Properties Defined on the Basis of Coincidence in GFO-Space

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Abstract. The General Formal Ontology (GFO) is a top-level ontology that includes theories of time and space, two domains of entities of fundamental character. In this connection the axiomatic theories of GFO are based on the relation of coincidence, which can apply to the boundaries of time and space entities. Two coincident, but distinct boundaries account for distinct temporal or spatial locations of no distance. This is favorable for modeling or grasping certain situations. At a second glance and especially concerning space, in combination with axioms on boundaries and mereology coincidence also gives rise to a larger variety of entities. This deserves ontological scrutiny.

In the present paper we introduce a new notion (based on coincidence) referred to as 'continuousness' in order to capture and become capable of addressing an aspect of space entities that, as we argue, is commonly implicitly assumed for basic kinds of entities (such as 'line' or 'surface'). Carving out that definition first leads us to adopting a few new axioms for GFO-Space. Together with the two further properties of connectedness and ordinariness we then systematically investigate kinds of space entities.

Keywords. top-level ontology, ontology of space, coincidence, GFO

1. Introduction

The General Formal Ontology (GFO) [1] is one of those top-level ontologies that aims at accounting for entities of time and space by means of axiomatic theories about these domains. In line with philosophical work of Franz Brentano [2], firstly GFO considers time and space as manifestations of the more abstract notion of a continuum. Furthermore, there is another idea by Brentano with deep impact on the theories of time and space developed for GFO: It is the idea that certain time or space entities can *coincide* / are *coincident*. Two entities coincide, if, intuitively speaking, they are compatible and they yield temporal or spatial co-locations (for entities that are in time and/or space). Put differently, they must be "congruent" and there is no distance between them.

In the case of time, the motivating idea for this relationship is that if a time interval is partitioned into two sub intervals that meet, then *both* of these intervals are equipped

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with their own two time points that start and end the respective sub interval. For example, if the period of an hour is split into two halves of 30 minutes each, each half is claimed to have a first and a last time point. In contrast to classical topology, the last point of the first 30 minutes and the starting point of the subsequent 30 minutes are understood as two *distinct* time entities, whereas measuring time yields zero distance between them – these two time points coincide.

The notion of coincidence is intimately tied to the notion of *boundaries*, which derives from the relation of one entity *being a boundary of* another one. Indeed, time points in GFO are understood as the boundaries of time intervals. Boundaries existentially depend on the entities they are boundaries of.

Regarding space entities, matters become substantially more complex. Not only two points on a line may coincide (in analogy to time intervals and time points), but lines themselves may be coincident. In that case they are boundaries of a partitioned surface. The possibility to coincide extends further to surfaces, two of which may coincide if they result from "slicing" a three-dimensional space entity, say, a sphere into an upper and a lower part. Again, each part is conceived to be fully bounded. Where the upper and the lower part touch each other, those (parts of the maximal) boundaries coincide – they are congruent, but distinct and have no measurable distance.

Corresponding theories for GFO have been developed and are most recently expounded in [3] for the domain of time (also referred to as "Brentano Time", thus abbreviated as $\mathcal{BT}$). Another work [4] presents an initial version of a GFO space theory $\mathcal{BS}$ (for "Brentano Space"). As it turns out, there are still kinds of entities that are conceivable in the theory $\mathcal{BS}$, e.g., based on common mereological assumptions in combination with the availability of coincidence, which should be scrutinized ontologically.

The current paper reports on work in progress to extend $\mathcal{BS}$, focusing on properties that are derived from coincidence and which shall allow for a finer-grained classification of space entities. Sect. 2 motivates our research on *continuousness*, as a newly determined property in the context of GFO-Space. The subsequent sect. 3 presents a solution exemplified at the level of zero- and one-dimensional entities, by defining continuous line entities on the basis of directionally compatible points. Then we gather systematically and illustrate the kinds of line entities that can be distinguished on the basis of continuousness, ordinariness and connectedness, in sect. 4. Besides concluding the paper, the final sect. 5 points to related work and indicates the continuation of the work presented herein.

