same and, specifically is given by $p_i = 1/N_i$, thus the shape information S_i is given by:

$$S_i = -K\sum_{1}^{N_i} p_i \log(p_i) = -K\ N_i \frac{1}{N_i}\log\left(\frac{1}{N_i}\right) = K\ \log(N_i) \tag{7}$$

This is the same quantity that we have called shape entropy in (4) and thus we can interpret the shape entropy as a measure of the information content of a given shape. It is interesting to note that the amount of information differs from class to class. The maximum information is embedded into a random shape, meaning with this a shape that has the same probability to be realized and whose class is populated by the whole configuration space. For this shape the amount of information is:

$$S_{\max} = -K\sum_{1}^{256} \frac{1}{256}\log\left(\frac{1}{256}\right) = 8 \tag{8}$$

Where we have assumed K=1 and the base 2 for the log function to get the information measured in bit units. The shape information in s_3 is just ln(36)=5.17 bits; the difference being a measure of the redundancy of the coding that we selected for the shape representation[7].

5 Shape is physical

In the second section we have introduced our idea of shape of an object by using four-marbles-in-an-egg-carton toy model. In the third section we have discussed the shape dynamics under the action of a random force. We have introduced the "shape entropy", i.e. a quantitative measure of the tendency of a shape to change into another spontaneously. In the fourth section we have shown that the shape entropy is

[7] The redundancy is a measure of the compressibility of the representation. If we want to achieve a loss-less compression we can reduce the number of bits used up to 6 bits for s_3 being 6 the lowest integer greater or equal to log(36).

equivalent to the Boltzmann entropy for a physical system at thermal equilibrium at a certain temperature and coincides with the amount of information that we can attribute to the shape according to the Shannon definition. We are now in position to discuss the energetic implications of the shape change. Specifically we would like to announce a special version of the second principle of the thermodynamic in the form:

No process is possible whose sole result is the change of shape of a physical system from a shape of larger shape entropy to a shape of smaller shape entropy.

The present phrasing of this principle is inspired by the original formulation given by Clausius[8] in 1865 and the "impossibility" has to be intended here in a probabilistic sense. In fact, in a macroscopic physical system the number of configurations is very large (of the order of the Avogadro number) and the probabilistic character of the second principle is less evident, however in a system with few configurations like the case we have treated here (256 possible configurations overall) also shapes with lesser probability can be easily observed in a shape change process.

The gist of this formulation of the second principle is that, much as shown by Bennet[10] and Landauer[11] for the information processing in the sixties, also shape (being a physical representation of the information embedded into the object) plays a physical role. An interesting consequence of that is that the shape change can affect the energy budget of the transformation: for an isolated system a shape change that implies an increase of shape entropy comes necessarily at expenses of a decrease of the free energy available. On the other hand if we want to perform a shape change that implies a net shape entropy change of ΔS>0 bits, this requires a minimum of

$$Q = K_B T \ln(2) \ \Delta S \tag{9}$$

energy to be dissipated during the transformation[12].

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[8] From [9]: *Es gibt keine Zustandsänderung, deren einziges Ergebnis die Übertragung von Wärme von einem Körper niederer auf einen Körper höherer Temperatur ist.*

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