Synthesising Elementary Net Systems with Interval Order Semantics

Maciej Koutny[1], Marta Pietkiewicz-Koutny[1,*]

[1] *School of Computing, Newcastle University*
Urban Sciences Building, 1 Science Square, Newcastle Helix, Newcastle upon Tyne, NE4 5TG, United Kingdom

Abstract
Elementary net systems are fundamental Petri net models with very simple markings which are sets of places. Their standard semantics is based on sequences of executed transitions, which can be understood as (labelled) total orders. In this paper, we consider a newly proposed semantics based on interval (partial) orders which allows one to describe behaviours where transitions have non-atomic duration. For such a semantical model, we consider the net synthesis problem, and show that the standard notion a region of a transition system can still be applied.

1. Introduction

Petri nets are a general model of concurrent systems which emerged in the 1960's as a counterpart to the state machines that were used so successfully to model sequential systems. A particular advantage of Petri nets is that the model allows one to both specify concurrent system designs, and the behaviours of such systems. It is generally acknowledged that concepts related to fundamental notions of concurrency theory, such as causality and independence, can be well explained using the framework provided by Petri nets [1, 2, 3, 4]. A fundamental class of Petri nets in that respect are Elementary Net systems (EN-systems) [5].

In general, the execution semantics of Petri nets (i.e., the representation of individual runs or observations) is captured by total orders of executed transitions (or, equivalently, by firing sequences), or stratified orders of executed transitions where simultaneity is transitive (or, equivalently, by step sequences). Having said that, it was argued by Wiener in 1914 [6] (and later, more formally, in [7]) that any execution that can be observed by a single observer must be an interval order, and so the most precise (qualitative) observational semantics is based on interval orders, where simultaneity is often non-transitive.

In this paper, using EN-systems as a system model, we first show how one can generate interval order observations of their executions in a direct way, without the need to modify the original system specification (e.g., splitting transitions into explicit beginnings and endings) as it was done, for example, in [8, 9, 10]. We also define interval reachability graphs (IR-graphs) which can

Algorithms & Theories for the Analysis of Event Data and Petri Nets for Twin Transition 2023, June 26-27, 2023, Lisbon, Portugal
*Corresponding author.
maciej.koutny@ncl.ac.uk (M. Koutny); marta.koutny@ncl.ac.uk (M. Pietkiewicz-Koutny)
0000-0003-4563-1378 (M. Koutny)

be seen as finite generators of potentially infinite sets of interval orders defined by EN-systems. IR-graphs are a subclass of interval transition systems (IT-systems) which differ from the standard transition systems since instead of having their arcs labelled by executed transitions, they have states labelled by sets of transitions (interpreted as transitions currently being executed). Then, assuming the interval order semantics of EN-systems, we consider the problem of synthesising EN-systems from given IT-systems.

We approach the new synthesis problem using the standard synthesis approach based on the theory of regions [11, 12, 13]. If one considers sequential behaviours of nets, a transition system is realised by a net *iff* it is isomorphic to the sequential reachability graph (or case graph) of this net. Ehrenfeucht and Rozenberg investigated the realisation of transition systems by elementary nets and produced an axiomatic characterisation of the realisable transition systems in terms of regions [11, 12]. As in the existing literature about Petri net synthesis, the net realisable IT-systems are characterised by adapted State Separation and Forward Closure axioms.

2. Preliminaries

Labelled partial orders with domain elements representing executed actions are commonly used in concurrency theory to formalise different notions of dynamic semantics.

A (*strict labelled*) *partial order* is a triple $po = \langle X, \prec, \ell \rangle$ such that $X(=X_{po})$ is a set, $\prec (=\prec_{po})$ is a binary relation over X which is irreflexive and transitive, and $\ell(=\ell_{po})$ is a labelling for the elements of X. The maximal elements of po are $\max_{po} = \{x \in X \mid \neg\exists y \in X : \ x \prec y\}$, and $\mathrm{nomax}_{po} = X \setminus \max_{po}$ are the non-maximal elements. For all $x \neq y \in X$, $x \frown y$ if $x \not\prec y \not\prec x$.

If $X = \varnothing$ then po is the empty partial order denoted by $\varnothing$.

The partial order is *total* whenever, for all $x \neq y \in X$, $x \prec y$ or $y \prec x$. Moreover, it is *interval* whenever, for all $x, y, z, w \in X$, if $x \prec z$ and $y \prec w$ then $x \prec w$ or $y \prec z$. The adjective 'interval' derives from the following result.

Theorem 1 (Fishburn [14])**.** *A countable partial order $\langle X, \prec, \ell \rangle$ is interval iff there exists a total order $\langle Y, \lhd, \ell' \rangle$ and two injective mappings $\beta, \varepsilon : X \to Y$ such that $\beta(X) \cap \varepsilon(X) = \varnothing$ and, for all $x, y \in X$, $\beta(x) \lhd \varepsilon(x)$ and $x \prec y \iff \varepsilon(x) \lhd \beta(y)$.*

The mappings β and ε above are usually interpreted as providing the 'beginnings of' and 'endings of' actions represented by the elements of X.

The relevance of interval orders in concurrency theory follows from an observation, credited to Wiener [6], that any execution of a physical system that can be observed by a single observer must be an interval order. It implies that the most precise observational semantics should be defined in terms of interval orders (cf. [7]).

3. Elementary net systems and their standard semantics

Definition 1 (EN-system)**.** *An* elementary net system *(or* EN-*system) is a tuple $en = \langle P, T, F, m_0 \rangle$, where P and T are disjoint finite sets of* nodes*, called respectively* places *and* transitions*, $F \subseteq (T \times P) \cup (P \times T)$ is the* flow relation*, and $m_0 \subseteq P$ is the* initial marking *(in general, any subset of places is a* marking*). Moreover, we have the following:*

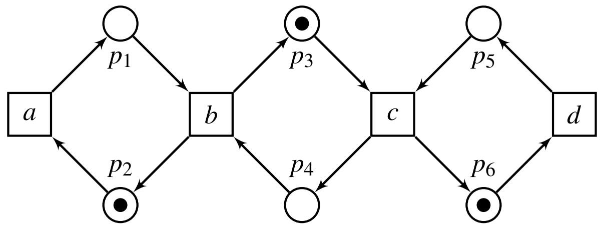

Figure 1: EN-system where $\overline{p_1} = p_2$, $\overline{p_3} = p_4$, and $\overline{p_5} = p_6$.

- *For every node x,* $^\bullet x = \{y \mid \langle y,x\rangle \in F\}$ *and* $x^\bullet = \{y \mid \langle x,y\rangle \in F\}$.
- *For every transition t,* $^\bullet t \neq \varnothing \neq t^\bullet$ *and* $^\bullet t \cap t^\bullet = \varnothing$.
- *For every place p, there is a unique* complement *place* $\overline{p}$ *such that:* $^\bullet p = \overline{p}^\bullet$, $p^\bullet = {}^\bullet\overline{p}$, *and* $p \in m_0 \iff \overline{p} \notin m_0$.

Note that the last part of Definition 1 is added to the standard one in order to simplify Definition 2.

In diagrams, places are represented by circles, transitions by rectangles, the flow relation by directed arcs, and a marking by small black dots drawn inside places belonging to the marking.

Example 1. *Figure 1 shows an* EN*-system. Intuitively, its three components represented by cyclic sub-nets progress independently, but any action shared by two components can be executed only if both of them do so.*

The dynamic behaviour of EN-systems is introduced by defining valid sequences of executed transitions, called *firing sequences*.

Definition 2 (firing sequence)**.** *Let* $en = \langle P,T,F,m_0\rangle$ *be an* EN*-system. The* firing sequences *of en, denoted by* SEQ_{en}*, are generated as follows.*

- *The empty sequence* λ *is a firing sequence of en, and it leads to marking* $\mathrm{mar}_\lambda = m_0$.
- *Let* σ *be a firing sequence of en leading to marking* mar_σ*, and t be a transition such that* $^\bullet t \subseteq \mathrm{mar}_\sigma$ *(t is enabled at* mar_σ*). Then* σt *is a firing sequence of en leading to marking* $\mathrm{mar}_{\sigma t} = \mathrm{mar}_\sigma \setminus {}^\bullet t \cup t^\bullet$.

