A Meta-Ontological Inquiry on Topics for Non-Topic-Neutral Logics

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Abstract
J.C. Beall in [1] proposes to read the middle-value of some Weak Kleene logics as *off-topic*. In such a frame, propositions can be *true-and-on-topic*, *false-and-on-topic* or (simply) *off-topic*. This reading has two obvious consequences. The first is that it is not true, as it has been argued for a long time, that logic is *topic-neutral*. The second is that it is crucial to know what a topic is to clarify the ontological commitments of a logical theory, and more generally to understand how logic interacts with ontology. Thus, the aim of this paper is to elaborate on Beall's notion of topic and compare it with the three main approaches to topics or subject matter that are currently available in the literature.

Keywords
Off-topic interpretation of the third truth-value, topic, subject matter, Weak Kleene Logics, Bochvar's logic

Introduction

Topic-neutrality is taken as a standard, implicit typical trait of logic. Such a feature is connected to various ideas and views about logic: the Kantian view that logic is constitutive of thought, Wittgenstein's view that logical truths are not properly meaningful – for they do not restrict the possibility space –, the view that logical truths are analytic – which is held, for example, by neo-empiricists like Carnap –, and so on.

A way to specify topic-neutrality is by arguing that logic is ontologically neutral. Take, for example, Varzi's reading of topic-neutrality as ontological neutrality: "[a]s a general theory of reasoning, and especially as a theory of what is true no matter what is the case (or in every 'possible world'), logic is supposed to be ontologically neutral. It should be free from any metaphysical presuppositions. It ought to have nothing substantive to say concerning what there is, or whether there is anything at all" (Varzi [2]).

Topic-neutrality of logic as ontological neutrality is usually considered a standard also in formal ontology. Consider, for example, what Guarino, Carrara and Giaretta wrote at the start of their paper on ontological commitments:

FOUST VII: 7th Workshop on Foundational Ontology, 9th Joint Ontology Workshops (JOWO 2023), co-located with FOIS 2023, 19-20 July, 2023, Sherbrooke, Québec, Canada.
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> First order logic is notoriously neutral with respect to ontological choices: when a logical language is used with the purpose of modelling a particular aspect of reality, the set **M** of all its models is usually much larger than the set **Mi** of the intended ones, which describe only those states of affairs which are compatible with some underlying ontological commitment.
>
> [Guarino et al. [3], p. 560]

However, nowadays topic-neutrality of logic is widely questioned. Here we focus on some logics where this problem is particularly evident because topicality can be considered as part of the very same logical theory: Weak Kleene logics.

Traditionally, Kleene's three-valued logics divide into two families: strong and weak.[1] Weak Kleene logics, K_3^w, originate from weak tables (see table 1 below). Arguably, the two most important K_3^w are Bochvar [5] and Halldén [6]'s logics (B and H, respectively), which differ in the designated values they take on.[2] B assumes that classical truth is the only value to be preserved by valid inferences. H includes also the non-classical value among the designated ones. In this paper we will focus on B.

What does the semantic non-classical value **0.5**[3] mean in B? Some different interpretations are available, such as *nonsense*, *meaninglessness*, and *undefined*.[4] Among them, a new proposal by Beall [1] suggests to read **0.5** as *off-topic*. Thus, a proposition that obtains this value should be understood as being off-topic. In general, each proposition could be *true-and-on-topic*, *false-and-on-topic* or (simply) *off-topic*. Such an interpretation of **0.5** has been criticised by Francez [12] pointing out that **0.5** as *off-topic* does not satisfy the pre-theoretic understanding of a truth-value in a truth-functional logic.[5] If Francez's criticism is sound, a characterization of topic (or a subject matter) is needed to evaluate pros and cons of Beall's proposal. But Beall [1] is silent about what a topic is.[6] Our goal here is to help to remedy this deficiency, giving a way of understanding the fact that in a certain logic a proposition is *in* or *out* of a certain topic. We do this by specifying what *topics* could be, specifically in B logic.

This paper is divided into four sections. In §1 we introduce Bochvar's logic. In §2 we briefly present the main approaches to topics that are currently available in the literature. In §3 we discuss and further develop Beall's interpretation of B. Finally, in §4 we elaborate on Beall's notion of topic and compare it with the three main approaches presented in §2.

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[1] See Kleene et al. [4].

[2] There is increasing interest in K_3^w. To give some examples, Coniglio and Corbalan [7] develop sequent calculi for K_3^w, Paoli and Pra Baldi [8] introduce a cut-free calculus (a hybrid system between a natural deduction calculus and a sequent calculus) for one of K_3^w, PWK, Ciuni [9] explores some connections between H and Graham Priest's Logic of Paradox, LP, and Ciuni and Carrara [10] focus on logical consequence in H.

[3] **0.5** means *undefined* as in Kleene et al. [4]. This value might also be referred to as the *third value*, the *middle value*, or **0.5**.

[4] For a survey on new interpretations of **0.5** see Ciuni and Carrara [11].

[5] Specifically, Francez [12] claims that any notion that aspires to qualify as an interpretation of a truth-value has to satisfy certain requirements, and that Beall's interpretation of **0.5** as *off-topic* does not do that.

