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Computational Exploration of the Degree Sequence of the Malyshev Polynomials*

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Abstract

Zolotarev's First Problem (ZFP) in Approximation Theory [1, 2, 15], which is one of Kaltofen's favorite open problems in symbolic computation [5], asks to select the one among all monic polynomials of fixed degree $n \geq 2$ and fixed 2nd leading coefficient $a_{n-1} = -ns$ $(s > \tan^2(\frac{\pi}{2n}))$ which deviates the least from zero on the interval $I = [-1, 1]$. It turns out that this extremal polynomial also deviates least from zero among the monic polynomials of fixed degree n on the set S which consists of two disjoint intervals, $S = I \cup [\alpha(s), \beta(s)]$, $1 < \alpha = \alpha(s) < \beta = \beta(s)$, and can be characterized uniquely by roots of bivariate integer polynomials $F = F(s, \alpha)$, $G = G(s, \beta)$. These polynomials were coined Malyshev polynomials in [11], since Malyshev was the first who systematically enumerated these polynomials in 2002 [8] up to degree 5. In this paper we investigate the degree sequence of F and G via symbolic computation up to degree 16 and seek for general patterns in the sequence. We analyse the obtained results by exploiting a connection of the Malyshev polynomials to Schiefermayr's (asymmetric) homogeneous 4-variate polynomials, whose zeros are so-called $\mathbb{T}_n$-tuples [13] and to the generalized Zolotarev polynomials of Lebedev [7]. Moreover, we sketch a recursive method for computing the degree sequence without the explicit knowledge of the coefficients of the Malyshev polynomials. For the computations we used the computer algebra systems Maple and Mathematica.

Keywords: Abel-Pell differential equation, degree sequence, extremal polynomial, genus, Lededev, Malyshev polynomial, Schiefermayr, symbolic computation, $\mathbb{T}_n$-tuple, Zolotarev's First Problem, Zolotarev polynomial

MSC: 41A10

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1. Introduction

Definition 1.1. Let $I = [-1, 1]$ and let $||.||_\infty$ denote the sup-norm on I. *Zolotarev's first problem* (ZFP) amounts, for a given $n \geq 2$, to the determination of

$$\min_{(a_0,\cdots,a_{n-2})\in\mathbb{R}^{n-1}} ||Z_{n,s}||_\infty = L_n(s), \text{ where} \tag{1.1}$$

$$Z_{n,s}(x) = \sum_{k=0}^{n-2} a_k\, x^k + (-ns)x^{n-1} + x^n,$$

and of the extremal polynomial, $Z^*_{n,s}$, where $s \in \mathbb{R}$ is assumed. Thus $a_n = 1$ and the second leading coefficient, $a_{n-1} = (-ns)$, although thought of as being fixed, may attain arbitrary values, so that we save the notation s_0 for a concrete prescribed number s. It suffices to consider the "complicated" cases $s > \tan^2(\frac{\pi}{2n})$, see [1, 2, 7, 15]. From these sources we adapt the following theorem:

Theorem 1.2. *For all $n \geq 2$ and $s > \tan^2(\frac{\pi}{2n})$ the solution $Z^*_{n,s}$ of* (1.1) *is unique and is called a* monic proper Zolotarev polynomial*.*

The above best-approximation problem was posed by Chebyshev to Zolotarev, see [15, p.2]. It is the first of four famous problems in Approximation Theory which were considered by Zolotarev [15], hence the name ZFP. Recently, with the advance of symbolic computation, Kaltofen ranked ZFP and the related computational (quantifier elimination) problem for $n > 5$ to one of his favorite open problems in symbolic computation [5]. From the quoted literature there follows:

Lemma 1.3. *For fixed $n \geq 2$ and varying $s \in (\tan^2(\frac{\pi}{2n}), \infty)$, $Z^*_{n,s}$ forms a one-parameter family of polynomials which may be parametrized, for instance, with s. A particular Z^*_{n,s_0}* equioscillates *on I n times and twice on the uniquely determined interval $[\alpha, \beta]$, where $1 < \alpha = \alpha(s_0) < \beta = \beta(s_0)$. Z^*_{n,s_0} also deviates the least from zero among the monic polynomials of fixed degree n on the set S which consists of two disjoint intervals, $S = I\dot\cup[\alpha, \beta]$, see Figure 1.*

