Decidability frontier for fragments of first-order logic with transitivity*

Lidia Tendera[0000−0003−0681−4040]

University of Opole, Poland
tendera@math.uni.opole.pl

Abstract. Several decidable fragments of first-order logic have been identified in the past as a generalisation of the standard translation of modal logic. These include: the fluted fragment, the two-variable fragment, the guarded fragment and the unary negation fragment; some of them have been recently generalised or combined to yield even more expressive decidable logics (guarded negation fragment or uniform one-dimensional fragment). None of the fragments allows one to express transitivity of a binary relation or related properties like being an equivalence, a linear or a partial order, that naturally appear in specifications or in verification. The question therefore arises what is the impact of adding transitivity to these fragments and, where the cost is too high, how can these languages be *tamed*.
In this talk we survey results concerning the decidability frontier of the above-mentioned fragments extended with transitivity. We discuss both, general and finite satisfiability, as presence of transitivity axioms often allows one to express axioms of infinity. Simultaneously, we locate in the picture known description logics, discuss relevant technics, admire a few exotic results and state some open questions.

Keywords: First-Order logic · Decidability · (Finite) Satisfiability · Transitivity · Complexity.

Introduction and Preliminaries

It is well-known that formulas of propositional modal logics under Kripke semantics can be naturally encoded in first-order logic, using the so-called *standard translation* (cf. [2]); the same applies to the basic description logic $\mathcal{ALC}$ and many of its variations. Since first-order logic is not so well-behaved as modal or description logics, in particular the (finite) satisfiability problems are undecidable, it was natural to ask what the right image of the standard translation is and 'Why is modal logic so robustly decidable?'. This question opened an interesting and fruitful research path in theoretical computer science, and in this talk we explore part of the path.

The formulas obtained under the standard translation evince some patterns: (i) variables appear in some fixed order and no rescoping of variables occurs, (ii)

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quantifiers are relativized by atomic formulas, (iii) negation is applied only to subformulas with a single free variable. Moreover, by properly reusing variables: (iv) the number of variables can be restricted to *two*.

The four syntactic restrictions considered separately give rise to four fragments of first order logic: (i) the fluted fragment $\mathcal{FL}$, (ii) the guarded fragment $\mathcal{GF}$, (iii) the unary negation fragment $\mathcal{UNF}$, and (iv) the two-variable fragment $\mathcal{FO}^2$. All of the four languages have the finite model property, and hence are decidable. As already mentioned in the abstract, none of them can express transitivity of a binary relation or related properties like being a partial order or an equivalence relation. Hence, their extensions are studied when it is assumed that some of the binary symbols from the signature are interpreted in a special way (as transitive relations, orderings, equivalences etc.).

In the talk we review main properties of the above mentioned fragments and describe the decidability frontier for their extensions with various transitive relations. The presentation includes joint work with Georg Gottlob, Emanuel Kieroński, Jakub Michaliszyn, Andreas Pieris, Ian Pratt-Hartmann, and Wiesław Szwast. In the next section we recall formal definitions of the four fragments mentioned above; more surveying material with several references can be found e.g. in [7, 4]. In Section 2 we present supplementary material that is featured in the talk but has not yet been published elsewhere.

1 Definitions

We assume relational signatures not containing constants or function symbols.