[6] "Topic" and "subject matter" are just synonyms, and throughout the paper we will use them interchangeably.

1. B Logic

The language of Bochvar's logic is the standard propositional language, L. Given a nonempty countable set Var $= \{p, q, r, \dots\}$ of atomic propositions, L is defined by the following Backus-Naur Form:

$$\Phi_L ::= p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi$$

We use $\phi, \psi, \gamma, \delta \dots$ to denote arbitrary formulas, and $\Gamma, \Phi, \Psi, \Sigma, \dots$ for sets of formulas. As usual, $\phi \supset \psi =_{def} \neg\phi \vee \psi$.

Propositional variables are interpreted by a valuation function $V_a : \text{Var} \longmapsto \{\mathbf{1}, \mathbf{0.5}, \mathbf{0}\}$ that assigns one out of three values to each atomic proposition in Var. The valuation extends to arbitrary formulas according to the following definition:

Definition 1.1 (Valuation)**.** A valuation $V : \Phi_L \longmapsto \{\mathbf{1}, \mathbf{0.5}, \mathbf{0}\}$ is the unique extension of a mapping $V_a : \text{Var} \longmapsto \{\mathbf{1}, \mathbf{0.5}, \mathbf{0}\}$ that is induced by the tables from table 1.

Table 1
Weak tables for logical connectives in L

	$\neg\phi$
1	**0**
0.5	**0.5**
0	**1**

$\phi \vee \psi$	**1**	**0.5**	**0**
1	**1**	**0.5**	**1**
0.5	**0.5**	**0.5**	**0.5**
0	**1**	**0.5**	**0**

$\phi \wedge \psi$	**1**	**0.5**	**0**
1	**1**	**0.5**	**0**
0.5	**0.5**	**0.5**	**0.5**
0	**0**	**0.5**	**0**

Table 1 provides the full weak tables as they can be derived from Bochvar and Bergmann [5]. The way **0.5** transmits is usually called *contamination* (or *infection*), since the value propagates from any $\phi \in \Phi_L$ to any construction $k(\phi, \psi)$, independently from the value of ψ (here, k is any complex formula made out of some occurrences of both ϕ and ψ and whatever well-formed combination of $\vee$, $\wedge$, and $\neg$). To better capture the way **0.5** works in combination with the other truth-values, let us introduce the following definition:

Definition 1.2. For any $\phi \in \Phi_L$, *var* is a mapping from Φ_L to the power set of Var, which can be defined inductively as follows:

- $var(p) = \{p\}$,
- $var(\neg\phi) = var(\phi)$,
- $var(\phi \vee \psi) = var(\phi) \bigcup var(\psi)$,
- $var(\phi \wedge \psi) = var(\phi) \bigcup var(\psi)$,

Then, $var(\phi)$ is the set of all and only the propositional variables occurring in ϕ, for all $\phi \in \Phi_L$. We can also apply *var* to sets of sentences by stipulating that for any set X of sentences of L, $var(X) = \bigcup\{var(\phi) \mid \phi \in X\}$.

Then, the following fact expresses *contamination* very clearly:

Fact 1.1 (Contamination)**.** For all formulas $\phi \in \Phi_L$ and any valuation V:

$$V(\phi) = \mathbf{0.5} \qquad \text{iff} \qquad V_a(p) = \mathbf{0.5} \text{ for some } p \in var(\phi)$$

The left-to-right direction is shared by all the most common three-valued logics; the right-to-left direction is a peculiar property of K_3^w.

Bochvar [13] interprets the third value **0.5** as *meaningless* – or *nonsensical*.[7] This interpretation goes along with the way **0.5** propagates to a compound formula from its components: the sense of a compound sentence depends on that of its components, and if some component makes no sense, the sentence as a whole will make no sense either.[8] In his jargon, nonsensical sentences are expressions that are syntactically well-formed and yet fail to convey a proposition, such as the sentences from which the Russell paradox and the Grelling–Nelson paradox are derived.[9]

The logical consequence relation of B is defined as preservation of truth – i.e., the only designated value is **1**. In other words:

Definition 1.3. $\Gamma \vDash_B \psi$ iff there is no valuation V such that $V(\phi) = \mathbf{1}$ for all $\phi \in \Gamma$, and $V(\psi) \neq \mathbf{1}$.

Let us now move on and provide a brief overview of the main conceptions of topic that are currently available.

2. Conceptions of Topic

In this section, we introduce the main approaches to the notion of topic that have been developed so far. We will basically follow Hawke [17], who distinguishes three main strategies to account for subject matter: the *way-based* approach (WBA), the *atom-based* approach (ABA) and the *subject-predicate* approach (SPA).[10] All the main theories of topic we can find in the literature fall into one of these three categories. To be short, we just select and present the basic version

[7] Bochvar [13] uses *meaningless* as an umbrella term including paradoxical statements such as the Liar and Russell's paradoxes, vague sentences, denotational failure, and ambiguity. Though, there are some issues concerning the interpretation of **0.5**. This is the reason why some alternative interpretations have been recently proposed, e.g. Beall [1], Boem and Bonzio [14], and Carrara and Zhu [15].

[8] The principle is also endorsed by Goddard [16].

