A NOTE ON THE BOOLEAN DIMENSION OF A GRAPH AND OTHER RELATED PARAMETERS

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ABSTRACT

We consider Boolean, binary and symplectic dimensions of a graph. We obtain an exact formula for the Boolean dimension of a tree in terms of a certain star decomposition. We relate the binary dimension to the $mrank_2$ of a graph.

Keywords Graphs · Tournaments

1 Preliminaries

Let $\mathbb{F}_2$ be the 2-element field, identified with the set $\{0, 1\}$. Let U be a vector space over $\mathbb{F}_2$, and B be a bilinear form over U. This form is *symmetric* if $B(x, y) = B(y, x)$ for all $x, y \in U$. A vector $x \in U \setminus \{0\}$ is *isotropic* if $B(x, x) = 0$; two vectors x, y are *orthogonal* if $B(x, y) = 0$. The form B is said to be *alternating* if each $x \in U$ is isotropic, in which case (U, B) is called a *symplectic space*. The form is a *scalar product* if U has an *orthonormal base* (made of non-isotropic and pairwise othogonal vectors). If U has finite dimension, say k, we identify it with $\mathbb{F}_2^k$, the set of all k-tuples over $\{0, 1\}$; we suppose that the scalar product of two vectors $x := (x_1, \dots, x_k)$ and $y := (y_1, \dots, y_k)$ is $\langle x \mid y \rangle := x_1 y_1 + \dots + x_k y_k$.

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The graphs we consider are undirected and have no loop. That is a graph is a pair (V, E) where E is a subset of $[V]^2$, the set of 2-element subsets of V. Elements of V are the *vertices* and elements of E are its *edges*. The graph G be given, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. For $u, v \in V(G)$, we write $u \sim v$ if there is an edge joining u and v. For a vertex $v \in V(G)$, we denote by $N(v)$ the set of vertices in G adjacent with v. We are going to define three notions of dimension of a graph. The graph does not need to be finite, but our main results are for finite graphs.

Definition 1.1. Let $B: U \times U \to \mathbb{F}_2$ be a symmetric bilinear form. Let G be a graph. We say that $\phi: V(G) \to U$ is a *representation* of G in (U, B) if for all $u, v \in V(G), u \neq v$, we have $u \sim v$ if and only if $B(\phi(u), \phi(v)) = 1$. The *binary dimension* of G is the least cardinal κ for which there exists a symmetric bilinear form B on a vector space U of dimension κ and exists a representation of G in (U, B). The *symplectic dimension* of G is the least cardinal κ for which there exists a symplectic space (U, B) in which G has a representation. When the bilinear form is a scalar product, a representation is called a *Boolean representation*. The *Boolean dimension* of G is the least cardinal κ for which G has a Boolean representation in a space of dimension κ equipped with a scalar product.

For the Boolean representation and the Boolean dimension, we have the following equivalent definition (Proposition 3.1 of [2]).

Definition 1.2. Let G be a graph. A *Boolean representation* is a family $\mathcal{V} := (V_i)_{i<\kappa}$ of subsets of V such that $u \sim v$ if and only u and v belong to an odd number of V_i's. The Boolean dimension is the minimum cardinality of the family $\mathcal{V}$ for which such a representation exists. The *Boolean dimension* of G is denoted by $b(G)$.

This notion of Boolean dimension has been considered by Belkhechine et al. [2, 3] (see also [1, 7]) The symplectic dimension has also been considered by other authors, for example, [5, 6].

2 Boolean dimension of trees

In this section, we show that there is a nice combinatorial interpretation for the Boolean dimension of trees.

We mention the following result [Belkhechine et al. [3]]

Lemma 2.1. *Let $G := (V, E)$ be a graph, with $V \neq \varnothing$. Let $f: V \to \mathbb{F}_2^m$ be a boolean representation of G. Let $S \subseteq V$ such that $S \neq \varnothing$. Suppose that for all $A \subseteq S, A \neq \varnothing$, there exists $v \in V \setminus A$ such that $|N(v) \cap A|$ is odd. Then $\{f(x) \mid x \in A\}$ is linearly independent.*

This suggests the following definition.

