Recursive Constructions of Graphs of Large Girth and Given Degree

Leonard Chidiebere Eze, Robert Jajcay

Department of Algebra and Geometry, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská Dolina 824 48, Bratislava, Slovakia

Abstract

This paper is devoted to recursive constructions of small regular graphs of given degree k and girth g, called (k,g)-graphs, using Cayley graphs, perfect matchings and/or voltage graph constructions. First, we consider an analog of the canonical double cover construction, which produces $(k,g+1)$-graphs of even girth from (k,g)-graphs of odd girth by relying on Cayley graphs and/or voltage lifts. By considering Moore bounds, we show that there is no universal recursive construction of $(k,g+1)$-graphs from (k,g)-graphs of even girth g that would produce graphs whose order is a constant multiple of the order of the original graph. Further, we introduce a novel approach for obtaining $(k+1,6)$-graphs from $(k,6)$-graphs using perfect matchings and the voltage graph construction and compare our results with previously known results. We conclude our paper with a discussion of the potential of computer assisted searches relying on our constructions, and present a link between $(k,6)$-graphs and applications in communication systems.

Keywords

Regular graphs, girth, voltage graph, Cayley graph, LDPC codes

1. Introduction

We define a (k,g)-*graph* as a k-regular graph of girth g. One of the fundamental problems addressed in Extremal Graph Theory is the so-called *Cage Problem*; the problem of finding a smallest (k,g)-graph for a given pair of parameters $k,g \geq 3$. It is a hard optimization problem over a well-defined infinite class of graphs. This problem has been widely studied since the pioneering work of Erdős and Sachs [7] and that of Hoffman and Singleton [15]. The first pair of authors showed that for any given integers $g \geq 3,\ k \geq 2$, there exist infinitely many (k,g)-graphs [7]. However, determining a smallest (k,g)-graph in this infinite class has proven to be a much harder problem.

In this paper, we consider recursive constructions of two kinds, both starting from a given (k,g)-graph and resulting in a new larger graph.

First, we attempt to increase the second parameter, the girth g, while keeping k fixed. This type of construction can be traced back to the work of Erdős and Sachs [7]. Using a construction called a canonical double cover, starting from a (k,g)-graph of odd girth g, they constructed a new graph of the same degree and higher girth, whose order is twice the order of the original graph, thereby proving the following:

Theorem 1 ([7]). *For any integer $k \geq 3$ and odd $g \geq 3$,*

$$n(k,g+1) \leq 2n(k,g).$$

In a recent development, Balbuena et al. [2] improved this bound as expressed below.

Theorem 2 ([2]). *Let $k \geq 3$ and $g \geq 5$ odd. Then,*

$$n(k,g+1) \leq \begin{cases} 2n(k,g) - 2\left(\frac{k(k-1)^{\frac{g-3}{4}}-2}{k-2}\right), \\ \quad if\ g \equiv 3 \pmod 4 \\ \\ 2n(k,g) - 4\left(\frac{2(k-1)^{\frac{g-1}{4}}-1}{k-2}\right), \\ \quad if\ g \equiv 1 \pmod 4. \end{cases} \tag{1}$$

Graphs whose orders match these results are also obtained using the canonical double cover construction (on a subgraph of the original (k,g)-graphs). In Section 2, we illustrate the canonical double cover of graphs with examples and provide two different constructions, which produce the same graphs as the canonical double cover but rely on Cayley graphs and voltage lifts.

It is important to note that the canonical double cover construction produces $(k,g+1)$-graphs from (k,g)-graphs of *odd* girth g which are always twice as large as the original graph. Unfortunately, this construction only works for odd girth g. For this reason, we investigate the possibility of a recursive construction starting from a (k,g)-graph of even girth using the parametric analysis of the Moore bound; a lower bound on the order of (k,g)-graphs: $M(k,g) \leq n(k,g)$, for all $k,g \geq 3$.

ITAT'22: Information technologies – Applications and Theory, September 23–27, 2022, Zuberec, Slovakia
leonard.eze@fmph.uniba.sk (L. C. Eze); robert.jajcay@fmph.uniba.sk (R. Jajcay)
© 2022 Copyright for this paper by its authors. Use permitted under Creative Commons License Attribution 4.0 International (CC BY 4.0). Authors supported in part by VEGA 1/0423/20, and APVV-19-0308.
CEUR Workshop Proceedings (CEUR-WS.org)

CEUR-WS.org/Vol-3226/paper24.pdf

Definition 1. *For $k \geq 2,\ g \geq 3$, the Moore bound $M(k,g)$ of the (k,g)-graph is given by*

$$M(k,g) = \begin{cases} 1 + \sum_{i=0}^{\frac{g-3}{2}} k(k-1)^i, & g \text{ odd}, \\ \\ 2\sum_{i=0}^{\frac{g-2}{2}} (k-1)^i, & g \text{ even}. \end{cases} \quad (2)$$

Our analysis shows that there is no constant α such that for any given pair $k, g \geq 3$, g even, one could start from a (k,g)-graph and obtain a $(k, g+1)$-graph whose order is at most an α multiple of the order of the original graph. For more details on the Moore bounds and cages we refer the reader to the dynamic cage survey paper by Exoo and Jajcay [8].