[9] Russell sentence: *The set of all sets which are not members of themselves is a member of itself.*
Grelling–Nelson sentence: *'Heterological' is heterological.*

[10] The assumption here is that the notion of topic can actually license a unified theory.

– or theory – of each approach: Perry [18] for the SPA, Hawke [17, §4.2.1] for the ABA, and Lewis [19] for the WBA.

Before presenting them, let us make some assumptions and define a simple notation to facilitate the discussion. We use bold letters for topics, such as $\mathbf{s}$, $\mathbf{t}$, etc. $\leq$ is the inclusion relation between topics, so that $\mathbf{s} \leq \mathbf{t}$ expresses that $\mathbf{s}$ is included into (or is a subtopic of) $\mathbf{t}$. Given that, we define a *degenerate* topic as one that is included in every topic.[11] Also, we define the overlap relation $\circ$ between topics as follows: $\mathbf{s} \circ \mathbf{t}$ iff there exists a non-degenerate topic $\mathbf{u}$ such that $\mathbf{u} \leq \mathbf{s}$ and $\mathbf{u} \leq \mathbf{t}$. Further, it is assumed that every meaningful sentence α comes with a *least* subject matter, represented by $\tau(\alpha)$. $\tau(\alpha)$ is the unique topic which α is about, such that for every topic α is about, $\tau(\alpha)$ is included into it. Thus, we say that α is *exactly* about $\tau(\alpha)$.[12] But α can also be *partly* or *entirely* about other topics: α is entirely about $\mathbf{t}$ iff $\tau(\alpha) \leq \mathbf{t}$, whereas α is partly about $\mathbf{t}$ iff $\tau(\alpha) \circ \mathbf{t}$. Lastly, $+$ (combination) is a – for now, unspecified – binary operator between topics that outputs another topic, i.e. their combination. Thus, not only does any given approach consist in defining what a topic is, but also provides a specific interpretation for $\leq$, $\circ$ and (if needed) $+$.

Also, for ease of comparison we introduce Hawke's criteria of adequacy which serve as constraints to evaluate different theories of topic.[13] They should be sufficiently clear and require no comment. But those who wish to know more about them can refer to Hawke [17, §3]. Thus, here are Hawke's conditions:

- H1) If ϕ is entirely about $\mathbf{s}$, then $\neg\phi$ is entirely about $\mathbf{s}$.
- H2) If Fa is entirely about $\mathbf{s}$ and Gb is entirely about $\mathbf{s}$, then $Fa \vee Gb$ is entirely about $\mathbf{s}$.
- H3) If Fa is entirely about $\mathbf{s}$, then $Fa \wedge Gb$ is partly about $\mathbf{s}$.
- H4) The subject matter of $Fa \wedge Gb$ includes that of $Fa \vee Gb$.
- H5) Expressions of the form $Fa \vee Fb$ are about something but not necessarily about everything.
- H6) If Fa and Gb have different subject matter and Fa is contingent, then Fa differs in topic to $Fa \wedge (b = b)$ and $Fa \wedge (Gb \vee \neg Gb)$.
- H7) If Fa and Gb have different subject matter and Fa is contingent, then Fa differs in topic to $Fa \vee (b \neq b)$ and $Fa \vee (Gb \wedge \neg Gb)$.
- H8) A claim of the form $Fa \vee \neg Fa$ is about something (for example, $\mathbf{a}$) but not about everything (at least if Fa is about something but not everything).
- H9) A claim of the form $Fa \wedge \neg Fa$ is about something (for example, $\mathbf{a}$) but not about everything (at least if Fa is about something but not everything).
- H10) Expressions of the form aRb and bRa are not necessarily about the same topic. Nor is the subject matter of Fa necessarily identical to that of an expression of the form Ga.
- H11) If a is part of b, then: if ϕ is entirely about $\mathbf{a}$, then ϕ is entirely about $\mathbf{b}$.
- H12) A question $\mathbf{Q}$ serves (in some sense) as a subject matter.

Thus, a theory of topic that aspires to be correct must meet (at least) all such criteria.

[11]The inclusion relation $\leq$ is usually taken to be reflexive, so that every topic includes itself.

[12]Throughout this paper, when we talk about the topic of a sentence we mean its least topic. In case we want to refer to one of its topics that is not the least one, we will make it clear.

[13]Though, they are not exhaustive: there might be other criteria a theory of topic must satisfy to be correct.

Table 2
Evaluation of Perry [18]'s theory of topic based on Hawke's twelve criteria of adequacy.