Note that the last part of Definition 1 implies that, for all places $p \neq q$ and reachable markings m, $^\bullet p \neq {}^\bullet q$ or $p^\bullet \neq q^\bullet$ or $m(p) \neq m(q)$ (markings are treated here as characteristic functions of sets they represent).

Proposition 1. *Let* $en = \langle P,T,F,m_0\rangle$ *be an* EN*-system,* σ *be a firing sequence of en, and* $t \in T$ *be a transition such that* $^\bullet t \subseteq \mathrm{mar}_\sigma$. *Then* $t^\bullet \cap \mathrm{mar}_\sigma = \varnothing$.

Proof. Suppose that there is $p \in P$ such that $p \in t^\bullet \cap \mathrm{mar}_\sigma$. The complement place $\overline{p}$ of p which belongs to P (Definition 1) is such that $\overline{p} \in {}^\bullet t \subseteq \mathrm{mar}_\sigma$. But $p \in \mathrm{mar}_\sigma$ as well, which is not possible as p and $\overline{p}$ are complementary places. Hence we obtained a contradiction. $\square$

An alternative way of defining the standard semantics of EN-systems is by associating labelled total orders of transition occurrences with the executed interleaving sequences of transitions. In what follows, the i-th occurrence of transition t will be denoted by $t^{(i)}$ and called *event*.

Definition 3 (total orders of EN-system)**.** *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system. The* total orders *of* en*, denoted by* TPO_{en}*, are generated as follows.*

- *The empty total order* $tpo_{\varnothing}$ *is a total order of* en*, and it leads to marking* $\mathrm{mar}_{tpo_{\varnothing}} = m_0$.
- *Let* $tpo = \langle X, \prec, \ell \rangle$ *be a total order of* en *leading to marking* mar_{tpo}*, and* t *be a transition such that* ${}^{\bullet}t \subseteq \mathrm{mar}_{tpo}$*. Then,*

$$tpo' = \langle X \cup \{x\}, \prec \cup X \times \{x\}, \ell' \rangle \tag{1}$$

is a total order of en *leading to marking* $\mathrm{mar}_{tpo'} = \mathrm{mar}_{tpo} \setminus {}^{\bullet}t \cup t^{\bullet}$*, where:*

  - $x = t^{(n+1)}$ *with* n *being the number of the elements of* X *labelled by* t.
  - $\ell'|_X = \ell$ *and* $\ell'(x) = t$.

Proposition 2. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system. Then* TPO_{en} *is a set of labelled total orders.*

Proof. Follows directly from the definitions. □

There is a canonical way of associating a labelled total order with a finite sequence of transitions $\sigma = t_1 \dots t_k$ ($k \geq 0$), namely

$$\mathrm{seq2tpo}(\sigma) = \langle \{x_1, \dots, x_k\}, \prec, \ell \rangle ,$$

where $x_1 \prec \cdots \prec x_k$ and, moreover, each $x_i = t_i^{(k_i)}$ is such that $\ell(x_i) = t_i$, and k_i is the number of occurrences of t_i in the sub-sequence $t_1 \dots t_i$.

Proposition 3. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system. Then* seq2tpo *induces a bijection between* SEQ_{en} *and* TPO_{en}.

Proof. Follows directly from the definitions. □

4. Interval order semantics of EN-systems

The standard execution semantics of EN-systems implicitly assumes that events are executed instantaneously, or that their duration is negligible. Let us now assume that transitions are fired over intervals of arbitrary duration. Moreover, the firing of a transition t is *transaction-like*. By this we mean that the places in $t^{\bullet}$ are locked when t starts its execution, and if the execution is finished, then the tokens present in these places become available for firing.

Our aim is to define an abstract interval order semantics for en in as simple as possible way. Hence, we start with the traditional way in which the firing of transitions is carried out by taking a firing sequence $t_1 \dots t_k$. In such a view, the firings are instantaneous (or at least non-overlapping), and so they are implicitly ordered $t_1{}^{(m_1)} \prec \cdots \prec t_k{}^{(m_k)}$. Here, there exists an easy way of relating the executed transitions, which only takes into account the direct causal dependencies resulting from creating and consuming resources (tokens), viz. ${t_i}^{(m_i)} \prec_{causal} {t_j}^{(m_j)}$ whenever $t_i^\bullet \cap {}^\bullet t_j \neq \varnothing$, for all $1 \leq i < j \leq k$.

When working towards a sound interval order semantics for *en* in the case of interval overlapping, we could proceed by noting down the beginnings and ends of all the executed transitions and convert the 'interval sequence' obtained in this way into the corresponding interval order. It is crucial now to observe that, in general, the result is in no way based on the fundamental *causality* relationship (i.e., $\prec_{causal}$) which is inherent in EN-systems.

Similarly as in Definitions 2 and 3, we will use an inductive approach to define interval order semantics of EN-systems. This leads to the following question: Having observed a hypothetical interval order execution *ipo*, resulting from extending the initial empty interval order by observed events ${t_1}^{(m_1)}, \ldots, {t_k}^{(m_k)}$, what could we say about the interval order obtained after firing of the beginning of another transition? In other words, what could we say about ipo' derived from *ipo* after adding a single event $x = t^{(n)}$? Our answer is based on the following key observations:

(i) If $v \in \text{nomax}_{ipo}$ is a non-maximal event in *ipo* then, for sure, we should have $v \prec_{ipo'} x$.

(ii) If $v \in \max_{ipo}$ is a maximal event in *ipo* such that $\ell(v)^\bullet \cap {}^\bullet t \neq \varnothing$, then v must have terminated before x started, and we should have $v \prec_{ipo'} x$.

(iii) If $v \in \max_{ipo}$ is a maximal event in *ipo* such that $\ell(v)^\bullet \cap {}^\bullet t = \varnothing$, then all we can be sure of is that v has not started after x finished, and so we should have either $v \prec_{ipo'} x$ or $v \frown_{ipo'} x$.

Intuitively, the maximal events in *ipo* can be considered 'unfinished' before starting x, and can either be ended 'just before' x started or continued to be finished after the execution of x has started. Note that the first two cases above, (i) and (ii), are deterministic. However, case (iii) is a source of non-determinism which is not present in the standard interleaving semantics of *en*.

In what follows, we will assume that each transition of an EN-system occurs in at least one firing sequence, i.e., there are no dead transitions.

4.1. Interval orders generated by EN-systems

Definition 4 (interval orders of EN-system)**.** *Let $en = \langle P, T, F, m_0\rangle$ be an* EN*-system. The* interval orders *of en, denoted by* IPO_{en}*, are generated as follows.*

- *The empty interval order $ipo_\varnothing$ is an interval order of en, and it leads to marking* $\text{mar}_{ipo_\varnothing} = m_0$.
- *Let $ipo = \langle X, \prec, \ell\rangle$ be an interval order of en leading to marking* mar_{ipo}*, and t be a transition such that ${}^\bullet t \subseteq \text{mar}_{ipo}$. Then*

$$ipo' = \langle X \cup \{x\}, \prec \cup\, (\text{nomax}_{ipo} \cup V \cup W) \times \{x\}, \ell'\rangle \tag{2}$$

  is an interval order of en, and it leads to marking $\text{mar}_{ipo'} = \text{mar}_{ipo} \setminus {}^\bullet t \cup t^\bullet$*, where:*
  - $x = t^{(n+1)}$ *with n being the number of the elements of X labelled by t.*
  - $V = \{v \in \max_{ipo} \mid \ell(v)^\bullet \cap {}^\bullet t \neq \varnothing\}$.
  - $W \subseteq \{v \in \max_{ipo} \mid \ell(v)^\bullet \cap {}^\bullet t = \varnothing\}$.
  - $\ell'|_X = \ell$ *and* $\ell'(x) = t$.

We also denote $ipo \rightarrow_{en} ipo'$ *and* $ipo \xrightarrow{t}_{en} ipo'$.