Next, we try to increase the degree k of the obtained graph while keeping its girth g constant. In the opposite direction, Gács and Héger in 2008 built on the work of Brown [4] and presented recursive constructions for $g = 6, 8, 12$, which resulted in graphs of smaller degrees [12]. Namely, they considered incidence graphs of generalized n-gons and obtained smaller regular subgraphs by removing/adding some edges and vertices. To achieve this, for any given generalized n-gon $(\mathcal{P}, \mathcal{L})$, they defined a *$t$-good structure* of lines and points to delete in order to obtain a smaller (k,g)-graph of smaller degree. They proved the following: (1) For any prime power q and $1 \leq t \leq q$, there is a $(q+1-t, 6)$-regular graph whose order is $2(q^2+q+1-(tq+1))$ with the size of the t-good structure $tq+1$. (2) For any q for which a projective plane exists, there exists a $(k,6)$-graph of order $2(q^2-1)$. (3) For any square prime power q and $1 \leq t \leq q - \sqrt{q}$, there is a $(k,6)$-graph whose order is $2(q^2+q+1-t(q+\sqrt{q}+1))$ with the size of the t-good structure equal to $t(q+\sqrt{q}+1)$. However, their result is restricted to degrees k which are prime powers, and the resulting graphs are of smaller degrees than the starting point-line incidence graphs. To the best of our knowledge, until now there was no universal approach proceeding from an arbitrary $(k,6)$-graph to a $(k+1,6)$-graph.

Proceeding through the paper, we present several recursive constructions. Two of them are shown to be equivalent to the canonical double cover construction but rely on Cayley graphs. Another construction gives $(k+1,6)$-graphs from $(k,6)$-graphs. This construction is a new approach to obtaining $(k,6)$-graphs of an increasing degree with a fixed girth. Our approach makes it possible to move from a smaller $(k,6)$-graph to a $(k+1,6)$-graph. It is opposite to the construction of Gács and Héger [12] which starts from a bigger $(k,6)$-graph and produces a smaller $(k-t,6)$-graph. We rely on the fact that every regular bipartite graph has a perfect matching and voltage graph construction to construct the desired graph of higher degree.

Our paper is organized as follows: Section 2 contains an explanation of the concept of voltage graph construction with examples. Section 3 contains two analogs of the canonical double cover construction starting with Cayley graphs with some examples again. In Section 4, we give a parametric analysis of the Moore bound. Our recursive construction using perfect matchings is presented in Section 5, while the comparison of our construction with existing constructions is presented in Section 6. Finally, in Sections 7 and 8 we discuss computer assisted methods and an application of $(k,6)$-graphs in communication systems.

2. Voltage Graph Construction

Let Γ be a finite graph, not necessarily simple (possibly with multiple edges and loops), and $D(\Gamma)$ the set of *darts* of Γ obtained by replacing each edge e of Γ by a pair of opposing darts (arcs) e and e^{-1}. A *voltage assignment* on Γ is any mapping $\alpha : D(\Gamma) \to G$ satisfying the condition that $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for all $e \in D(\Gamma)$, where G is a group called the *voltage group*. For the purposes of our construction, we will always assume that G is finite. The *voltage graph* (also called the *derived graph* or the *lift*) of Γ with respect to α, denoted by Γ^α, is a new graph with vertex set $V(\Gamma^\alpha) = V(\Gamma) \times G$ and edge set $E(\Gamma^\alpha)$ defined by making vertices u_a and v_b adjacent in Γ^α if $e = (u,v) \in D(\Gamma)$ and $b = a\alpha(e)$. For any voltage graph Γ^α, we define the *net voltage* of a walk in Γ^α as the product of the group elements assigned to the edges of the walk in the corresponding order.

We say that $\tilde{\Gamma}$ is a *covering graph* of Γ if there exists a map $\psi : V(\tilde{\Gamma}) \longrightarrow V(\Gamma)$ called the *covering map of* Γ such that for every $v \in V(\tilde{\Gamma})$, the set of neighbours of v denoted by $\mathcal{N}_{\tilde{\Gamma}}(v)$ is mapped one-to-one onto the neighborhood $\mathcal{N}_\Gamma(\psi(v))$. We also say that $\tilde{\Gamma}$ is a lift of Γ if there exists a covering map from $V(\tilde{\Gamma})$ to $V(\Gamma)$, and we call the lift an n-lift of Γ if the preimages $\psi^{-1}(v)$ consist of n elements. Clearly, the voltage lift Γ^α is a $|G|$-lift of Γ.

We say that Γ^α is a *canonical double cover* of Γ if the voltage group is $\mathbb{Z}_2$ and each edge of Γ receives the voltage assignment $1 \in \mathbb{Z}_2$. This is a very special type of voltage graph construction, and it has been used by many authors [2, 11, 16, 19]. In what follows, we shall apply the canonical double cover to illustrate how to obtain a larger graph of even girth from a graph of odd girth.

Let Γ be a graph with odd girth g. Take $\mathbb{Z}_2 = \{0,1\}$ as the voltage group and define $\alpha(e) = 1$, for all $e \in E(\Gamma)$.

Example 1. *For C_4 and Pentagon, canonical double covers are respectively given below.*

Example 2. *Let Γ be the base graph of five vertices and girth three shown in Figure 2. The canonical double cover*

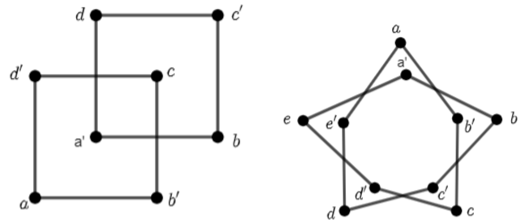

Figure 1: Canonical Double Covers of C_4 and Pentagon

of Γ *has 10 vertices and girth 4.*

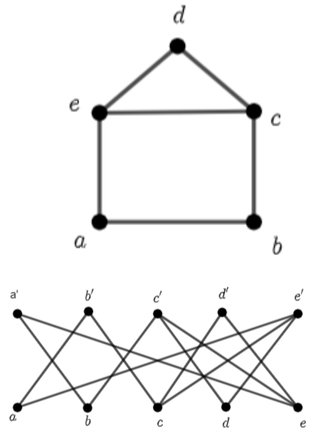

Figure 2: Canonical Double Cover of Γ

Example 3. *The canonical double cover of the Petersen graph of girth* 5 *is the Desargues graph, which has 20 vertices and girth 6.*

Based on the above examples, it is easy to deduce the following [11].