Theory	H1	H2	H3	H4	H5	H6	H7	H8	H9	H10	H11	H12
Perry [18]	no	yes	yes	yes	no	no	yes	yes	no	no	yes	yes

2.1. The Subject-Predicate Approach

On the SPA, a topic is a set of entities. Therefore, the relations we have previously mentioned correspond to their set-theoretic counterparts: $\mathbf{s} \leq \mathbf{t}$ is $\mathbf{s} \subseteq \mathbf{t}$, $\mathbf{s} \circ \mathbf{t}$ is $\mathbf{s} \cap \mathbf{t}$ and $\mathbf{s} + \mathbf{t}$ is $\mathbf{s} \cup \mathbf{t}$. $\emptyset$ is a degenerate topic. More specifically, the topic of an arbitrary sentence, ϕ, is the set of all and only those objects of which a property or a relation is predicated by ϕ. As an example, consider the following sentences:

(p) logic is fun
(q) logic is more fun than topology

Thus, according to the SPA $\tau(p) = \{\text{logic}\}$ and $\tau(q) = \{\text{logic}, \text{topology}\}$. Also, since **mathematics** $= \{\text{logic}, \text{algebra}, \text{topology}, \ldots\}$ then $\tau(p) \leq$ **mathematics** and $\tau(q) \leq$ **mathematics**, that is both p and q are entirely about **mathematics**. And since **philosophy** $= \{\text{logic}, \text{ethic}, \ldots\}$ then $\tau(q) \circ$ **philosophy**, that is q is partly about **philosophy**.

Perry [18]'s theory of topic makes use of the SPA in combination with situation theory. For a situation S to be the case is for certain objects to stand in certain relations and certain objects to fail to stand in certain relations. Therefore, S may be represented by a partial valuation ρ_S, assigning **1** (true), **0** (false), or nothing (undetermined) to every atomic claim, in accord with S. Then, an arbitrary claim is verified or falsified by S as follows:

- S verifies atomic p iff $\rho_S(p) = \mathbf{1}$. S falsifies atomic p iff $\rho_S(p) = \mathbf{0}$.
- S verifies $\neg\phi$ iff S falsifies ϕ. S falsifies $\neg\phi$ iff S verifies ϕ.
- S verifies $\phi \wedge \psi$ iff S verifies ϕ and verifies ψ. S falsifies $\phi \wedge \psi$ iff S falsifies ϕ or falsifies ψ.
- S verifies $\phi \vee \psi$ iff S verifies ϕ or verifies ψ. S falsifies $\phi \vee \psi$ iff S falsifies ϕ and falsifies ψ.

Then, Perry proposes that an object x is in $\tau(\alpha)$ iff x is part of every situation that verifies α. As an example, this results in the following intuitive consequences: $\tau(Fa \wedge Gb) = \{a, b\}$; $\tau(Fa) = \{a\} = \tau(\neg Fa)$. But Hawke [17, pp. 705-706] shows that some of the conditions H1-12 are not satisfied. We summarize his evaluation in table 2.

2.2. The Atom-Based Approach

In general, the ABA of topic resorts to (i) a class of objects (e.g. sets of worlds), U, and (ii) a *topic function*, τ. A subject matter is any subclass of U. Again, $\emptyset$ is a degenerate topic, $\leq$ is $\subseteq$, and $\circ$ is $\cap$. τ is a function that assigns a topic to every atomic formula, p. The topic of complex formulas is a combination ($+$) of the topics of their atomic components.

Table 3
Evaluation of Hawke [17, §4.2.1]'s theory of topic based on Hawke's twelve criteria of adequacy.

Theory	H1	H2	H3	H4	H5	H6	H7	H8	H9	H10	H11	H12
Hawke [17]	yes	yes	yes	yes	yes	yes	yes	no	no	yes	yes	yes

A particular atom-based theory of topic specifies U, $+$ and τ. As an example, a basic atom-based theory[14] takes U to be the class of all sets of possible worlds,[15] so that a member of U can be interpreted as a *piece of information* or a *truth set*.[16] Thus, a topic is a set of sets of worlds – i.e. a set of pieces of information. In particular, the topics of atomic sentences are sets of just one set of worlds; instead, the topics of complex sentences are sets of more than one sets of worlds. Then, τ assigns a topic $\tau(p) = \{P\}$ to every atomic sentence, p, where P is the truth set of p – i.e. P is the set of the worlds at which p holds. Finally, $+$ is $\cup$ between the topics of atomic formulas. Therefore, the subject matter of, say, $p \wedge (q \vee r)$ is $\{P\} \cup \{Q\} \cup \{R\} = \{P, Q, R\}$, where P, Q and R are the sets of worlds at which p, q and r are respectively true.

As shown in table 3, such a theory meets many but not all the H1-12 criteria.

2.3. The Way-Based Approach

According to the WBA, a topic is a comprehensive set of *ways things can be*, where "a way things can be" is again a truth set or a piece of information – i.e. a set of possible worlds. Therefore, this is essentially the same situation we have in the atom-based approach, but with one crucial difference: for a set of sets of possible worlds to be a subject matter it must be *comprehensive*. A set of ways $\mathbf{W}$ is comprehensive if the union of the members of $\mathbf{W}$ is equal to the set of possible worlds, W. In other words, a subject matter essentially corresponds to a partition of W.

In general, $\leq$ is not $\subseteq$, but some sort of refinement relation between topics: $\mathbf{s} \leq \mathbf{t}$ iff $\mathbf{s}$ *refines* $\mathbf{t}$. Such a refinement relation can be precisely defined in different ways. For instance, let us consider Lewis [19]'s theory. Here, $\mathbf{s}$ *refines* $\mathbf{t}$ means that every member of $\mathbf{t}$ is the union of some members of $\mathbf{s}$. In other words, every member of $\mathbf{s}$ is a subset of some members of $\mathbf{t}$ – i.e. no members of $\mathbf{s}$ cuts across any member of $\mathbf{t}$. In particular, given any arbitrary sentence ϕ (atomic or complex) Lewis [19] takes $\tau(\phi)$ to be the binary partition consisting of the truth set of ϕ and its complement.[17]

As for the previous accounts of topic, we show which constraints are met by Lewis [19]'s theory in table 4.