Definition 2.2 (Belkhechine et al. [3])**.** Let $G := (V, E)$ be a graph. A set $U \subset V$ is called *independent* $\pmod 2$ if for all $B \subseteq U, B \neq \varnothing$, there exists $u \in V \setminus B$ such that $|N_G(u) \cap B|$ is odd, where $N_G(u)$ denotes the neighbourhood of u in G; otherwise U is said to be *dependent* $\pmod 2$. Let $a(G)$ denote the maximum size of an independent set $\pmod 2$ in G. **From now, we omit $\pmod 2$ unless it is necessary to talk about independence in the graph theoretic sense.**

Definition 2.3. Let $T := (V, E)$ be a tree. A *star decomposition* Σ of T is a family $\{S_1, \dots, S_k\}$ of subtrees of T such that each S_i is isomorphic to $K_{1,m}$ (a star) for some $m \geq 1$, the stars are mutually edge-disjoint, and their union is T. For a star decomposition Σ, let $t(\Sigma)$ be the number of trivial stars in Σ (stars that are isomorphic to $K_{1,1}$), and let $s(\Sigma)$ be the number of nontrivial stars in Σ (stars that are isomorphic to $K_{1,m}$ for some $m > 1$). We define the parameter $m(T) := \min_\Sigma\{t(\Sigma) + 2s(\Sigma)\}$ over all star decompositions Σ of T. A star decomposition Σ of T for which $t(\Sigma) + 2s(\Sigma) = m(T)$ is called an *optimal star decomposition* of T.

Theorem 2.4. *For all trees T, we have $a(T) = b(T) = m(T)$.*

We know that $a(G) \leq b(G)$ for all graphs G, and $b(T) \leq m(T)$ for all trees T. See Belkhechine et al. [3] for details.

The proof of Theorem 2.4 will depend on the following propositions.

Definition 2.5. A *cherry* in a tree T is a maximal subtree S isomorphic to $K_{1,m}$ for some $m > 1$ that contains m end vertices of T. We refer to a cherry with m edges as an m-cherry.

Proposition 2.6. *Let $T := (V, E)$ be a tree that contains a cherry. If all proper subtrees T' of T satisfy $a(T') = m(T')$, then $a(T) = m(T)$.*

Proof. Let $x \in V$ be the center of a k-cherry in T, with $N_T(x) = \{u_1, \dots, u_k, w_1, \dots, w_\ell\}$, where $d(u_i) = 1$ for all i, and $d(w_i) > 1$ for all i. Here $d(x)$ denotes the degree of vertex x. For each $i = 1$ to ℓ, let T_i be the maximal subtree that contains w_i but does not contain x.

First, we show that any optimal star decomposition of T in which x is not the center of a star can be transformed into an optimal star decomposition in which x is the center of a star. Consider an optimal star decomposition Σ in which x is not the center of a star. Therefore, edges xu_i are trivial stars of Σ. Now if $k > 2$ or if there is a trivial star xw_i in Σ, then we could have improved $t(\Sigma) + 2s(\Sigma)$ by replacing all trivial stars containing x by their union, which is a star centered at x. Hence, assume that $k = 2$ and each w_i is the center of a nontrivial star S_i, which contains the edge xw_i. Now replace each S_i by $S_i' \coloneqq S_i - xw_i$, and add a new star centered at x with edge set $\{xw_1, \dots, xw_\ell, xu_1, xu_2\}$. The new decomposition is also optimal.

Now consider an optimal star decomposition Σ in which x is the center of a star. The induced decompositions on T_i are all optimal since Σ is optimal. Let for each $i \in \{1, \dots, \ell\}$, let A_i be a maximum size independent set in T_i. Hence $|A_i| = a(T_i) = m(T_i)$ for all i, and $m(T) = 2 + \sum_i m(T_i) = 2 + \sum_i a(T_i)$. We show that $A \coloneqq \{x, u_1\} \cup (\cup_i A_i)$ is a maximum size independent set in T.

Consider a non-empty set $B \subseteq A$. We show that there exists $v \in V$ such that $|N_T(v) \cap B|$ is odd. If $x \in B$, then we take $v = u_2$. If $B = \{u_1\}$, then we take $v = x$. In all other cases, $B_i \coloneqq B \cap V_i$ is non-empty for some i, and $x \notin B$. We find $v \in V_i \setminus B_i$ such that $|N_{T_i} \cap B_i|$ is odd. Now $|N_T(v) \cap B|$ is odd since $x \notin B$ and v is not adjacent to u_1. Moreover, $|A| = m(T)$. □

Proposition 2.7. *Let $T \coloneqq (V, E)$ be a tree that contains a vertex y of degree 2 adjacent to a vertex z of degree 1. If $a(T - z) = m(T - z)$, then $a(T) = m(T)$.*

Proof. First, we show that $m(T) = m(T - z) + 1$. If there is an optimal star decomposition of $T - z - y$ in which x is the center of a star, then $m(t - z) = m(T - z - y)$ and $m(T) = m(T - z) + 1$, else $m(T - z) = m(T - z - y) + 1$ and $m(T) = m(T - z - y) + 2$.