We lack space in this paper in order to recapitulate all relevant content of the theory $\mathcal{BS}$ and thus we mainly refer to [4] for this purpose. However, the most uncommon notion of the coincidence of spatial boundaries, which is a primitive in $\mathcal{BS}$, is already introduced above. For all remaining notions of major relevance, we believe that GFO-Space is rather close to an intuitive-commonsensical understanding of phenomenal space. This concerns the remaining three primitive notions of *space region*, of *being a spatial boundary* of an entity and of *spatial parthood*, i.e., the latter two are relations. All other categories and relations are defined. Space entities are understood to be of distinct dimensions, referred to as either zero- to three-dimensional entities or as point, line and surface entities for lower dimensions and space region for three-dimensional ones. We mention in addition only the relation of being a *hyper part* of a space entity. Given that spatial parthood applies only among entities of the same dimension, the term 'hyper part' is used to refer to "parts" of entities with co-dimension ≥ 1. For example, a point or a line segment "within" a space region are called hyper parts of that region. A final, technical note concerns the numbering of formulas: we presuppose the identifiers of all definitions, axioms and theorems as in [4]. Thus, formula identifiers herein do not start at 1, but rather continue or refer to the counting in accordance with [4].

2. Motivation – The Quest for Continuousness

On the one hand, coincidence is clearly a useful "tool" for modeling certain situations of space entities, such as allowing for fully bounded entities that meet / touch each other. This applies all the more in connection with material entities that occupy space entities, cf. [4, sect. 3 and 5]. On the other hand, coincidence in combination with mereological principles leads to a richer typology of entities compared to an ontology that lacks coincidence. In the theory $\mathcal{BS}$ there are two central aspects of mereological summation. First, summation is restricted to equidimensional entities, which follows from the stronger requirement that already spatial parthood implies equidimensionality [4, A7]. Secondly, the formation of mereological sums is not only unconstrained for equidimensional entities, but it is even enforced as a principle of existence.

A16. $eqdim(x, y) \rightarrow \exists z\ sum(x, y, z)$ (existence of sum)

In combination with coincidence, this leads easily to kinds of entities that are uncommon compared to intuitive notions such as points or lines, for instance. Imagine two lines, which happen to be coincident with one another. Each of these lines is a one-dimensional entity, hence by the axiom just given there is the mereological sum of those two lines. In contrast to a "common" line entity, there are distinct parts of the entity under consideration that coincide. This kind of entities has already been singled out in $\mathcal{BS}$ (as presented in [4]) by distinguishing *ordinary* from *extraordinary* entities.

D22. $ExOrd(x) := \exists yz\ (spart(y, x) \wedge spart(z, x) \wedge \neg sov(y, z) \wedge scoinc(y, z))$ (x is an extraordinary space entity)

D23. $Ord(x) := \neg ExOrd(x)$ (x is an ordinary space entity)

In the course of further studies another case of entities occurred that exhibits features that are similarly surprising compared to common intuitions, that is not covered by the notion of extraordinariness and that turns out to be substantially harder to grasp – ideally in the form of a definition, but at least by an axiomatic characterization. We refer to this notion as *continuous* or, nominalized, as *continuousness*. Our prototypical example that is not continuous or *discontinuous* in the realm of one-dimensional entities is the line entity consisting of the components y' and x' in Fig. 1. Notably, speaking of lines or drawing lines such as y' and x' (each considered on its own), we argue that these are implicitly assumed to be continuous in the sense that we intend to understand and believe to have captured in sect. 3 below.

In its four 'scenes' (or sections) identified by the numbered scrolls, Fig. 1 provides material in order to illustrate and intuitively explain the discontinuous line entity in scene 3. In slight abuse of formal notation, we may use names of entities in illustrations in argument positions of predicates of $\mathcal{BS}$, such as writing $1D(y')$ for the fact that the entity y' in Fig. 1 stands for a line, i.e. (so far), a one-dimensional entity that is connected and ordinary.