[14] See Hawke [17, §4.2.1]
[15] Let us call the set of possible worlds W. Thus, U is the power set (or class) of W, $U = \mathcal{P}(W)$.
[16] A set of worlds can be viewed as a truth set to the extent that the members of such a set are the worlds at which a/some given proposition(s) is/are true. The piece of information represented by this set of worlds is the conjunction of the propositions that are true at those worlds.
[17] This is not entirely correct, but we can gloss over that. See Hawke [17, fn. 24].

Table 4
Evaluation of Lewis [19]'s theory of topic based on Hawke's twelve criteria of adequacy.

Theory	H1	H2	H3	H4	H5	H6	H7	H8	H9	H10	H11	H12
Lewis [19]	yes	yes	no	no	yes	no	no	no	no	yes	yes	yes

3. Beall's Interpretation of B

As we already mentioned, according to Halldén [6] and Bochvar and Bergmann [5], the third value in K_3^w is to be interpreted as *meaningless* or *nonsense*. However, such interpretations seem to suffer from some problems. For example, it is not at all obvious that we can make the conjunction or the disjunction of a meaningless sentence with one with a traditional truth-value. Then, Beall [1] proposes to "[...] read the value [**1**] not simply as *true* but rather as *true and on-topic*, and similarly [**0**] as *false and on-topic*. Finally, read the third value [**0.5**] as *off-topic*" [1, p. 140]. Thus, an arbitrary sentence ϕ is either true and on-topic, false and on-topic, or off-topic. And note that since both on-topic and off-topic sentences are arguably meaningful, the problem concerning the conjunction/disjunction of meaningless sentences vanishes.

Following Beall's proposal, *What is a topic?* becomes a crucial question to be addressed. For depending on how we answer this question we may have different consequences on such a reading. Unfortunately, Beall [1] is silent about that. But he gives some constraints we can use to explore how topics behave and how they relate to the K_3^w truth-values.

Beall [1]'s new interpretation starts from setting a terminology concerning a *theory*, T. T is a set of sentences closed under a consequence relation, Cn. That is, $T = Cn(X) = \{\phi \mid X \vdash \phi\}$ where X is a given set of sentences and $\vdash$ is the consequence relation of the logic we are working with. As for B, theories (or B-theories) are sets of sentences closed under B's logical consequence. Then, Beall puts forward the following motivating ideas for his proposal:

1. A theory is about all and only what its elements – that is, the claims in the theory – are about.
2. Conjunctions, disjunctions and negations are about exactly whatever their respective subsentences are about:
   a) Conjunction $\phi \wedge \psi$ is about exactly whatever ϕ and ψ are about.
   b) Disjunction $\phi \vee \psi$ is about exactly whatever ϕ and ψ are about.
   c) Negation $\neg\phi$ is about exactly whatever ϕ is about.

[1, p. 139][18]

[18] To be precise, Beall states also a third condition concerning a theory being about every topic. We do not report it here since it is not relevant for our discussion.

We can arguably formalize Beall's motivating ideas as follows:[19]

Definition 3.1. Let T be a B-theory and $\tau(T)$ be its topic. The following conditions regiment how the topics of the compound $\phi \in \Phi_L$ and $\tau(T)$ behave:

1. $\tau(T) = \bigcup\{\tau(\phi) \mid \phi \in T\}$.
2. a) $\tau(\phi \wedge \psi) = \tau(\phi) \cup \tau(\psi)$.
   b) $\tau(\phi \vee \psi) = \tau(\phi) \cup \tau(\psi)$.
   c) $\tau(\neg\phi) = \tau(\phi)$.

From definition 3.1 we can get some further results.

Corollary 3.1. For any $\phi \in \Phi_L$, $\tau(\phi) = \bigcup\{\tau(p) \mid p \in var(\phi)\}$.

Proof. We can prove it by induction.

1. Let ϕ be an atomic sentence p. Then $\tau(\phi) = \tau(p)$.
2. Let $\phi = \neg\psi$ and $\tau(\psi) = \bigcup\{\tau(p) \mid p \in var(\psi)\}$. Since $var(\phi) = var(\neg\psi) = var(\psi)$, we have $\tau(\neg\psi) = \tau(\psi) = \bigcup\{\tau(p) \mid p \in var(\phi)\}$.
3. Let $\phi = \gamma \wedge \delta$, $\tau(\gamma) = \bigcup\{\tau(p) \mid p \in var(\gamma)\}$, and $\tau(\delta) = \bigcup\{\tau(q) \mid q \in var(\delta)\}$. Since $var(\phi) = var(\gamma) \cup var(\delta)$, we can derive $\tau(\phi) = \tau(\gamma) \cup \tau(\delta) = \bigcup\{\tau(r) \mid r \in (var(\gamma) \cup var(\delta))\}$. That is, $\tau(\phi) = \bigcup\{\tau(r) \mid r \in var(\phi)\}$.
4. For $\phi = \gamma \vee \delta$, we can prove that $\tau(\phi) = \bigcup\{\tau(r) \mid r \in var(\phi)\}$ in the same way as above.