Example 1.4. For $n = 4$ and $s = s_0 = 5/18\,(> \tan^2(\frac{\pi}{8}) = 3 - 2\sqrt{2})$ the explicit power form solution to (1.1) with least deviation $L_4(5/18) = \frac{6400}{19683}$ is

$$Z^*_{4,5/18}(x) = \frac{53}{243} + \frac{15470}{19683}x - \frac{296}{243}x^2 - \frac{10}{9}x^3 + x^4 \text{ with } [\alpha_0, \beta_0] = \left[\frac{37}{27}, \frac{43}{27}\right].$$

Remark 1.5. The coefficients of the extremal polynomial $Z^*_{4,5/18}$ in Example 1.4 are all rationals. However, for $n > 4$ no rational solution is known and in fact one can prove that for $4 < n < 14$, all parametrizations of the monic proper Zolotarev polynomials must be non-rational ones, for $n = 5$ see [4], and for $n = 6$ see [12] and see also the genuses in Table 3.

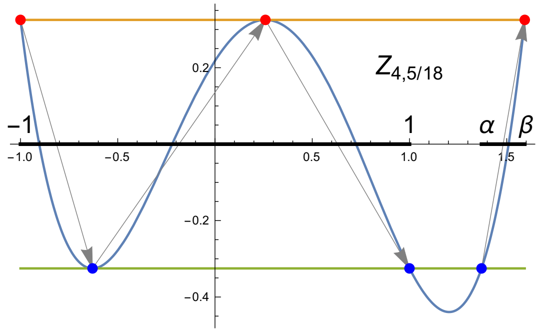

Figure 1: The equioscillation property of $Z^*_{4,5/18}$.
Note that $|Z^*_{4,5/18}(-1)| = |Z^*_{4,5/18}(-17/27)| = |Z^*_{4,5/18}(7/27)| = |Z^*_{4,5/18}(1)| = |Z^*_{4,5/18}(\alpha_0)| = |Z^*_{4,5/18}(\beta_0)| = L_4(5/18)$.

Surprisingly, Zolotarev in 1877 solved ZFP with the aid of elliptic functions. However, this solution is “too complicated to be useful in practice”, see [12], and indeed even for the simplest interesting case $n = 2$, the transformation of this solution formula to a pure algebraic power form is highly nontrivial, see [2]. We also note that numerical (approximate) solutions of ZFP for a particular $s = s_0$ can be obtained via the Remez-exchange algorithm [10].

However, in this article, we use neither the elliptic solution-formulae nor the approximate solutions for a possible reconstruction of the exact algebraic symbolic solution. For our purposes, that is, to derive a generic symbolic algebraic solution for a particular n, but for an arbitrary s, we found that the most useful characterization of the solution $Z^*_{n,s}$ is given by the Abel-Pell differential equation, see [1, p.17]. For the description of the solution of ZFP, we make use of the bivariate *Malyshev polynomials*. We will introduce them via a suitable form of the Abel-Pell differential equation in the next sections.

We note that recent research papers solved ZFP symbolically and algebraically completely for $n \leq 12$ ([6, 11]), however, our explicit investigation of the degree sequence of the Malyshev polynomials for $1 < n \leq 16$ and of their intrinsic characteristic properties seems to be novel.

ZFP can be formulated as a real quantifier elimination problem, see [3, 14]. If we exploit the equioscillation property of the sought-for best-approximating polynomial, then the formula matrix of the quantifier elimination problem consist of mainly (nonlinear) polynomial equations and only a few polynomial inequalities which can be considered as side conditions of the solutions of the equation system. Since for a particular s and n, the equation system has only finitely many solutions, our computational strategy, which first solves the polynomial equation system via Groebner Basis and then selects the proper solution of ZFP, proves to be the most promising one.

Still, somewhat surprisingly, not all the known descriptions of the algebraic solutions in the literature use the Abel-Pell differential equation representation and the Malyshev polynomials as we propose, see e.g. [3, 6].

2. The degree sequence of the Malyshev polynomials

Definition 2.1. The uniquely determined endpoints of the interval $[\alpha, \beta] = [\alpha(s), \beta(s)]$, which is given in Lemma 1.3, can be characterized by roots of integer bivariate polynomials $F^n_{m(n)}(s, \alpha)$ and $G^n_{m(n)}(s, \beta)$ of degree $m(n)$. We coin these polynomials $F^n_{m(n)}$, $G^n_{m(n)}$ *Malyshev polynomials* in view of [8], see also [11].

The main subject of this article is to determine the degree sequence $m(n)$ of the Malyshev polynomials for small n's and to explore some patterns in this sequence. The here computed values $m(n)$ for $n > 12$ are new.