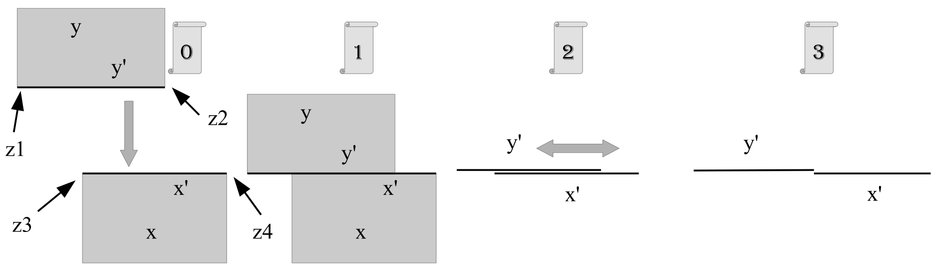

Figure 1. “Constructing” an extraordinary (2) and a discontinuous (3) line entity.

Scene 0 shows two unconnected spatial rectangles x and y. Each is a fully bounded entity, but the relation sb of being a spatial boundary of an entity neither assumes nor entails maximality. Hence (and among other segments of the maximal boundaries), the two lines x' and y' are spatial boundaries of x and y, respectively. These lines, in turn, have their own zero-dimensional boundaries, such that $sb(z1, y')$, $sb(z2, y')$, $sb(z3, x')$ and $sb(z4, x')$ apply.

Next, consider scene 1 with two alike rectangles x and y (and their boundaries x' and y'), but which are in contact with / touch each other. x' and y' are still distinct and do not even overlap, but now they coincide (which in $\mathcal{BS}$ accounts for the fact that x and y touch). Accordingly, the sum of x' and y' is an extraordinary entity. The latter remains the case in scene 2, where x' and y' are laterally offset, albeit x' and y' themselves do not coincide. But they have parts that do so, namely those parts directly opposed to each other in the area where x and y still touch. (Note that x and y are not displayed in scene 2 any more, but are still there, not at least because boundaries must have an entity that they are a boundary of.)

Eventually, in scene 3 (x and y and thus) x' and y' are offset in such a way that (1) there are no more parts of x' and y' that coincide, but (2) it is the case that $z2$ coincides with $z3$. The mereological sum of x' and y' is no longer extraordinary, but ordinary, due to (1). Moreover, it is a connected entity because of (2). Up to this point, there is no notion in $\mathcal{BS}$ that would allow us to distinguish that mereological sum from an “intuitively proper” line, such as x' in itself.

We refer to x' and y' as *continuous* lines, whereas the sum of the two does not exhibit such continuousness. The sum contains a kind of “crack” due to switching from one part originating from a surface above it (y) to another part originating from a surface below it (x). Considering the level of zero-dimensional entities, we perceive a difference between an arbitrary pair of coincident points within the line x' (or within the line y') and the special situation of $z2$ and $z3$. The latter two points do coincide, but they are boundaries of parts of the sum of x' and y' that originate from surfaces that differ in the specific way just indicated. Insofar we say that $z2$ and $z3$ are not *directionally compatible*, whereas pairs of coincident points within, say, x' are called directionally compatible.

Besides in itself raising this problem, the main aims of this paper are to capture these intuitive notions of continuousness and directional compatibility formally precisely and to apply continuousness together with extraordinariness in classifying line entities.

3. Directional Compatibility and Continuousness

The development of the definitions of directional compatibility and continuousness cannot reasonably be reflected here in terms of various intermediate versions that failed in one way or another prior to reaching the results presented in this section. Therefore, the section gathers only newly required preliminaries for those definitions, actually introduces and discusses them and concludes with a few new axioms and theorems that played a role in dealing with continuousness and that serve as an extension of the theory $\mathcal{BS}$ as in [4].

3.1. Distinguishing Strict and Weak Boundaries of Entities

The primitive relation sb of being a spatial boundary turned out to be not sufficiently discriminating in order establish the targeted definitions. Returning to Fig. 1, note that in all four scenes it is the case that (1) $sb(z1, y')$, $sb(z2, y')$, $sb(z3, x')$ and $sb(z4, x')$ as well as (2) each of the points $z1, \ldots, z4$ is a boundary of the mereological sum of x' and y'. However, referring to scenes 2 and 3, we need to distinguish between boundaries such as $z1$ and $z4$, on the one hand, and $z2$ and $z3$ on the other hand. The former are called *strict boundaries* (of the sum), the latter *weak boundaries*.