□

Corollary 3.2. $\tau(T) = \bigcup\{\tau(p) \mid p \in var(T)\}$.

This corollary follows from the definition 3.1 and corollary 3.1. It states that what a theory T is about boils down to the union of what the atomic components of each claims in T are about. Moreover, even if Beall does not mention what an arbitrary set (i.e. not necessarily a theory) is about, we opt for the following very plausible definition:

Definition 3.2. Let X be a set of sentences. Such a set is about all and only what its elements are about. That is, $\tau(X) = \bigcup\{\tau(\phi) \mid \phi \in X\}$.

Thus, the following corollary follows from definition 3.2 and corollary 3.2:

Corollary 3.3. Let X be a set of sentences. Then $\tau(X) = \bigcup\{\tau(p) \mid p \in var(X)\}$.

[19]Note that we use the same notation, $\tau(\ldots)$, both for the topic of a sentence and the topic of a theory – i.e. a set of sentences.

By virtue of corollary 3.3 and definition 3.1, we get:

Corollary 3.4. For any $\phi, \psi \in \Phi_L$, $\tau(k(\phi,\psi)) = \tau(\{\phi,\psi\})$.

Proof. By induction and definition 3.1, $\tau(k(\phi,\psi)) = \tau(\phi) \cup \tau(\psi)$.[20] According to definition 3.2, $\tau(\{\phi,\psi\}) = \tau(\phi) \cup \tau(\psi)$. Hence, $\tau(k(\phi,\psi)) = \tau(\{\phi,\psi\})$. □

Corollary 3.5. For any $\phi, \psi \in \Phi_L$, $\tau(k(\phi,\psi)) = \tau(var(\phi)) \cup \tau(var(\psi))$.

Proof. According to corollary 3.4, $\tau(k(\phi,\psi)) = \tau(\{\phi,\psi\})$. By virtue of corollary 3.3, we can derive $\tau(k(\phi,\psi)) = \bigcup\{\tau(p) \mid p \in var(\{\phi,\psi\})\}$. According to definition 1.2, $var(\{\phi,\psi\}) = var(\phi) \cup var(\psi)$. Hence, $\tau(k(\phi,\psi)) = \bigcup\{\tau(p) \mid p \in (var(\phi) \cup var(\psi))\} = \bigcup\{\tau(p) \mid p \in var(\phi)\} \cup \bigcup\{\tau(q) \mid q \in var(\psi)\}$. Since $var(\phi) = var(var(\phi))$ and $var(\psi) = var(var(\psi))$, we can derive $\tau(k(\phi,\psi)) = \tau(var(\phi)) \cup \tau(var(\psi))$ by definition 3.2. □

Corollary 3.6. For any $\phi \in \Phi_L$ and B-theory T, $\tau(T) = \bigcup\{\tau(var(\phi)) \mid \phi \in T\}$.

Proof. From definition 3.2 and $var(\phi) = var(var(\phi))$, we can derive $\tau(\phi) = \tau(var(\phi))$. Since definition 3.1 claims that $\tau(T) = \bigcup\{\tau(\phi) \mid \phi \in T\}$, we can derive $\tau(T) = \bigcup\{\tau(var(\phi)) \mid \phi \in T\}$ by substituting $\tau(\phi)$ with $\tau(var(\phi))$. □

Lemma 3.1. For any $p \in \Phi_L$ and B-theory T, if $var(p) \subseteq var(T)$, then $\tau(p) \subseteq \tau(T)$.

Proof. Since for any set X of sentences of L, $var(X) = \bigcup\{var(\phi) \mid \phi \in X\}$, then $var(T) = \bigcup\{var(\phi) \mid \phi \in T\}$, $var(var(\phi)) = \bigcup\{var(p) \mid p \in var(\phi)\}$. Then $var(T) = \bigcup\{var(p) \mid p \in var(T)\}$. If $var(p) \subset var(T)$, then $p \in var(T)$. Since $\tau(T) = \bigcup\{\tau(p) \mid p \in var(T)\}$, then $\tau(p) \subseteq \tau(T)$. □

However, the result does not hold in the opposite direction. That is, if $\tau(p) \subseteq \tau(T)$, it might not be the case that $var(p) \subseteq var(T)$. To understand this point, consider the following counterexample.

Example 3.1. For any $r, q \in \Phi_L$ and B-theory T, let $var(r) \nsubseteq var(T)$, $var(q) \subseteq var(T)$, and $\tau(r) = \tau(q) \subseteq \tau(T)$. Therefore, even though $\tau(r) \subseteq \tau(T)$, $var(r) \nsubseteq var(T)$.

This counterexample is possible because τ is not necessarily bijective. As a justification, consider the following line of reasoning. Suppose that if $\tau(r) \subseteq \tau(T)$, then $var(r) \subseteq var(T)$. In that case, we get that whatever sentence is in the theory, it is also on-topic; and that whatever sentence is not in the theory, it is also off-topic. But then, Beall's reading of the truth-value **0** as

[20] Read $k(\phi,\psi)$ as defined on page 3.

false-and-on-topic is not available anymore. In other words, allowing for the bijection results in a conflict between Beall's conception of topic and his reading of B's truth-values.