Definition 1. The fluted fragment *[5]: Let $\bar{x}_\omega = x_1, x_2, \ldots$ be a fixed sequence of variables. We define the sets of formulas $\mathcal{FL}^{[k]}$ (for $k \geq 0$) by structural induction as follows: (i) any atom $\alpha(x_\ell, \ldots, x_k)$, where $x_\ell, \ldots, x_k$ is a contiguous subsequence of $\bar{x}_\omega$, is in $\mathcal{FL}^{[k]}$; (ii) $\mathcal{FL}^{[k]}$ is closed under boolean combinations; (iii) if φ is in $\mathcal{FL}^{[k+1]}$, then $\exists x_{k+1}\varphi$ and $\forall x_{k+1}\varphi$ are in $\mathcal{FL}^{[k]}$. The set of fluted formulas is defined as $\mathcal{FL} = \bigcup_{k\geq 0}\mathcal{FL}^{[k]}$. A fluted sentence is a fluted formula over an empty set of variables, i.e. an element of $\mathcal{FL}^{[0]}$. Thus, when forming Boolean combinations in the fluted fragment, all the combined formulas must have as their free variables some suffix of some prefix $x_1, \ldots, x_k$ of $\bar{x}_\omega$; and when quantifying, only the last variable in this sequence may be bound.*

Definition 2. The guarded fragment *[1], $\mathcal{GF}$, is defined as the least set of formulas such that: (i) every atomic formula belongs to $\mathcal{GF}$; (ii) $\mathcal{GF}$ is closed under logical connectives $\neg, \vee, \wedge, \rightarrow$; and (iii) quantifiers are appropriately relativised by atoms. More specifically, in $\mathcal{GF}$, condition (iii) is understood as follows: if φ is a formula of $\mathcal{GF}$, α is an atomic formula featuring all the free variables of φ, and $\bar{x}$ is any sequence of variables in α, then the formulas $\forall\bar{x}(\alpha \rightarrow \varphi)$ and $\exists\bar{x}(\alpha \wedge \varphi)$ belong to $\mathcal{GF}$. In this context, the atom α is called a* guard. *The equality symbol when present in the signature is also allowed in guards.*

Definition 3. The unary negation fragment *[6], $\mathcal{UNF}$, consists of formulas in which the use of negation is restricted only to subformulas with at most one free variable. More precisely, $\mathcal{UNF}$ is defined as the least set of formulas such that: (i) every atomic formula of the form $R(\bar{x})$ or $x = y$ belongs to $\mathcal{UNF}$; (ii) $\mathcal{UNF}$ is closed under logical connectives $\vee$, $\wedge$ and under existential quantification; (iii) if $\varphi(x)$ is a formula of $\mathcal{UNF}$ featuring no free variables besides (possibly) x, then $\neg\varphi(x)$ belongs to $\mathcal{UNF}$.*

Definition 4. The two variable fragment*: By the k-variable fragment of a logic $\mathcal{L}$, denoted $\mathcal{L}^k$, we mean the set of formulas of $\mathcal{L}$ featuring at most k distinct variables. In particular $\mathcal{FO}^k$ denotes the set of all first-order formulas with at most k variables.*

All of the above languages are incomparable in terms of expressive power, have the finite model property and the satisfiability problem (=finite satisfiability problem) is decidable.

2 Fluted Fragment with Transitive Relations

We show two undecidability results for the fluted fragment with two variables, $\mathcal{FL}^2$, extended with transitive relations. We employ the apparatus of tiling systems (cf. e.g. [3]).

A *tiling system* is a tuple $\boldsymbol{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_H, \mathcal{C}_V)$, where $\mathcal{C}$ is a finite set of *tiles*, and $\mathcal{C}_H$, $\mathcal{C}_V \subseteq \mathcal{C} \times \mathcal{C}$ are the *horizontal* and *vertical* constraints. Let S be any of the spaces $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}_t \times \mathbb{Z}_t$. A tiling system $\boldsymbol{\mathcal{C}}$ *tiles* S, if there exists a function $\rho : S \to \mathcal{C}$ such that for all $(p, q) \in S$: $(\rho(p, q), \rho(p + 1, q)) \in \mathcal{C}_H$ and $(\rho(p, q), \rho(p, q + 1)) \in \mathcal{C}_V$. It is known that the problem whether a given tiling system tiles any of the spaces $\mathbb{N}\times\mathbb{N}$, $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{Z}_t\times\mathbb{Z}_t$ is undecidable. To reduce one of the problems to the (finite) satisfiability problem for some logic $\mathcal{L}$ we proceed as follows.