Note that in Definition 4 we need to consider all possible sets W. Even when $\{v \in \max_{ipo} \mid \ell(v)^\bullet \cap {}^\bullet t = \varnothing\} = \{w\}$, we have two possibilities: $W = \varnothing$ and $W = \{w\}$. The nondeterministic execution results from having $2^{|\{v \in \max_{ipo} | \ell(v)^\bullet \cap {}^\bullet t = \varnothing\}|}$ choices of W.

Proposition 4. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system. If* $ipo \rightarrow_{en} ipo'$ *then there is a unique transition* t *such that*

$$\ell_{ipo'}(\max_{ipo'}) \setminus \ell_{ipo}(\max_{ipo}) = \{t\} \quad \textit{and} \quad \ell_{ipo'}(\max_{ipo'}) \setminus \{t\} \subseteq \ell_{ipo}(\max_{ipo}) \ .$$

Proof. Follows directly from Definition 4. □

Proposition 5. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system,* $t \in T$*, and* $ipo \in \mathsf{IPO}_{en}$*. Moreover, let*

$$\{ipo_1, \ldots, ipo_k\} = \{ipo' \mid ipo \xrightarrow{t}_{en} ipo'\} \neq \varnothing \ ,$$

where $ipo_i \neq ipo_j$*, for all* $1 \leq i < j \leq k$*. Then:*

$$\{\ell_{ipo_1}(\max_{ipo_1}) \setminus \{t\}, \ldots, \ell_{ipo_k}(\max_{ipo_k}) \setminus \{t\}\} = 2^{(\ell_{ipo_1}(\max_{ipo_1}) \cup \cdots \cup \ell_{ipo_k}(\max_{ipo_k})) \setminus \{t\}} \ .$$

Also, for every $u \in (\ell_{ipo}(\max_{ipo}) \setminus (\ell_{ipo_1}(\max_{ipo_1}) \cup \cdots \cup \ell_{ipo_k}(\max_{ipo_k}))$*, we have* $u^\bullet \cap {}^\bullet t \neq \varnothing$.

Proof. Follows directly from Definition 4 and Proposition 4. □

Proposition 6. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system. Then* IPO_{en} *is a set of labelled interval orders such that* $\mathsf{TPO}_{en} \subseteq \mathsf{IPO}_{en}$*.*

Proof. Clearly, as we can always set $W = \max_{ipo} \setminus V$, we have $\mathsf{TPO}_{en} \subseteq \mathsf{IPO}_{en}$.

To show that IPO_{en} is a set of interval orders, we use Theorem 1. More precisely, for every $ipo \in \mathsf{IPO}_{en}$, we will show there exist suitable integer-valued functions β_{ipo} and ε_{ipo} such that there is $k_{ipo} \geq 0$ such that $\varepsilon_{ipo}(y) \leq k_{ipo}$, for all $y \in \mathrm{nomax}_{ipo}$, and $\varepsilon_{ipo}(z) = k_{ipo} + 1$, for all $z \in \max_{ipo}$. We proceed by induction on the derivation of ipo.

In the base case there is nothing to show, and if ipo has one element x we set $\beta_{ipo}(x) = 0$ and $\varepsilon_{ipo}(x) = 2$.

In the inductive case, we assume that ipo, x, and ipo' are as in Definition 4. We then set $\beta_{ipo'}(x) = k_{ipo} + 2$, $\varepsilon_{ipo'}(x) = k_{ipo} + 3$, and keep β and ε unchanged except for re-setting $\varepsilon_{ipo'}(z) = k_{ipo} + 3$, for all $z \in \max_{ipo'} \setminus \{x\}$. Moreover, $k_{ipo'} = k_{ipo} + 2$. □

Remark 1. *The construction of interval orders in Definition 4 is not only a natural generalisation of that for the total orders of en, but it also can capture other semantics which might be used to define execution semantics of* EN*-systems. Below we describe some of more obvious possibilities, depending on the choice of* W*. (Note that the same option is taken at all the stages of an execution.)*

1. $W = \max_{ipo} \setminus V$*. Then the construction generates all the total orders of en, as in Definition 3.*

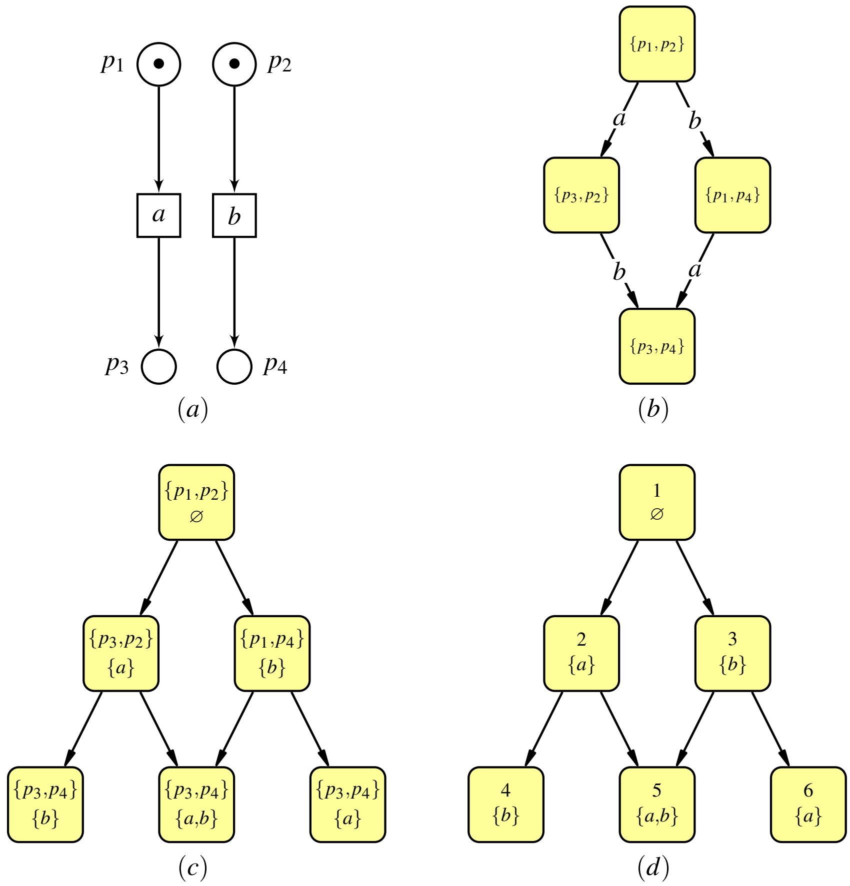

Figure 2: (a) EN-system; (b) its interleaving reachability graph; (c) its IR-graph; and (d) an isomorphic IT-system.

2. $W = V = \varnothing$ *or* $W = \max_{ipo} \setminus V$. *Then the construction generates all the stratified orders*[1] *included in* IPO_{en}. *The result is the step sequence semantics of en expressed using partial orders corresponding to step sequences.*
3. $W = V = \varnothing$ *or* $(W = \max_{ipo} \setminus V) \wedge (\forall u \in T : {}^\bullet u \subseteq \mathrm{mar}_{ipo} \implies \ell(\max_{ipo})^\bullet \cap {}^\bullet u \neq \varnothing)$. *Then the construction generates all the stratified orders included in* IPO_{en} *such that no stratum can be extended to yield a valid stratified order. The result is the 'maximal concurrency' semantics of en.*
4. $W = \varnothing$. *To our knowledge this option does not correspond to any semantical framework considered in the literature. Intuitively, it can be regarded as 'maximally prolonged' execution model.*

[1] A partial order $po = \langle X, \prec, \ell \rangle$ is stratified if $\{\langle x, y\rangle \in X \times X \mid x \not\prec y \not\prec x\}$ is an equivalence relation.