To sum up: by Beall's ideas and the way B works we get that (1) the topic of a sentence is completely determined by (is the union of the topics of) its propositional variables, and (2) the topic of a theory is completely determined by (is the union of the topics of) the propositional variables of its sentences. Moreover, we get also the following important result:

Theorem 3.1. For any $\phi \in \Phi_L$ and B-theory T, if $var(\phi) \subseteq var(T)$, then $\tau(\phi) \subseteq \tau(T)$.

Proof. We can prove it by induction.

1. If ϕ is an atomic sentence, this theorem holds for ϕ by virture of Lemma 3.1.
2. If $\phi = \neg\psi$ and this theorem holds for $\neg\psi$. We can derive that this theorem holds for ϕ, because $var(\neg\psi) = var(\psi)$.
3. If $\phi = \gamma \vee \delta$, and this theorem holds for γ and δ. We can derive this theorem holds for ϕ, because $var(\gamma \vee \delta) = var(\gamma) \cup var(\delta)$.
4. We can prove this holds for $\phi = \gamma \wedge \delta$ in the same way.

□

By virtue of theorem 3.1, we can clarify Beall's on-topic/off-topic interpretation in the following way.

Corollary 3.7. Let T be a B-theory and $\tau(T)$ be its topic. For any $\phi \in \Phi_L$,

1. ϕ is on-topic iff $\tau(\phi) \subseteq \tau(T)$. But note that this does not guarantee that $\phi \in T$. However, if $\phi \in T$, by definition 3.1, ϕ is definitively on-topic.
2. ϕ is off-topic iff $\tau(\phi) \nsubseteq \tau(T)$. This suffices to say that $\phi \notin T$.

Finally, we can note that such an interpretation fits Beall's conditions as well as B semantics. To see this, let's conjoin two propositional variables, p and q, to get $p \wedge q$. Suppose that both are on-topic, i.e. $\tau(p) \subseteq \tau(T)$ and $\tau(q) \subseteq \tau(T)$. According to 2(a), $\tau(p \wedge q) = \tau(p) \cup \tau(q)$. Thus, $\tau(p \wedge q) \subseteq \tau(T)$, that is $p \wedge q$ is on-topic, which is in line with B semantics. Now, suppose that at least one of the conjuncts is off-topic, say q. Thus, $\tau(q) \nsubseteq \tau(T)$. Therefore, $\tau(p \wedge q) \nsubseteq \tau(T)$, which is also in line with B semantics. Alternatively, we might also be tempted to consider the following different interpretation: for p to be on-topic means that $\tau(p) \cap \tau(T) \neq \emptyset$, whereas to be off-topic means that $\tau(p) \cap \tau(T) = \emptyset$. For instance, this is exactly what Hawke [17, p. 700] suggests: "[t]o say that a claim is *somewhat* on-topic is to say that its subject matter overlaps with the discourse topic".[21] However, such an interpretation is not compatible with Beall's constraints. For condition 2(a) clashes with B semantics. To see this, suppose that $\tau(p) \cap \tau(T) \neq \emptyset$ but $\tau(q) \cap \tau(T) = \emptyset$. Thus, since $\tau(p \wedge q) = \tau(p) \cup \tau(q)$, it follows that $\tau(p \wedge q) \cap \tau(T) \neq \emptyset$ – i.e. $p \wedge q$ is on-topic. This contradicts B semantics – namely, contamination. Moreover, our observations

[21] Here, the discourse topic is what we call the topic of reference.

match [1, fn. 5]: "[a]n alternative account might explore 'partially off-topic', but I do not see this as delivering a natural interpretation of K_3^{w}". Here, Beall is suggesting to distinguish two notions: *off-topic* and *partially off-topic*. The latter might be legitimately taken to correspond to the alternative reading in terms of overlap between topics that we rejected – as indeed he does.

Now that Beall's alternative interpretation of B and his preliminary remarks about how topics should behave have been clarified, we can make a deeper analysis of his intended notion of topic and a comparison between that and the conceptions of topic described in §2.

4. Beall's Conception of Topic: Evaluation and Comparison

Here, we evaluate Beall's conception of topic[22] based on the criteria H1-12, and compare it with the other theories we have introduced in §2. Thus, in what follows we go through H1-12 and see whether Beall's constraints meet them.

