$\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ are Harder than $\mathcal{SHOIQ}$*

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Abstract. We identify the complexity of (finite model) reasoning in the DL $\mathcal{SROIQ}$ to be N2ExpTime-complete. We also prove that (finite model) reasoning in the DL $\mathcal{SR}$—a fragment of $\mathcal{SROIQ}$ without nominals, number restrictions, and inverse roles—is 2ExpTime-hard.

1 From $\mathcal{SHIQ}$ to $\mathcal{SROIQ}$

In this paper we study the complexity of reasoning in the DL $\mathcal{SROIQ}$—the logic chosen as a candidate for OWL 2.[1] $\mathcal{SROIQ}$ has been introduced in [1] as an extension of $\mathcal{SRIQ}$, which itself was introduced previously in [2] as an extension of $\mathcal{RIQ}$ [3]. These papers present tableau-based procedures for the respective DLs and prove their soundness, completeness and termination.

In contrast to sub-languages of $\mathcal{SHOIQ}$ whose computational complexities are currently well understood [4], almost nothing was known, up until now, about the complexity of $\mathcal{SROIQ}$, $\mathcal{SRIQ}$ and $\mathcal{RIQ}$ except for the hardness results inherited from their sub-lanbuages: $\mathcal{SROIQ}$ is NExpTime-hard as an extension of $\mathcal{SHOIQ}$, $\mathcal{SRIQ}$ and $\mathcal{RIQ}$ are ExpTime-hard as extensions of $\mathcal{SHIQ}$. The difficulty was caused by complex role inclusion axioms $R_1 \circ \cdots \circ R_n \sqsubseteq R$, which cause exponential blowup in the tableau procedure. In this paper we demonstrate that this blowup was essentially unavoidable by proving that reasoning in $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ is exponentially harder than in $\mathcal{SHIQ}$ and $\mathcal{SHOIQ}$.

We assume that the reader is familiar with the DL $\mathcal{SHOIQ}$ [5]. A $\mathcal{SHOIQ}$ *signature* is a tuple $\Sigma = (C_\Sigma, R_\Sigma, I_\Sigma)$ consisting of the sets of *atomic concepts* C_Σ, *atomic roles* R_Σ and *individuals* I_Σ. A $\mathcal{SHOIQ}$ *interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ is a non-empty set called the *domain* of $\mathcal{I}$, and $\cdot^\mathcal{I}$ is the *interpretation function*, which assigns to every $A \in C_\Sigma$ a subset $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, to every $r \in R_\Sigma$ a relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and to every $a \in I_\Sigma$, an element $a^\mathcal{I} \in \Delta^\mathcal{I}$. The interpretation $\mathcal{I}$ is *finite* iff $\Delta^\mathcal{I}$ is finite.

A *role* is either some $r \in R_\Sigma$ or an *inverse role* r^-. For each $r \in R_\Sigma$, we set $\mathsf{Inv}(r) = r^-$ and $\mathsf{Inv}(r^-) = r$. A $\mathcal{SHOIQ}$ *RBox* is a finite set $\mathcal{R}$ of *role inclusion axioms* (RIA) $R_1 \sqsubseteq R$, *transitivity axioms* $\mathsf{Tra}(R)$ and *functionality axioms* $\mathsf{Fun}(R)$ where R_1 and R are roles. Let $\sqsubseteq^*_\mathcal{R}$ be the reflexive transitive closure of the relation $\sqsubseteq_\mathcal{R}$ on roles defined by $R_1 \sqsubseteq_\mathcal{R} R$ iff $R_1 \sqsubseteq R \in \mathcal{R}$ or

* Unless 2ExpTime = NExpTime, in which case just $\mathcal{SROIQ}$ is harder than $\mathcal{SHOIQ}$

[1] A.k.a. OWL 1.1: http://www.webont.org/owl/1.1

$\mathsf{Inv}(R_1) \sqsubseteq \mathsf{Inv}(R) \in \mathcal{R}$. A role S is called *simple* (w.r.t. $\mathcal{R}$) if there is no role R such that $R \sqsubseteq^*_{\mathcal{R}} S$ and either $\mathsf{Tra}(R) \in \mathcal{R}$ or $\mathsf{Tra}(\mathsf{Inv}(R)) \in \mathcal{R}$.

Given an RBox $\mathcal{R}$, the set of $\mathcal{SHOIQ}$ concepts is the smallest set containing $\top$, $\bot$, A, $\{a\}$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\exists R.C$, $\forall R.C$, $\geqslant n\, S.C$, and $\leqslant n\, S.C$, where A is an atomic concept, a an individual, C and D concepts, R a role, S a simple role w.r.t. $\mathcal{R}$, and n a non-negative integer. A $\mathcal{SHOIQ}$ *TBox* is a finite set $\mathcal{T}$ of *generalized concept inclusion axioms* (GCIs) $C \sqsubseteq D$ where C and D are concepts. We write $C \equiv D$ as an abbreviation for $C \sqsubseteq D$ and $D \sqsubseteq C$. A $\mathcal{SHOIQ}$ *ABox* is a finite set consisting of *concept assertions* $C(a)$ and *role assertions* $R(a, b)$ where a and b are individuals from I_Σ. A $\mathcal{SHOIQ}$ *ontology* is a triple $\mathcal{O} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$, where $\mathcal{R}$ is a $\mathcal{SHOIQ}$ RBox, $\mathcal{T}$ a $\mathcal{SHOIQ}$ TBox, and $\mathcal{A}$ a $\mathcal{SHOIQ}$ ABox.

