Holes, Knots and Shapes: A Spatial Ontology of a Puzzle

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Abstract. In this paper we propose a spatial ontology for reasoning about holes, rigid objects and strings, taking a classical puzzle as a motivating example. In this ontology the domain is composed of spatial regions whereby a theory about holes is defined over a mereological basis. We also assume primitives for representing shapes of objects (including the string). From these primitives we propose a sufficient condition for object's penetrability through holes. Additionally, a string is represented as a data structure defined upon a sequence of sections limited by points where the string crosses itself or points where it passes through a hole. This paper first appeared as as technical report [7] and presents the initial framework that was further developed in [8, 2].

1 Introduction

Real life situations where we must deal with strings tying objects and passing through holes appear from time to time in very different contexts. Examples range from tying shoelaces, to handling ropes in a sailboat or organising the cable connections map inside an office, a building or a whole city. Although humans show an amazing intuition for solving problems of this nature, a formal representation for reasoning about holes and strings is still a relatively unexplored area. To understand the problem, note for instance that using a fully detailed mathematical model of the involved objects does not seem feasible for computational purposes, let alone when we consider deformable objects like a string. Moreover, humans typically describe solutions to spatial reasoning problems in terms of *qualitative* descriptions instead. This is, in fact, the orientation followed by *Qualitative Spatial Reasoning* (QSR) [9], a field that attempts the logical formalisation of spatial knowledge based on primitive relations defined over elementary spatial entities.

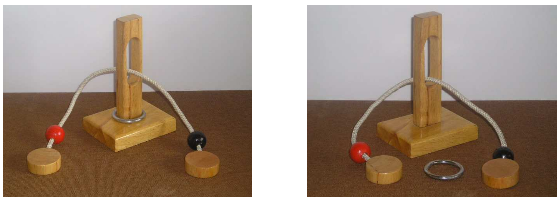
To obtain a suitable representation for strings and holes we have adopted the following methodology. We begin from specific formalisations of particular scenarios, which usually imply a more abstract and simplified description level, and advance them towards more general representations to cover different domains, implying a more fine-grained ontology. As a starting point, puzzle-like examples constitute a good test bed, as they offer a small number of objects while keeping enough complexity for a challenging problem of KR.

Following this line, we take as a starting point the work developed in [1], which presented an automated solution to a classical string puzzle called *Fisherman's Folly* (see Figure 1). To solve the problem, the authors applied several strong assumptions like identifying a single-holed object with its unique hole, assuming that each hole always has two entry boundaries or ignoring that strings may form knots. The treatment of knots, for instance, is unnecessary for the final solution, but it may be reasonably objected that there is no direct justification for discarding knots from the very beginning, or that a slight change in the puzzle goal could easily require handling knots.

In this paper we go one step further and remove these assumptions to propose a more general ontology applicable to other scenarios. On the one hand, we describe a theory about holes defined over a mereological basis, proposing a sufficient condition for object's penetrability through holes. In this way, we can derive information of which objects can pass through a given hole, something that was taken as given in [1].

2 The Fisherman's Folly

The elements of the puzzle are a holed post (P) fixed to a wooden base (B), a string (Str), a ring (R), a pair of spheres (S_1, S_2) and a pair of disks (D_1, D_2). The spheres can be moved along the string, whereas the disks are fixed at each string endpoint. The string passes through the post's hole in a way that one sphere and one disk remain on each side of the post. It is worth pointing out that the spheres are larger than the post's hole, therefore the string cannot be separated from the post without cutting either the post, or the string, or destroying one of the spheres. The disks and the ring, in contrast, can pass through the post's hole. In this work we assume that neither the length nor the thickness of the string constrain any solution to the puzzle, i.e. the string is infinitely extensible and one-dimensional. Relaxing these assumptions is a matter for future work.



(a) Initial (b) Goal

Fig. 1. A spatial puzzle: the Fisherman's Folly.

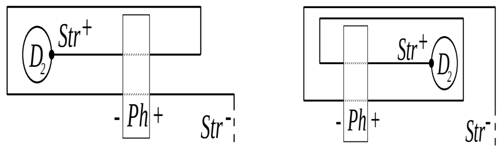
In the initial state (shown in Figure 1(a)) the post is in the middle of the ring, which in its turn is supported on the post's base. The goal of this puzzle is to find a sequence of transformations that, while maintaining the physical integrity of the domain objects, allow us to free the ring from the rest of objects, regardless their final configuration. Figure 1(b) shows one possible goal state. As we shall see, the representation for initial and (one possible) goal states is shown, respectively, on line 0 and line 5 of Figure 2.