Example 2.2. $m(4) = 4$ and the polynomials $F^4_{m(4)}$ and $G^4_{m(4)}$ (see also [8]) are given as

$$F^4_{m(4)}(s,\alpha) = F^4_4(s,\alpha) = (-13 - 136s - 448s^2 - 896s^3 + 256s^4)+$$
$$(44 + 184s + 128s^2 - 640s^3)\alpha + (-22+168s + 576s^2)\alpha^2 + (-36-216s)\alpha^3 + 27\alpha^4,$$

$$G^4_{m(4)}(s,\beta) = G^4_4(s,\beta) = F^4_4(-s,-\beta) = (-13 + 136s - 448s^2 + 896s^3 + 256s^4)+$$
$$(-44 + 184s - 128s^2 - 640s^3)\beta + (-22-168s + 576s^2)\beta^2 + (36-216s)\beta^3 + 27\beta^4.$$

Lemma 2.3. *The solution $p = p(x)$ of (1.1) satisfies the Abel-Pell differential equation*

$$(1 - x^2)(x - \alpha)(x - \beta)(p')^2(x) = n^2(L^2_n(s) - p^2(x))(x - (\alpha + \beta)/2 + s)^2, \quad (2.1)$$

where the intended meaning of $L_n(s)$ is given in Definition 1.1 and α and β are given in Definition 2.1.

For a proof, see [1, 13].

Lemma 2.4. *Relying on Lemma 2.3, a coefficient comparison will transform the problem of solving ZFP to a solution of a nonlinear polynomial system NPS for each particular n. This NPS is then analysed by Groebner basis techniques. From the finitely many solutions of this NPS the desired one, which yields the proper monic Zolotarev polynomial, can be strategically selected with the aid of equality and inequality constraints. Considering s as an indeterminate in the above NPS, the computation of $F^n_{m(n)}$ and $G^n_{m(n)}$ $(n \leq 16)$ is accomplished with the aid of the computer algebra systems* Maple *[9] and* Mathematica *[17]. For the concrete coefficients of the Malyshev polynomials we refer, because of the bulkiness of the formulae, to the web-based repository [16].*

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(n)$	1	2	4	6	8	12	16	18	24	30	32	42	48	48	64

Table 1: Elements of the degree sequence $m(n)$ of the Malyshev polynomials $F^n_{m(n)}$ and $G^n_{m(n)}$ for $2 \leq n \leq 16$

3. Computational results

Theorem 3.1. *Table 1 shows the first 15 elements of the degree sequence $m(n)$ of the Malyshev polynomials F and G. These elements were computed according to Lemma 2.3 and 2.4.*

Remark 3.2. We note that for a fixed (rational) $s = s_0$, a suitable real root of the then univariate $F^n_{m(n)}$ and $G^n_{m(n)}$ describes the endpoints of the interval $[\alpha, \beta]$. An alternative characterization would be to provide the bivariate polynomial $H^n_{m(n)}(\alpha, \beta)$ where the points of a suitable part of the planar curve $H^n_{m(n)}(\alpha, \beta) = 0$ correspond to the ordered pair (α, β), see [4, 11]. The bivariate polynomial H has the same degree as the Malyshev polynomials (H equals to the factor of degree $m(n)$ of the resultant $res_s(F^n_{m(n)}, G^n_{m(n)})$). For reference purposes, they also have been put to the repository [16], and we point out that they are novel for $n \geq 8$. $H^6_{m(6)} = H^6_8$ and $H^7_{m(8)} = H^7_{12}$ are given in [11]. The planar curves for $n = 2, 3, 4, 5$ are displayed in Figure 2.