The interpretation $\mathcal{I}$ is extended to complex role, complex concepts, axioms, and assertions in the usual way [5]. $\mathcal{I}$ is a *model* of a $\mathcal{SHOIQ}$ ontology $\mathcal{O}$, if every axiom and assertion in $\mathcal{O}$ is satisfied in $\mathcal{I}$. A concept C is *(finitely) satisfiable* w.r.t. $\mathcal{O}$ if $C^\mathcal{I} \neq \emptyset$ for some (finite) model $\mathcal{I}$ of $\mathcal{O}$. It is well-known [6, 4] that the problem of concept satisfiability for $\mathcal{SHOIQ}$ is NExpTime-complete.

$\mathcal{SROIQ}$ [1] extends $\mathcal{SHOIQ}$ in several ways. (1) It provides for the *universal role* U, which is interpreted as the total relation: $U^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. (2) It allows for *negative role assertions* $\neg R(a, b)$. (3) It introduces a concept constructor $\exists S.\mathsf{Self}$ interpreted as $\{x \in \Delta^\mathcal{I} \mid \langle x, x\rangle \in S^\mathcal{I}\}$ where S is a simple role. (4) It allows for new role axioms $\mathsf{Sym}(R)$, $\mathsf{Ref}(R)$, $\mathsf{Asy}(S)$, $\mathsf{Irr}(S)$, $\mathsf{Disj}(S_1, S_2)$ where $S_{(i)}$ are simple roles, which restrict $R^\mathcal{I}$ to be *symmetric* or *reflexive*, $S^\mathcal{I}$ to be *asymmetric* or *irreflexive*, or $S_1^\mathcal{I}$ and $S_2^\mathcal{I}$ to be *disjoint*. (5) Finally, it allows for *complex role inclusion axioms* of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$, which require that $R_1^\mathcal{I} \circ \cdots \circ R_n^\mathcal{I} \subseteq R^\mathcal{I}$ where $\circ$ is the usual composition of binary relations. The notion of simple roles is adjusted to make sure that no simple role can be implied by a role composition. $\mathcal{SRIQ}$ [2] is the fragment of $\mathcal{SROIQ}$ without nominals.

The constructors (1)–(4) do not introduce too many difficulties in $\mathcal{SROIQ}$—the existing tableau procedure for $\mathcal{SHOIQ}$ [5] can be relatively easily adapted to support the new constructors. Dealing with complex role inclusion axioms in DLs turned out to be more difficult. First, with an exception of the DL $\mathcal{EL}^{++}$ [7], the unrestricted usage of complex RIAs easily leads to undecidability of modal and description logics [8, 3]. Therefore special syntactic restrictions have been introduced in $\mathcal{SROIQ}$ to regain decidability. A *regular order on roles* is an irreflexive transitive binary relation $\prec$ on roles such that $R_1 \prec R_2$ iff $\mathsf{Inv}(R_1) \prec R_2$. A RIA $R_1 \circ \cdots \circ R_n \sqsubseteq R$ is said to be $\prec$*-regular*, if it does not contain the universal role U and either: (i) $n = 2$ and $R_1 = R_2 = R$, or (ii) $n = 1$ and $R_1 = \mathsf{Inv}(R)$, or (iii) $R_i \prec R$ for $1 \leq i \leq n$, or (iv) $R_1 = R$ and $R_i \prec R$ for $1 < i \leq n$, or (v) $R_n = R$ and $R_i \prec R$ for $1 \leq i < n$.

Example 1. Consider the complex RIA (1). This RIA is not $\prec$-regular regardless of the choice for the ordering $\prec$. Indeed, (1) does not satisfy (i)–(ii) since $n = 3$, and does not satisfy (iii)–(iv) since $v = R_2 \not\prec R = v$.

$$r \circ v \circ r \sqsubseteq v \tag{1}$$

$$v_i \circ v_i \sqsubseteq v_{i+1}, \quad 0 \leq i < n \tag{2}$$

As an example of $\prec$-regular complex RIAs, consider axioms (2) over the atomic roles $v_0, \ldots, v_n$. It is easy to see that these axioms satisfy condition (*iii*) of $\prec$-regularity for every ordering $\prec$ such that $v_i \prec v_j$, for every $0 \leq i < j \leq n$.

Although Example 1 does not demonstrate the usage of the conditions (*i*), (*ii*), (*iv*) and (*v*) for $\prec$-regularity of RIAs, as will be shown soon, axioms that satisfy just the condition (*iii*) already make reasoning in $\mathcal{SROIQ}$ hard.

The syntactic restrictions on the set of RIAs of an RBox $\mathcal{R}$ ensure that $\mathcal{R}$ is *regular* in the following sense. Given a role R, let $L_{\mathcal{R}}(R)$ be the language consisting of the words over roles defined by:

$$L_{\mathcal{R}}(R) := \{R_1 R_2 \ldots R_n \mid \mathcal{R} \models (R_1 \circ \cdots \circ R_n \sqsubseteq R)\}$$

It has been shown in [3] that if the RIAs of $\mathcal{R}$ are $\prec$-regular for some ordering $\prec$, then for every role R, the language $L_{\mathcal{R}}(R)$ is regular. The tableau procedure for $\mathcal{SROIQ}$ presented in [1], utilizes the non-deterministic finite automata (NFA) corresponding to $L_{\mathcal{R}}(R)$ to ensure that only finitely many states are produced by tableau expansion rules. Unfortunately, the NFA for $L_{\mathcal{R}}(R)$ can be exponentially large in the size of $\mathcal{R}$, which results in exponential blowup in the number of states produced in the worst case by the procedure for $\mathcal{SROIQ}$ compared to the procedure for $\mathcal{SHOIQ}$. It was conjectured in [1] that without further restrictions on RIAs such blowup is unavoidable. In Example 2, we demonstrate that minimal automata for regular RBoxes can be exponentially large.

Example 2 (Example 1 continued). Let $\mathcal{R}$ be an RBox consisting of the single axiom (1). It is easy to see that $L_{\mathcal{R}}(s) = \{r^i v r^i \mid i \geq 0\}$, where r^i denotes the word consisting of i letters r. The language $L_{\mathcal{R}}(v)$ is non-regular, which can be shown, e.g., by using the pumping lemma for regular languages (see, e.g., [9]).