A crucial observation is that the puzzle actually deals with four holes: the post hole (Ph), the ring hole Rh and the two sphere holes Sh_1 and Sh_2. Note that in a natural language description we would probably identify holes with their host objects, saying that "the string passes through the sphere" (hole) or that "the post passes through the ring" (hole). Furthermore, we would talk about "sliding the ring upwards the post," rather than "moving the post downwards the ring hole."

In [1] the puzzle entities were classified into three different sorts: *long objects*, *regular objects* and *holes*, corresponding in the puzzle to the sets $\{P, Str\}$, $\{R, S_1, S_2, D_1, D_2, B\}$ and $\{Ph, Rh, Sh_1, Sh_2\}$, respectively. A distinguishing feature of a long object x is that we usually identify the two opposite extremities of its major axis. These extremities, denoted as x^- and x^+, respectively receive the names of *negative terminal* and *positive terminal* of x. As a thumb rule, when not stated, we assume in all figures that rightmost or topmost extremities are positive, whereas leftmost or bottom are negative (where the left-right dichotomy dominates the top-bottom one to solve any ambiguity). To put an example of this notation, the right disk D_2 is linked to Str^+, while the post base B is linked to P^-.

A second important observation is that a long object may be simultaneously crossing several holes. In fact, although the post just crosses Rh, when executing the puzzle's solution, the string may be simultaneously passing through all the holes and, moreover, it may cross the same hole several times. Thus, we associate to each long object x a list $Chain(x)$ collecting the sequence of all hole crossings made by x following, for instance, the arbitrary ordering from x^- to x^+. Furthermore, the *direction* in which the string crosses the hole is also relevant in order to provide an unambiguous description of distinct puzzle states. To understand why, just consider the partial configuration represented in Figure 2(a). If we represent this situation using $Chain(Str) = [\ldots, Ph, Ph]$, then it would not be possible to distinguish it from the state shown in Figure 2(b), which clearly represents a substantially different situation: the disk D_2 is now to the *right* (or positive side) of the post hole Ph. Therefore, in an analogous way to long object terminals, we also denote two *poles*[1], h^- and h^+ per each hole h in the puzzle, considering that in this case holes have only two entry boundaries. Thus, we can describe $Chain(x)$ as a list of (outgoing) hole poles – those through which x *exits* when going from x^- to x^+. As a result, $Chain(Str) = [\ldots, Ph^-, Ph^-]$ would represent Figure 2(a) whereas $Chain(Str) = [\ldots, Ph^+, Ph^+]$ would correspond to the crossings in Figure 2(b).

[1] We follow the same thumb rule criterion of identifying right or top as positive.



(a) (b)

Fig. 2. Two distinct puzzle states.

Under this setting, the solution deals with two elementary actions: an action $PassOb(t, p)$ for passing an object terminal t towards a hole pole p, and an action $PassH(h, p)$ for passing the (object containing) hole h towards the hole pole p. For brevity sake, we omit here the detailed effects of these actions (see [1]), although they can be guessed from the description of the puzzle's solution in Figure 3. In that figure, each state is identified by its sequence number plus the pair of lists $Chain(P)$ and $Chain(Str)$ in this order. The performed actions in each transition are interlaced between each state i and the next one $i + 1$.

$$
\begin{array}{l}
0 : [Rh^+],\ [Sh_1^+, Ph^+, Sh_2^+] \\
\qquad\qquad\qquad PassOb(Str^+, Ph^-) \\
1 : [Rh^+],\ [Sh_1^+, Ph^+, Sh_2^+, Ph^-] \\
\qquad\qquad\qquad PassOb(P^+, R^-)\ \&\ PassH(Ph, R^-) \\
2 : [\,],\ [Sh_1^+, Rh^-, Ph^+, Rh^+, Sh_2^+, Rh^-, Ph^-, Rh^+] \\
\qquad\qquad\qquad PassH(Sh_2, Rh^-) \\
3 : [\,],\ [Sh_1^+, Rh^-, Ph^+, Sh_2^+, Ph^-, Rh^+] \\
\qquad\qquad\qquad PassH(Rh, Ph^+) \\
4 : [\,],\ [Sh_1^+, Ph^+, Rh^-, Sh_2^+, Rh^+, Ph^-] \\
\qquad\qquad\qquad PassH(Sh_2, Rh^+) \\
5 : [\,],\ [Sh_1^+, Ph^+, Sh_2^+, Ph^-]
\end{array}
$$

Fig. 3. A formal solution for the Fisherman's puzzle.