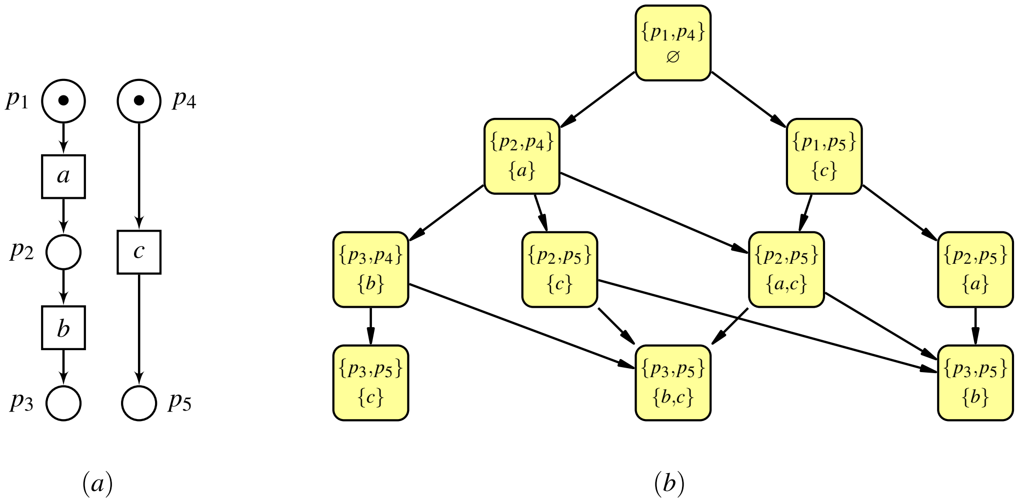

Figure 3: (a) EN-system (some of complement places are omitted); and (b) its IR-graph.

4.2. Reachable states and interval reachability graphs

In the standard semantics of EN-systems, one usually associates the notion of a 'a reachable system state' with that of the marking reached after executing a firing sequence. This, in turn, leads to the notion of the reachability graph of an EN-system. Such graphs can be seen, in particular, as generators of all the firing sequences that can be executed.

It is not difficult to see that markings alone are insufficient to identify states of EN-systems under the interval order semantics. Consider, for example, the EN-system *en* depicted in Figure 3(a). It generates two interval orders, ipo_1 and ipo_2, both with the domain $\{a^{(1)}, c^{(1)}\}$ and such that $c^{(1)} \prec_{ipo_1} a^{(1)}$ and $a^{(1)} \frown_{ipo_2} c^{(1)}$. Both orders lead to the same marking $\{p_2, p_5\}$ which enables transition b. Furthermore, following Definition 4, ipo_1 can only be extended in one way (with $a^{(1)} \prec b^{(1)}$ as in Figure 5(c)), whereas ipo_2 can be extended in two ways (one with $a^{(1)} \prec b^{(1)}$ and $c^{(1)} \frown b^{(1)}$ as in Figure 5(f), and the other with $a^{(1)} \prec b^{(1)}$ and $c^{(1)} \prec b^{(1)}$ as in Figure 5(e)).

Clearly, each $ipo \in \mathsf{IPO}_{en}$ leads to a 'state'. However, associating a state with each individual interval order would generate a huge (infinite) state space. This is not the way to go. It turns out that we can associate a state of *en* with all those interval orders which lead to the same marking, and have the same set of labels of maximal events. The reason is that all the 'continuations' for such interval orders are the same.

Definition 5 (equivalent interval orders)**.** *Let* $en = \langle P, T, F, m_0\rangle$ *be an* EN*-system. Then* $\sim$ *be a binary relation on* IPO_{en} *such that, for all* $ipo, ipo' \in \mathsf{IPO}_{en}$*, we have* $ipo \sim ipo'$ *if* $\mathrm{mar}_{ipo} = \mathrm{mar}_{ipo'}$ *and* $\ell_{ipo}(\max_{ipo}) = \ell_{ipo'}(\max_{ipo'})$.

The above relation is an equivalence relation. Moreover, as the next result states, it can be used to define system states.

Proposition 7. *Let* $en = \langle P, T, F, m_0\rangle$ *be an* EN*-system. If* $ipo_1 \sim ipo_2$ *and* $ipo_1 \rightarrow_{en} ipo'_1$*, then there is* ipo'_2 *such that* $ipo_2 \rightarrow_{en} ipo'_2$ *and* $ipo'_1 \sim ipo'_2$.

Proof. It follows directly from Definition 4. □

We can then define the reachability graph of an EN-system.

Definition 6 (interval reachability graph)**.** *The* interval reachability graph (or IR-graph) *of an* EN*-system* $en = \langle P,T,F,m_0\rangle$ *is* $irg_{en} = \langle Q,A,q_0,i\rangle$*, where:*

1. $Q = \{\langle \mathrm{mar}_{ipo}, \ell_{ipo}(\mathrm{max}_{ipo})\rangle \mid ipo \in \mathsf{IPO}_{en}\}$ *are the states.*
2. $A = \{\langle\langle \mathrm{mar}_{ipo}, \ell_{ipo}(\mathrm{max}_{ipo})\rangle, \langle \mathrm{mar}_{ipo'}, \ell_{ipo'}(\mathrm{max}_{ipo'})\rangle\rangle \mid ipo \rightarrow_{en} ipo'\}$ *are the arcs.*
3. $q_0 = \langle m_0, \varnothing\rangle$ *is the initial state.*
4. $i : Q \rightarrow 2^T$ *is the labelling such that* $i(q) = V$*, for every* $q = \langle m,V\rangle \in Q$.

In the next section, we will show that irg_{en} is a generator of all the interval orders of *en*.

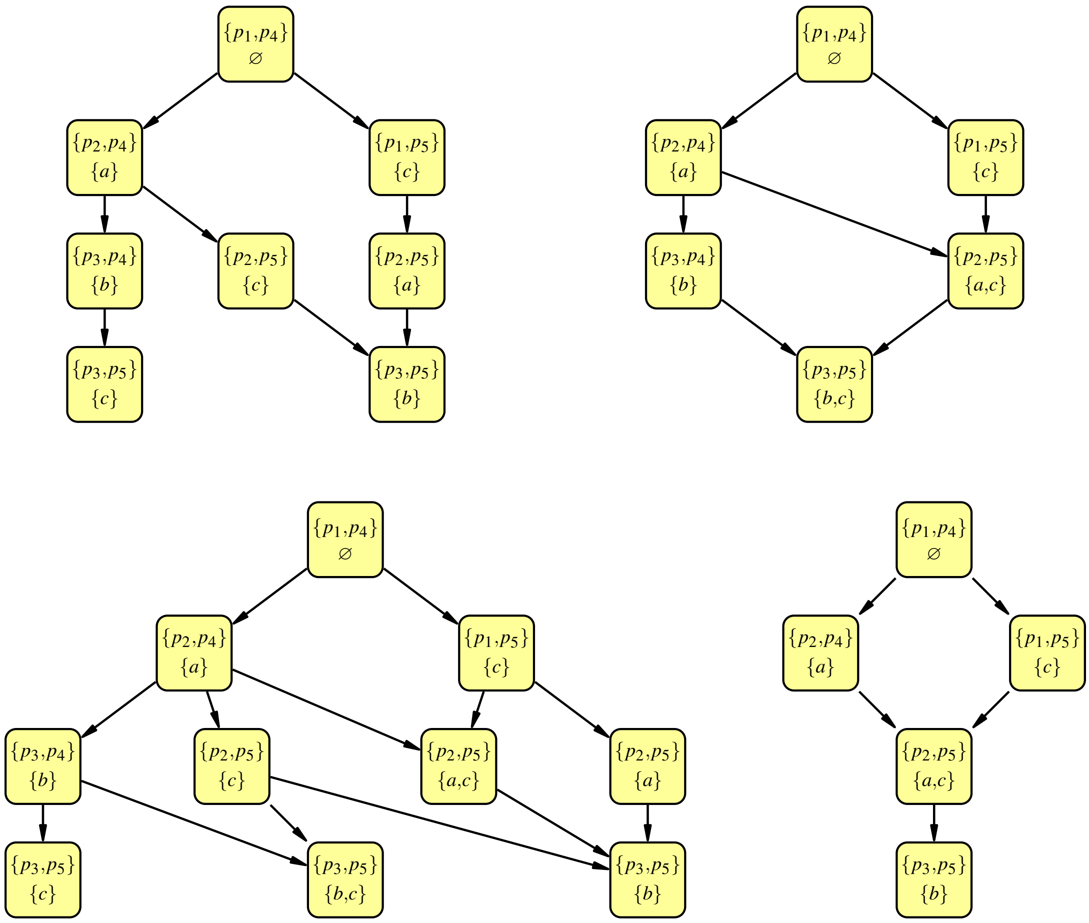

Figure 4: Reachability graphs for different semantical models of the EN-system in Figure 3(a): interleaving semantics as in Remark 1(1) (top-left); maximally prolonged semantics as in Remark 1(4) (top-right); step sequence semantics as in Remark 1(2) (bottom-left); and maximally concurrent semantics as in Remark 1(3) (bottom-right).