On the other hand, the RBox $\mathcal{R}$ consisting of the axioms (2) gives regular languages. It is easy to show by induction on i that $L_{\mathcal{R}}(v_i)$ consist of finitely many words, and hence, are regular. Moreover, by induction on i it is easy to show that $v_0^j \in L_{\mathcal{R}}(v_i)$ iff $j = 2^i$. Let $B_{\mathcal{R}}(v_i)$ be an NFA for $L_{\mathcal{R}}(v_i)$ and $q_0, \ldots, q_{2^i}$ a run in $B_{\mathcal{R}}(v_i)$ accepting $v_0^{2^i}$. Then all states in this run are different, since otherwise there is a cycle, which would mean that $B_{\mathcal{R}}(v_i)$ accepts infinitely many words. Hence $B_{\mathcal{R}}(v_i)$ has at least $2^i + 1$ states.

2 The Lower Complexity Bounds

In this section, we prove that reasoning in $\mathcal{SROIF}$—a fragment of $\mathcal{SROIQ}$ that includes functional roles instead of number restrictions—is N2ExpTime-hard. The proof is by reduction from the doubly-exponential Domino tiling problem. We also demonstrate that reasoning in $\mathcal{SR}$—a fragment of $\mathcal{SRIQ}$ that does not use counting and inverse roles—is 2ExpTime-hard by reduction from the word problem for an exponential-space alternating Turing machine.

The main idea of our reductions is to enforce double-exponentially long chains using $\mathcal{SR}$ axioms. Single-exponentially long chains can be enforced using a well-known "integer counting" technique [6]. A *counter* $c^{\mathcal{I}}(x)$ is an integer between

0 and $2^n - 1$ that is assigned to an element x of the interpretation $\mathcal{I}$ using n atomic concepts $B_1, \ldots, B_n$ as follows: the i-th bit of $c^{\mathcal{I}}(x)$ is equal to 1 if and only if $x \in B_i^{\mathcal{I}}$. It is easy to see that axioms (3)–(7) induce an exponentially long r-chain by initializing the counter and incrementing it over the role r.

$$Z \equiv \neg B_1 \sqcap \cdots \sqcap \neg B_n \tag{3}$$

$$E \equiv B_1 \sqcap \cdots \sqcap B_n \tag{4}$$

$$\neg E \equiv \exists r.\top \tag{5}$$

$$\top \equiv (B_1 \sqcap \forall r.\neg B_1) \sqcup (\neg B_1 \sqcap \forall r.B_1) \tag{6}$$

$$B_{i-1} \sqcap \forall r.\neg B_{i-1} \equiv (B_i \sqcap \forall r.\neg B_i) \sqcup (\neg B_i \sqcap \forall r.B_i), \quad 1 < i \leq n \tag{7}$$

Axiom (3) is responsible for initializing the counter to zero using the atomic concept Z. Axiom (4) can be used to detect whether the counter has reached the final value $2^n - 1$, by checking whether E holds. Thus, using axiom (5), we can express that an element has an r-successor if and only if its counter has not reached the final value. Axioms (6) and (7) express how the counter is incremented over r: axiom (6) expresses that the lowest bit of the counter is always flipped; axioms (7) express that any other bit of the counter is flipped if and only if the lower bit is changed from 1 to 0.

Lemma 1. *Let $\mathcal{O}$ be an ontology containing axioms (3)–(7). Then for every model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{O}$ and $x \in Z^{\mathcal{I}}$ there exist $x_i \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ such that $x = x_0$ and $\langle x_{i-1}, x_i \rangle \in r^{\mathcal{I}}$ for every i with $1 \leq i < 2^n$, and $c^{\mathcal{I}}(x_i) = i$.*

Now we use similar ideas to enforce double-exponentially long chains in the model. This time, however, we cannot use just atomic concepts to encode the bits of the counter since there are exponentially many bits. Therefore, we assign a counter not to elements but to exponentially long r-chains induced by axioms (3)–(7) using one atomic concept X: the i-th bit of the counter corresponds to the value of X at the i-th element of the chain. In Figure 1(a) we have depicted a doubly-exponential chain formed for the sake of presentation as a "zig-zag" that we are going to induce using $\mathcal{SR}$ axioms. The chain consists of 2^{2^n} r-chains, each having exactly 2^n elements, that are joined together using a role v—the last element of every r-chain, except for the final chain, is v-connected to the first element of the next r-chain. The tricky part is to ensure that the counters corresponding to r-chains are properly incremented. This is achieved by using the regular role inclusion axioms from (2), which allow us to propagate information using a role v_n across chains of exactly 2^n roles. The structure in Figure 1(a) is enforced using axioms (8)–(15) in addition to axioms (2)–(7).