Let $\boldsymbol{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_H, \mathcal{C}_V)$ be a tiling system. We write an $\mathcal{L}$-formula $\eta_{\boldsymbol{\mathcal{C}}}$ such that the following reduction properties hold

(i) $\eta_{\boldsymbol{\mathcal{C}}}$ is satisfiable iff $\boldsymbol{\mathcal{C}}$ tiles $\mathbb{N} \times \mathbb{N}$ or $\mathbb{Z} \times \mathbb{Z}$, and
(ii) $\eta_{\boldsymbol{\mathcal{C}}}$ is finitely satisfiable iff $\boldsymbol{\mathcal{C}}$ tiles $\mathbb{Z}_t \times \mathbb{Z}_t$, for some $t \geq 1$.

Properties (i) and (ii) above yield undecidability of the satisfiability, respectively, finite satisfiability, problem for $\mathcal{L}$.

In order to encode tilings using two-variable logics it is useful to define structures that are grid-like. A structure $\mathfrak{G} = (G, h, v)$ with two binary relation h and v is *grid-like*, if one of the standard grids $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}_t \times \mathbb{Z}_t$ can be homomorphically embedded into $\mathfrak{G}$. To show that a structure $\mathfrak{G}$ is grid-like it suffices to require that $\mathfrak{G} \models \forall x(\exists y h(x, y) \wedge \exists y v(x, y))$ (note that this is an $\mathcal{FL}^2$-formula) and the following *confluence* property holds

$$\mathfrak{G} \models \forall x, x', y, y'((h(x, y) \wedge v(x, x') \wedge v(y, y') \to h(x', y')). \qquad (*)$$

The confluence property above uses four variables and is not fluted. We will enforce it (or a variation of it) in $\mathcal{FL}^2$ using transitive relations.

2.1 Undecidability of $\mathcal{FL}^2$ with two transitive relations and equality

Suppose the signature contains two transitive relations b (blue) and r (red), and additional unary predictes $c_{i,j}$ ($0 \leq i \leq 3$, $0 \leq j \leq 3$) called *colours*. We write a formula φ_{grid} capturing several properties of the intended expansion of the $\mathbb{Z} \times \mathbb{Z}$ grid as shown in Figure 1. There, each element with coordinates (k, l) satisfies $c_{i,j}$, where $i = k \mod 4$ and $j = l \mod 4$ and the transitive relations connect only some elements that are close in the grid. The formula φ_{grid} is a conjunction of the statements (1)-(7) described below; in the formulas in this subsection we assume that addition in subscripts of the c_{ij}s is always understood modulo 4.

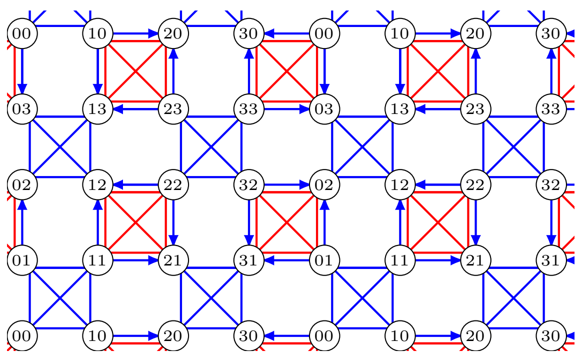

Fig. 1. Intended expansion of the $\mathbb{Z} \times \mathbb{Z}$ grid with transitive relations b and r. Edges without arrows represent connections in both direction. Nodes are marked by the indices of the c_{ij}s they satisfy.

(1) there is an initial element: $\exists x c_{00}(x)$.

(2) the c_{ij}s enforce a partition of the universe: $\forall x \dot{\bigvee}_{0 \leq i \leq 3} \dot{\bigvee}_{0 \leq j \leq 3} c_{i,j}(x)$.

