Many-valued Temporal Weighted Knowledge Bases with Typicality for Explainability

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Abstract

In this paper, we develop a many-valued semantics for the description logic $LTL_{\mathcal{ALC}}$, a temporal extension of description logic $\mathcal{ALC}$, based on Linear-time Temporal Logic (LTL). We add a typicality operator to represent defeasible properties, and discuss the use of the (many-valued) temporal conditional logic and of weighted KBs for explaining the dynamic behaviour of a network.

Keywords

Preferential Logics, Temporal Logics, Many-valued Description Logics, Explainability

1. Introduction

Preferential extensions of Description Logics (DLs) allow reasoning with exceptions through the identification of *prototypical properties* of individuals or classes of individuals. *Defeasible inclusions* are allowed in the knowledge base, to model typical, defeasible, non-strict properties of individuals. Their semantics extends DL semantics with a preference relation among domain individuals, along the lines of the preferential semantics introduced by Kraus, Lehmann and Magidor [1, 2] (KLM for short). Preferential extensions and rational extensions of the description logic $\mathcal{ALC}$ [3] have been studied [4, 5, 6], and several different closure constructions have been developed [7, 8, 9, 10, 11, 12], inspired by Lehmann and Magidor's rational closure [2] and Lehmann's lexicographic closure [13]. More recently, *multi-preferential* extensions of DLs have been developed, by allowing multiple preference relations with respect to different concepts [14, 15, 16], as the semantic for ranked and weighted knowledge bases with typicality.

LTL extensions of Description Logics are very well-studied in DLs literature, and we refer to [17, 18] for surveys on temporal DLs and their complexity and decidability. While preferential extensions of LTL with defeasible temporal operators have been recently studied [19, 20, 21] to enrich temporal formalisms with non-monotonic reasoning features, a preferential extension of a temporal DL has been proposed in [22], based on the approach proposed in [5] to define a description logic with typicality. More specifically, in [22] we build over a temporal extension of $\mathcal{ALC}$, $LTL_{\mathcal{ALC}}$ [17], based on Linear Time Temporal Logic (LTL), to develop a temporal $\mathcal{ALC}$ with typicality, $LTL^{\mathbf{T}}_{\mathcal{ALC}}$. Generalizing the approach in [5], a typicality operator $\mathbf{T}$ (that selects the most typical instances of a concept) is added to $LTL_{\mathcal{ALC}}$ to represent temporal properties of concepts which admit exceptions.

It is proven that the preferential extension of $LTL^{\mathbf{T}}_{\mathcal{ALC}}$ can be polynomially encoded into $LTL_{\mathcal{ALC}}$, and this approach allows borrowing decidability and complexity results from $LTL_{\mathcal{ALC}}$. A similar encoding

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can be given for a multi-preferential extension of $LTL^{\mathbf{T}}_{\mathcal{ALC}}$, by allowing a concept-wise preferential semantics, where different preferences are associated to different concepts.

In this paper, we aim at developing a many-valued extension of $LTL_{\mathcal{ALC}}$ with typicality, which makes it possible to represent a concept inclusions such as

$$\exists lives\_in.Town \sqcap Young \sqsubseteq \mathbf{T}(\Diamond Granted\_Loan),$$

(meaning that who lives in town and is young, normally is eventually granted a loan), where the interpretation of some concepts (e.g., $Young$) may be non-crisp.

In the paper we first recall fuzzy extensions of $\mathcal{ALC}$ and temporal extensions of $\mathcal{ALC}$. Then, we develop a many-valued extension of $LTL_{\mathcal{ALC}}$, by building on many-valued DLs, which are widely studied in the literature, both for the fuzzy case [23, 24, 25, 26, 27] and for the finitely-valued case [28, 29, 30, 31]. Then we add a typicality operator to the language of the many-valued $LTL_{\mathcal{ALC}}$, to get a many-valued temporal extension of $\mathcal{ALC}$ with typicality.

We discuss extensions of the closure constructions for weighted knowledge bases with typicality [15, 32, 33] to the temporal case. This allows for a finer grained representation of the plausibility of prototypical properties of a concept, including temporal properties, by assigning weights to the different typicality properties. We discuss how the preferential temporal logic can be used to provide a logical interpretation of the transient behaviour of (recurrent) neural networks.

2. Fuzzy $\mathcal{ALC}$

Fuzzy description logics have been widely studied in the literature for representing vagueness in DLs [23, 24, 25, 26, 27], based on the idea that concepts and roles can be interpreted as fuzzy sets. Formulas in Mathematical Fuzzy Logic [34] have a degree of truth in an interpretation rather than being true or false; similarly, axioms in a fuzzy DL have a degree of truth, usually in the interval $[0, 1]$. The finitely many-valued case is also well studied for DLs [28, 29, 30, 31]. We first recall the semantics of a fuzzy extension of $\mathcal{ALC}$, following [25]; then we will consider the finitely-valued case.

Let N_C be a set of concept names, N_R a set of role names and N_I a set of individual names. The set of $\mathcal{ALC}$ *concepts* (or, simply, concepts) can be defined inductively as follows:

(i) $A \in N_C$, $\top$ and $\bot$ are concepts;

(ii) if C and D are concepts, then $r \in N_R$, then $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$, $\exists r.C$ are concepts.

A *fuzzy interpretation* for $\mathcal{ALC}$ is a pair $I = \langle \Delta, \cdot^I \rangle$ where: Δ is a non-empty domain and $\cdot^I$ is *fuzzy interpretation function* that assigns to each concept name $A \in N_C$ a function $A^I : \Delta \rightarrow [0, 1]$, to each role name $r \in N_R$ a function $r^I : \Delta \times \Delta \rightarrow [0, 1]$, and to each individual name $a \in N_I$ an element $a^I \in \Delta$. A domain element $x \in \Delta$ belongs to the extension of A to some degree in $[0, 1]$, i.e., A^I is a fuzzy set.