$$O \sqsubseteq Z \sqcap Z_v \sqcap E_v \tag{8}$$

$$\top \sqsubseteq \forall v.(Z \sqcap E_v) \tag{9}$$

$$Z_v \sqsubseteq \neg X \sqcap \forall r.Z_v \tag{10}$$

$$E_v \sqcap X \sqsubseteq \forall r.E_v \tag{11}$$

$$\neg(E_v \sqcap X) \sqsubseteq \forall r.\neg E_v \tag{12}$$

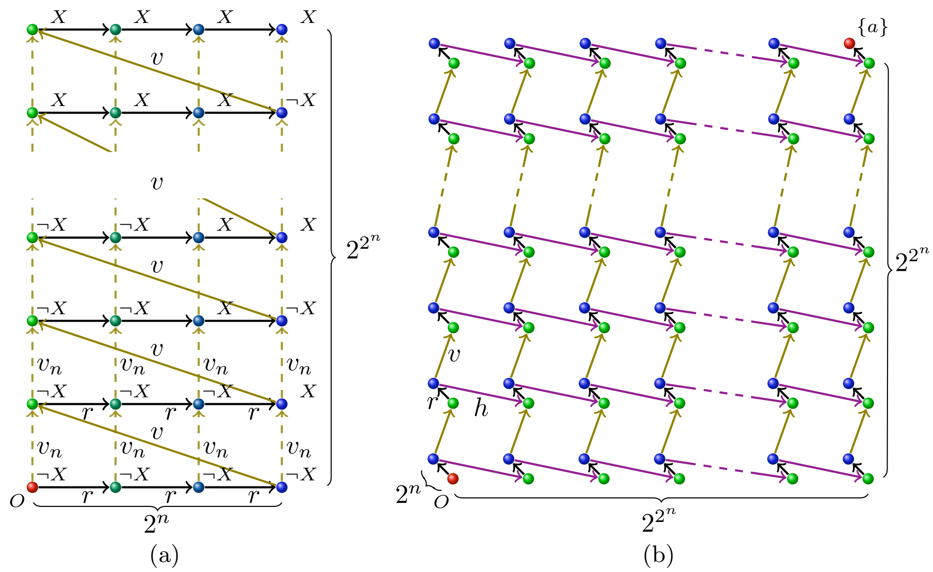

Fig. 1. (a) Using $\mathcal{SR}$ axioms to encode double-exponentially long chains; (b) Using $\mathcal{SROIF}$ axioms to encode double-exponentially large grids

$$E \sqcap \neg(E_v \sqcap X) \sqsubseteq \exists v.\top \tag{13}$$

$$r \sqsubseteq v_0, \quad v \sqsubseteq v_0 \tag{14}$$

$$\forall r.(X \sqcap \forall v_n.\neg X) \equiv (X \sqcap \forall v_n.\neg X) \sqcup (\neg X \sqcap \forall v_n.X) \tag{15}$$

The atomic concept O corresponds to the origin of our structure. Axioms (8) and (9) express that O and every v-successor start a new 2^n-long r-chain because of the atomic concept Z and axioms (3)–(7). In addition, the r-chain starting from O should be initialized to "zero" using Z_v and axiom (10). In order to identify the final chain, we use the atomic concept E_v which should hold on an element of an r-chain iff X holds on all preceding elements of this r-chain. Axioms (8) and (9) say that E_v holds at the first element of every r-chain. Axioms (11) and (12) propagate the value of E_v over the elements of the r-chain. Now, axiom (13) says that the last element of every non-final r-chain has a v-successor.

Axioms (14) and (15) together with axioms (2) are responsible for incrementing the counter between r-chains. Recall that axioms from (2) imply $(v_0)^i \sqsubseteq v_n$ if and only if $i = 2^n$, where $(v_0)^i$ denotes the composition of the role v_0 i times. Now, using axioms (14) we make sure that exactly the corresponding elements of the consequent r-chains are connected by the role v_n. Finally, axiom (15) expresses the transformation of bits in a similar way as axioms (6) and (7): the value of X for the last element of every r-chain is always flipped over v_n since there is no r-successor due to axiom (5); the value of X for every other element is flipped if and only if for all its r-successors X is changed to $\neg X$ over v_n.

Lemma 2. *For every model* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *of every ontology* $\mathcal{O}$ *containing axioms (2)–(15), and every* $x \in O^{\mathcal{I}}$ *there exist* $x_{(i,j)} \in \Delta^{\mathcal{I}}$ *with* $0 \le i < 2^n$ *and* $0 \le j < 2^{2^n}$ *such that (i)* $x = x_{(0,0)}$*, (ii)* $\langle x_{(i-1,j)}, x_{(i,j)} \rangle \in r^{\mathcal{I}}$ *when* $i \ge 1$*, and (iii)* $\langle x_{(2^n-1,j-1)}, x_{(0,j)} \rangle \in v^{\mathcal{I}}$ *when* $j \ge 1$.

Now we demonstrate that using $\mathcal{SROIF}$ axioms one can express the grid-like structure in Figure 1(b). Our construction is like for $\mathcal{ALCOIQ}$ in [6], which uses a pair of counters to encode the coordinates of the grid and a nominal with inverse functionality to join the elements with the same coordinates. The only difference is that we use the counters up to 2^{2^n} instead of just up to 2^n.

The grid-like structure in Figure 1(b) consists of $2^{2^n} \times 2^{2^n}$ 2^n-long r-chains which are joined vertically using the role v and horizontally using the role h in the same way as in Figure 1(a). Every r-chain stores information about two counters. The first counter uses the concept name X and corresponds to the vertical coordinate of the r-chain; the second counter uses Y and corresponds to the horizontal coordinate of the r-chain. The axioms (2)–(15) are now used to express that the vertical counter for r-chains is initialized in O and is incremented over v. A copy of these axioms (16)–(24) expresses the analogous property for the horizontal counter.

$$O \sqsubseteq Z \sqcap Z_h \sqcap E_h \tag{16}$$
$$\top \sqsubseteq \forall v.(Z \sqcap E_h) \tag{17}$$
$$Z_h \sqsubseteq \neg Y \sqcap \forall r.Z_h \tag{18}$$
$$E_h \sqcap Y \sqsubseteq \forall r.E_h \tag{19}$$
$$\neg(E_h \sqcap Y) \sqsubseteq \forall r.\neg E_h \tag{20}$$
$$E \sqcap \neg(E_h \sqcap Y) \sqsubseteq \exists v.\top \tag{21}$$
$$r \sqsubseteq h_0, \quad h \sqsubseteq h_0 \tag{22}$$
$$h_i \circ h_i \sqsubseteq h_{i+1}, \quad 0 \le i < n \tag{23}$$
$$\forall r.(Y \sqcap \forall h_n.\neg Y) \equiv (Y \sqcap \forall h_n.\neg Y) \sqcup (\neg Y \sqcap \forall h_n.Y) \tag{24}$$