D34. $strictsb(x, y) := sb(x, y) \wedge \forall x'(hypp(x', y) \wedge scoinc(x, x') \rightarrow x = x')$
(x is a strict spatial boundary)

D35. $weaksb(x, y) := sb(x, y) \wedge \neg strictsb(x, y)$ (x is a weak spatial boundary)

We illustrate the effects of the definitions a bit further with Fig. 2. Let us denote by s' the sum of x' and y'. Then, again, $z1, \ldots, z4$ are boundaries of s'. $z2$ is a weak spatial boundary of s', because there are exactly two distinct zero-dimensional hyper parts h'_1 and h'_2 (of x' and thus also of s') which coincide with $z2$ (and are also each distinct from $z2$). Analogously, there are two hyper parts of y' that coincide with $z3$, the second weak boundary of s'. The boundaries $z1$ and $z4$ are indeed strict boundaries, since all points coincident with one of the two are "outside" of x' and y', respectively, i.e. they are no hyper parts of them.

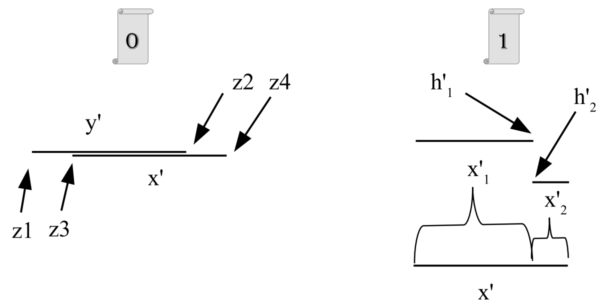

Figure 2. Weak and Strict Boundaries.

3.2. Definitions of Directional Compatibility and Continuousness

Let us first state the definitions of directional compatibility and continuousness formally. Herein, we focus on continuousness of one-dimensional entities (considered as embedded in a surface), which requires 1-directional compatibility of zero-dimensional entities, but confer the outlook in sect. 5 for remarks on the general case.

D36. $1dircomp(x, y) := scoinc(x, y) \wedge$
$\exists x'y'z'\ (\ Ord(x') \wedge Ord(y') \wedge sb(x, x') \wedge sb(y, y') \wedge \neg sov(x', y') \wedge$
$(SB(x') \wedge SB(y') \rightarrow$
$\exists x''y''z''g'\ (\ sb(x', x'') \wedge sb(y', y'') \wedge \neg sov(x'', y'') \wedge$
$sum(x'', y'', z'') \wedge grsb(g', z'') \wedge 0dhypp(x, g') \wedge 0dhypp(y, g') \wedge$
$\forall z(0dhypp(z, g') \rightarrow$
$((scoinc(x, z) \wedge \neg sov(x, z) \rightarrow y = z) \wedge$
$(scoinc(y, z) \wedge \neg sov(y, z) \rightarrow x = z)\ )))))$

(x and y are 1-directionally compatible)

D37. $1cont(x) := 1DE(x) \wedge$
$\forall y\ (\ (0dhypp(y, x) \wedge \forall y'(spart(y', y) \rightarrow \neg strictsb(y', x))) \rightarrow$
$\exists z(0dhypp(z, x) \wedge \neg sov(z, y) \wedge 1dircomp(y, z)))$

(x is 1-continuous)

A space entity x is 1-continuous iff it is a 1-dimensional entity and for each of its zero-dimensional hyper parts y that (*) has no strict boundary of x among its parts, there is another zero-dimensional hyper part that (**) does not overlap with y, but is 1-directionally compatible with it. Note that in the case of y being a point, (*) boils down to not *being* a strict boundary of x, while (**) then just means that there must be another, 1-directionally compatible point within x. Rephrased, except for its strict boundaries, all points within a continuous line entity must be complemented by a 1-directionally compatible point that is also *within* the line entity. This is the gist of the definition, while the more general formulations of (*) and (**) take mereological sums of points into account.

Returning to Fig. 2, point $z2$ witnesses that the sum entity s' is not 1-continuous. The points h'_1 and h'_2 within x' (and s') are mutually 1-directionally compatible, but none of the two is 1-directionally compatible with $z2$. Since h'_1 and h'_2 are the only points within s' that coincide with z_2, but are distinct from it, there is no complementary, 1-directionally compatible point for $z2$ in s' – thus s' is discontinuous. An analogous argument applies to $z2$ in scene 3 of Fig. 1, where the only candidate within the sum entity that is distinct from $z2$, but coincides with it, is $z3$ – which is not 1-directionally compatible with $z2$.