The interpretation function $\cdot^I$ is extended to complex concepts as follows:

$\top^I(x) = 1$, $\quad\bot^I(x) = 0$,
$(\neg C)^I(x) = \ominus C^I(x)$,
$(C \sqcap D)^I(x) = C^I(x) \otimes D^I(x)$,
$(C \sqcup D)^I(x) = C^I(x) \oplus D^I(x)$,
$(\exists r.C)^I(x) = \sup_{y \in \Delta} \; r^I(x, y) \otimes C^I(y)$,
$(\forall r.C)^I(x) = \inf_{y \in \Delta} \; r^I(x, y) \rhd C^I(y)$,

where $x \in \Delta$, and $\otimes$, $\oplus$, $\rhd$ and $\ominus$ are arbitrary but fixed *t-norm*, *s-norm*, implication function, and negation function, chosen among the combination functions of some fuzzy logic. In particular, in Gödel logic $a \otimes b = min\{a, b\}$, $a \oplus b = max\{a, b\}$, $a \rhd b = 1$ *if* $a \leq b$ *and* b *otherwise*; $\ominus a = 1$ *if* $a = 0$ *and* 0 *otherwise*. In Łukasiewicz logic, $a \otimes b = max\{a + b - 1, 0\}$, $a \oplus b = min\{a + b, 1\}$, $a \rhd b = min\{1 - a + b, 1\}$ and $\ominus a = 1 - a$. Following [25], we will not commit to a specific choice of combination functions, A *fuzzy $\mathcal{ALC}$ knowledge base* K is a pair $(\mathcal{T}, \mathcal{A})$ where $\mathcal{T}$ is a fuzzy TBox and $\mathcal{A}$ a fuzzy ABox. A fuzzy TBox is a set of *fuzzy concept inclusions* of the form $C \sqsubseteq D\ \theta\ n$, where $C \sqsubseteq D$ is an $\mathcal{ALC}$ concept inclusion axiom, $\theta \in \{\geq, \leq, >, <\}$ and $n \in [0, 1]$. A fuzzy ABox $\mathcal{A}$ is a set of *fuzzy assertions* of the form $C(a)\theta n$ or $r(a, b)\theta n$, where C is an $\mathcal{ALC}$ concept, $r \in N_R$, $a, b \in N_I$, $\theta \in \{\geq, \leq, >, <\}$ and $n \in [0, 1]$. Following Bobillo and Straccia [27], we assume that fuzzy interpretations are *witnessed*, i.e., the sup and inf are attained at some point of the involved domain. The interpretation function $\cdot^I$ is also extended to axioms as follows:

$$(C \sqsubseteq D)^I = inf_{x\in\Delta^I} C^I(x) \rhd D^I(x) \qquad (C(a))^I = C^I(a^I)$$

Definition 1 (Satisfiability and entailment for $\mathcal{LC}_n$ knowledge bases). *Let $K = (\mathcal{T}, \mathcal{A})$ be a weighted $\mathcal{LC}_n$ knowledge base, and I be an interpretation. The satisfiability relation $\models$ is defined as follows:*

- $I \models C \sqsubseteq D\ \theta\alpha$ *if* $(C \sqsubseteq D)^I\ \theta\alpha$;
- $I \models C(a)\ \theta\alpha$ *if* $C^I(a^I)\ \theta\alpha$;
- $I \models r(a, b)\ \theta\ n$ *if* $r^I(a^I, b^I)\theta\ n$.
- *for a set S of axioms,* $I \models S$ *if* $I \models E$ *for all* $E \in S$;
- $I \models K$ *if* $I \models \mathcal{T}$ *and* $I \models \mathcal{A}$.

If $I \models \Gamma$, we say that I satisfies Γ *or that I is a* model *of* Γ *(for Γ being an axiom, a set of axioms, or a KB). An axiom E is* entailed *by K, written $K \models E$, if $I \models E$ holds for all models I of K.*

For the finitely many-valued case, we assume the truth space to be $\mathcal{C}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\}$, for an integer $n \geq 1$ [28, 29, 30]. In the following, we will use $\mathcal{ALC}_n$ to refer to a finitely-valued extension of $\mathcal{ALC}$ interpreted over the truth space $\mathcal{C}_n$, without committing to a specific choice of combination functions.

3. The temporal Description Logic $LTL_{\mathcal{ALC}}$

The temporal Description Logic $LTL_{\mathcal{ALC}}$ is a temporal extension of $\mathcal{ALC}$ based on linear time temporal logic (LTL) The concepts of $LTL_{\mathcal{ALC}}$ can be formed by adding to the constructors of $\mathcal{ALC}$ the temporal operators $\bigcirc$ (next), $\mathcal{U}$ (until), $\Diamond$ (eventually) and $\Box$ (always) of LTL. Temporal extensions of Description Logics are very well-studied in the literature; see, for instance, the survey on temporal DLs and their complexity and decidability by Lutz et al. [17].

The set of temporally extended concepts is the following:

$$C ::= A \mid \top \mid \bot \mid C \sqcap D \mid C \sqcup D\ \mid \neg C \mid \exists r.C \mid \forall r.C \mid \bigcirc C \mid C\mathcal{U}D \mid \Diamond C \mid \Box C$$

where $A \in N_C$, and C and D are temporally extended concepts.