Now we consider a maximum size independent set A' in $T - z$. We have $|A'| = a(T - z) = m(T - z)$. We define $A \coloneqq A' \cup \{y\}$ if $y \notin A'$; and $A \coloneqq A' \cup \{z\}$ if $y \in A'$. We show that A is independent in T.

Case 1: $y \notin A'$, hence $y \in A$ and $z \notin A$. Let $B \subseteq A, B \neq \varnothing$.

If $y \in B$, then $|N_T(z) \cap B|$ is odd.

If $y \notin B$, then $B \subseteq A'$, hence there exists $v \in V(T - z)$ such that $|N_{T-z}(v) \cap B|$ is odd, and $|N_T(v) \cap B|$ is odd.

Case 2: $y \in A'$, hence $z \in A$. Let $B \subseteq A, B \neq \varnothing$.

If $z \notin B$, then $B \subseteq A'$. Find $v \in V(T - z) \setminus B$ such that $|N_{T-z}(v) \cap B|$ is odd. Hence $|N_T(v) \cap B|$ is odd.

Now suppose that $z \in B$. If $B = \{z\}$, then $N_T(y) \cap B$ is odd. Otherwise, consider $(B \setminus \{z\})$, which is a subset of A'. Find $v \in V(T - z) \setminus (B \setminus \{z\})$ such that $|N_{T-z}(v) \cap (B \setminus \{z\})|$ is odd. If $v \neq y$, then $|N_T(v) \cap B|$ is odd. and $x \in B$. In this case, let $B' \coloneqq (B \setminus \{z\}) \cup \{y\}$. This is a subset of A'. Find $u \in V(T - z) \setminus B'$ such that $|N_{T-z}(u) \cap B'|$ is odd. Since B' contains x and y, we conclude that u is not adjacent to any of y and z, hence $|N_T(u) \cap B|$ is odd.

Thus we have shown that A is independent. We have $a(T) \geq |A| = |A'| + 1 = m(T - z) + 1 = m(T)$. Since $a(T)$ cannot be more than $m(T)$, we have $a(T) = m(T)$. □

Proof of Theorem 2.4. If a tree T has 2 vertices, then $a(T) = m(T) = 1$. Each tree with at least 3 vertices contains a cherry or a vertex of degree 2 adjacent to a vertex of degree 1. (This is seen by considering the second-to-last vertex of a longest path in a tree.) Now induction on the number of vertices, using Propositions 2.6 and 2.7 implies the result. □

Remark 2.8. Fallat and Hogben [4] consider the problem of minimum rank of graphs, and obtain a combinatorial description for the minimum rank of trees. The connection between minimum rank and the binary dimension is made clear in the next section for arbitrary graphs. Here we only state that in case of trees, the Boolean dimension, binary dimension and the minimum rank coincide, thus the formula given above for the Boolean dimension gives yet another combinatorial description for the minimum rank of a tree.

3 Binary and symplectic dimensions

A graph G is called *reduced* if it has no isolated vertices and no two vertices have the same neighbourhood. Our definition is that from Godsil and Royle [6], where it is noted that there are slightly different definitions of 'reduced' in the literature.

Let $A(G)$ denote the adjacency matrix of G. We denote the rank of a matrix M over $\mathbb{F}_2$ by $\mathrm{rank}_2(M)$, and define $\mathrm{rank}_2(G) := \mathrm{rank}_2(A(G))$. Let $\mathcal{D}_n$ be the set of $n \times n$ matrices with non-diagonal entries 0 and diagonal entries 0 or 1. Suppose that $|V(G)| = n$. We define $\mathrm{mrank}_2(G) := \min\{\mathrm{rank}_2(D + A(G)) \mid D \in \mathcal{D}_n\}$. In the following propositions, we relate the binary and symplectic dimensions of a graph G to its rank and mrank, respectively.

Proposition 3.1. *Let G be a reduced graph on n vertices with adjacency matrix $A(G)$. The symplectic dimension of G is equal to* $\mathrm{rank}_2(G)$.

Proof. The argument is essentially based on [6], where it is shown that there exists a symplectic representation in a vector space over $\mathbb{F}_2$ of dimension $r := \mathrm{rank}_2(G)$.