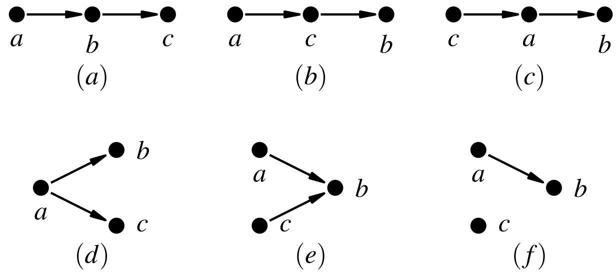

Figure 5: Interval orders generated by different paths in the transition system of Figure 3(b).

Remark 2. *Remark 1 mentioned four different kinds of semantics which can be applied to* EN*-systems. Figure 4 shows reachability graphs generated by these four semantics applied to the* EN*-system in Figure 3(a) using the equivalence relation on interval orders introduced in Definition 5.*

4.3. Transition systems generating interval orders

In general, we are interested in transition systems which are capable of generating interval orders.

Definition 7 (interval transition system)**.** *An* interval transition system over T (or IT-system) *is* $its = \langle S, \to, s_0, \iota \rangle$*, where* S *is a finite set of states,* $\to \subseteq S \times S$ *is the set of arcs,* $s_0 \in S$ *is the initial state, and* $\iota : S \to 2^T$ *is the labelling of states. It is assumed that the following hold, for all* $s \in S$*:*

1. *All states are reachable from* s_0.
2. $\iota(s) = \varnothing$ *iff* $s = s_0$.
3. *If* $s \to r$*, then there is* $t \in T$ *such that* $\iota(r) \setminus \iota(s) = \{t\}$*, and* $\iota(r) \setminus \{t\} \subseteq \iota(s)$*. We then also denote* $s \xrightarrow{t} r$.
   Moreover, we denote $s \xrightarrow{t}$ *if there is at least one* $r \in S$ *such that* $s \xrightarrow{t} r$.
4. *For every* $t \in T$*, there is at least one* $r \in S$ *such that* $r \xrightarrow{t}$.
5. *If* $s \to r$ *and* $s \to q$ *are such that* $\iota(r) = \iota(q)$*, then* $r = q$.
6. *If* $s \xrightarrow{t} r$ *and* $V \subseteq \iota(r) \setminus \{t\}$*, then there is* $q \in S$ *such that* $s \xrightarrow{t} q$ *and* $\iota(q) = V \cup \{t\}$.
7. *If* $s \xrightarrow{t} r$ *and* $s \xrightarrow{t} r'$ *then there is* $r'' \in S$ *such that* $s \xrightarrow{t} r''$ *and* $\iota(r'') = \iota(r) \cup \iota(r')$.

Proposition 8. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system. Then* irg_{en} *is an* IT*-system over* T.

Proof. It follows from Definitions 4, 6 and 7 as well as Proposition 4. □

Note that Definition 7(5) reflects the deterministic nature of EN-systems, and Definition 7(6,7) captures the non-deterministic aspect of the interval order semantics of EN-systems.

IT-systems are generators of interval orders.

Definition 8 (interval orders of IT-system)**.** *The* interval orders *of an* IT*-system* $its = \langle S, \to, s_0, \iota \rangle$*, denoted by* IPO_{its}*, are the interval orders* ipo_π *generated by its paths* π *originating at the initial state. They are generated as follows:*

- ipo_{s_0} *is the empty interval order of its.*
- *Let* $\pi = s_0 \ldots s_k$ *be a path in its such that* $ipo = ipo_{s_0 \ldots s_{k-1}} = \langle X, \prec, \ell \rangle$ *and* $s_{k-1} \xrightarrow{t} s_k$*. Then*

$$ipo_{\pi} = \langle X \cup \{x\}, \prec \cup R \times \{x\}, \ell' \rangle \tag{3}$$

  is an interval order of its, where:
  - $x = t^{(n+1)}$ *with* n *being the number of the elements of* X *labelled by* t*.*
  - $R = X \setminus \{y \in \max_{ipo} \mid \ell(y) \in \iota(s_k)\}$*.*
  - $\ell'|_X = \ell$ *and* $\ell'(x) = t$*.*

Proposition 9. *Let* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *be an* IT*-system over* T*,* $t \in T$*, and* $s \in S$*. Moreover, let*

$$\{s_1, \ldots, s_k\} = \{s' \mid s \xrightarrow{t} s'\} \neq \varnothing ,$$

where $s_i \neq s_j$*, for all* $1 \leq i < j \leq k$*. Then:*

$$\{\iota(s_1) \setminus \{t\}, \ldots, \iota(s_k) \setminus \{t\}\} = 2^{(\iota(s_1) \cup \cdots \cup \iota(s_k)) \setminus \{t\}} .$$

Also, for every $u \in \iota(s) \setminus (\iota(s_1) \cup \cdots \cup \iota(s_k))$*, we have* $u \prec t$*, where* $\prec$ *is as in Definition 8.*

Proof. Follows directly from Definition 7(3,5,6,7). □

Theorem 2. *Let* $en = \langle P, T, F, m_0 \rangle$ *be an* EN*-system.* $\mathsf{IPO}_{en} = \mathsf{IPO}_{irg_{en}}$*.*

Proof. Follows directly from Definitions 4 and 8 as well as Proposition 8. □

Definition 9. *Two* IT*-systems,* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *and* $its' = \langle S', \rightarrow', s'_0, \iota' \rangle$*, are* isomorphic *if there is a bijection* $\psi : S \rightarrow S'$ *such that:*

- $\psi(s_0) = s'_0$*.*
- *For all* $s, s' \in S$*,* $s \rightarrow s'$ *if and only if* $\psi(s) \rightarrow' \psi(s')$*.*
- *For every* $s \in S$*,* $\iota(s) = \iota'(\psi(s))$*.*

5. Synthesis

Our synthesis procedure follows the standard approach applied in [11, 12, 13, 15, 16, 17, 18, 19], where a transition system with its global states is used as an initial specification from which local states (places of Petri nets) are inferred in the form of regions. In our case, transitions systems are IT-systems. The verification that a given IT-system is realisable by an EN-system with interval order semantics is done by checking whether the interval reachability graph of the synthesised net is isomorphic to the initial IT-system.

Definition 10 (region)**.** *A* region *of an* IT*-system over* T*,* $its = \langle S, \rightarrow, s_0, \iota \rangle$*, is a triple* $\rho = \langle In_\rho, Out_\rho, \sigma_\rho \rangle$*, where* $In_\rho, Out_\rho \subseteq T$ *and* $\sigma_\rho : S \rightarrow \{0, 1\}$ *are such that, for every* $s \xrightarrow{t} r$*, the following hold:*

1. *If* $t \in In_\rho$ *then* $\sigma_\rho(s) = 0$ *and* $\sigma_\rho(r) = 1$*.*

2. *If* $t \in Out_\rho$ *then* $\sigma_\rho(s) = 1$ *and* $\sigma_\rho(r) = 0$.
3. *If* $\sigma_\rho(s) = 1$ *and* $\sigma_\rho(r) = 0$*, then* $t \in Out_\rho$.
4. *If* $\sigma_\rho(s) = 0$ *and* $\sigma_\rho(r) = 1$*, then* $t \in In_\rho$.

There are two trivial regions, $\langle \varnothing, \varnothing, \widehat{1} \rangle$ and $\langle \varnothing, \varnothing, \widehat{0} \rangle$, where $\widehat{1}$ and $\widehat{0}$ denote constant functions returning 1 and 0, respectively. The set of all non-trivial regions of *its* will be denoted by $\mathscr{R}_{its}$. We also denote, for every $t \in T$:

$$t^\circ = \{\rho \in \mathscr{R}_{its} \mid t \in In_\rho\} \quad \textit{and} \quad {}^\circ t = \{\rho \in \mathscr{R}_{its} \mid t \in Out_\rho\} .$$

The set of states associated with a region $\rho = \langle In_\rho, Out_\rho, \sigma_\rho \rangle$ is given by $S_\rho = \sigma_\rho^{-1}(\{1\})$. Also, we denote $\mathscr{R}_s = \{\rho \in \mathscr{R}_{its} \mid s \in S_\rho\}$.