- (H1) Negation preserves subject matter, and corresponds to Beall's condition 2(c). Therefore, H1 is met.
- (H2) *Fa* and *Gb* cannot be expressed in B, since L is a propositional language. However, we can consider a generalization of H2: if ϕ is entirely about **s** and ψ is entirely about **s**, then $\phi \vee \psi$ is entirely about **s**. This says that disjunction preserves shared subject matters. Now, because of 2(b) and union idempotence,[23] such a generalization is met. Therefore, since H2 is just an instance of that, H2 is also met.
- (H3) The same strategy applies here. We can take a generalization of H3: if ϕ is entirely about **s**, then $\phi \wedge \psi$ is partly about **s**. This states that conjunction results in a topic expansion, so that the topics of the conjuncts overlap the topic of the conjunction. Now, because of 2(a) and the basic set-theoretical theorem $X \cap (X \cup Y) \neq \emptyset$, provided $X = \emptyset$, this is true for Beall's conception. Therefore, H3 is met.
- (H4) H4 is met. Consider the following version of this constraint: the topic of $\phi \wedge \psi$ includes that of $\phi \vee \psi$. Since the inclusion relation ($\leq$) between topics is interpreted as the subset relation, this says that $\tau(\phi \wedge \psi) \subseteq \tau(\phi \vee \psi)$. Because of 2(a) and 2(b), $\tau(\phi \wedge \psi) = \tau(\phi \vee \psi)$. Therefore, $\tau(\phi \wedge \psi) \subseteq \tau(\phi \vee \psi)$,[24] and then H4.
- (H5) Again, let us reason in the propositional context of B. Recall that for a sentence ϕ to be about everything means that $\tau(\phi) = \emptyset$. If the sentence is a disjunction, $\phi \vee \psi$, by 2(b) we have $\tau(\phi \vee \psi) = \tau(\phi) \cup \tau(\psi)$. Thus, for $\tau(\phi \vee \psi) = \emptyset$ we need $\tau(\phi) = \tau(\psi) = \emptyset$. But this is not necessary, since it is very plausible to claim that there is at least one sentence of the form ϕ such that $\tau(\phi) \neq \emptyset$. For example, consider the sentence "Maths is fun". Arguably, this is about **mathematics**, but is not about **literature**. Thus, we can conclude that **mathematics** $\neq \emptyset$. Therefore, H5 is met.

[22] To be clear, we reiterate that Beall [1] dos not offer a specific conception of topic. For he does not say what a topic is. It may be represented by a set – we claim –, but Beall is silent about what we should take its members to be (e.g. sets of worlds, objects, etc). Nevertheless, we can partially develop how Beall's conception of topic works based on his 1-3 conditions.

[23] That is, $X \cup X = X$.

[24] This would not be true if $\leq$ were taken to be the strict subset relation, $\subset$.

Table 5
A comparison between Beall's conception of topic (BT) and Hawke [17, §4.2.1] basic version of the ABA.

Theory	H1	H2	H3	H4	H5	H6	H7	H8	H9	H10	H11	H12
Hawke [17]	yes	yes	yes	yes	yes	yes	yes	no	no	yes	yes	yes
BT	yes	yes	yes	yes	yes	yes	yes	yes	yes	?	?	?

(H6-9) As for H6 (H7), note that the only way for Beall to violate it is to take the topic of tautologies (contradictions) to be $\emptyset$. But this is not possible. As an example, consider $\phi \vee \neg\phi$. According to Beall, $\tau(p \vee \neg p) = \tau(p) \neq \emptyset$. In other words, the subject matter of tautologies (contradictions) is not degenerate – i.e. is not $\emptyset$. And this is also in line with H8 and H9.

(H10) We cannot express *aRb*, *bRa*, *Fa* and *Ga* in B. Thus, Beall's interpretation is silent about H10.

(H11) Beall's proposal is also silent about H11. For the language of B does not contain any individual constant.

(H12) H12 depends on what a topic is taken to be (a set of objects, a set of worlds, a partition on *W*, etc). But Beall does not say anything about what a topic is. Therefore, this issue remains open.

We report the results of our analysis in table 5. Here, we also show a quick comparison between Beall's conception of topic and Hawke [17, §4.2.1] basic version of the ABA. Thus, as should be clear, this is the closest theory to Beall's proposal among those we have considered. Essentially, this suggests that Beall's conception of topic points in the direction of an atom-based approach.

Conclusion

In this paper we have formalized Beall's ideas about topic and drawn some facts from them, to see what conception of topic is the best one for his off/on-topic reading of the truth-values of B. The result is that, in Beall's perspective, for a claim ϕ to be on-topic means that $\tau(\phi) \subseteq \tau(T)$, where T is the B-theory at stake; whereas, for ϕ to be off-topic means that $\tau(\phi) \nsubseteq \tau(T)$. This is in line with B semantics. Further, even if Beall is silent about what exactly a topic is, the basic atom-based conception of topic does seem to be the closest notion of topic to his conception.

Some final questions, to be answered in future works. How does the topicality of a logic interact with the ontological commitments of the theory described? And what contribution does the topic of a logical frame make in terms of the characterisation of the meta-ontological categories of a specific theory? For instance, consider a tool such as OntoClean, i.e. a methodology that by means of certain meta-properties can validate ontologies.[25] What is the role given by the topicality of the logical frame in the validation of ontologies?

[25]See Staab and Studer [20, pp. 201–221] for an overview on OntoClean.

Acknowledgments

Massimiliano Carrara's research is partially funded by the CARIPARO Foundation Excellence Project (CARR_ECCE20_01): *Polarization of irrational collective beliefs in post-truth societies. How anti-scientific opinions resist expert advice, with an analysis of the antivaccination campaign (PolPost).*

Filippo Mancini's research, including this work, is supported by the DFG (Deutsche Forschungsgemeinschaft), research unit FOR 2495.

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