Lemma 1. *Let* Γ^α *be the canonical double cover of a graph* Γ*, then*

1. $|V(\Gamma^\alpha)| = 2 \times |V(\Gamma)|$, $|E(\Gamma^\alpha)| = 2 \times |E(\Gamma)|$;
2. Γ^α *is a bipartite graph;*
3. Γ^α *is connected if and only if* Γ *is connected and non-bipartite;*
4. *If* C *is a cycle in* Γ *of odd length* $2r+1$*, the preimage of* C *in* Γ^α *(the lift of* C *in* Γ^α*) is a cycle of the double length* $4r+2$*;*
5. *If* C *is a cycle in* Γ *of even length* $2r$*, the lift of* C *in* Γ^α *is a pair of cycles of length* $2r$*;*
6. *If* Γ *is* k*-regular,* Γ^α *is also* k*-regular.*

To this end, we see that Theorem 1 of Erdős and Sachs is a consequence of the canonical double cover construction. In other words, the canonical double cover of a k-regular graph of odd girth g is a k-regular graph of even girth greater than g.

Next, we present two analogs of the canonical double cover construction by utilizing the idea of a Cayley graph and voltage lift.

3. Recursive Constructions from Odd to Even Girth Starting from Cayley Graphs

Let G be a finite group with a generating set $S = S^{-1}$ not containing the identity and closed under inverses. The *Cayley graph* $\Gamma = \mathcal{C}(G, S)$ is a regular graph of degree $|S|$ that has G as its set of vertices, and its adjacency is defined by making each vertex $g \in G$ adjacent to all the vertices in the set $gS = \{gs \mid s \in S\}$. Alternatively, for any two vertices $g, h \in G$, g is adjacent to h if and only if $h^{-1}g \in S$. Note that the fact that S is closed under inverses makes the resulting graph undirected. The graph $\Gamma = \mathcal{C}(G, S)$ is connected if and only if S generates G.

3.1. Direct Product Construction

Our first construction is essentially just another way of looking at the canonical double cover of a Cayley graph, which might sometimes prove useful. It shows that the canonical double cover of a Cayley graph is a Cayley graph again. The proof is left to the reader.

Theorem 3. *Let* $\Gamma = \mathcal{C}(G, S)$, $S = \{s_1, s_2, \ldots, s_k\}$. *The Cayley graph* $\Gamma = \mathcal{C}(G \times \mathbb{Z}_2, \{(s_1, 1), (s_2, 1), \ldots, (s_k, 1)\})$ *is the canonical double cover of* Γ.

3.2. Dipole Lift Construction

Let $\mathcal{C}(G, S)$ be a Cayley graph. Consider the base graph $\Gamma = (V, E)$ which is a dipole consisting of two vertices and $|S|$ multiple parallel edges. Take G to be the voltage group, and let α assign to each edge of Γ a unique element of S. Consider the lift Γ^α.

Example 4. *Consider a complete graph* K_4 *which is a Cayley graph* $\mathcal{C}(G, S)$ *with* $G = \mathbb{Z}_2^2$ *and* $S = \{(1,0), (0,1), (1,1)\}$. *The dipole lift graph* Γ^α *is shown in Figure 3.*

Example 5. *Let* $\mathcal{C}(G, S)$ *a be Cayley graph with* $G = S_3$ *and* $S = \{(12), (13), (23)\}$. *Then, the dipole lift graph* Γ^α *is shown in Figure 4.*

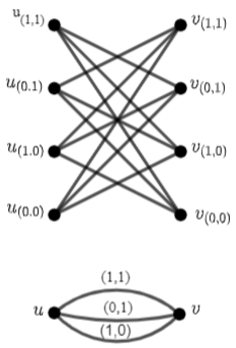

Figure 3: Lift Graph of K_4 Viewed as a Cayley Graph

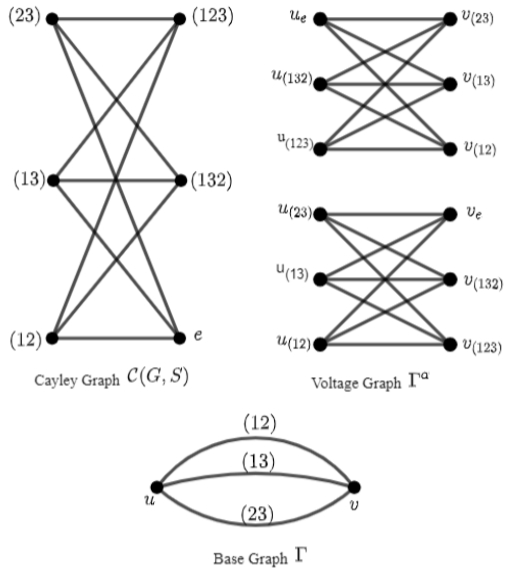

Figure 4: Lift Graph of Cayley Graph Using $G = S_3$

Theorem 4. *Let $\mathcal{C}(G,S)$ be a Cayley graph. The lift of the dipole graph with $|S|$ parallel edges obtained via Construction 3.2 is isomorphic to the canonical double cover of $\mathcal{C}(G,S)$.*

Proof. The proof is again straightforward. It is easy to see that the preimage sets $\{u_g \mid g \in G\}$, $\{v_g \mid g \in G\}$, of the two vertices u, v of the dipole (called *fibers*) are independent and that the edges of the constructed graph connect u_g to v_h if and only $h = gs$, for some $s \in S$. □

To conclude the section, it is interesting to note that the connectedness of the graphs constructed via Construction 3.2 is expressed differently from that of the canonical double cover stated in Condition 3 of Lemma 1.

Lemma 2. *Let $\mathcal{C}(G,S)$ be a Cayley graph. The lift of the dipole graph with $|S|$ parallel edges obtained via Construction 3.2 is connected if and only if the subgroup of all elements of G that can be expressed as a product of elements from S consisting of an even number of generators is equal to G.*

4. Parametric Analysis of the Moore Bound

All the preceding constructions start with an odd-girth graph and construct an even-girth graph of larger girth and order twice the order of the original graph. In view of the usefulness of this construction, it is natural to ask whether there might exist an analogous universal construction starting from an even-girth graph and resulting in an odd-girth graph of larger girth and order an α-multiple of the order of the original graph.