A *temporal interpretation* for $LTL_{\mathcal{ALC}}$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a nonempty domain; $\cdot^{\mathcal{I}}$ is an extension function that maps each concept name $C \in N_C$ to a set $C^{\mathcal{I}} \subseteq \mathbb{N} \times \Delta^{\mathcal{I}}$, each role name $r \in N_R$ to a relation $r^{\mathcal{I}} \subseteq \mathbb{N} \times \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and each individual name $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. Following [17] we assume individual names to be *rigid*, i.e., having the same interpretation at any time point. In a pair $(n, d) \in \mathbb{N} \times \Delta^{\mathcal{I}}$, n represents a time point and d a domain element; $(n, d) \in C^{\mathcal{I}}$ means that d is an instance of concept C at time point n, and similarly for $(n, d_1, d_2) \in r^{\mathcal{I}}$. Function $\cdot^{\mathcal{I}}$ is extended to complex concepts as follows:

$$\top^{\mathcal{I}} = \mathbb{N} \times \Delta^{\mathcal{I}} \qquad \bot^{\mathcal{I}} = \emptyset \qquad (\neg C)^{\mathcal{I}} = (\mathbb{N} \times \Delta^{\mathcal{I}}) \backslash C^{\mathcal{I}}$$
$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}} \qquad (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$(\exists r.C)^{\mathcal{I}} = \{(n, x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \exists y.(n, x, y) \in r^{\mathcal{I}} \text{ and } (n, y) \in C^{\mathcal{I}}\}$$
$$(\forall r.C)^{\mathcal{I}} = \{(n, x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \forall y.(n, x, y) \in r^{\mathcal{I}} \Rightarrow (n, y) \in C^{\mathcal{I}}\}$$

$$(\bigcirc C)^{\mathcal{I}} = \{(n, x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid (n+1, x) \in C^{\mathcal{I}}\}$$
$$(\Diamond C)^{\mathcal{I}} = \{(n, x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \exists m \geq n \text{ such that } (m, x) \in C^{\mathcal{I}}\}$$
$$(\Box C)^{\mathcal{I}} = \{(n, x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \forall m \geq n, \ \ (m, x) \in C^{\mathcal{I}}\}$$
$$(C\mathcal{U}D)^{\mathcal{I}} = \{(n, x) \in \mathbb{N} \times \Delta^{\mathcal{I}} \mid \exists m \geq n \text{ s.t. } (m, x) \in D^{\mathcal{I}} \text{ and } (k, x) \in C^{\mathcal{I}}, \forall k \ (n \leq k < m)\}$$

While the definition above assumes a *constant domain* (i.e., that the domain elements are the same at all time points), in the following we will also consider the case with *expanding domains*, when there is a sequence of increasing domains $\Delta_0^{\mathcal{I}} \subseteq \Delta_1^{\mathcal{I}} \subseteq \ldots$, one for each time point.

For simplicity, in the following we will focus on the case of non-temporal TBox, i.e., to a TBox containing a set of concept inclusions $C \sqsubseteq D$, where C, D are temporally extended concepts, but without temporal operator applied to the concept inclusions themselves.

The notions of satisfiability and model of a knowledge base can be easily extended to $LTL^{\mathbf{T}}_{\mathcal{ALC}}$ with non-temporal TBox. All inclusions in the (non-temporal) TBox $\mathcal{T}$ are regarded as global temporal constraints, and have to be satisfied at all time points, i.a., a concept inclusion $C \sqsubseteq D$ is satisfied in an interpretation $\mathcal{I}$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

It has been proven that, for non-temporal TBoxes, concept satisfiability in $LTL_{\mathcal{ALC}}$ w.r.t. non-temporal TBoxes is ExpTime-complete, both with expanding domains [35] and with constant domains [17]. The complexity of other cases and, specifically, the cases of temporal ABoxes [36] and temporal TBoxes (which allow temporal operators over concept inclusions), have as well been studied in the literature, and we refer to [17] for a discussion of the result and algorithms for satisfiability checking.

In [22] we have shown that, in the two-valued case, a typicality operator can be added to $LTL_{\mathcal{ALC}}$ and that a preferential extension of $LTL_{\mathcal{ALC}}$ with typicality can be polynomially encoded into $LTL_{\mathcal{ALC}}$. The encoding allows borrowing some decidability and complexity results from $LTL_{\mathcal{ALC}}$ to its preferential version with typicality.

In the following section, we first develop a many-valued semantics for $LTL_{\mathcal{ALC}}$ and, then, we define the typicality operator. Finally, we extend the notion of weighted KBs to the temporal, many-valued case.

4. A many-valued semantics for $LTL_{\mathcal{ALC}}$

Let us now move to the many-valued case. To define a temporal extension of $LTL_{\mathcal{ALC}}$ with typicality, we develop a many-valued semantics for $LTL_{\mathcal{ALC}}$, by interpreting, at each time point, all concepts and role names over a *truth degree set* $\mathcal{S}$ equipped with a preorder relation $\leq^{\mathcal{S}}$, a bottom element $0^{\mathcal{S}}$, and a top element $1^{\mathcal{S}}$. We denote by $<^{\mathcal{S}}$ and $\sim^{\mathcal{S}}$ the related strict preference relation and equivalence relation. In the following we will assume $\mathcal{S}$ to be the unit interval $[0, 1]$ or the finite set $\mathcal{C}_n$, for an integer $n \geq 1$, and that $\otimes$, $\oplus$, $\rhd$ and $\ominus$ are a t-norm, an s-norm, an implication function, and a negation function in some well known system of many-valued logic. In particular, in the following we will restrict to *continuous* t-norms.