Proposition 10. *Let* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *be an* IT*-system over* T *and* $t \in T$ *be such that there is* $s \in S$ *with* $s \xrightarrow{t}$*. Then* $t^\circ \cap {}^\circ t = \varnothing$.

Proof. Suppose $s \xrightarrow{t} s'$ and $\rho \in t^\circ \cap {}^\circ t$. Then $t \in In_\rho \cap Out_\rho$. But, from Definition 10(1,2), we have $\sigma_\rho(s) = 0$ and $\sigma_\rho(s) = 1$, a contradiction. □

Proposition 11. *Let* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *be an* IT*-system over* T *and* $s \xrightarrow{t} s'$*. Then*

1. $\rho \in {}^\circ t$ *implies* $s \in S_\rho$ *(*$\sigma_\rho(s) = 1$*) and* $s' \notin S_\rho$ *(*$\sigma_\rho(s') = 0$*).*
2. $\rho \in t^\circ$ *implies* $s \notin S_\rho$ *(*$\sigma_\rho(s) = 0$*) and* $s' \in S_\rho$ *(*$\sigma_\rho(s') = 1$*).*

Proof. Follows directly from the definitions of t° and ${}^\circ t$ and Definition 10(1,2). □

Proposition 12. *Let* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *be an* IT*-system over* T *and* $s \xrightarrow{t} s'$*. Then* $\mathscr{R}_s \setminus \mathscr{R}_{s'} = {}^\circ t$ *and* $\mathscr{R}_{s'} \setminus \mathscr{R}_s = t^\circ$.

Proof. We show $\mathscr{R}_s \setminus \mathscr{R}_{s'} = {}^\circ t$ as the second part can be shown in a similar way.

By Proposition 11(1), ${}^\circ t \subseteq \mathscr{R}_s$ and ${}^\circ t \cap \mathscr{R}_{s'} = \varnothing$. Hence ${}^\circ t \subseteq \mathscr{R}_s \setminus \mathscr{R}_{s'}$. Suppose now that $\rho \in \mathscr{R}_s \setminus \mathscr{R}_{s'}$. This implies $s \in S_\rho$ and $s' \notin S_\rho$. Hence, by Definition 10(3) and $s \xrightarrow{t} s'$, we have $t \in Out_\rho$, and so $\rho \in {}^\circ t$. Thus, $\mathscr{R}_s \setminus \mathscr{R}_{s'} \subseteq {}^\circ t$, and so $\mathscr{R}_s \setminus \mathscr{R}_{s'} = {}^\circ t$. □

Definition 11. *The tuple associated with an* IT*-system over* T*,* $its = \langle S, \rightarrow, s_0, \iota \rangle$*, is given as*

$$en_{its} = \langle \mathscr{R}_{its}, T, F_{its}, \mathscr{R}_{s_0} \rangle$$

where $F_{its} = \{(\rho, t) \in \mathscr{R}_{its} \times T \mid t \in Out_\rho\} \cup \{(t, \rho) \in T \times \mathscr{R}_{its} \mid t \in In_\rho\}$.

Proposition 13. *Let* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *be an* IT*-system over* T*, and* $en_{its} = \langle \mathscr{R}_{its}, T, F_{its}, \mathscr{R}_{s_0} \rangle$ *be the tuple associated with it. Then, for every* $t \in T$*,* ${}^\bullet t = {}^\circ t$ *and* $t^\bullet = t^\circ$.

Proof. Follows from Definition 11 and the definitions of ${}^\circ t$ and t°. □

Proposition 14. *Let* $its = \langle S, \rightarrow, s_0, \iota \rangle$ *be an* IT*-system over* T*, and* $\rho = \langle In_\rho, Out_\rho, \sigma_\rho \rangle \in \mathscr{R}_{its}$*. Then* $\overline{\rho} = \langle Out_\rho, In_\rho, \overline{\sigma}_\rho \rangle \in \mathscr{R}_{its}$*, where* $\overline{\sigma}_\rho(s) = 1 - \sigma_\rho(s)$*, for every* $s \in S$.

Proof. Follows from Definition 10. □

The region $\overline{\rho}$ is called the *complement* of ρ.

Definition 12 (ENIT-system)**.** *Let $its = \langle S, \to, s_0, \iota \rangle$ be an* IT*-system over T. Then its is an* ENIT*-system over T if the following hold, for all $t \in T$ and $s, r \in S$:*

1. $t^\circ \neq \varnothing \neq {}^\circ t$.
2. *If $s \neq r$ and $\iota(s) = \iota(r)$, then there is a region $\rho \in \mathscr{R}_{its}$ such that* *(state separation)*
$$\sigma_\rho(s) \neq \sigma_\rho(r) .$$
3. *If $s \overset{t}{\not\to}$, then there is a region $\langle In_\rho, Out_\rho, \sigma_\rho \rangle \in \mathscr{R}_{its}$ such that* *(forward closure)*
$$(t \in In_\rho \wedge \sigma_\rho(s) = 1) \quad or \quad (t \in Out_\rho \wedge \sigma_\rho(s) = 0) .$$

The above three 'axioms' characterise the EN-system realisable IT-systems. State separation requires that if two distinct states are not distinguished by at least one region, then they are distinguished by the labels of the maximal elements of their associated interval orders. Forward closure requires that for each action (i.e., transition label), a state at which the considered action has no occurrence is separated by a region from all states where this action occurs. While the forward closure axiom has similar form as 'forward closure' axioms that can be found in the literature for solving other synthesis problems [13, 15, 16, 17, 18, 19], the state separation axiom for ENIT-systems differs from its standard form as here the separation of states does not rely only on regions.

Proposition 15. *Let $its = \langle S, \to, s_0, \iota \rangle$ be an* ENIT*-system over T. Then $en_{its} = \langle \mathscr{R}_{its}, T, F_{its}, \mathscr{R}_{s_0} \rangle$ is an* EN*-system.*

Proof. Let $t \in T$. From Proposition 13 and Definition 12(1) we have that ${}^\bullet t \neq \varnothing \neq t^\bullet$.

From Definition 7(4) we have that there is $s \in S$ such that $s \xrightarrow{t} r$, for some $r \in S$. Suppose that ${}^\circ t \cap t^\circ \neq \varnothing$. Then there is region $\rho \in {}^\circ t \cap t^\circ$ and so $t \in In_\rho \cap Out_\rho$, yielding a contradiction with Definition 10. Hence ${}^\circ t \cap t^\circ = \varnothing$. This, together with Proposition 13, implies ${}^\bullet t \cap t^\bullet = \varnothing$. Moreover, Proposition 14 implies that $\rho \in \mathscr{R}_{its} \iff \overline{\rho} \in \mathscr{R}_{its}$, so in en_{its} every place has a unique complement. Hence en_{its} is an EN-system. $\square$

Proposition 16. *Let $its = \langle S, \to, s_0, \iota \rangle$ be an* ENIT*-system over T, and $en = en_{its} = \langle \mathscr{R}_{its}, T, F_{its}, \mathscr{R}_{s_0} \rangle$ be the tuple associated with it. Then the states, arcs, and the labelling function of $irg_{en} = \langle Q, A, q_0, i \rangle$ are as follows:*

1. $Q = \{\langle \mathscr{R}_s, \iota(s) \rangle \mid s \in S\}$.
2. $A = \{\langle \langle \mathscr{R}_s, \iota(s) \rangle, \langle \mathscr{R}_{s'}, \iota(s') \rangle \rangle \mid s \to s'\}$.
3. *For every $s \in S$, $i(\langle \mathscr{R}_s, \iota(s) \rangle) = \iota(s)$.*

Proof. First, from Proposition 15 it follows that en is an EN-system. Note also that from Definition 7(1) all the states of its are reachable from s_0. Moreover, all the states of irg_{en} are reachable from q_0, which follows from the construction of irg_{en} and the inductive approach of Definition 4. Furthermore, from Definition 6(3) we have $q_0 = \langle \mathscr{R}_{s_0}, \varnothing \rangle$, and from Definition 7(2) we have $\iota(s_0) = \varnothing$. Hence, $q_0 = \langle \mathscr{R}_{s_0}, \iota(s_0) \rangle$.