The grid structure in Figure 1(b) is now enforced by adding axioms (25)–(28).

$$\top \sqsubseteq (X \sqcap \forall h_n.X) \sqcup (\neg X \sqcap \forall h_n.\neg X) \tag{25}$$
$$\top \sqsubseteq (Y \sqcap \forall v_n.Y) \sqcup (\neg Y \sqcap \forall v_n.\neg Y) \tag{26}$$
$$E_v \sqcap X \sqcap E_h \sqcap Y \sqsubseteq \{a\} \tag{27}$$
$$\mathsf{Fun}(r^-),\ \mathsf{Fun}(h^-),\ \mathsf{Fun}(v^-) \tag{28}$$

Axioms (25) and (26) express that the values of the vertical / horizontal counters are copied across h / respectively v. Axiom (27) expresses that the last element of the r-chain with the final coordinates is unique. Together with axiom (28) expressing that the roles r, h and v are inverse functional, this ensures that no two different r-chains have the same coordinates. Note that the roles r, h and v are simple since they do not occur at the right hand side of RIAs (2), (14), (22), and (23). The following analogue of Lemmas 1 and 2 claims that the models of our axioms that satisfy O correspond to the grid in Figure 1(b).

Lemma 3. *For every model* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *of every ontology* $\mathcal{O}$ *containing axioms* (2)–(28)*, and every* $x \in O^{\mathcal{I}}$ *there exist* $x_{(i,j,k)} \in \Delta^{\mathcal{I}}$ *with* $0 \leq i < 2^n$, $0 \leq j, k < 2^{2^n}$ *such that* (i) $x = x_{(0,0,0)}$, (ii) $\langle x_{(i-1,j,k)}, x_{(i,j,k)} \rangle \in r^{\mathcal{I}}$ *when* $i \geq 1$, (iii) $\langle x_{(2^n-1,j-1,k)}, x_{(0,j,k)} \rangle \in v^{\mathcal{I}}$ *when* $j \geq 1$*, and* (iv) $\langle x_{(2^n-1,j,k-1)}, x_{(0,j,k)} \rangle \in h^{\mathcal{I}}$ *when* $k \geq 1$.

Our complexity result for $\mathcal{SROIF}$ is obtained by a reduction from the bounded domino tiling problem. A *domino system* is a triple $\mathcal{D} = (T, H, V)$, where $T = \{1, \dots, k\}$ is a finite set of *tiles* and $H, V \subseteq T \times T$ are *horizontal* and *vertical matching relations*. A *tiling* of $m \times m$ for a domino system $\mathcal{D}$ with *initial condition* $c^0 = \langle t_1^0, \dots, t_n^0 \rangle$, $t_i^0 \in T$, $1 \leq i \leq n$, is a mapping $t : \{1, \dots, m\} \times \{1, \dots, m\} \to T$ such that $\langle t(i-1, j), t(i, j) \rangle \in H$, $1 < i \leq m$, $1 \leq j \leq m$, $\langle t(i, j-1), t(i, j) \rangle \in V$, $1 < i \leq m$, $1 \leq j \leq m$, and $t(i, 1) = t_i^0$, $1 \leq i \leq n$. It is well known [10] that there exists a domino system $\mathcal{D}_0$ that is N2ExpTime-complete for the following decision problem: given an initial condition c^0 of the size n, check if $\mathcal{D}_0$ admits the tiling of $2^{2^n} \times 2^{2^n}$ for c^0. Axioms (29)–(34) in addition to axioms (2)–(28) provide a reduction from this problem to the problem of concept satisfiability in $\mathcal{SROIF}$.

$$\top \sqsubseteq D_1 \sqcup \dots \sqcup D_k \tag{29}$$

$$D_i \sqcap D_j \sqsubseteq \bot, \quad 1 \leq i < j \leq k \tag{30}$$

$$D_i \sqsubseteq \forall r.D_i, \quad 1 \leq i \leq k \tag{31}$$

$$D_i \sqcap \exists h.D_j \sqsubseteq \bot, \quad \langle i, j \rangle \notin H \tag{32}$$

$$D_i \sqcap \exists v.D_j \sqsubseteq \bot, \quad \langle i, j \rangle \notin V \tag{33}$$

$$O \sqsubseteq D_{t_1^0} \sqcap \forall h_n.(D_{t_2^0} \sqcap \forall h_n.(D_{t_3^0} \sqcap \cdots (\forall h_n.D_{t_n^0}) \cdots)) \tag{34}$$

The atomic concepts $D_1, \dots, D_k$ correspond to the tiles of the domino system $\mathcal{D}_0$. Axioms (29) and (30) express that every element in the model is assigned with a unique tile D_i. Axiom (31) expresses that the elements of the same r-chain are assigned with the same tile. Axioms (32) and (33) express the horizontal and vertical matching properties. Finally, axiom (34) expresses the initial condition. It is easy to see that this reduction is polynomial in n ($\mathcal{D}_0$ is fixed).

Theorem 1. *Let* c^0 *be an initial condition of size* n *for the domino system* $\mathcal{D}_0$ *and* $\mathcal{O}$ *an ontology consisting of the axioms* (2)–(34)*. Then* $\mathcal{D}_0$ *admits the tiling of* $2^{2^n} \times 2^{2^n}$ *for* c^0 *if and only if* O *is (finitely) satisfiable in* $\mathcal{O}$.