Admittedly, the definition of (1-)directional compatibility is the more challenging one, based on the following key ideas. First, only coincident points x and y can be 1-directionally compatible. In order to verify the latter, in addition there must be ordinary, non-overlapping line entities x' and y' as well as non-overlapping surface entities x'' and y'',[2] such that x "derives" from x' and thus x'' (i.e, $sb(x, x')$, $sb(x', x'')$) and likewise for y, y' and y''. The next idea is to consider the mereological sum of those two surfaces and inspect it's greatest spatial boundary g' for additional points that coincide with x and y. If there is no such point, then x and y are indeed 1-directionally compatible, whereas if there is a third coincident point within g', they are not.

Reconsider first scene 1 of Fig. 2 to see where we drew the inspiration for this definition from. The boundaries h'_1 and h'_2 are (intuitively) directionally compatible. We

[2]The precondition of x' and y' being spatial boundaries is satisfied by any ordinary line entity. That condition becomes relevant if the definitional scheme is applied to define 3-directional compatibility of surfaces, cf. sect. 5.

can select x'_1 and x'_2 with h'_1 and h'_2 as boundaries, and then select two non-overlapping, but *touching* surfaces below x'_1 and x'_2, respectively, such that the sum of those surfaces has x' as part of its greatest boundary. *Within* x', h'_1 and h'_2 remain the only points that coincide. There are further points that coincide with the two, for example one on a boundary (of the assumed left-hand surface), which leads downwards from h'_1. However, since those surfaces below x'_1 and x'_2 touch, all those points do not belong to x' (nor the greatest boundary of the sum of the surfaces), but are inside them.

In contrast, returning to scene 3 of Fig. 1, where $z2$ and $z3$ are (intuitively) not directionally compatible, there it is not possible, starting from $z2$ and $z3$, to find corresponding lines and (non-overlapping) surfaces, such that those surfaces touch (in order to achieve the same effect as above, that additional points coinciding with $z2$ and $z3$ "disappear" within the sum of those surfaces). The reason for that impossibility is that any such surface at the side of $z2$ extends upwards from y' (or a part of y'), whereas any surface at the side of $z3$ extends downwards from x'. The greatest boundary of the sum of such surfaces will therefore have four line segments as parts, all of which meet at (also) $z2$ and $z3$, which means that $z2$ and $z3$ coincide with two further points *within* that greatest boundary.

Sect. 4 comprises additional examples that demonstrate the effects of the definitions here given. Beforehand, we present new axioms introduced to the theory $\mathcal{BS}$ together with further theorems derived in the course of determining those definitions.

3.3. New Axioms and New Theorems

The claim near the end of the previous sub section, that "any surface at the side of $z3$ extends downwards from x'" in scene 3 of Fig. 1 is centrally based on one of three new axioms adopted for $\mathcal{BS}$ (namely A31). Indeed, we stipulate that boundaries are not boundaries of arbitrary objects. To the contrary, there is "local information" inherent in each boundary about the entities that it is a boundary of.

A31. $sb(x,y) \wedge sb(x,z) \rightarrow \exists u(sb(x,u) \wedge sppart(u,y) \wedge sppart(u,z))$
(entities with a common boundary share a proper part at that boundary)

A32. $sb(x,y) \wedge sb(u,v) \wedge x \neq u \rightarrow$
$\exists y'v' \,(\, spart(y',y) \wedge sb(x,y') \wedge spart(v',v) \wedge sb(u,v') \wedge \neg sov(y',v'))$
(entities with distinct boundaries have parts at those boundaries that do not overlap)

A33. $LDE(x) \wedge Ord(x) \rightarrow SB(x)$
(ordinary lower-dimensional entities are spatial boundaries)

A33 is added to exclude intuitively strange cases of ordinary entities that could not be derived from – eventually – space regions by the sb relation. But A31 and A32 are much more insightful regarding the nature of boundaries and coincidence, in that the dependence of boundaries on the entities they are boundaries of involves a kind of exclusiveness concerning the question which entities they actually can be boundaries of. Interestingly, we further observe that A31 can be seen as a generalization of A28 and A29 of the time theory $\mathcal{BT}^{\mathcal{C}}$ in [3].