A *many-valued temporal interpretations for* $LTL_{\mathcal{ALC}}$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty domain; $\cdot^{\mathcal{I}}$ is an interpretation function that maps each concept name $A \in N_C$ to a function $A^{\mathcal{I}} : \mathbb{N} \times \Delta^{\mathcal{I}} \rightarrow \mathcal{S}$, each role name $r \in N_R$ to a function $r^{\mathcal{I}} : \mathbb{N} \times \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow \mathcal{S}$, and each individual name $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. Again, in the following definition we assume individual names to be *rigid*, i.e., having the same interpretation at any time point n. Given a time point $n \in \mathbb{N}$ and a domain element $d \in \Delta^{\mathcal{I}}$, the interpretation $A^{\mathcal{I}}$ of a concept name A assigns to the pair (n, d) a value $A^{\mathcal{I}}(n, d) \in \mathcal{S}$ representing the degree of membership of d in concept A at time point n; and similarly for roles.

The interpretation function $\cdot^{I}$ is extended to complex concepts as follows (where, for the semantics of the temporal operators, we adapt a formulation from [37]):

$$\bot^{\mathcal{I}}(n,x) = 0,\ \top^{\mathcal{I}}(n,x) = 1$$
$$(\neg C)^{\mathcal{I}}(n,x) = \ominus C^{\mathcal{I}}(n,x)$$
$$(C \sqcap D)^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n,x) \otimes D^{\mathcal{I}}(n,x)$$
$$(C \sqcup D)^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n,x) \oplus D^{\mathcal{I}}(n,x)$$
$$(\exists r.C)^{\mathcal{I}}(n,x) = sup_{y \in \Delta}\ r^{\mathcal{I}}(n,x,y) \otimes C^{\mathcal{I}}(n,y)$$
$$(\forall r.C)^{\mathcal{I}}(n,x) = inf_{y \in \Delta}\ \ r^{\mathcal{I}}(n,x,y) \rhd C^{\mathcal{I}}(n,y)$$
$$(\bigcirc C)^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n+1,x)$$
$$(\Diamond C)^{\mathcal{I}}(n,x) = \bigoplus_{m \geq n} C^{\mathcal{I}}(m,x)$$
$$(\Box C)^{\mathcal{I}}(n,x) = \bigotimes_{m \geq n} C^{\mathcal{I}}(m,x)$$
$$(C\mathcal{U}D)^{\mathcal{I}}(n,x) = \bigoplus_{m \geq n}(D^{\mathcal{I}}(m,x) \otimes \bigotimes_{k=n}^{m-1} C^{\mathcal{I}}(k,x))$$

The semantics of $\Diamond$, $\Box$ and $\mathcal{U}$ requires a passage to the limit. Following [37], one can introduce a bounded version for $\Diamond$, $\Box$ and $\mathcal{U}$, by adding new temporal operators $\Diamond_t$ (eventually in the next t time points), $\Box_t$ (always within t time points) and $\mathcal{U}_t$, with the interpretation:

$$(\Diamond_t C)^{\mathcal{I}}(n,x) = \bigoplus_{m=n}^{n+t} C^{\mathcal{I}}(m,x)$$
$$(\Box_t C)^{\mathcal{I}}(n,x) = \bigotimes_{m=n}^{n+t} C^{\mathcal{I}}(m,x)$$
$$(C\mathcal{U}_t D)^{\mathcal{I}}(n,x) = \bigoplus_{m=n}^{n+t}(D^{\mathcal{I}}(m,x) \otimes \bigotimes_{k=n}^{m-1} C^{\mathcal{I}}(k,x))$$

so that $(\Diamond C)^{\mathcal{I}}(n,x) = lim_{t \to +\infty}(\Diamond_t C)^{\mathcal{I}}(n,x)$ and $(\Box C)^{\mathcal{I}}(n,x) = lim_{t \to +\infty}(\Box_t C)^{\mathcal{I}}(n,x)$ and $(C\mathcal{U}D)^{\mathcal{I}}(n,x) = lim_{t \to +\infty}(C\mathcal{U}_t D)^{\mathcal{I}}(n,x)$. The existence of the limits is ensured by the fact that $(\Diamond C)^{\mathcal{I}}(n,x)$ and $(C\mathcal{U}D)^{\mathcal{I}}(n,x)$ are increasing in n, while $(\Box C)^{\mathcal{I}}(n,x)$ is decreasing in n.

Note that, here, we have not considered the additional temporal operators ("soon", "almost always", etc.) introduced by Frigeri et al. [37] for representing vagueness in the temporal dimension. As a consequence, for the case $\mathcal{S} = [0,1]$, the semantics above is an extension to $\mathcal{ALC}$ of the FLTL (Fuzzy Linear-time Temporal Logic) semantics by Lamine and Kabanza [38].

Proposition 1. *For all concepts C and D, and for all time points n, the following properties hold:*
$(\Diamond C)^{\mathcal{I}}(n,x) = C^{I}(n,x) \oplus (\Diamond C)^{\mathcal{I}}(n+1,x)$
$(\Box C)^{\mathcal{I}}(n,x) = C^{I}(n,x) \otimes (\Box C)^{\mathcal{I}}(n+1,x)$
$(C\mathcal{U}D)^{\mathcal{I}}(n,x) = D^{I}(n,x) \oplus (C^{I}(n,x) \otimes (C\mathcal{U}D)^{\mathcal{I}}(n+1,x))$

Note that, although in this section we have considered a constant domain $\Delta^{\mathcal{I}}$, for a many-valued preferential temporal interpretation $\mathcal{I}$, expanding domains could have been considered as well, considering a domain $\Delta^{\mathcal{I}}_n$ for each time point n, with condition $\Delta^{\mathcal{I}}_0 \subseteq \Delta^{\mathcal{I}}_1 \subseteq \ldots$, as for $LTL_{\mathcal{ALC}}$ in the the two-valued case [17].