As shown in [6], it is possible to write

$$A(G) = \begin{pmatrix} M & H^T \\ H & N \end{pmatrix},$$

where the matrix M is the adjacency matrix of a reduced r-vertex graph of rank r, and $H = RM$, and $N = RH^T = RMR^T$, which expresses the rows of the $(n-r) \times n$ matrix $\begin{pmatrix} H & N \end{pmatrix}$ as a linear combination of the rows of the $r \times n$ matrix $\begin{pmatrix} M & H^T \end{pmatrix}$. Rewriting, we have

$$A(G) = \begin{pmatrix} M & MR^T \\ RM & RMR^T \end{pmatrix} = \begin{pmatrix} I \\ R \end{pmatrix} M \begin{pmatrix} I & R^T \end{pmatrix},$$

where the matrix I is the $r \times r$ identity matrix. Thus M determines a non-degenerate symplectic form on $\mathbb{F}_2^r$ given by $B(x,y) := x^T M y$. Taking the columns of the $r \times n$ matrix $\begin{pmatrix} I & R^T \end{pmatrix}$ as the vertices of G, we obtain a representation of G in $(\mathbb{F}_2^r, B)$. Hence the symplectic dimension of G is at most r.

Now suppose that there is a symplectic representation ϕ of G in $(\mathbb{F}_2^k, B)$ for some symplectic form B on $\mathbb{F}_2^k$. We show that $k \geq r$.

Writing $\phi(V(G)) := \{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are column vectors representing the vertices of G with respect to the standard basis, we can write $A(G) = X^T M X$, where M is the symmetric $k \times k$ matrix of the form B with respect to the standard basis $\{e_1, \ldots, e_k\}$ (i.e., $M_{ij} = B(e_i, e_j)$), and $X := (x_1 \cdots x_n)$.

Now let $X := \begin{pmatrix} P & Q \end{pmatrix}$, where P is a $k \times k$ matrix (the first k columns of X) and Q is a $k \times (n-k)$ matrix (the last $n-k$ columns of X). Therefore,

$$A(G) = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} M \begin{pmatrix} P & Q \end{pmatrix} = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} \begin{pmatrix} MP & MQ \end{pmatrix}.$$

Thus we have expressed the rows of $A(G)$ as linear combinations of the rows of the $k \times n$ matrix $\begin{pmatrix} MP & MQ \end{pmatrix}$, which implies that $k \geq r$. □

Proposition 3.2. *Let G be a reduced graph on n vertices with adjacency matrix $A(G)$. The binary dimension of G is equal to* $\mathrm{mrank}_2(G)$.

Proof. The proof of this proposition is similar to that of Proposition 3.1.

Let $D \in \mathcal{D}_n$. Suppose that the rank of $D + A(G) = r$. As in Proposition 3.1, we write

$$D + A(G) = \begin{pmatrix} M & H^T \\ H & N \end{pmatrix},$$

where the matrix M is a symmetric matrix of rank r (it is the adjacency matrix of a graph which possibly has loops but no multiple edges), and $H = RM$, and $N = RH^T = RMR^T$, which expresses the rows of $\begin{pmatrix} H & N \end{pmatrix}$ as a linear combination of the rows of $\begin{pmatrix} M & H^T \end{pmatrix}$. Rewriting, we have

$$D + A(G) = \begin{pmatrix} M & MR^T \\ RM & RMR^T \end{pmatrix} = \begin{pmatrix} I \\ R \end{pmatrix} M \begin{pmatrix} I & R^T \end{pmatrix},$$

where the matrix I is the $r \times r$ identity matrix. Thus M determines a non-degenerate bilinear form on $\mathbb{F}_2^r$ given by $B(x,y) := x^T M y$. Taking the columns of $\begin{pmatrix} I & R^T \end{pmatrix}$ as the vertices of G, we obtain a representation of G in $(\mathbb{F}_2^r, B)$. Hence the binary dimension of G is at most r, which further implies that the binary dimension of G is at most $\mathrm{mrank}_2(G)$ (by taking D that minimises $\mathrm{rank}_2(D + A(G))$).

Next we show that the binary dimension is at least $\mathrm{mrank}_2(G)$.

Let B be a bilinear form on $\mathbb{F}_2^k$, and suppose that there exists a representation ϕ of G in $(\mathbb{F}_2^k, B)$. We write $\phi(V(G)) \coloneqq \{x_1, \ldots, x_n\}$, where x_i are column vectors with respect to the standard basis of $\mathbb{F}_2^k$. Hence, for some D, we have $D + A(G) = X^T M X$, where M is the symmetric matrix of the bilinear form B. As in Proposition 3.1, we write

$$D + A(G) = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} M \begin{pmatrix} P & Q \end{pmatrix} = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} \begin{pmatrix} MP & MQ \end{pmatrix},$$

where P and Q are obtained from X as before.

Thus we have expressed the rows of $D + A(G)$ as linear combinations of the rows of the $k \times n$ matrix $(MP \;\; MQ)$, which implies that $k \geq \mathrm{rank}_2(D + A(G)) \geq \mathrm{mrank}_2(G)$. Hence the binary dimension of G is at least $\mathrm{mrank}_2(G)$. □

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