Note that State 5 has actually reached the goal since, at this point, the ring hole Rh does not occur in any list, i.e., it is not crossed by any long object. Rather than the particular mechanics of the puzzle, our main concern in this work is to analyse in more detail the spatial knowledge representation used to obtain this solution. For instance, it must be noticed that [1] used additional information to constrain the possible actions to be performed, including a predicate $CannotPass(x, h)$ to describe when an object x cannot pass through a hole h. The information for this predicate was assumed as given and had the form of an explicit list of ground atoms. We claim that, with a suitable representation for holes and objects, this predicate should be derived. Furthermore, it should also account for *groups* of objects rather than for a single object x. To understand why, note that if we just consider that the sphere S_2 can pass through the ring hole Rh, the action $PassOb(Sh_2, Rh^-)$ (that is, moving S_2 down the ring hole) could be performed in the initial situation. But this movement is physically *impossible*, since there would be a moment in which the post P and the sphere S_2 would cross Rh, and both objects *altogether* cannot pass through the ring.

3 A theory about holes

In this section we follow the guidelines proposed in [11, 3] and construct a basic ontology about holes using mereological relations.

The domain objects in this work are identified with their occupancy regions. Holes are defined as the spatial region that is part of the portion of an object's complement that lies inside that object's occupancy region. We name this object the *host* of a hole.

There are at least three distinct types of holes: *cavities*, i.e. holes that are entirely hidden inside their hosts; *hollows*, which are superficial depressions on the host; and, perforating holes (or tunnels), which are holes that have at least two distinct entrance boundaries. In this paper we shall deal only with perforating holes, since only these are relevant to the puzzle's solutions[2].

In the formalisation described below, holes are assumed as open regions whose boundaries belong to their host objects. The relationship between holes and their hosts is formalised using the elementary relation: $H(h, x)$, meaning "h is a hole in the object x" (conversely, "x is the host of h") [3]. For example, it is a fact about the puzzle domain described above that $H(Rh, R)$.

As we assumed that the space is only populated by spatial regions, apart from the relation $H/2$, it is convenient to include in the basic theory about holes a set of mereological relations accounting for the degree of connectedness between regions. In this work we assume RCC-8 ([5]) which is a first-order axiomatisation of spatial relations based on a dyadic primitive relation of *connectivity* ($C/2$) between two regions. Informally, assuming two regions x and y, the relation $C(x, y)$, read as *"x is connected with y"*, is true if and only if the *closures* of x and y have a point in common. Assuming the $C/2$ relation as primitive, and that x, y and z are variables for spatial regions, the following mereological relations can be defined: $DC(x, y)$, which stands for "x is disconnected from y"; $EQ(x, y)$, for "x is equal to y"; $PO(x, y)$, for "x partially overlaps y"; $EC(x, y)$, for " the closure of x and y are externally connected"; $TPP(x, y)$, for "x is a tangential proper part of y"; $NTPP(x, y)$, for "x is a non-tangential proper part of y"; and, $TPPi/2$ and $NTPPi/2$ are the inverse relations of $TPP/2$ and $NTPP/2$ respectively.

Assuming RCC, the relation $H(h, x)$ can be constrained by the axioms (1) and (2) below. Axiom (1) guarantees that the host of a hole is not itself a hole; whereas Axioms (2) states that the hole and it's host object are externally

[2] Therefore, in the remainder of this paper we will use the words: *tunnels*, *perforating holes* and *holes* interchangeably.

connected.

$$H(h, x) \rightarrow \neg H(x, y) \tag{1}$$
$$H(h, x) \rightarrow EC(h, x) \tag{2}$$

Moreover, Axiom 1 implies that the relation H is irreflexive (meaning that no hole hosts itself) and anti-symmetric (i.e., the host cannot be a hole of its hole).

An essential characteristic of holes is that they can be interpenetrated by other objects. Therefore, the hole ontology has to include relations about the relative location of a hole wrt the penetrating object. In a world uniquely populated by spatial regions, relative location can be expressed by mereological relations. In order to define relative location wrt a hole, we need the concept of a hole *entry boundary* (EB) that is defined in [3] by the relation $EB(h_i, h, x)$, read as "h_i is the maximally connected part of the hole h (fiat) boundary that is nowhere a boundary of the host x." If a hole h has n entry boundaries, we denote them as h_i with $1 \leq i \leq n$ (as we deal with tunnels, $n \geq 2$).