Here, we present a parametric analysis of the Moore bound with a view to understanding the feasibility of obtaining a recursive construction from even to odd girth. The aim is to understand the rate at which $n(k,g)$ varies with a continuous change in g or k or both. In other words, we are looking for the best possible α and β for which these inequalities hold

$$n(k, g+1) \leq \alpha n(k,g) \ \ and \ \ n(k+1, g) \leq \beta n(k,g)$$

in case of even g.

Since $M(k,g) \leq n(k,g)$, $\forall\ k, g \geq 3$, we approach this question by considering the Moore bound. Obtaining constants α', β' for which $M(k, g+1) \leq \alpha' M(k,g)$ or $M(k+1, g) \leq \beta' M(k,g)$ is sufficient for determining lower bounds for the rate of growth of $n(k,g)$.

To analyze the parameters k and g, we first simplify the expression in Equation (2) by substituting the formula for the sum of the nth term of the geometric progression to obtain

$$M(k,g) = \begin{cases} \frac{k(k-1)^{\frac{g-1}{2}}-2}{k-2}, & g \ odd \\ \\ \frac{2(k-1)^{\frac{g}{2}}-2}{k-2}, & g \ even. \end{cases} \tag{3}$$

For brevity, we use Equation 3 in our analysis, and the following fundamental cases are considered.

Case 1. *Let k be fixed and $g = 2r-1$. Define $\alpha_{k,g}$ as the ratio between two successive Moore bounds for g and $g+1$. Then,*

$$\begin{aligned} \alpha_{k,g} = \frac{M(k,g+1)}{M(k,g)} &= \frac{2(k-1)^r - 2}{k(k-1)^{r-1} - 2} \\ &> \frac{2(k-1)^r - 2(k-1)}{k(k-1)^{r-1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2(k-1)}{k} - \frac{2}{k(k-1)^{r-2}} \\
&\geq \frac{2(k-1)}{k} - 1
\end{aligned}
$$

This result is in line with the canonical double cover construction from odd to even girth, and the bound $n(k, g+1) \leq 2n(k,g)$, g odd, as presented by Erdős and Sachs in 1963. It is also easy to see why it was possible to improve this original bound as was done by Balbuena et al. in [2]. For completeness, we compare the results of Balbuena et al. as stated in in Theorem 2 with the known record (k, g)-graphs listed in [8] in Table 4. The wide gap between the record graph and the upper bound from Theorem 2 suggests that a further improvement might be possible.

Table 1
Comparison of the best known $(3, g)$-graphs and the upper bound by Balbuena et al. [2].

Girth g	Record $n(3,g)$	[2]
12	126	204
14	384	474
16	960	1196
18	2560	4228
20	5376	8556
22	16206	31804
24	35640	88076

Case 2. *Let k be fixed and $g = 2r$. Define $\overline{\alpha_{k,g}}$ as the ratio between two successive Moore bounds for g and $g+1$. Then,*

$$
\begin{aligned}
\overline{\alpha_{k,g}} = \frac{M(k,g+1)}{M(k,g)} &= \frac{k(k-1)^r - 2}{2(k-1)^r - 2} \\
&> \frac{k(k-1)^r - 2}{2(k-1)^r} \\
&= \frac{k}{2} - \frac{2}{2(k-1)^r} \\
&> \frac{k}{2} - 1
\end{aligned}
$$

Comparing the lower bound for $\underline{\alpha_{k,g}}$ to that of $\overline{\alpha_{k,g}}$, it is immediately clear that the transition between the odd and even girth g and that of the even and odd girth g are of a different character. As a consequence, we obtain the following theorem.

Theorem 5. *There is no $\alpha \in \mathbb{R}$ such that for any $k \geq 3$ and even $g \geq 4$, $n(k, g+1) \leq \alpha n(k,g)$.*

Proof. Suppose for a contradiction, that there exists an $\alpha \in \mathbb{R}$ such that $n(k, g+1) \leq \alpha n(k,g)$, for all $k \geq 3$ and all even $g \geq 4$. If that were the case, an easy induction would yield the upper bound:

$$n(k, 2r) \leq 2^{r-2}\alpha^{r-2} n(k,4) = 2^{r-2}\alpha^{r-2} 2k,$$

for all $k \geq 3$. At the same time, the above analysis yields the lower bound:

$$
\begin{aligned}
M(k,2r) &= \underline{\alpha_{k,2r-1}}\,\overline{\alpha_{k,2r-2}} \dots \underline{\alpha_{k,5}}\,\overline{\alpha_{k,4}}\, n(k,4) \\
&\geq \left(\frac{2(k-1)}{k} - 1\right)^{r-2} \left(\frac{k}{2} - 1\right)^{r-2} 2k.
\end{aligned}
$$

It is easy to see that for sufficiently large k, $2^{r-2}\alpha^{r-2} < \left(\frac{2(k-1)}{k} - 1\right)^{r-2} \left(\frac{k}{2} - 1\right)^{r-2}$, and hence, for sufficiently large k and sufficiently large g, $n(k,g) < M(k,g)$, which is a contradiction. □

In summary, there is no analogue to the canonical double cover construction starting from an even girth graph. As suggested in the above Case 2, any universal construction of an odd girth graph from an even girth graph has to have the property that the order of the resulting graph should be roughly proportional to the $\frac{k}{2}$ multiple of the original graph. However, the precise ratio also depends on how close is the order of the smallest (k, g)-graph, g even, to the corresponding Moore bound $M(k, g)$.

5. A Recursive Construction of $(k+1, 6)$-Graphs from $(k, 6)$-Graphs Using Perfect Matching

Our last construction is again recursive. It starts with a k-regular bipartite $\Gamma = (V, E)$ and a selected perfect matching for Γ whose existence is guaranteed by the following well-known result.