Proof (sketch). It is easy to show that if $\mathcal{D}_0$ admits the tiling of $2^{2^n} \times 2^{2^n}$ for c^0 then the structure in Figure 1(b) (which finitely satisfies O) can be expanded to a model of $\mathcal{O}$ by interpreting D_i accordingly. On the other hand, it is easy to show using Lemma 3 that any model of $\mathcal{O}$ that satisfies O witnesses a tiling of $2^{2^n} \times 2^{2^n}$ for c^0. □

Corollary 1. *The problem of (finite) concept satisfiability in the DL* $\mathcal{SROIF}$ *is* N2ExpTime*-hard (and so are all the standard reasoning problems).*

In the remainder of this section, we prove that (finite model) reasoning in $\mathcal{SR}$ is 2ExpTime-hard. The proof is by reduction from the word problem of an exponential-space alternating Turing machine. The main idea of our reduction is to use the zig-zag-like structures in Figure 1(a) to simulate a computation of an alternating Turing machine.

An *alternating Turning machine* (ATM) is a tuple $M = (\Gamma, \Sigma, Q, q_0, \delta_1, \delta_2)$ where Γ is a finite *working alphabet* containing a *blank symbol* $\boxdot$; $\Sigma \subseteq \Gamma \setminus \{\boxdot\}$ is the *input alphabet*; $Q = Q_\exists \uplus Q_\forall \uplus \{q_a\} \uplus \{q_r\}$ is a finite set of *states* partitioned into *existential states* $Q_\exists$, *universal states* $Q_\forall$, an *accepting state* q_a and a *rejecting state* q_r; $q_0 \in Q_\exists$ is the *starting state*, and $\delta_1, \delta_2 : (Q_\exists \cup Q_\forall) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ are *transition functions*. A *configuration* of M is a word $c = w_1 q w_2$ where $w_1, w_2 \in \Gamma^*$ and $q \in Q$. An *initial configuration* is $c^0 = q_0 w^0$ where $w^0 \in \Sigma^*$. The *size* $|c|$ of a configuration c is the number of symbols in c. The *successor configurations* $\delta_1(c)$ and $\delta_2(c)$ of a configuration $c = w_1 q w_2$ with $q \neq q_a, q_r$ over the transition functions δ_1 and δ_2 are defined like for deterministic Turing machines (see, e.g., [9]). The sets $C_a(M)$ of *accepting configurations* and $C_r(M)$ of *rejecting configurations* of M are the smallest sets such that (i) $c = w_1 q w_2 \in C_a(M)$ if either $q = q_a$, or $q \in Q_\forall$ and $\delta_1(c), \delta_2(c) \in C_a(M)$, or $q \in Q_\exists$ and $\delta_1(c) \in C_a(M)$ or $\delta_2(c) \in C_a(M)$, and (ii) $c = w_1 q w_2 \in C_r(M)$ if either $q = q_r$, or $q \in Q_\exists$ and $\delta_1(c), \delta_2(c) \in C_r(M)$, or $q \in Q_\forall$ and $\delta_1(c) \in C_r(M)$ or $\delta_2(c) \in C_r(M)$. The set of configurations *reachable from* an initial configuration c^0 in M is the smallest set $M(c^0)$ such that $c^0 \in M(c^0)$ and $\delta_1(c), \delta_2(c) \in M(c^0)$ for every $c \in M(c^0)$. M is $g(n)$ *space bounded* if for every initial configuration c^0 we have: (i) $c^0 \in C_a(M) \cup C_r(M)$, and (ii) $|c| \leq g(|c^0|)$ for every $c \in M(c^0)$. A classical result AExpSpace = 2ExpTime [11] implies that there exists a 2^n space bounded ATM M_0 for which the following decision problem is 2ExpTime-complete: given an initial configuration c^0 decide whether $c^0 \in C_a(M_0)$.

Let c^0 be an initial configuration of M_0 and $n = |c^0|$ (w.l.o.g., assume that $n > 2$). In order to decide whether $c^0 \in C_a(M_0)$, we try to build all the required accepting successor configurations of c^0 from $M_0(c^0)$. We encode the configurations of $M_0(c^0)$ on 2^n-long r-chains. An r-chain corresponding to c is connected to r-chains corresponding to $\delta_1(c)$ and $\delta_2(c)$ via the roles v and h in a similar way as in Figure 1(a). It is a well-known property of the transition functions of Turing machines that the symbols c_i^1 and c_i^2 at the position i of $\delta_1(c)$ and $\delta_2(c)$ are uniquely determined by the symbols c_{i-1}, c_i, c_{i+1}, and c_{i+2} of c at the positions $i-1$, i, $i+1$, and $i+2$.[2] We assume that this correspondence is given by the (partial) functions γ_1 and γ_2 such that $\gamma_1(c_{i-1}, c_i, c_{i+1}, c_{i+2}) = c_i^1$ and $\gamma_2(c_{i-1}, c_i, c_{i+1}, c_{i+2}) = c_i^2$. The computation of M_0 from c^0 can be encoded using axioms (35)–(47) in addition to axioms (2)–(24).

$$\top \sqsubseteq \bigsqcup_{s \in Q \cup \Gamma} A_s \tag{35}$$

$$A_{s_1} \sqcap A_{s_2} \sqsubseteq \bot, \quad s_1 \neq s_2 \tag{36}$$

[2] If any of the indexes $i-1$, $i+1$, or $i+2$ are out of range for the configuration c, we assume that the correspondent symbols c_{i-1}, c_{i+1}, or c_{i+2} are the blank symbol $\boxdot$

$$Z \sqcap A_{s_2} \sqcap \exists r.(A_{s_3} \sqcap \exists r.A_{s_4}) \sqsubseteq S_{\boxdot s_2 s_3 s_4} \quad (37)$$

$$A_{s_1} \sqcap \exists r.(A_{s_2} \sqcap \exists r.(A_{s_3} \sqcap \exists r.A_{s_4})) \sqsubseteq \forall r.S_{s_1 s_2 s_3 s_4} \quad (38)$$

$$A_{s_1} \sqcap \exists r.(A_{s_2} \sqcap \exists r.(A_{s_3} \sqcap E)) \sqsubseteq \forall r.S_{s_1 s_2 s_3 \boxdot} \quad (39)$$