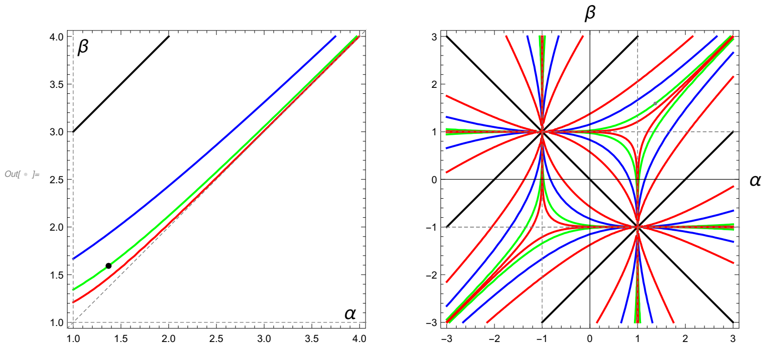

Figure 2: (left) Real parts of the planar curves $H^2_{m(2)} = H^2_1 = 0$ (black), $H^3_{m(3)} = H^3_2 = 0$ (blue), $H^4_{m(4)} = H^4_4 = 0$ (green) and $H^5_{m(5)} = H^5_6 = 0$ (red) with the $(\alpha_0, \beta_0) = \left(\frac{37}{27}, \frac{43}{27}\right)$ point from Example 1.4 in the (α, β)-plane. (right) The (α, β) points corresponding to all possible solutions of (2.1) without the natural side conditions, $(n = 2, 3, 4, 5)$.

To analyse the obtained computational results for $m(n)$, it is useful to consider the prime decomposition of $n = p_1^{\epsilon_1} \dots p_\nu^{\epsilon_\nu}$, $\epsilon_i > 0$, where the p_i's are the prime factors of the n in ascending order and in particular, to consider the odd and even cases of n separately.

Lemma 3.3. *If, in Table 1, n is an odd prime, then we have*

$$m(n) = m(2l+1) = \frac{n^2-1}{4} = l^2 + l, \tag{3.1}$$

and for 2-powers in Table 1 we have

$$m(n) = m(2l+2) = \frac{n^2}{4} = (l+1)^2. \tag{3.2}$$

However, for composite numbers, neither of the simple formulae (3.1), (3.2) *works. Rather, the following Theorem 3.4 holds, which explains the gap between the formulae and the computed values.*

Theorem 3.4 (Lebedev [7])**.** *For a composite number n, the Abel-Pell differential equation (2.1) without side conditions has polynomial solution(s) different from the proper Zolotarev polynomial $Z^*_{n,s}$. With the natural side conditions $p(-1) = (-1)^n L_n, p(1) = -L_n, p(\alpha) = -L_n, p(\beta) = L_n, (\alpha \neq \beta \neq \pm 1)$ one can rule out some of these solutions, but not all of them. The additional polynomial solutions satisfying the natural side conditions are generalized Zolotarev polynomials. They have the form $T_l(Z^*_{k,s})$, where T_l is the l-th Chebyshev polynomial of the first kind on I and $n = l \cdot k, k > 1$. It turns out that if $2 \not| l$, then $T_l(Z^*_{k,s})$ solves* (2.1) *with the natural side conditions above (and no other polynomial solution exists). Therefore, for some composite n, the bivariate polynomial $Q_1 = Q_1(s, \beta)$ in the variable s and β or $Q_2 = Q_2(\alpha, \beta)$ in the variable α and β, in the elimination ideal defined by the nonlinear polynomial system NPS and the natural side conditions, decomposes into several factors. However, the sum of their total degrees is actually*

$$ds(n) = \left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2-1}{4}, & \textit{if } n \textit{ is odd,} \\ \frac{n^2}{4}, & \textit{if } n \textit{ is even.} \end{cases} \tag{3.3}$$

Table 2 shows the degree sequences $m(n)$ and $ds(n)$. By underscoring we highlight the cases where there is a positive gap between the two sequences.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(n)$	1	2	4	6	$\underline{8}$	12	16	$\underline{18}$	$\underline{24}$	30	$\underline{32}$	42	$\underline{48}$	$\underline{48}$	64
$ds(n)$	1	2	4	6	9	12	16	20	25	30	36	42	49	56	64

Table 2: The degree sequences $m(n)$ and $ds(n)$ for $2 \leq n \leq 16$

Remark 3.5. We note that in [13], Schiefermayr defines 4-variate homogeneous polynomials $P \in \mathbb{C}[a, b, c, d]$ in a constructive way via determinants which characterize whether a slightly generalized version of the Abel-Pell equation (2.1), namely

$$(x-a)(x-b)(x-c)(x-d)(p')^2(x) = n^2(p(x)^2 - \Lambda_n^2(s))(x - (a+b+c+d)/2 + s)^2,$$

(depending on the complex numbers a, b, c, d), has a (polynomial) solution $p = \Lambda_n \mathbb{T}_n$. It is proved there that the solution exists, which is not a solution for $n/2$, if and only if the polynomial inverse image $\mathbb{T}_n^{-1}[-1, 1]$ consists of two Jordan arcs with endpoints a, b, c, d. The (a, b, c, d)-quadruples formed by the endpoints are called $\mathbb{T}_n$-tuples. Then these endpoints are described purely algebraically.