Theorem 6 ([17]). *Every regular bipartite graph has a perfect matching.*

Select a perfect matching for a bipartite k-regular Γ, and let $\tilde{\Gamma}$ be the multigraph obtained from Γ by adding a parallel edge to each of the edges of the perfect matching of Γ. It follows from the properties of a perfect matching that $\tilde{\Gamma}$ is a $(k+1)$-regular multigraph. In what follows, we shall refer to the added edges of $\tilde{\Gamma}$ as the *new edges* and to the original edges of Γ as the *old edges*. To use the voltage graph construction, let us further replace each edge of $\tilde{\Gamma}$ with a pair of opposing darts. Let $V(\tilde{\Gamma}) = V_1 \cup V_2$ be the bipartite division of the vertices of $\tilde{\Gamma}$, and let $D(\tilde{\Gamma})$ denote the set of darts of $\tilde{\Gamma}$. Let our selected voltage group be $G = \mathbb{Z}_3$, assign the voltage 0 to all darts originating from the old edges of $\tilde{\Gamma}$ and assign the voltage 1 to either of the two darts formed from the new edges of $\tilde{\Gamma}$ (and 2 to the opposed dart). Consider the voltage graph $\tilde{\Gamma}^{\alpha}$.

Lemma 3. *$\tilde{\Gamma}^{\alpha}$ is a $(k+1)$-regular bipartite graph.*

Proof. Let V_1^{α} and V_2^{α} be the vertex sets of $\tilde{\Gamma}^{\alpha}$ formed as unions of fibers of the elements of V_1 and V_2, respectively. The two sets are clearly disjoint, while no two vertices belonging to V_1^{α} and no two vertices belonging to V_2^{α} are adjacent. As the degree of the vertices in the voltage lift is equal to the degree of the corresponding vertices in the $(k+1)$-regular base graph, the result follows. □

Example 6. *Let us apply the first step of the above voltage lift construction to the bipartite cubic Heawood graph of order 14. Selecting a particular perfect matching and adding the new parallel edges results in the multigraph shown in Figure 5.*

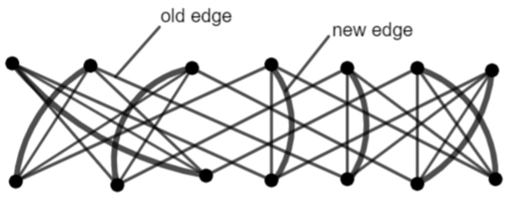

Figure 5: Heawood Graph with Doubled Perfect Matching

The following lemma is the key to obtaining a recursive construction of graphs of girth 6.

Lemma 4. *If Γ is a bipartite k-regular graph of girth 6, then $\tilde{\Gamma}^{\alpha}$ is a $(k+1)$-regular graph of girth 6 and order the 3-multiple of the order of Γ.*

Proof. Let α be as defined above. We begin by describing the edges of Γ^{α} as shown in Figure 6.

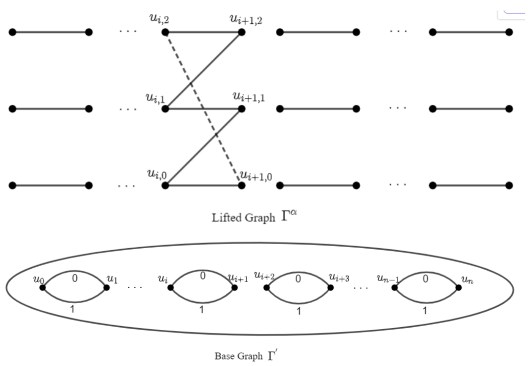

Figure 6: Base graph $\tilde{\Gamma}$ and lift graph $\tilde{\Gamma}^{\alpha}$

The edges of Γ^{α} are of two types: the lifts of the old edges, which we shall refer to as the *horizontal edges*, and the lifts of the new edges, which shall be called *vertical edges*. Since our voltage group is $\mathbb{Z}_3$, and all the old edges of $\tilde{\Gamma}$ received the voltage $0 \in \mathbb{Z}_3$, $\tilde{\Gamma}^{\alpha}$ contains three parallel horizontal copies of $V(\Gamma)$. We call each of such copy a *layer*. While the horizontal edges connect vertices belonging to the same layer, vertical edges connect vertices between distinct layers. If we denote the vertices of Γ by $u_0, u_1, \ldots, u_{n-1}$ (with n representing the order of Γ), all vertices of Γ^{α} are of the form $u_{i,j}$, where i indicates the position of the vertex in a layer and $j \in \mathbb{Z}_3$ indicates the specific layer. Let us assume without loss of generality that the new edges of the perfect matching in $\tilde{\Gamma}$ connect the vertices u_i, u_{i+1}, with i even. Under these assumptions, all the vertical edges in Γ^{α} are of the form $u_{i,j}u_{i,j+1}$, with i even, and $j+1$ calculated modulo 3. The three layers of $\tilde{\Gamma}^{\alpha}$, its vertical edges, and the corresponding base graph $\tilde{\Gamma}$ are pictured in Figure 6.