We can now express the following relations wrt an object x and a hole h:

- x is *wholly outside* h ($WOut(x, h)$) iff $DC(x, h)$;
- x is *just outside* h wrt the hole entry boundary h_i ($JOut(x, h, h_i)$) iff

$$\exists y(H(h, y) \wedge EB(h_i, h, y)) \wedge EC(x, h_i) \wedge \neg TPP(x, h);$$

- x is *partially outside* h wrt the EB h_i ($POut(x, h, h_i)$) iff

$$\exists y(H(h, y) \wedge EB(h_i, h, y)) \wedge PO(x, h) \wedge \neg PO(x, y);$$

- x is *just inside* h wrt the EB h_i ($JIn(x, h, h_i)$) iff

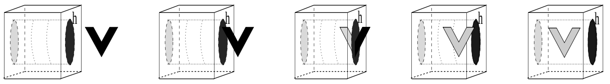
$$\exists y(H(h, y) \wedge EB(h_i, h, y)) \wedge EC(x, h_i) \wedge TPP(x, h).$$

- x is *wholly inside* h ($WIn(x, h)$) iff

$$TPP(x, h) \vee NTPP(x, h)) \wedge \neg \exists h_i JIn(x, h, h_i).$$

$WOut$, $JOut$, $POut$, WIn and JIn are schematised in Figure 4, where the host object is the cuboid, the hole is the cylindrical figure inside the cuboid and the penetrating object is the **v**-shaped figure.

It is worth pointing out that, in contrast to [3], encoding the relative location of an object wrt a hole using RCC relations allowed us to include both $JOut$ and JIn into the same formalism since RCC is defined over the *closure* of regions. Therefore, the concepts of just inside and just outside can coexist with the initial assumption of holes as open regions. Another difference between the formalism presented above wrt that proposed in [3] is the inclusion of the hole entry boundary in the definitions of $JOut$, $POut$ and JIn, in order to account for the action of an object passing through a particular hole entry.



(a) WOut (b) JOut (c) POut (d) JIn (e) WIn

Fig. 4. Relative location of an object v wrt a hole h

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Figure 4 can be understood as a sequence of continuous transitions from the relation *wholly outside* to *wholly inside*. In order to provide a formal solution to the Fisherman's Folly, however, we need to be able to locate an object in space that is $WOut$, with respect to every hole, but that is near a particular entry boundary of a tunnel. In effect, tunnels are important qualitative landmarks that could be used as local reference frames. This idea is developed in the next section.

3.1 Hole subspaces

It is not unusual in the common language to characterise sections of a road by the sections *before* and *after* a tunnel. In a domestic domain, we decide where to locate (non-wireless) electronic objects according to the nearby plugs (which are, in fact, tunnel entry boundaries). The issue of reasoning about tunnels becomes quite critical when the problem is to locate buried infrastructure so that repairs can be conducted on a particular network of pipes and cables underground. However, to the best of our knowledge, there are no references that account to the potential use of hole entry boundaries as local reference frames. This section describes an initial attempt to cope with this issue.

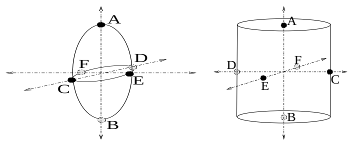
Following the previously introduced notation, each entry boundary is uniquely identified by a symbol referring to its host hole plus a subscript number. If a global reference frame is assumed in the domain, the entry boundaries can be identified by the 3D Cartesian coordinates of their respective centre points; thus, the local reference provided by the entry boundaries could be associated to a global reference frame. In this work, however, objects are only located with respect to the near neighbourhoods of hole entry boundaries. An object x is in the near neighbourhood of a hole EB h_i iff x is just outside h_i or there is either another object y connected to x or a hole hx in x connected to y and y is just outside h_i or partially outside h_i. More formally: $NN(x, h_i)$ iff $JOut(x, h, hi) \lor \exists\, y\, hx\ (H(hx, x) \land (C(x, y) \lor C(hx, y)) \land (JOut(y, h, h_i) \lor POut(y, h, h_i)))$. Therefore, we can say that (in the situation of Figure 1(a)) the left sphere, left disk and a part of the string are in the NN of Ph^-. It is worth noting also that, for an object x and a hole h, $NN(x, h^-)$ and $NN(x, h^+)$ are not inconsistent as there are feasible situations where a hole has two entry boundaries closer to each other. In effect this is equivalent to describing the position of an object wrt various distinct local reference frames.

We are now capable of expressing formally that an object is near a tunnel (e.g., a car is parked outside the Eurotunnel entrance) or that objects are related to a complex arrangement of objects and holes (which is the case of the puzzle in question). However, in order to account for the main issues involved in the Fisherman's Folly, the theory has to include some basic ideas about object's shape so that it is capable of expressing object's penetrability through holes. The next section discusses some insights about this issue.