As in [22], for simplicity, we consider knowledge bases with non-temporal TBox and ABox, where a non-temporal TBox $\mathcal{T}$ is a set of concept inclusions $C \sqsubseteq D$, where (as in the two-valued case) C, D are temporally extended concepts, but no temporal operator is applied in front of concept inclusions themselves. The notions of satisfiability and model of a knowledge base can be easily generalized to a many-valued $LTL_{\mathcal{ALC}}$ knowledge base with non-temporal ABox and TBox. As $\mathcal{A}$ is a non-temporal ABox, the assertions in $\mathcal{A}$ are evaluated at time point 0. On the other hand, concept inclusions in the (non-temporal) TBox $\mathcal{T}$ are evaluated by considering all time points n.

Given a many-valued temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, the interpretation function $\cdot^{I}$ is extended to inclusion axioms as follows:

$$(C \sqsubseteq D)^{I} = inf_{x \in \Delta^{I}, n \in \mathbb{N}}(C^{I}(n,x) \rhd D^{I}(n,x))$$

Let K be an $LTL_{\mathcal{ALC}}$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ with non-temporal ABox and TBox.

Definition 2 (Satisfiability in many-valued $LTL_{\mathcal{ALC}}$). *Given a many-valued temporal interpretation for $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, satisfiability of an axiom in $\mathcal{I}$ is defined as follows:*

- $\mathcal{I} \models C \sqsubseteq D \ \theta\alpha$ *if* $(C \sqsubseteq D)^{\mathcal{I}} \ \theta\alpha$;
- $\mathcal{I} \models C(a) \ \theta\alpha$ *if* $C^{\mathcal{I}}(0, a^{\mathcal{I}}) \ \theta\alpha$;
- $\mathcal{I} \models r(a, b) \ \theta \ \alpha$ *if* $r^{\mathcal{I}}(0, a^{\mathcal{I}}, b^{\mathcal{I}})\theta \ \alpha$.

The interpretation $\mathcal{I}$ is a model *of $K = (\mathcal{T}, \mathcal{A})$ if $\mathcal{I}$ satisfies all concept inclusions in $\mathcal{T}$ and all assertions in $\mathcal{A}$. A knowledge base $K = (\mathcal{T}, \mathcal{A})$ is* satisfiable *in the many-valued extension of $LTL_{\mathcal{ALC}}$ if a many-valued temporal model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of K exists.*

5. A many-valued $LTL_{\mathcal{ALC}}$ withTypicality

As in the two-valued case [22], the language of a many-valued $LTL_{\mathcal{ALC}}$ can be extended with *typicality concepts* of the form $\mathbf{T}(C)$ representing the set of typical instances of concept C. The typicality operator $\mathbf{T}$ may occur both in concepts of TBox and ABox, but it cannot be nested. Unlike [5, 10], where a typicality operator was introduced for $\mathcal{ALC}$, here we do not require that the typicality operator only occurs on the left hand side of concept inclusions of the form $\mathbf{T}(C) \sqsubseteq D$, and this choice is in agreement with [39, 40]. As usual, we assume that the typicality operator $\mathbf{T}$ cannot be nested. *Extended concepts* can be built by combining the concept constructors in $LTL_{\mathcal{ALC}}$ with the typicality operator. They can freely occur in concept inclusions, such as, for instance, the following ones (adapted from [22]):

$$\mathbf{T}(Professor) \sqsubseteq (\exists teaches.Course) \mathcal{U} Retired$$
$$\exists lives\_in.Town \sqcap Young \sqsubseteq \mathbf{T}(\Diamond Granted\_Loan)$$

Note that, while the semantics in [22] was two-valued, in this example, the interpretation of some concepts (e.g., $Young$ and $Granted\_Loan$) may have a non-crisp value in $[0, 1]$. Indeed, being young is a fuzzy concept and $Granted\_Loan$ may have a degree of truth, for the different domain individuals (depending, e.g., on the outcome of some classifier on input exemplars).

From the semantic side, in the many valued case, the degree of membership of domain individuals in concept C at the different time points n induces a preference relation $\prec_C^n$ over the domain. Such preference relations are used to define the typical C-elements at the different time points.

Given a temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ over a truth degree set $\mathcal{S}$, a preference relation $\prec_C^n$ on $\Delta^{\mathcal{I}}$ can be associated to any concept C and time point $n \in \mathbb{N}$, based on the many valued interpretation of concepts in $\mathcal{I}$ and on the the strict partial order $<^{\mathcal{S}}$: for all $x, y \in \Delta^{\mathcal{I}}$,

$$x \prec_C^n y \text{ if and only if } C^{\mathcal{I}}(n, y) <^{\mathcal{S}} C^{\mathcal{I}}(n, x),$$

where $x \prec_C^n y$ means that x *is preferred to* y *wrt* C *at time point* n.

The many-valued temporal semantics introduced in the previous section easily extends to the language with typicality. Note that this semantics is inherently multi-preferential.

We regard typical C-elements (at time point n) as the domain elements x which are preferred with respect to $\prec_C^n$ among all domain elements (and such that $C^{\mathcal{I}}(x) \neq 0$). The interpretation of typicality concepts $\mathbf{T}(C)$ can be defined as follows:

Definition 3. *Given an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, for all $n \in \mathbb{N}$, $x \in \Delta^{\mathcal{I}}$, $(\mathbf{T}(C))^{\mathcal{I}}(n, x) = C^{\mathcal{I}}(x)$, if there is no $y \in \Delta^{\mathcal{I}}$ such that $y \prec_C^n x$; $(\mathbf{T}(C))^{\mathcal{I}}(n, x) = 0$, otherwise.*

When $(\mathbf{T}(C))^{\mathcal{I}}(x) > 0$, x is said to be a *typical C-element* in $\mathcal{I}$. Note that, when $\leq^{\mathcal{S}}$ is a total preorder (as it is in the cases $\mathcal{S} = [0, 1]$ and $\mathcal{S} = \mathcal{C}_n$), relation $\prec_C^n$ is an irreflexive, transitive and modular relation over $\Delta^{\mathcal{I}}$, like ranked preference relations in KLM-style ranked interpretations by Lehmann and Magidor [2]. For finitely-many truth values, $\prec_C^n$ is also *well-founded*.