(3) transitive paths do not connect distinct elements of the same colour:

$$\bigwedge_{0 \leq i,j \leq 3} \forall x (c_{ij}(x) \to \forall y ((b(x,y) \vee r(x,y)) \wedge c_{ij}(y) \to x = y))$$

(4) each element belongs to a 4-element blue clique; we write the following conjuncts for each $i, j \in \{0, 2\}$:

$$\forall x (c_{ij}(x) \to \exists y (b(x,y) \wedge c_{i+1,j}(y)))$$
$$\forall x (c_{i+1,j}(x) \to \exists y (b(x,y) \wedge c_{i+1,j+1}(y)))$$
$$\forall x (c_{i+1,j+1}(x) \to \exists y (b(x,y) \wedge c_{i,j+1}(y)))$$
$$\forall x (c_{i,j+1}(x) \to \exists y (b(x,y) \wedge c_{ij}(y)))$$

(5) each element belongs to a 4-element red clique; we write the following conjuncts for each $i, j \in \{1, 3\}$:

$$\forall x(c_{ij}(x) \to \exists y(r(x,y) \wedge c_{i+1,j}(y)))$$
$$\forall x(c_{i+1,j}(x) \to \exists y(r(x,y) \wedge c_{i+1,j+1}(y)))$$
$$\forall x(c_{i+1,j+1}(x) \to \exists y(r(x,y) \wedge c_{i,j+1}(y)))$$
$$\forall x(c_{i,j+1}(x) \to \exists y(r(x,y) \wedge c_{ij}(y)))$$

(6) A group of formulas saying that some pairs of elements connected by r are also connected by b:

$$\bigwedge_{i=0,2} \forall x(c_{ii}(x) \to \forall y(r(x,y) \wedge (c_{i,i-1}(y) \vee c_{i-1,i}(y)) \to b(x,y))) \quad (6a)$$

$$\bigwedge_{i=1,3} \forall x(c_{ii}(x) \to \forall y(r(x,y) \wedge (c_{i,i+1}(y) \vee c_{i+1,i}(y)) \to b(x,y))) \quad (6b)$$

$$\bigwedge_{i=0,2} \forall x(c_{i,i+1}(x) \to \forall y(r(x,y) \wedge (c_{i,i+2}(y) \vee c_{i-1,i+1}(y)) \to b(x,y))) \quad (6c)$$

$$\bigwedge_{i=1,3} \forall x(c_{i,i-1}(x) \to \forall y(r(x,y) \wedge (c_{ii}(y) \vee c_{i,i-2}(y)) \to b(x,y))) \quad (6d)$$

(7) A group of formulas saying that some pairs of elements connected by b are also connected by r:

$$\bigwedge_{i=0,2} \forall x(c_{ii}(x) \to \forall y(b(x,y) \wedge (c_{i,i-1}(y) \vee c_{i-1,i}(y)) \to r(x,y))) \quad (7a)$$

$$\bigwedge_{i=1,3} \forall x(c_{ii}(x) \to \forall y(b(x,y) \wedge (c_{i,i+1}(y) \vee c_{i+1,i}(y)) \to r(x,y))) \quad (7b)$$

$$\bigwedge_{i=0,2} \forall x(c_{i,i+1}(x) \to \forall y(b(x,y) \wedge (c_{i,i+2}(y) \vee c_{i-1,i+1}(y)) \to r(x,y))) \quad (7c)$$

$$\bigwedge_{i=1,3} \forall x(c_{i,i-1}(x) \to \forall y(b(x,y) \wedge (c_{ii}(y) \vee c_{i,i-2}(y)) \to r(x,y))) \quad (7d)$$

It should be clear that the structure shown in Figure 1 is a model of φ_{grid}. One can note that φ_{grid} has also finite models expanding a toroidal grid structure $\mathbb{Z}_{4m} \times \mathbb{Z}_{4m}$ ($m > 0$) obtained by identifying elements from columns 0 and $4m$ and from rows 0 and $4m$.