4 The shapes of objects

Representing and reasoning about objects' shape are, at the same time, the most elusive and the most important issues in reasoning about the common sense space [4]. In this paper we cannot escape from taking into account objects' shape, since the solution of puzzles such as that shown in Figure 1 involves passing an object of a particular shape and size, through a hole entry boundary, also of a particular shape and size. This section presents some primitives to account for the shapes of rigid objects and strings.

We assume in this work ellipsoids and elliptic cylinders (Figures 5(a) and 5(b), respectively) as the basic primitives to describe the shapes of rigid objects.



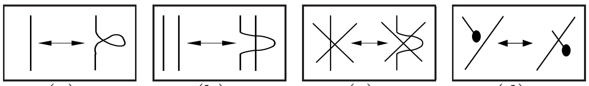
(a) (b)

Fig. 5. Base shape primitive.

An ellipsoid (Figure 5(a)) is a 3D figure which every planar cross section is an ellipsis. This figure has three symmetry axes: $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ (cf. Figure 5(a)) that are called, respectively, *major*, *mean* and *minor* axes. Thus, the spheres in our puzzle have ellipsoid shapes whose three symmetry axes are of the same length. The post is ellipsoid shaped, where its major axis is much greater than both its mean and minor axes.

In an analogous way, we use an elliptic cylinder (Figure 5(b)) to account for the shapes of the puzzle objects not represented by the ellipsoid. An elliptic cylinder is a cylinder whose base is an ellipsis. This figure also has a *major*, *mean* and *minor* axes (respectively axes $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ in Figure 5(b)). Thus, the shape of the puzzle's disks and ring can be approximated to cylinders whose axis $\overline{AB}$ is much smaller than $\overline{CD}$ and $\overline{EF}$, and the last two are of equal length. The shape of the post base can also be approximated to a cylinder.

The string's shape has an extra complication which is related to this object's intrinsic flexibility. Consequently, every different shape resulting from non-destructive deformations of a string is also a shape of the string. In order to cope with this issue we define the string's shape as that of an elongated elliptical cylinder, where $\overline{AB}$ is much greater than $\overline{CD}$ and $\overline{EF}$, or the shape resulting from the application of any sequence of the transformations depicted in Figure 6 on such elongated elliptical cylinder.



(a) (b) (c) (d)

Fig. 6. The Reidemeister moves and the cross move.

Figure 6 shows the three Reidemeister moves [6] (Figures 6(a), 6(b) and 6(c)) and the cross move [10] (Figure 6(d)), that can be described as follows: Reidemeister move I (Figure 6(a)) adds or deletes a simple twist in the string; Reidemeister move II (Figure 6(b)) allows the inclusion (or exclusion) of two crossings in the string; Reidemeister move III (Figure 6(c)) slides a strand of the string from one side of a crossing to the other; the cross move (Figure 6(d)) is defined on simple open curves and adds or removes a string crossing by sliding an open end of it over a continuous part of the string.

In the next section we use the shape primitives above to propose a sufficient condition for an object to pass through an entry boundary.

4.1 Passing an object through a hole

We first assume that every object is conducted through a hole in the direction of the largest semi-line connecting any two points of its boundary, this semi-line we call *conducting line*. Thus, the post is always conducted through the ring hole in the direction of its major axis; similarly, the disks are conducted through the post hole via their diameters (i.e., via their mean or minor axes).

Let's define the region of the orthogonal projection of an object o (taken through the object's conducting line) as $pl(o)$ and the region defined by the orthogonal projection of a hole entry boundary h_i as $p(h_i)$. Now we say that the object can pass through the hole if it is possible to superimpose $pl(o)$ and $p(h_i)$ so that

$$TPP(pl(o), p(h_i)) \vee NTPP(pl(o), p(h_i)) \vee EQ(pl(o), p(h_i)). \tag{3}$$

This condition can be extended for the case of a group of objects passing through a hole by simply considering, instead of the object o in (3), the convex hull of the group of objects.

Now that we have a way of checking whether a particular object can pass through a determined hole, the next section defines a suitable representation for expressing the various states of the puzzle.

5 Concluding remarks

In this work we investigated knowledge representation issues regarding the spatial aspects of a puzzle. The puzzle chosen is called Fisherman's Folly and is constituted by an arrangement of rigid objects and non-trivial elements such as holes and a string. The goal of this paper is to define the basic elements of an ontology about rigid, flexible and holed object, therefore we leave for future work the problem of representing actions and change in this domain as well as the investigation of the possible consequences of the resulting ontology.

Acknowledgements

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