Now we show that $\langle q,q'\rangle \in A$ and $q=\langle \mathscr{R}_s, \iota(s)\rangle$, for some $s\in S$, imply that there is $s'\in S$ such that $s\to s'$ and $q'=\langle \mathscr{R}_{s'}, \iota(s')\rangle$. By $\langle q,q'\rangle \in A$ we have, from Definition 6(2) and Definition 4, that there are two interval orders, ipo_q and $ipo_{q'}$, such that $ipo_q \to_{en} ipo_{q'}$, and that there is $t\in T$ such that ${}^\bullet t \subseteq \mathrm{mar}_{ipo_q}$. Furthermore, from Proposition 1, $t^\bullet \cap \mathrm{mar}_{ipo_q} = \varnothing$. Also, as $q=\langle \mathscr{R}_s, \iota(s)\rangle$ by the assumption, we have ${}^\bullet t \subseteq \mathscr{R}_s = \mathrm{mar}_{ipo_q}$ and $\ell_{ipo_q}(\max_{ipo_q}) = \iota(s)$.

From Proposition 13, we have then that

$$^\circ t = \{\rho \in \mathscr{R}_{its} \mid t \in Out_\rho\} \subseteq \mathscr{R}_s \quad and \quad t^\circ \cap \mathscr{R}_s = \varnothing \quad (t^\circ = \{\rho \in \mathscr{R}_{its} \mid t \in In_\rho\})\ .$$

Therefore, we have that for all $\rho \in \mathscr{R}_{its}$:

1. $t\in Out_\rho$ implies $\sigma_\rho(s)=1$ and
2. $t\in In_\rho$ implies $\sigma_\rho(s)=0$.

So, from Definition 12(3), as *its* is an ENIT-system, we have $s \xrightarrow{t}$.

Therefore (see Definition 7(3)), there is at least one $s'\in S$ satisfying $s\xrightarrow{t} s'$ (there can be more than one such a state). Hence, by Proposition 12, we have $\mathscr{R}_{s'} = (\mathscr{R}_s \setminus {}^\circ t)\cup t^\circ$. At the same time we have, from Definition 4,

$$\mathrm{mar}_{ipo_{q'}} = (\mathrm{mar}_{ipo_q} \setminus {}^\bullet t)\cup t^\bullet = (\mathscr{R}_s \setminus {}^\circ t)\cup t^\circ = \mathscr{R}_{s'}\ .$$

Hence $\mathrm{mar}_{ipo_{q'}} = \mathscr{R}_{s'}$.

If there are several states like s', then the last conclusion will be true for all of them.

Furthermore, we have $\ell_{ipo_q}(\max_{ipo_q}) = \iota(s)$. If in irg_{en} we have several 'children' of the vertex $q=\langle \mathscr{R}_s, \iota(s)\rangle$ such that $q\xrightarrow{t} q_i$ ($1\le i\le k$) and q' is one of them, then from Proposition 5 we have that their $\ell_{ipo_{q_i}}(\max_{ipo_{q_i}})$ will form the set of all subsets of the set

$$\ell_{ipo_q}(\max_{ipo_q}) \setminus \{u\in T \mid \text{ the execution of } t \text{ must follow the execution of } u\}\ .$$

From Proposition 9 we have that the 'children' of the vertex s in *its* (s_i such that $s\xrightarrow{t} s_i$) will be labelled with the sets of transitions that are subsets of the same set of transitions as $\ell_{ipo_q}(\max_{ipo_q}) = \iota(s)$ and so we will have the same number of children of q in irg_{en} and children of s in *its*. Hence, if q' is one of the q_i's then we will be able to find s' among the s_i's such that $\ell_{ipo_{q'}}(\max_{ipo_{q'}}) = \iota(s')$.

So, we proved that

$$Q \subseteq \{\langle \mathscr{R}_s, \iota(s)\rangle \mid s\in S\} \quad and \quad A \subseteq \{\langle\langle \mathscr{R}_s, \iota(s)\rangle, \langle \mathscr{R}_{s'}, \iota(s')\rangle\rangle \mid s\to s'\}\ .$$

We now prove the reverse inclusions. By Definitions 6(3) and 7(2), $\langle \mathscr{R}_{s_0}, \iota(s_0)\rangle = \langle \mathscr{R}_{s_0}, \varnothing\rangle \in Q$. It is enough to show that if $s\to s'$ and $\langle \mathscr{R}_s, \iota(s)\rangle \in Q$, then $\langle \mathscr{R}_{s'}, \iota(s')\rangle \in Q$ and $\langle\langle \mathscr{R}_s, \iota(s)\rangle, \langle \mathscr{R}_{s'}, \iota(s')\rangle\rangle \in A$.

By Definition 7(3) and $s\to s'$, we have that there exists t satisfying $s\xrightarrow{t} s'$. From $s\xrightarrow{t} s'$ and Proposition 12, we have $\mathscr{R}_s \setminus \mathscr{R}_{s'} = {}^\circ t$ and $\mathscr{R}_{s'} \setminus \mathscr{R}_s = t^\circ$.

From Definition 6(1) and $\langle \mathscr{R}_s, \iota(s)\rangle \in Q$, we have that there exists $ipo \in \mathsf{IPO}_{en}$ such that $\mathrm{mar}_{ipo} = \mathscr{R}_s$ and $\ell_{ipo}(\max_{ipo}) = \iota(s)$. So, as ${}^\bullet t = {}^\circ t \subseteq \mathscr{R}_s$ and $t^\bullet \cap \mathscr{R}_s = t^\circ \cap \mathscr{R}_s = \varnothing$ (see Proposition 13) we have that t is enabled at mar_{ipo} in *en*. Then, according to Definition 4 there is an interval order ipo' such that $ipo \rightarrow_{en} ipo'$ and $\mathrm{mar}_{ipo'} = (\mathrm{mar}_{ipo} \setminus {}^\bullet t) \cup t^\bullet$. At the same time, we have $\mathscr{R}_{s'} = (\mathscr{R}_s \setminus {}^\bullet t) \cup t^\bullet)$ (see Proposition 13) and $\mathrm{mar}_{ipo} = \mathscr{R}_s$, and so $\mathrm{mar}_{ipo'} = \mathscr{R}_{s'}$. Hence $\langle\langle \mathscr{R}_s, \iota(s)\rangle, \langle \mathscr{R}_{s'}, \ell_{ipo'}(\max_{ipo'}\rangle\rangle \in A$.

If we have many 'children' of s in *its* (s_i such that $s \xrightarrow{t} s_i$) and s' is one of them, then the conclusion $\mathrm{mar}_{ipo'} = \mathscr{R}_{s'}$ will be true for all of them. Furthermore, we can use Propositions 5 and 9 to find the 'child' of the state $\langle \mathscr{R}_s, \iota(s)\rangle \in Q$ in irg_{en} such that $\ell_{ipo'}(\max_{ipo'}) = \iota(s')$. Hence, we have $\langle \mathscr{R}_{s'}, \iota(s')\rangle \in Q$ and $\langle\langle \mathscr{R}_s, \iota(s)\rangle, \langle \mathscr{R}_{s'}, \iota(s')\rangle\rangle \in A$.

Finally, we observe that part (3) follows from part (1) and Definition 6(4). □

Theorem 3. *Let $its = \langle S, \rightarrow, s_0, \iota\rangle$ be an* ENIT*-system over T, and $en = en_{its} = \langle \mathscr{R}_{its}, T, F_{its}, \mathscr{R}_{s_0}\rangle$ be the tuple associated with it. Then its is isomorphic to the* IR*-graph of en, $irg_{en} = \langle Q, A, q_0, i\rangle$. Moreover, the unique isomorphism ψ between its and irg_{en} is given by $\psi(s) = \langle \mathscr{R}_s, \iota(s)\rangle$, for every $s \in S$.*

Proof. Note that, by Proposition 16(1), $\psi : S \rightarrow Q$ such that $\psi(s) = \langle \mathscr{R}_s, \iota(s)\rangle$ is a well defined mapping satisfying $\psi(s_0) = \langle \mathscr{R}_{s_0}, \iota(s_0)\rangle = q_0$.