For $LTL_{\mathcal{ALC}}$ with typicality, the notion of satisfiability of an axiom in a multi-preferential temporal interpretation $\mathcal{I}$ and the notion of model of a KB, are the ones given in Definition 2 (again for non-temporal KBs).

In the following, we will denote with $LTL_{\mathcal{ALC}}{}^{n}\mathbf{T}$ the many-valued extension of $LTL_{\mathcal{ALC}}$ with typicality, with truth degree set $\mathcal{S} = \mathcal{C}_n$, for $n \geq 1$, and with $LTL_{\mathcal{ALC}}{}^{\mathbf{F}}\mathbf{T}$ the fuzzy extension of $LTL_{\mathcal{ALC}}$ with typicality (where $\mathcal{S} = [0, 1]$).

6. Weighted temporal knowledge bases

Besides a set of *strict* concept inclusions in the TBox, weighted KBs also allow a set of *typicality inclusions* (or *defeasible inclusions*), each one with a weight. *Weighted typicality inclusions* for a concept C_i have the form $(\mathbf{T}(C_i) \sqsubseteq D_j, w_{ij})$, and describe the *prototypical properties of* C_i*-elements* (where D_j is a concept, and the weight w_{ij} is a real number). A concept C_i for which weighted typicality inclusions are provided is said to be a *distinguished concept*.

A *weighted* $\mathcal{LC}_n\mathbf{T}$ *knowledge base* is a tuple $\langle \mathcal{T}, \mathcal{D}, \mathcal{A} \rangle$, where the (strict) TBox $\mathcal{T}$ is a set of concept inclusions, the defeasible TBox $\mathcal{D}$ is a set of weighted typicality inclusions for the distinguished concepts C_i, and $\mathcal{A}$ is a set of assertions.

Consider the weighted $\mathcal{ALC}\mathbf{T}$ knowledge base $K = \langle \mathcal{T}, \mathcal{D}, \ \mathcal{A} \rangle$, over the set of distinguished concepts $\{Student, Employee, Person, \ldots\}$, with $\mathcal{T}$ containing, for instance, the inclusion $Student \sqsubseteq Person \geq 1$.

The set $\mathcal{D}$ of weighted typicality inclusions may contain, e.g., the following inclusions, describing the *prototypical properties* of concept *Student*:

$(\mathbf{T}(Student) \sqsubseteq Has\_Classes$, +50),
$(\mathbf{T}(Student) \sqsubseteq Active$,+35) ,
$(\mathbf{T}(Student) \sqsubseteq \exists has\_Boss.\top$, -70),

That is, a student normally has classes and is active, but she usually does not have a boss (negative weight). Accordingly, a student having classes, but not a boss, is more typical than an active student having classes and a boss. In the two valued case, one can evaluate how typical are two domain individuals $mary$ and tom as students, by considering their weight with respect of concept $Student$, i.e., by summing the (positive or negative) weights of the defeasible inclusions satisfied by $mary$ and tom, and comparing them. The higher the weight the more typical is the individual. In the many-value case, in defining the weight of a domain element x with respect to a distinguished concept C_i, we have to consider that, in an interpretation $\mathcal{I}$, at time point n, element x may belong to other concepts to some degree (e.g., at time point n, $mary$ may be active with degree 0.8, i.e., $Active^{\mathcal{I}}(n, mary) = 0.8$).

Given a many-valued temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, the *weight of* $x \in \Delta^{\mathcal{I}}$ *with respect to a distinguished concept* C_i *at time point* n is given by

$$W^{\mathcal{I}}_{i,n}(x) = \textstyle\sum_{(\mathbf{T}(C_i) \sqsubseteq D_j, w_{ij}) \in \mathcal{D}} \ w_{ij} \ D^{\mathcal{I}}_j(n, x).$$

Intuitively, the higher the value of $W^{\mathcal{I}}_{i,n}(x)$, the more typical is x as an instance of C_i), at time point n (considering the defeasible properties of C_i). Here, the membership degree $D^{\mathcal{I}}_j(n, x)$ of x in each concept D_j at time point n is considered.

The notions of faithful, coherent and φ-coherent semantics introduced for many-valued weighted KBs [41, 15, 16] can be smoothly extended to the temporal case. Generalizing from the non-temporal case, we expect the membership degree of a domain element x in a concept C_i at a time point n to be in agreement with the weight of x with respect to concept C_i, at the same time point n. We consider some different *agreement conditions at time point* n, as follows.

A many-valued temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ is *faithful at* n if, for all $x, y \in \Delta^{\mathcal{I}}$,

$$x \prec^n_{C_i} y \ \ \Rightarrow \ \ W^{\mathcal{I}}_{i,n}(x) > W^{\mathcal{I}}_{i,n}(y)$$

The interpretation $\mathcal{I}$ is *coherent at* n if, for all $x, y \in \Delta^{\mathcal{I}}$,

$$x \prec^n_{C_i} y \ \ \text{iff} \ \ W^{\mathcal{I}}_{i,n}(x) > W^{\mathcal{I}}_{i,n}(y)$$

Given a collection of monotonically non-decreasing functions $\varphi_i : \mathbb{R} \rightarrow \mathcal{S}$, one for each concept $C_i \in \mathcal{C}$:

- the interpretation $\mathcal{I}$ is *φ-coherent at n* if, for all $x \in \Delta^{\mathcal{I}}$,

$$C_i^{\mathcal{I}}(n,x) = \varphi_i(W_{i,n}^{\mathcal{I}}(x))$$

- the interpretation $\mathcal{I}$ is *transient φ-coherent at n* if, for all $x \in \Delta^{\mathcal{I}}$,

$$C_i^{\mathcal{I}}(n+1,x) = \varphi_i(W_{i,n}^{\mathcal{I}}(x))$$

It is easy to see that a many-valued temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ determines, at each time point n, a (non-temporal) many-valued interpretation $J^n = \langle \Delta^{J^n}, \cdot^{J^n} \rangle$, where $\Delta^{J^n} = \Delta^{\mathcal{I}}$, (for $A \in N_C$), and $r^{J^n}(x,y) = r^{\mathcal{I}}(n,x,y)$ (for $r \in N_R$). Letting the interpretation of typicality in J^n exploit the preference relations $\prec_{C_i}^n$ for each C_i (see Section 5), i.e., the preference relation induced by the many-valued interpretation of concept C_i in J^n, a many-valued temporal interpretation $\mathcal{I}$ can be regarded as a sequence $J^0, J^1, J^2, \ldots$ of many-valued preferential interpretations, as the ones considered in [33]. At each single time point the KLM properties of preferential consequence relation are then be expected to hold.

When considering the single time point n, the condition that the interpretation $\mathcal{I}$ is coherent (resp., faithful, φ-coherent) at n, means that the preferential interpretation J^n is coherent (resp., faithful, φ-coherent) according to their definition in [33]. Different notions of *agreement* at different time points can then be combined to give rise to different semantics of a temporal weighted KB, and different notions of entailment (based on different closure constructions).

7. Temporal weighted KBs and the transient behaviour of a neural network

In [33] it has been shown that many-valued Weighted KBs with typicality can provide a logical interpretation to some neural network model. Specifically, the φ-coherent semantics allows to capture the stationary states of multilayer networks as well as of networks with cyclic dependencies. In this section, we are interested in the transient behavior of a network.

Let us first recall from [42] the model of a *neuron* as an information-processing unit in an (artificial) neural network. A neuron k can be described by the following pair of equations: $u_k = \sum_{j=1}^{n} w_{kj} x_j$, and $y_k = \varphi(u_k + b_k)$, where $x_1, \ldots, x_n$ are the input signals, $w_{K1}, \ldots, w_{kn}$ are synaptic weights; b_k is the bias, φ an activation function, and y_k is the output signal of unit k. By adding a new synapse with input $x_0 = +1$ and synaptic weight $w_{k0} = b_k$, one can write: $u_k = \sum_{j=0}^{n} w_{kj} x_j$, and $y_k = \varphi(u_k)$, where u_k is called the *induced local field* of the neuron. The neuron can be represented as a directed graph, where the input signals $x_1, \ldots, x_n$ and the output signal y_k of neuron k are nodes of the graph. An edge from x_j to y_k, labelled w_{kj}, means that x_j is an input signal of neuron k with synaptic weight w_{kj}.

A neural network can then be seen as “a directed graph consisting of nodes with interconnecting synaptic and activation links" [42]: nodes in the graph are the neurons (the processing units) and the weight w_{ij} on the edge from node j to node i represents “the strength of the connection [..] by which unit j transmits information to unit i" [43]. Source nodes (i.e., nodes without incoming edges) produce the input signals to the graph. Neural network models are classified by their synaptic connection topology. In a *feedforward* network the architectural graph is acyclic, while in a *recurrent* network it contains cycles. In a recurrent network at least one feedback exists, so that “the output of a node in the system influences in part the input applied to that particular element" [42]. A time delay may be associated to feedback connections.

Let us consider a trained network $\mathcal{N}$. We do not put restrictions on the topology the network. Following the approach in [33], $\mathcal{N}$ can be mapped into a (non-temporal) weighted conditional knowledge base $K^{\mathcal{N}}$ [15, 33], by regarding the units in the network as concept names and the synaptic connections between units as weighted inclusions.

If C_k is the concept name associated to unit k and $C_{j_1}, \ldots, C_{j_m}$ are the concept names associated to units $j_1, \ldots, j_m$, whose output signals are the input signals for unit k, with synaptic weights $w_{k,j_1}, \ldots, w_{k,j_m}$, then unit k can be associated a set $\mathcal{T}_{C_k}$ of weighted typicality inclusions: $\mathbf{T}(C_k) \sqsubseteq C_{j_1}$ with $w_{k,j_1}, \ldots, \mathbf{T}(C_k) \sqsubseteq C_{j_m}$ with w_{k,j_m}.

It has been proven that the input-output behavior of a multilayer network $\mathcal{N}$ can be captured by a preferential interpretation $I^{\Delta}_{\mathcal{N}}$ built over a set of input stimuli Δ (e.g., the test set), through a simple construction, which exploits the activity level of units for the input stimuli.

A logical characterization of a trained multi-layer network $\mathcal{N}$ is established [33] by proving that the preferential interpretation $I^{\Delta}_{\mathcal{N}}$, describing the network behavior over a set Δ of input stimuli, is indeed a φ-coherent model of the weighted knowledge base $K^{\mathcal{N}}$ and, vice-versa, that any φ-coherent model of the knowledge base $K^{\mathcal{N}}$ captures the behavior of the network over some set Δ of input stimuli. Also in the case the network is not feedforward, the φ-coherent semantics allows the *stationary states* of the network $\mathcal{N}$ to be captured.