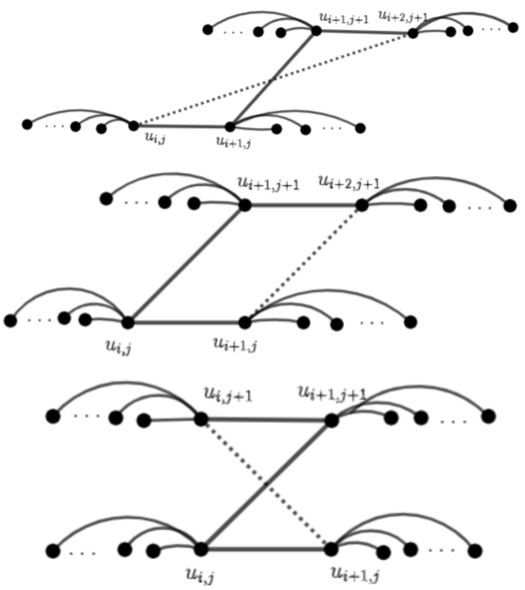

Figure 7: Possible cycles of length 4

Next, we prove that the girth of $\tilde{\Gamma}^{\alpha}$ is 6. Since, for i even, the vertices $(u_{i+1,0})$, $(u_{i,0})$, $(u_{i+1,1})$, $(u_{i,1})$, $(u_{i+1,2})$, $(u_{i,2})$, $(u_{i+1,0})$ form a 6-cycle, the girth of $\tilde{\Gamma}^{\alpha}$ is at most 6 (of course, all the original 6-cycles of Γ also lift into 6-cycles of $\tilde{\Gamma}^{\alpha}$). Thus, it suffices to show that the girth of $\tilde{\Gamma}^{\alpha}$ is not smaller. Note that since $\tilde{\Gamma}^{\alpha}$ is bipartite, it contains no cycle of odd length.

To complete this proof, we show that $\tilde{\Gamma}^{\alpha}$ contains no cycles of length 4. Suppose the existence of a 4-cycle $\mathcal{C}$ in $\tilde{\Gamma}^{\alpha}$. Then, $\mathcal{C}$ can either be contained in one of the layers or contains vertices from two or more layers. Since each layer is a copy of Γ, which is of girth 6, no cycle fully contained in a horizontal layer is of length 4. If $\mathcal{C}$ is contained in two or more layers, then $\mathcal{C}$ must be of one of the three forms $\{(u_{i,j}), (u_{i+1,j}), (u_{i+1,j+1}), (u_{i+2,j+1}), (u_{i,j})\}$, or $\{(u_{i+1,j}), (u_{i,j}), (u_{i+1,j+1}), (u_{i+2,j+1}), (u_{i+1,j})\}$, or $\{(u_{i+1,j}), (u_{i,j}), (u_{i+1,j+1}), (u_{i,j+1}), (u_{i+1,j})\}$; pictured in Figure 8.

Suppose $\mathcal{C}$ is a cycle of the first type. In this case, two incident edges (u_i, u_{i+1}) and (u_{i+1}, u_{i+2}) belong to the perfect matching, which is a contradiction. Similarly, the second case implies that two edges (u_i, u_{i+1}) and (u_{i+1}, u_{i+2}) with a common vertex u_{i+1} are in the perfect matching, which is again impossible. The last case would result in forcing a cycle $(u_{i-1}, u_i, u_{i+1}, u_{i+2})$ in the base graph. This is impossible since Γ is assumed to be of girth 6. It follows that the girth of $\tilde{\Gamma}^{\alpha}$ is indeed 6. □

Example 7. *Consider the graph in Figure 5 obtained from Heawood graph. Applying Lemma 4 yields the graph below.*

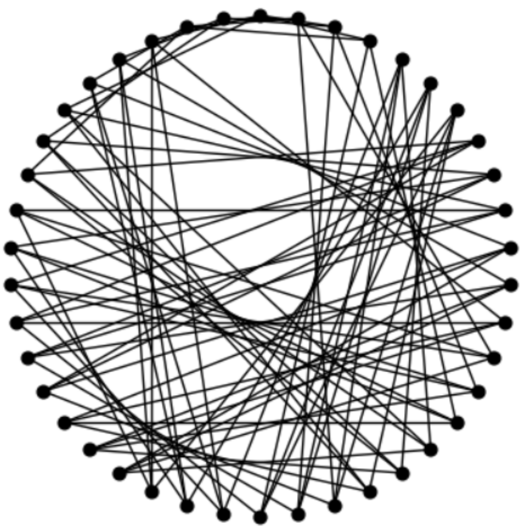

Figure 8: $\tilde{\Gamma}^{\alpha}$ of $\tilde{\Gamma}$ in Figure 5

Recall that Heawood graph is a bipartite 3-regular graph of girth 6 and order 14. Note also that the graph $\tilde{\Gamma}^{\alpha}$ constructed from a bipartite k-regular graph of girth 6 is bipartite $(k+1)$-regular of girth 6. Thus, relying repeatedly on Theorem 6 and starting from the Heawood graph, we obtain the following.

Theorem 7. *Let $k \geq 3$. Then,*

$$n(k, 6) \leq 3^{k-3} \cdot 14.$$

Clearly, the above upper bound is not particularly strong. This is due to the fact that we do not know whether (k, g)-cages for even g must necessarily be bipartite. If that was the case, we would obtain the stronger result $n(k+1, g) \leq 3n(k, g)$, for all $k \geq 3$ and even $g \geq 4$. Interestingly, the bipartiteness of the (k, g)-cages for even g has been repeatedly conjectured by various authors [8].

Even performing a detailed analysis of the rate of growth of the Moore bound in case of a fixed even girth g yields a much weaker rate of growth for $n(k, g)$ than the one stated in Theorem 7:

Case 3. *Let $g = 2r$ be fixed. Define $\gamma_{k,g}$ as the following ratio:*

$$\begin{aligned} \gamma_{k,g} = \frac{M(k+1, g)}{M(k, g)} &= \frac{(2k^r - 2)(k-2)}{(2(k-1)^r - 2)(k-1)} \\ &\geq \frac{(k^r - 1)(k-2)}{((k-1)^r - 1)(k-1)} \end{aligned}$$

Thus, regardless of the girth, $\lim_{k\to\infty} \gamma_{k,g} = 1$, and unlike the case of the transition from even girth to odd, there might exist a constant γ and a universal construction of a $(k+1, g)$-graph from a (k, g)-graph which yields $(k+1, g)$-graphs of orders proportional to a γ multiple of the order of the original (k, g)-graph with γ smaller than 3.