By Proposition 16(1), ψ is onto. Moreover, by Definition 12(2) (as *its* is an ENIT-system) it is injective. Hence ψ is a bijection.

We then observe that, by Proposition 16(2), we have $s \rightarrow s'$ if and only if $\langle \psi(s), \psi(s')\rangle \in A$. Furthermore, by Proposition 16(3), $i(\langle \mathscr{R}_s, \iota(s)\rangle) = \iota(s)$, for every $s \in S$. Hence ψ is an isomorphism for *its* and irg_{en}. □

6. Conclusion

In this paper, we introduced a new class of transition systems (interval transition systems, or IT-systems) whose paths are associated with interval partial orders and labelled with the maximal elements of these interval orders. Also, we demonstrated how EN-systems can generate interval orders and produce their interval reachability graphs. In this paper, EN-systems are executed sequentially, but there is an assumption that every transition execution takes time and may (partially) overlap with transitions that started earlier, but not yet finished. We provided an axiomatisation of IT-systems that can be synthesised to EN-systems with interval order semantics (Definition 12 of ENIT-system). We showed that there is a possibility to adapt the synthesis approach based on the concept of regions of standard sequential transition systems to work also for the ENIT-systems.

Our approach to constructing a system from 'interval data' differs from other approaches pursued in the area of Petri net synthesis and process mining.

As already mentioned, in this paper we proposed a solution to the problem of synthesising EN-systems from IT-systems, whose states are labelled by sets of currently active transitions (there might be more than one), and whose arcs are implicitly labelled by single transitions representing new actions/activities responsible for the changes of states. As with any type of initialised transition systems, they capture the behaviour a system starting from its initial state and show its progress from state to state when transitions are executed.

The IT-systems (including IR-graphs) *do not* directly show the temporal relationships between transitions/actions, but these relationships can be inferred from them during the synthesis procedure and become evident in the synthesised EN-systems. However, these relationships are not as precise as in other approaches found in the literature, where systems are discovered/synthesised from behavioural information about the activities that are treated as non-instantaneous (i.e., taking some time to complete). For example, Context-Aware Temporal Network Representation (TNR) graphs of [20] that are extracted from event logs capture the global relationships between different non-instantaneous activities/actions and use 13 relationships to relate the intervals of any two activities as described by Allen's Interval Algebra [21]. In our approach, we use an abstraction that recognises only two relationships between the intervals related to two transitions, viz. one can precede the other or they can overlap. As a result, at every state of an IT-system, a new active transition can follow the previously active transitions or join some of them to form a new set of active transitions.

In essence, the approaches of [20, 21] are semantically close to real-time semantics whereas the approach pursued in this paper is more abstract. For similar reasons, the interval order semantics used in this paper and the 'interval semantics' or 'interval time semantics' of, e.g., [22, 23], are incomparable.

Acknowledgments

The authors gratefully acknowledge three anonymous referees, whose comments significantly contributed to the final version of this paper.

References

[1] J. Desel, W. Reisig, Place/transition Petri nets, in: W. Reisig, G. Rozenberg (Eds.), Lectures on Petri Nets I: Basic Models, Advances in Petri Nets, the volumes are based on the Advanced Course on Petri Nets, held in Dagstuhl, September 1996, volume 1491 of *Lecture Notes in Computer Science*, Springer, 1996, pp. 122–173.

[2] K. Jensen, Coloured Petri Nets. Basic Concepts, Analysis Methods and Practical Use. Volume 1, Basic Concepts, Monographs in Theoretical Computer Science, Springer-Verlag, 1997.

[3] C. A. Petri, Concepts of net theory, in: Mathematical Foundations of Computer Science: Proceedings of Symposium and Summer School, Strbské Pleso, High Tatras, Czechoslovakia, September 3-8, 1973, Mathematical Institute of the Slovak Academy of Sciences, 1973, pp. 137–146.

[4] W. Reisig, Understanding Petri Nets - Modeling Techniques, Analysis Methods, Case Studies, Springer, 2013.

[5] G. Rozenberg, J. Engelfriet, Elementary net systems, in: W. Reisig, G. Rozenberg (Eds.), Petri Nets, volume 1491 of *Lecture Notes in Computer Science*, Springer, 1996, pp. 12–121.

[6] N. Wiener, A contribution to the theory of relative position, Proc. of the Cambridge Philosophical Society 17 (1914) 441–449.

[7] R. Janicki, M. Koutny, Structure of concurrency, Theoretical Computer Science 112 (1993) 5–52.

[8] E. Best, M. Koutny, Petri net semantics of priority systems, Theoretical Computer Science 96 (1992) 175–174.
[9] W. M. Zuberek, Timed Petri nets and preliminary performance evaluation, in: J. Lenfant, B. R. Borgerson, D. E. Atkins, K. B. Irani, D. Kinniment, H. Aiso (Eds.), Proceedings of the 7th Annual Symposium on Computer Architecture, La Baule, France, May 6-8, 1980, ACM, 1980, pp. 88–96.
[10] R. Janicki, X. Yin, Modeling concurrency with interval traces, Information and Computation 253 (2017) 78–108.
[11] A. Ehrenfeucht, G. Rozenberg, Theory of 2-structures, part I: clans, basic subclasses, and morphisms, Theoretical Computer Science 70 (1990) 277–303.
[12] A. Ehrenfeucht, G. Rozenberg, Theory of 2-structures, part II: representation through labeled tree families, Theoretical Computer Science 70 (1990) 305–342.
[13] É. Badouel, L. Bernardinello, P. Darondeau, Petri Net Synthesis, Texts in Theoretical Computer Science. An EATCS Series, Springer, 2015.
[14] P. C. Fishburn, Intransitive indifference with unequal indifference intervals, Journal of Mathematical Psychology 7 (1970) 144–149.
[15] É. Badouel, L. Bernardinello, P. Darondeau, Polynomial algorithms for the synthesis of bounded nets, in: P. D. Mosses, M. Nielsen, M. I. Schwartzbach (Eds.), TAPSOFT'95: Theory and Practice of Software Development, 6th International Joint Conference CAAP/FASE, Aarhus, Denmark, May 22-26, 1995, Proceedings, volume 915 of *Lecture Notes in Computer Science*, Springer, 1995, pp. 364–378.
[16] L. Bernardinello, G. D. Michelis, K. Petruni, S. Vigna, On the synchronic structure of transition systems, in: J. Desel (Ed.), Proceedings of the International Workshop on Structures in Concurrency Theory, STRICT 1995, Berlin, Germany, May 11-13, 1995, Workshops in Computing, Springer, 1995, pp. 69–84.
[17] M. Mukund, Petri nets and step transition systems, International Journal of Foundations of Computer Science 3 (1992) 443–478.
[18] N. Busi, G. M. Pinna, Synthesis of nets with inhibitor arcs, in: A. W. Mazurkiewicz, J. Winkowski (Eds.), CONCUR '97: Concurrency Theory, 8th International Conference, Warsaw, Poland, July 1-4, 1997, Proceedings, volume 1243 of *Lecture Notes in Computer Science*, Springer, 1997, pp. 151–165.
[19] M. Pietkiewicz-Koutny, The synthesis problem for elementary net systems with inhibitor arcs, Fundamenta Informaticae 40 (1999) 251–283.
[20] A. Senderovich, M. Weidlich, A. Gal, Context-aware temporal network representation of event logs: Model and methods for process performance analysis, Information Systems 84 (2019) 240–254.
[21] J. F. Allen, Maintaining knowledge about temporal intervals, Communications of ACM 26 (1983) 832–843.
[22] E. Pelz, Full axiomatisation of timed processes of interval-timed Petri nets, Fundamenta Informaticae 157 (2018) 427–442.
[23] L. Popova-Zeugmann, E. Pelz, Algebraical characterisation of interval-timed Petri nets with discrete delays, Fundamenta Informaticae 120 (2012) 341–357.