The elements in the quadruple (a_0, b_0, c_0, d_0) can occur as endpoints of the curves if and only if $P(a_0, b_0, c_0, d_0) = 0$. The polynomial P is of degree $ds(n)$ as given in (3.3), and with the special choice $a = \alpha, b = 1, c = -1, d = \beta$ (see [13, Section 4.2]) (which corresponds to normalizing one of the curves), P specializes to $Q_2(\alpha, \beta)$ and for odd primes and 2-powers to $H^n_{m(n)}(\alpha, \beta)$.

Summarizing our results, we sketch a simple (recursive) algorithm for the computation of $m(n)$ without the explicit knowledge of the Malyshev polynomials $F^n_{m(n)}, G^n_{m(n)}$.

Lemma 3.6. *If $n > 1$ is an odd prime or a 2-power, then $m(n) = ds(n)$ as given in* (3.3)*. Otherwise, assume that $m(k)$ is computed, if $k < n$.*

If n is even, let $n' = n/2^{\epsilon_1}$, that is, the product of the odd prime powers in n. Assume that n' decomposes into two factors, $n' = n_1 \cdot n_2$, where the first factor is nontrivial. Then $T_{n_1}(Z^*_{2^{\epsilon_1} \cdot n_2}, s)$ is also a solution of (2.1) *with the natural side conditions and we have to subtract $m(2^{\epsilon_1} \cdot n_2)$ from $ds(n)$.*

In a similar way, if n is an odd composite number and thus $n' = n$, then both factors n_1 and n_2 should be nontrivial.

Example 3.7.

$$m(17) = ds(17) = \frac{17^2 - 1}{4} = 72,$$

$$m(18) = ds(18) - m(2) - m(6) = \frac{18^2}{4} - 1 - 8 = 72,$$

because in the latter case $n' = 18/2 = 9$ and n' factors into $9 = 9 \cdot 1 = 3 \cdot 3$. Since $T_9(Z^*_{2,s})$ and $T_3(Z^*_{6,s})$ also solves (2.1) with the natural side conditions, we have to subtract $m(2) = 1$ and $m(6) = 8$ from $ds(18) = 81$.

Remark 3.8. The computed elements of the sequence $m(n)$ may also play a role in the analysis of the coefficients of the polynomials $H^n_{m(n)}$. For instance, while the coefficient of $\alpha^{m(n)}$ is 1 in $H^n_{m(n)}$, the constant term of $H^n_{m(n)}$ seems to be $2^{m(n)}$.

Remark 3.9. For the H-polynomials, we also computed the genus of the curve $H^n_{m(n)} = 0$, up to degree 13. This information may be used for the parametrization of the Zolotarev polynomials. Table 3 shows the result, which confirms and extends the data given in [4, p. 179] and [12]. The $n = 2, 3, 4$ cases are classical results. The $n = 5, 7, 8, 11$ cases were first given in [4] and $n = 6$ case in [12]. The $n = 9, 10, 12, 13$ cases seem to be new.

n	2	3	4	5	6	7	8	9	10	11	12	13
$g(n)$	0	0	0	1	1	4	5	7	9	16	13	25

Table 3: The genus sequence $g(n)$ for $H^n_{m(n)}$ for $2 \leq n \leq 13$

Connections to known integer sequences in the OEIS database We observe that the sequence $ds(n)$ does coincide with the infinite sequence A002620 in the OEIS database (see `oeis.org`). It was also observed in [11] that the finite sequence $\{m(n)\}_{n=2}^{12}$ coincides with the first 11 elements in the infinite sequence A055932. As the particular case $n = 13$ now shows, this coincidence breaks down for $n \geq 13$.

4. Conclusion

Based on the Abel-Pell differential equation and deploying Groebner basis techniques, we computed symbolically the bivariate Malyshev polynomials F, G and the polynomials H (defining a reduced relation curve) up to degree $n = 16$. All of them play a crucial role in the purely algebraic description of the solutions to ZFP.

The bulky expressions for F, G, H have been stored in the ZFP web-based repository [16]. By analysing the patterns in the degree sequence of the Malyshev polynomials and consulting the current literature, we gave a recursive algorithm for computing an arbitrary element of the degree sequence without the explicit knowledge of the Malyshev polynomials. We also computed the genuses of the curves $H = 0$ for $n \leq 13$. The computational results contribute to the analysis of the classical and generalized Zolotarev polynomials.

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