$$A_{s_1} \sqcap \exists r.(A_{s_2} \sqcap E) \sqsubseteq \forall r.S_{s_1 s_2 \boxdot\boxdot} \quad (40)$$

$$S_{s_1 s_2 s_3 s_4} \sqsubseteq \forall v_n.A_{\gamma_1(s_1,s_2,s_3,s_4)} \sqcap \forall h_n.A_{\gamma_2(s_1,s_2,s_3,s_4)} \quad (41)$$

$$O \sqsubseteq A \sqcap A_{c_1^0} \sqcap \forall r.(A_{c_2^0} \sqcap \cdots (\forall r.A_{c_n^0} \sqcap \forall r.Z_{\boxdot}) \cdots) \quad (42)$$

$$Z_{\boxdot} \sqsubseteq A_{\boxdot} \sqcap \forall r.Z_{\boxdot} \quad (43)$$

$$A \sqsubseteq \forall r.A, \ \neg A \sqsubseteq \forall r.\neg A \quad (44)$$

$$A \sqcap A_q \sqsubseteq \forall v_n.A \sqcup \forall h_n.A, \quad q \in Q_{\exists} \quad (45)$$

$$A \sqcap A_q \sqsubseteq \forall v_n.A \sqcap \forall h_n.A, \quad q \in Q_{\forall} \quad (46)$$

$$A \sqcap A_{q_r} \sqsubseteq \bot \quad (47)$$

We introduce an atomic concept A_s for every s from the set of states Q and the working alphabet Γ. Axioms (35) and (36) express that every element of the model is assigned with a unique symbol $s \in Q \cup \Gamma$. Axioms (37)–(41) express how the successor configurations $\delta_1(c)$ and $\delta_2(c)$ of a configuration c are computed: the concept names $S_{s_1 s_2 s_3 s_4}$ initialized by axioms (37)–(40) express that the current element of the r-chain is assigned with the symbol s_2, its r-predecessor with s_1 and its next two r-successors with s_3 and s_4 (s_1, s_3, and s_4 are $\boxdot$ if there are no such elements); (41) then expresses how the corresponding symbols in the successor r-chains are computed using the functions γ_1 and γ_2. Axioms (42) and (43) initialize c^0. In axioms (42) and (44)–(47), we use the atomic concept A to express that the configuration encoded by the r-chain is accepting. Axiom (44) ensures that all elements of the same r-chain have the same values of A. Axioms (45) and (46) express which successor configurations are accepting depending on whether the current state is existential or universal. Finally, axiom (47) expresses that a configuration with the rejecting state cannot be accepting.

Theorem 2. *Let c^0 be an initial configuration for the ATM M_0 and $\mathcal{O}$ an ontology consisting of the axioms* (2)–(24) *and* (35)–(47)*. Then $c^0 \in C_a(M_0)$ if and only if O is (finitely) satisfiable in $\mathcal{O}$.*

Corollary 2. *The problem of (finite) concept satisfiability in the DL $\mathcal{SR}$ is 2ExpTime-hard (and so are all the standard reasoning problems).*

3 The Upper Complexity Bound

In this section we prove that complexity of $\mathcal{SROIQ}$ is in N2ExpTime using an exponential-time translation into the two variable fragment with counting $\mathcal{C}^2$.

Let $\mathcal{O}$ be a $\mathcal{SROIQ}$ ontology for which we need to test satisfiability. By Theorem 9 from [1], w.l.o.g., we can assume that $\mathcal{O}$ does not contain concept and role assertions, the universal role, or axioms of the form $\mathsf{Irr}(S)$, $\mathsf{Tra}(R)$ or $\mathsf{Sym}(R)$. We also replace $\mathsf{Asy}(S)$ with $\mathsf{Disj}(S, \mathsf{Inv}(S))$ and $\mathsf{Ref}(R)$ with $s \sqsubseteq R$

1	$A \sqsubseteq \forall r.B$	$\forall x.(A(x) \rightarrow \forall y.[r(x,y) \rightarrow B(y)])$
2	$A \sqsubseteq\, \geqslant n\, s.B$	$\forall x.(A(x) \rightarrow \exists^{\geq n} y.[s(x,y) \wedge B(y)])$
3	$A \sqsubseteq\, \leqslant n\, s.B$	$\forall x.(A(x) \rightarrow \exists^{\leq n} y.[s(x,y) \wedge B(y)])$
4	$A \equiv \exists s.\mathsf{Self}$	$\forall x.(A(x) \leftrightarrow s(x,x))$
5	$A_a \equiv \{a\}$	$\exists^{=1} y.A_a(y)$
6	$\bigsqcap A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigvee \neg A_i(x) \vee \bigvee B_j(x))$
7	$\mathsf{Disj}(s_1, s_2)$	$\forall xy.(s_1(x,y) \wedge s_2(x,y) \rightarrow \bot)$
8	$s_1 \sqsubseteq s_2$	$\forall xy.(s_1(x,y) \rightarrow s_2(x,y))$
9	$s_1 \sqsubseteq s_2^-$	$\forall xy.(s_1(x,y) \rightarrow s_2(y,x))$
10	$r_1 \circ \cdots \circ r_n \sqsubseteq v, \quad n \geq 1$	

Table 1. Translation of simplified $\mathcal{SROIQ}$ axioms into $\mathcal{C}^2$

and $\top \sqsubseteq \exists s.\mathsf{Self}$, where s is a fresh (simple) role. Next, we convert $\mathcal{O}$ into the *simplified form* which contains only axioms of the form given in the first column of Table 1, where $A_{(i)}$ and $B_{(j)}$ are atomic concepts, $r_{(i)}$ atomic roles, $s_{(i)}$ simple atomic roles, and v a non-simple atomic role. The transformation can be done in polynomial time using the standard structural transformation which iteratively introduces definitions for compound sub-concept and sub-roles (see, e.g. [12]).