6. Comparison with Existing Constructions

In literature, one can find several constructions of $(k-1, 6)$-graphs starting with a known $(k, 6)$-graph [1, 2, 4, 12]. In this section, we compare our construction with a recursive construction for girth 6 introduced by Gács and Héger [12] which is the best recursive construction for $(k, 6)$-graphs. Like our construction, their construction generates an infinite family of $(k, 6)$-graphs using the idea of the t-good structure. However, Gács and Héger's construction works in a way that is opposite to ours. Namely, it starts from a point-line incidence graph of a projective plane which is a $(p^e + 1, 6)$ Moore graph of degree equal to a prime power plus 1, and recursively constructs graphs of smaller degrees by removing or adding vertices and edges (as we discussed earlier in the Introduction). As it is well known, the gap between two consecutive prime powers can be arbitrarily large. Since the ratio of the number of prime powers to that of prime numbers converges to 1, prime powers grow asymptotically at the same rate. Thus, the orders of $(k, 6)$-graphs constructed by removing t-good structures for k's which are only slightly larger than a prime power but are quite a bit smaller than the next prime power may be very far from the orders of corresponding cages.

In comparison, our constructions produce $(k+1,6)$-graphs from $(k,6)$-graphs for any degree k. This means that, especially in the case of k's described at the end of the previous paragraph, our constructions require a smaller number of recursions (as we may start from the closest smaller prime power than k). Nevertheless, our construction does not outperform that of Gács and Héger, and should be therefore viewed as the basis for another future construction that might eventually produce smaller graphs; at least in cases of k's just a bit larger than a prime power.

7. Computer Assisted Methods

All the construction methods considered so far were deterministic in the somewhat loose sense that they did not require searching a large space of possible constituents. In this section, we will briefly discuss methods for improving the girth of a voltage lift graph using computer assisted searches. The key lemma we will rely on is the following:

Lemma 5. *[9, Lemma 2.1.] Let Γ be a finite graph and $\alpha : D(\Gamma) \to G$ be a voltage assignment of Γ. The girth of the voltage graph lift Γ^α is equal to the length of a shortest closed non-backtracking walk $\mathcal{W}$ in Γ of net voltage 1_G.*

A *closed non-backtracking walk* in G is a closed walk which does not contain an edge travelled in one direction followed immediately by the same edge travelled in the opposite direction. It is easy to see that the length of a closed non-backtracking walk in G of girth g must be at least g, and also that a closed non-backtracking walk in G of length g must in fact be a cycle. The *net voltage* of a closed walk $e_1e_2\ldots e_n$, $e_i \in D(G)$ is the product of the voltages of its darts in the order determined by the walk, i.e., the net voltage of the above walk is the product $\alpha(e_1)\alpha(e_2)\ldots\alpha(e_n) \in G$. Our observations together with Lemma 5 yield the following corollary.

Corollary 1. *Let Γ be a finite graph of girth g and $\alpha : D(\Gamma) \to G$ be a voltage assignment for Γ. The girth of the lift Γ^α is greater than the girth of Γ if and only if the net voltage of every g-cycle of Γ is different from 1_G.*

When applying the above corollary to the canonical double cover of an odd-girth base graph Γ, we quickly observe that the net voltage of any g-cycle (as well as of any odd-length cycle) in Γ is equal to 1, the non-identity element of $\mathbb{Z}_2$. This is the basis of the proof of Lemma 1 as well as the argument that shows that the canonical double cover of an odd-girth graph Γ has larger girth than Γ itself. Applying Corollary 1 to graphs of even girth is quite a bit trickier. Namely, the canonical double cover voltage assignment clearly assigns the net voltage $0 \in \mathbb{Z}_2$ to all cycles of even length, and hence, the lift of an even-girth base graph is of the same girth as the original girth. This does not necessarily mean that no $\mathbb{Z}_2$-voltage assignment for an even girth base graph can lead to a lift graph of larger girth. As long as the order of the lift of a (k,g)-graph Γ does not violate the Moore bound for k and $g' > g$, a $\mathbb{Z}_2$-voltage assignment having the property that the net voltage of no g-cycle of Γ equals 0 might exist. Obviously, any such voltage assignment would lead to a (k,g')-graph of twice the order of the base graph. However, one must not forget the results of Section 4 where we proved that there must exist even girth (k,g)-graphs for which no $\mathbb{Z}_2$-voltage assignments have the above property (as otherwise there would exist a 'universal' construction doubling the order and increasing the girth of all even girth graphs). More specifically, the results of Section 4 suggest that in order to obtain a voltage graph lift of an even-girth k-regular graph of larger girth, one has to use voltage assignments using groups of orders proportional to $\frac{k}{2}$. This suggests the following computational approach to obtaining (k,g')-graphs from (k,g)-graphs using the voltage graph construction (we will assume that g is even and $g' > g$).

Let Γ be a k-regular graph of even girth g, and let G be a finite group of order close to $\frac{k}{2}$ or larger. A brute force approach to answering the question whether there exists a voltage assignment $\alpha : D(\Gamma) \to G$ that leads to a (k,g')-graph Γ^α of girth $g' > g$ is to consider all possible G-voltage assignments of Γ and for each of them to test whether the net voltages of all girth cycles in Γ differ from 1_G. Should one find such a voltage assignment, the answer to the above question would be positive, otherwise the answer would be a no. In case of a no answer, one might next consider other groups of the same order as G or groups of orders larger than $|G|$. It is useful to point out that considering larger and larger groups will eventually lead to a voltage assignment for which the net voltages of all girth cycles in Γ differ from 1_G. An easy proof of this fact uses the voltage assignment $\alpha : D(\Gamma) \to \mathbb{Z}_2^{|E(\Gamma)|}$ assigning to each pair of opposing darts of Γ a different unit basis vector $\vec{e}_i$ and the fact that there are no repeated edges in girth cycles of Γ (and hence all girth cycles in Γ have a non-trivial voltage). Unfortunately, the number of possible voltage assignments $\alpha : D(\Gamma) \to G$ is equal to $|G|^{|E(\Gamma)|}$, and using brute force becomes very quickly infeasible.