Let us introduce the following definitions:

$$\mathsf{h}(x,y) := b(x,y) \wedge \bigvee_{\substack{i=0,2 \\ j=0,1,2,3}} (c_{ij}(x) \wedge c_{i+1,j}(y)) \vee r(x,y) \wedge \bigvee_{\substack{i=1,3 \\ j=0,1,2,3}} (c_{ij}(x) \wedge c_{i+1,j}(y))$$

$$\mathsf{v}(x,y) := b(x,y) \wedge \bigvee_{\substack{i=0,1,2,3 \\ j=0,2}} (c_{ij}(x) \wedge c_{i,j+1}(y)) \vee r(x,y) \wedge \bigvee_{\substack{i=0,1,2,3 \\ j=1,3}} (c_{ij}(x) \wedge c_{i,j+1}(y))$$

Let $\mathfrak{A} \models \varphi_{grid}$. We show that every model $(\mathfrak{A}, \mathsf{h}, \mathsf{v})$ of φ_{grid} is grid-like. We first show that $\mathfrak{A}$ satisfies $\forall x(\exists y \mathsf{h}(x,y) \wedge \exists y \mathsf{v}(x,y))$. One considers several cases depending on the values of the unary predicates.

Let $a \in A$ and assume $\mathfrak{A} \models c_{00}(a)$. By (4) there are $a_1, a_2, a_3, a_4 \in A$ such that $\mathfrak{A} \models b(a, a_1) \wedge c_{10}(a_1) \wedge b(a_1, a_2) \wedge c_{11}(a_2) \wedge b(a_2, a_3) \wedge c_{01}(a_3) \wedge b(a_3, a_4) \wedge c_{00}(a_4)$. By (3) $a = a_4$ and by transitivity of b the elements a, a_1, a_2, a_3 form a blue clique in $\mathfrak{A}$. Hence, $\mathfrak{A} \models \mathsf{h}(a, a_1) \wedge \mathsf{v}(a, a_3)$.

The same argument works if a has one of the colours c_{02}, c_{20} or c_{22} and, similarly, applying (5) instead of (4) when a has the colours c_{11}, c_{31}, c_{13} or c_{33}.

Consider now the case $\mathfrak{A} \models c_{10}(a)$. By (4) there is $a' \in A$ such that $\mathfrak{A} \models b(a, a') \wedge c_{11}(a')$, hence $\mathfrak{A} \models \mathsf{v}(a, a')$. Moreover, by (5) there are $a_1, a_2, a_3, a_4 \in A$ such that $\mathfrak{A} \models r(a, a_1) \wedge c_{13}(a_1) \wedge r(a_1, a_2) \wedge c_{23}(a_2) \wedge r(a_2, a_3) \wedge c_{20}(a_3) \wedge r(a_3, a_4) \wedge c_{10}(a_4)$. By (3) $a = a_4$ and by transitivity of r the elements a, a_1, a_2, a_3 form a red clique in $\mathfrak{A}$. By (6d) $\mathfrak{A} \models b(a, a_1)$, hence $\mathfrak{A} \models \mathsf{h}(a, a_1)$.

Remaining cases are shown similarly.