The remainder of this section lists new sentences that emerged while working on the definitions of continuousness and directional compatibility, which can be shown to follow from $\mathcal{BS}$ (including A31–A33). For each theorem, Table 1 gathers a set of axioms and definitions from which it can be proved.

T17. $SReg(x) \rightarrow \neg SB(x)$ (space regions are no spatial boundaries)

T18. $SReg(x) \rightarrow Ord(x)$ (space regions are ordinary)

T19. $LDE(x) \rightarrow (Ord(x) \leftrightarrow SB(x))$
(for lower-dimensional ent. no difference between spatial boundary and ordinariness)

T20. $scoinc(x, y) \rightarrow Ord(x) \wedge Ord(y)$ (coincidence requires ordinariness)

T21. $1dircomp(x, y) \rightarrow 1dircomp(y, x)$ (symmetry of 1-directional compatibility)

T22. $\neg 1dircomp(x, x)$ (irreflexivity of 1-directional compatibility)

Theorem	*Sub Theory*
T17	D4, A2, A26
T18	D22, A2, A22
T19	A2, A22, A33, T18
T20	A22, T19
T21	D36
T22	D36, A31

Table 1. Theorems and sub theories from which they are provable.

4. Classification of Line Entities

In this section we distinguish line entities w.r.t. the properties of ordinariness, continuousness and connectedness, which implies that the section also demonstrates further effects of the definition of continuousness as formally introduced above. It turns out that all three properties are independent, i.e., any combination of them has witnessing examples. The subsequent illustrations provide for examples all of which are connected, while we discuss disconnected variants below.

All sub figures in this section are constructed in a way that is meant to support the verification of continuousness or discontinuousness of the sum entity that is shown in scence 3 of each sub figure, i.e., on the right-hand side. The scenes to the left allow one to explain from which line entities (x', y') that sum can be derived (scenes 1 and 2), as well as which surface entities have those lines as boundaries (x, y; scene 0). The analyses of those cases yield the classifications specified in the figure captions.

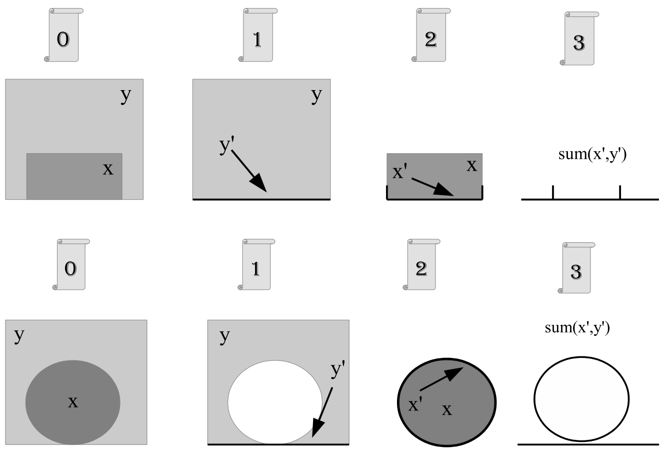

Figure 3. Two cases, where x', y' and their sum are each ordinary, continuous and connected.

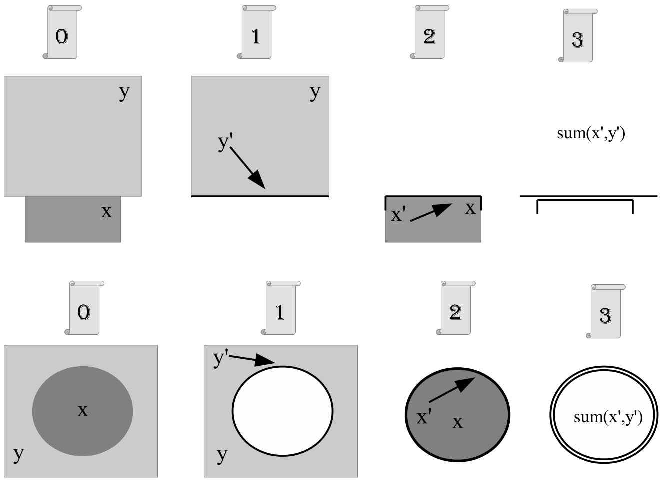

Figure 4. Two cases in which the sum of x' and y' is extraordinary, continuous and connected.