8. Application of $(k,6)$-Graphs in Communication Systems

The problem of communicating reliably over a noisy channel has been in existence for many decades. One approach to solving this problem is using *error-correcting codes*. An error-correcting code (ECC) is an encoding scheme that transmits messages as binary strings, in such a way that the message can be recovered even if some bits are erroneously flipped. It is done by introducing redundant (parity check) bits to the message to minimize the effect of the noise. Codes that form a subspace of the vector space $\mathbb{Z}_2^n$ are called *linear codes* and are of particular importance. A key concept in the study of linear error correcting codes is the parity-check matrix. A *parity-check matrix* is a matrix that, when multiplied by a binary string of appropriate length viewed as a vector, produces as the result of the multiplication the zero vector if and only if the string represents a code word (equivalently, the rows of a parity-check matrix of a linear code constitute a basis for the space orthogonal to the code space). It can also be used in decoding a message as well as in deciding whether a particular vector is a codeword. If H is a parity check matrix, a code word c belongs to a linear code block C if and only if $cH^T = 0$ (for further details, the reader might consult the standard textbook [14]).

We say that a code C is a *low-density parity-check (LDPC) code* if its parity-check matrix H contains only a small number of 1's. LDPC codes have an excellent performance with iterative decoding, which is very close to the Shannon limit over Additive White Gaussian Noise (AWGN) channels. These codes are constructed using bipartite graphs called Tanner graphs [18]. The Tanner graph of an LDPC code is composed of two sets of vertices (nodes); namely, variable vertices and check vertices. Each variable and check vertex correspond to the number of a codeword and a parity symbol, respectively. If a variable vertex is constrained by a check vertex, then there is an edge connecting the two vertices. In addition, Tanner graphs are used to construct longer codes from smaller ones.

Example 8. *We consider the parity check matrix H used in the study of the girth of a Tanner graph of an LDPC code from [6]. By studying the Tanner graph of H, we discovered that the graph of H is the $(3,6)$-cage, which is the Heawood graph.*

$$H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Importantly, the girth of a Tanner graph giving rise to an LDPC code is an important factor that determines how good an LDPC code is, especially for regular LDPC codes, Quasicyclic LDPC codes, etc. For the regular LDPC codes, the girth of its Tanner graph is a lower bound of its minimum distance. In other words, it is a threshold to overcome the noise. Therefore, the bigger the girth, the better the LDPC codes. Recently, researchers have studied a class of LDPC codes with large girths, which they called *Tanner (J, L)-regular QC-LDPC codes* [20]. Their results showed that most Tanner $(3,5)$-, $(5,7)$-, $(5,11)$- and $(5,13)$-regular QC-LDPC codes have girths 6, 8 and 10 [20].

References

[1] Araujo-Pardo G., Balbuena C., and Héger T.: Finding small regular graphs of girths 6, 8 and 12 as subgraphs of cages. Discrete Math. **310** (2010) 1301–1306.

[2] Balbuena C., González-Moreno D., and Montellano-Ballesteros J. J.: A note on the upper bound and girth pair of (k,g)-cages. Discrete Appl. Math. **161 (6)** (2013) 853–857.

[3] Biggs N. L.: Constructions for cubic graphs of large girth. Electron. J. Combin. **5** (1998).

[4] Brown W. G.: On Hamiltonian regular graphs of girth 6. J. London Math. Soc. **42** (1967) 514–520.

[5] Conder M., Exoo G., and Jajcay R.: On the limitations of the use of solvable groups in Cayley graph cage constructions. Eur. J. Comb. **31 (7)** (2010) 1819–28.

[6] Dehghani H., Ahmadi M., Alikhani S., and Hasni R.: Calculation of Girth of Tanner Graph in LDPC Codes. Trends in Applied Sciences Researchn **7** (2012) 929–934.

[7] Erdős P. and Sachs H.: Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. Wiss. Z. Uni. Halle (Math. Nat.) **12** (1963) 251–257.

[8] Exoo G. and Jajcay R.: Dynamic cage survey. Electron. J. Comb., Dynamic Survey **16** (2013).

[9] Exoo G. and Jajcay R.: On the girth of voltage graph lifts. Eur. J. Comb. **32 (4)** (2011) 554–562.

[10] Feit W. and Higman G.: The nonexistence of certain generalized polygons. J. Algebra **1** (1964) 114–131.

[11] Feng Y.Q., Kutnar K., Malnič A., and Marušič D.: On 2-fold covers of graphs. J. Comb. Theory, Ser. B **98 (2)** (2008) 324–341.

[12] Gács A. and Héger T.: On geometric constructions of (k,g)-graphs. Contrib. Discrete Math. **3** (2008) 63–80.

[13] Gross J. L. and Tucker T. W.: Topological Graph Theory. Dover Mineola, New York (2001).

[14] Hill R.: A first course in coding theory. Oxford Applied Mathematics and Computing Science Series, Oxford, Clarendon Press. XII (1986).

[15] Hoffman A. J. and Singleton R. R.: On Moore graphs with diameters 2 and 3. IBM J. Res. Develop. **4** (1960) 497–504.

[16] Imrich W. and Pisanski T.: Multiple Kronecker covering graphs. Eur. J. Comb. **29 (5)** (2008) 1116–1122.
[17] König D.: Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. Math. Ann. **77** (1916) 453–465.
[18] Tanner R. M.: A recursive approach to low complexity codes. IEEE Trans. Inform. Theory **27** (1981) 533-547.
[19] Waller D. A.: Double covers of graphs. Bull. Aust. Math. Soc. **14 (2)**(1976) 233–248.
[20] Xu H., Zhu H., Miao X., Xu M., Luo Z., and Li H.: A Short Note on the Girth of Tanner (5,13)-Regular QC-LDPC Codes. 2021 17th International Conference on Computational Intelligence and Security (CIS) (2021) 449–453. doi: 10.1109/CIS54983.2021.00099.