After the transformation, we eliminate RIAs of the form 10 using a technique from [13]. Axioms of the form 10 can cause unsatisfiability of $\mathcal{O}$ only through axioms of the form 1, since other axioms do not contain non-simple roles. Given an axiom $A \sqsubseteq \forall v.B$ of the form 1 where v is non-simple, and an NFA for $L_{\mathcal{R}}(v)$ with the (possibly exponential) set of states Q, starting state $q_0 \in Q$, accepting states $F \subseteq Q$, and transition relation $\delta \subseteq Q \times R_{\Sigma} \times Q$, we replace this axiom with axioms (48)–(50) where A_q^r is a fresh atomic concept for every $q \in Q$.

$$A \sqsubseteq A_q^r, \qquad q = q_0 \tag{48}$$

$$A_{q_1}^r \sqsubseteq \forall s.A_{q_2}^r, \qquad (q_1, s, q_2) \in \delta \tag{49}$$

$$A_q^r \sqsubseteq B, \qquad q \in F \tag{50}$$

Lemma 4. *Let $\mathcal{O}$ be an ontology consisting of axioms of the form* 1–10 *from Table 1, and $\mathcal{O}'$ obtained from $\mathcal{O}$ by replacing every axiom $A \sqsubseteq \forall v.B$ with axioms* (48)–(50) *and removing all axioms of the form* 10*. Then* (1) *every model of $\mathcal{O}$ can be expanded to a model of $\mathcal{O}'$ by interpreting A_q^r, and* (2) *every model of $\mathcal{O}'$ can be expanded to a model of $\mathcal{O}$ by interpreting the non-simple roles.*

Theorem 3. *(Finite) satisfiability of $\mathcal{SROIQ}$ ontologies is in* N2ExpTime *(and so are all the standard reasoning problems).*

Proof. The input $\mathcal{SROIQ}$ ontology $\mathcal{O}$ can be translated in exponential time preserving (finite) satisfiability into a simplified ontology containing only axioms of the form 1–9 from Table 1, which can be translated into the two variable fragment with counting quantifiers $\mathcal{C}^2$ according to the second column of Table 1. Since (finite) satisfiability of $\mathcal{C}^2$ is NExpTime-complete [14], our reduction proves that (finite) satisfiability of $\mathcal{SROIQ}$ is in N2ExpTime. □

4 Conclusions

In this paper we have identified the exact computational complexity of (finite model) reasoning in $\mathcal{SROIQ}$ to be N2ExpTime—that is, exponentially harder than for $\mathcal{SHOIQ}$. The complexity blowup is due to complex role inclusion axioms, and in particular due to their ability to "chain" a fixed exponential number of roles. Indeed, the complexity blowup occurs even when no other complex constructors such as nominals, number restrictions and inverse roles are used: $\mathcal{SR}$ and therefore $\mathcal{SRIQ}$ is 2ExpTime-hard, whereas $\mathcal{SHIQ}$ is merely in ExpTime. Our complexity results prove that the exponential blowups in the tableau procedures for $\mathcal{SRIQ}$ [2] and $\mathcal{SROIQ}$ [1] are unavoidable.

A few open questions are left for the future work. First, we did not obtain the upper complexity bound for $\mathcal{SRIQ}$. We conjecture that $\mathcal{SRIQ}$ is 2ExpTime-complete. Second, our proofs do not work for linear RIAs of the form $R_1 \circ R_2 \sqsubseteq R_1$ or $R_2 \circ R_1 \sqsubseteq R_1$ that have been originally used in $\mathcal{RIQ}$ [15]. It would be interesting to know if such RIAs also result in a higher complexity.

References

1. Horrocks, I., Kutz, O., Sattler, U.: The even more irresistible $\mathcal{SROIQ}$. In: KR. (2006) 57–67
2. Horrocks, I., Kutz, O., Sattler, U.: The irresistible $\mathcal{SRIQ}$. In: Proc. of the First OWL Experiences and Directions Workshop. (2005)
3. Horrocks, I., Sattler, U.: Decidability of $\mathcal{SHIQ}$ with complex role inclusion axioms. Artif. Intell. **160**(1-2) (2004) 79–104
4. Tobies, S.: Complexity Results and Practical Algorithms for Logics in Knowledge Representation. PhD thesis, RWTH Aachen, Germany (2001)
5. Horrocks, I., Sattler, U.: A tableau decision procedure for $\mathcal{SHOIQ}$. J. Autom. Reasoning **39**(3) (2007) 249–276
6. Tobies, S.: The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. J. Artif. Intell. Res. (JAIR) **12** (2000) 199–217
7. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ envelope. In: Proc. IJCAI-05. (2005) 364–369
8. Demri, S.: The complexity of regularity in grammar logics and related modal logics. J. Log. Comput. **11**(6) (2001) 933–960
9. Sipser, M.: Introduction to the Theory of Computation, Second Edition. Course Technology (February 2005)
10. Börger, E., Grädel, E., Gurevich, Y.: The Classical Decision Problem. Perspectives of Mathematical Logic. Springer (1997) Second printing (Universitext) 2001.
11. Chandra, A.K., Kozen, D.C., Stockmeyer, L.J.: Alternation. J. ACM **28**(1) (1981) 114–133
12. Kazakov, Y., Motik, B.: A resolution-based decision procedure for $\mathcal{SHOIQ}$. Journal of Automated Reasoning **40**(2-3) (2008) 89–116
13. Demri, S., de Nivelle, H.: Deciding regular grammar logics with converse through first-order logic. Journal of Logic, Language and Information **14**(3) (2005) 289–329
14. Pratt-Hartmann, I.: Complexity of the two-variable fragment with counting quantifiers. Journal of Logic, Language and Information **14**(3) (2005) 369–395
15. Horrocks, I., Sattler, U.: Decidability of $\mathcal{SHIQ}$ with complex role inclusion axioms. In: IJCAI. (2003) 343–348