Considering the lower part of Fig. 4, imagine that we punch a circle out of a square (scene 0). The two resulting objects y, the negative of the circle, and x, the circle itself, possess coinciding spatial boundaries y' and x' (scenes 1 and 2). The mereological sum of them is an example of an extraordinary and continuous line entity (scene 3), where the extraordinariness should be obvious. In order to see that the sum is continuous, it suffices to realize that any single point located on y' or x' possesses exactly one directionally compatible counterpart on y' or x', respectively.

Disconnected versions of all cases above can be obtained by considering space entities that correspond to "copies" of each other, but which do neither overlap nor share any hyper parts. Such "copies" are insensible to ordinariness and continuousness, i.e., these properties apply likewise to the sum of two such copies as they do to each "copy".

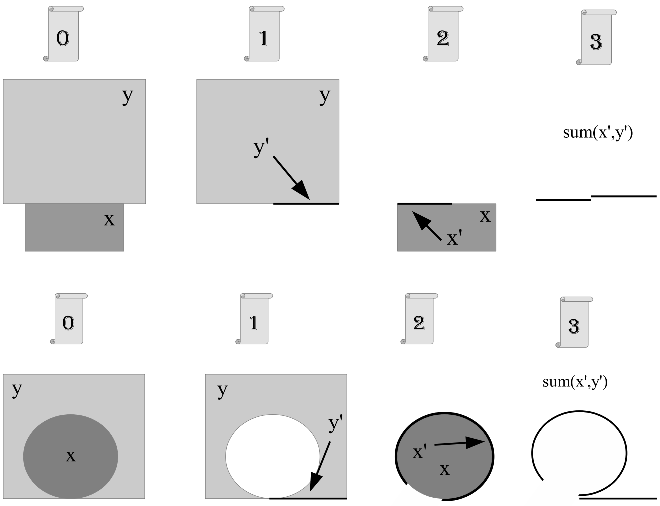

Figure 5. Two cases in which the sum of x' and y' is ordinary, discontinuous and connected.

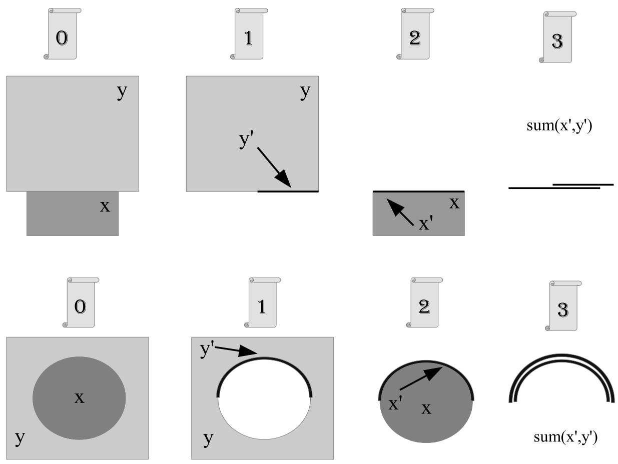

Figure 6. Two cases in which the sum of x' and y' is extraordinary, discontinuous and connected.

5. Conclusions

This paper is a work-in-progress report on dealing with specific issues identified in connection with the relation of coincidence in space ontologies, exemplified by the development of the space ontology [4] of GFO, the General Formal Ontology [1].

Concerning closely related work, we must admit not to be aware of other works that deal with a space ontology that involves coincidence, and even less the problem of continuousness. With a wider focus, space is clearly an important topic, cf. e.g. [5,6,7], albeit we see no closer connection to continousness, as well.

Our work continues to generalized definitions of directional compatibility as well as of continuousness, of which the definitions D36 and D37 form instances. Currently, we test these definitions further with entities of higher dimension, i.e., with lines, surface and space regions, preparing a classification extended to those types of entities.

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