This approach allows for the verification of conditional properties of the network (of the form $\mathbf{T}(C) \sqsubset D \geq \theta$) by *model checking* over the preferential interpretation $I^{\Delta}_{\mathcal{N}}$, or by using *entailment* from the conditional knowledge base $K^{\mathcal{N}}$ (e.g., in an ASP encoding of a finitely-valued semantics[32]). Both the model checking and entailment approach have been used in the verification of properties of feedforward neural networks for the recognition of basic emotions.

In the temporal case, when we consider a temporal preferential model $\mathcal{I}$ of the weighted knowledge base $K^{\mathcal{N}}$, we may represent different states of the network at different time points.

When $\mathcal{I}$ is φ-coherent at time point n, the condition (stated above) that, for all $x \in \Delta^{\mathcal{I}}$,

$$C_i^{\mathcal{I}}(n, x) = \varphi_i(\sum_h w_{ih}\ D_h^{\mathcal{I}}(n, x))$$

imposes that the (non-temporal) interpretation J^n at time point n represents a stationary state of network $\mathcal{N}$. In such a case, φ_i plays the role of the activation function, and the sum $\sum_h w_{ih}\ D_h^{\mathcal{I}}(n, x)$ plays the role of the induced local field.

However, the temporal formalism also allows to capture the dynamic behavior of the network beyond stationary states, and this is especially interesting when the network $\mathcal{N}$ is recurrent. In this case, the knowledge base $K^{\mathcal{N}}$ contains cyclic dependencies in DBox.

By imposing the condition that $\mathcal{I}$ is a *transient φ-coherent interpretation at all time points* n, one can enforce that the interpretations $J^0, J^1, J^2, \ldots$ at successive time points describe the dynamic evolution of the activity of units in the network (where the activity of each unit at time point $n+1$ depends on the activity of incoming units at time point n). The temporal formalism provides a semantics for capturing the trajectories of the network state. Alternatively, time delayed feedback connections can be easily captured by temporal operators in $K^{\mathcal{N}}$.

Once a trained neural network has been represented as a weighted defeasible knowledge base $K^{\mathcal{N}}$, entailment allows for temporal properties to be proved over the runs representing the evolution of the network, an approach which may be computationally quite costly, depending on the size of the neural network and on the length of the runs. The non-temporal case is already challenging, and we refer to complexity results and to an experimentation of some different ASP based encodings of defeasible entailment for the verification of properties of a neural network, both in the feedforward case and in the cyclic case [33, 44]. The model checking approach, on the other hand, does not require to consider in the model $\mathcal{I}^{\Delta}_{\mathcal{N}}$ the activity of all units, but only of the units involved in the properties to be verified. Similarly, not all time points need to be considered, but only those corresponding to the states of interest.

An interesting direction for future work, is an extension to the temporal case of the model-checking approach developed in Datalog [45, 33] for the verification of conditional properties of a network, for post-hoc verification.

8. Conclusions

In this paper, we develop a many-valued, temporal description logic with typicality, extending $LTL_{\mathcal{ALC}}$ to deal with defeasible reasoning. Our extension of $LTL_{\mathcal{ALC}}$ builds, on the one hand, on fuzzy and many-valued DLs, and, on the other hand, on preferential DLs with typicality. We have first developed a many-valued semantics for $LTL_{\mathcal{ALC}}$, and then added to the language a typicality operator, based on a (multi-) preferential semantics. Finally, we have defined an extension of weighted knowledge bases with typicality to the temporal many-valued case, for representing prototypical properties of different classes in the temporal case.

On a different route, a preferential LTL with defeasible temporal operators has been studied in [20, 21], where the decidability of meaningful fragments of the logic is proven, and tableaux based proof methods for such fragments is developed [19, 21]. Our approach does not consider defeasible temporal operators nor preferences over time points, but combines standard LTL operators with the typicality operator in a many-valued temporal $\mathcal{ALC}$. Preferences are over domain elements, but they change over time.

In previous work, we have developed a preferential temporal description logics with typicality $LTL^{\mathbf{T}}_{\mathcal{ALC}}$ [22]. The monotonic logic $LTL^{\mathbf{T}}_{\mathcal{ALC}}$ is further extended with multiple preferences. Such extensions show that the concept-wise multi-preferential semantic in [14] adapts smoothly to the temporal case. In the two-valued case, the semantics for rank and weighted $\mathcal{ALC}$ knowledge bases has been defined based on semantic closure constructions [14, 15], developed in the spirit of Lehmann's lexicographic closure [13], Kern-Isberner's c-representations [46, 47] and Weydert's algebraic semi-qualitative approach [48], Casini and Straccia's fuzzy rational closure [49], but allowing for multiple preferences defining a ranking on individuals for each concept. In this paper, we have considered the temporal many-valued case and developed a semantics for weighted knowledge bases that deals with different agreement conditions at the different time points, leading to different *closure constructions* for the temporal conditional logics.

Much work has been recently devoted to the combination of neural networks and symbolic reasoning [50, 51, 52]. While conditional weighted KBs have been shown to capture (in the many-valued case) the stationary states of a neural network (or its finite approximation) [15, 33], and allow for combining empirical knowledge with elicited knowledge for reasoning and for post-hoc verification, adding a temporal dimension opens to the possibility of verifying properties concerning the dynamic behaviour of the network, based on a model checking approach or an entailment based approach.

A different approach for dealing with defeasibility in temporal DL formalism has been proposed in [53], by combining a (dynamic) temporal action logic [54] for reasoning about actions (whose semantics is based on a notion of temporal answer set) and an $\mathcal{EL}^{\perp}$ ontology. The temporal action logic allows for complex actions, and the proof methods are based on ASP encodings of bounded model checking [54].

Extending the above mentioned ASP encodings to deal with model checking in temporal preferential interpretations is a direction of future work. Future work also includes studying the decidability for fragments of the logic and exploiting the formalism for explainability.

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