Now, we show the confluence property (*). Let $a, a', b, b' \in A$ and $\mathfrak{A} \models \mathsf{h}(a, b) \wedge \mathsf{v}(a, a') \wedge \mathsf{h}(b, b')$. One needs to consider several cases depending on the colour of a, in each of them showing that $\mathfrak{A} \models \mathsf{h}(a', b')$. For instance:

- $\mathfrak{A} \models c_{00}(a)$. Then $\mathfrak{A} \models c_{01}(a') \wedge b(a, a') \wedge b(a, b) \wedge c_{10}(b) \wedge b(b, b') \wedge c_{11}(b')$. By (4) b' is a member of a blue clique containing elements of colours $c_{11}, c_{01}, c_{00}, c_{10}$. Since by (3) the relation b does not connect distinct elements of the same colour, a' belongs to the blue clique of b' and $\mathfrak{A} \models b(a', a)$. Now, by transitivity of b, $\mathfrak{A} \models \mathsf{h}(a', b')$.
- $\mathfrak{A} \models c_{30}(a)$. Then $\mathfrak{A} \models c_{31}(a') \wedge b(a, a') \wedge r(a, b) \wedge c_{00}(b) \wedge b(b, b') \wedge c_{01}(b')$. Similarly as above, b belongs to a red clique of a, hence $\mathfrak{A} \models r(b, a)$. By (6a), $\mathfrak{A} \models b(b, a)$. Moreover, b' is in a blue clique of b, and so $\mathfrak{A} \models b(b', b)$. By transitivity of b, $\mathfrak{A} \models b(b', a')$. Now, by (7c), $\mathfrak{A} \models r(b', a')$. By (5), a' is a member of a red clique that, by (3), must contain b'. Hence $\mathsf{h}(a', b')$ holds.

Remaining cases can be shown in a similar way. Hence, every model of φ_{grid} is grid-like. Now we ensure that we also can assign tiles to elements of the grid-like models using fluted formulas. The task in $\mathcal{FO}^2$ is easy, it suffices to require that

(8) each node encodes precisely one tile: $\forall x(\dot{\bigvee}_{C \in \mathcal{C}} Cx)$,

(9) adjacent tiles respect $\mathcal{C}_H$ and $\mathcal{C}_V$; the first condition can be written as follows:

$$\bigwedge_{C \in \mathcal{C}} \forall x(Cx \rightarrow \forall y(\mathsf{h}(x,y) \rightarrow \bigvee_{C' : (C, C') \in \mathcal{C}_H} C'y)),, \tag{9a}$$

and similarly for $\mathcal{C}_V$.

Formula (8) is fluted, formulas in (9) are not, but can be written as fluted using first-order tautologies. E.g. each conjunct in (9a) can be written as follows:

$$\bigwedge_{i=0,2, j=0,1,2,3} \forall x(Cx \wedge c_{ij}(x) \rightarrow \forall y(b(x,y) \wedge c_{i+1,j}(y) \rightarrow \bigvee_{C' : (C, C') \in \mathcal{C}_H} C'y)) \wedge$$
$$\bigwedge_{i=1,3, j=0,1,2,3} \forall x(Cx \wedge c_{ij}(x) \rightarrow \forall y(r(x,y) \wedge c_{i+1,j}(y) \rightarrow \bigvee_{C' : (C, C') \in \mathcal{C}_H} C'y)).$$

Let $\eta_{\mathcal{C}}$ be the conjunction of φ_{grid} with the properties (8) and (9) written in $\mathcal{FL}^2$, as explained. Now it is routine to show that the reduction properties (i) and (ii) hold. If $\mathcal{C}$ tiles any of the spaces $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}_t \times \mathbb{Z}_t$, for some t, we expand the grids to our intended models. In the opposite direction, when $\mathfrak{G} \models \eta_{\mathcal{C}}$ and $\mathfrak{G}$ is infinite we obtain a tiling of $\mathbb{Z} \times \mathbb{Z}$; in case $\mathfrak{G}$ is finite we obtain a tiling of $\mathbb{Z}_t \times \mathbb{Z}_t$ with t divisible by 4. As a result we simultaneously conclude that the satisfiability and the finite satisfiability problems for $\mathcal{FL}^2$ with two transitive relations are undecidable.

One can also observe that the above formulas are guarded or can easily be rewritten as guarded. Hence we also get the following

Corollary 1. *The (finite) satisfiability problem for the intersection of the fluted fragment with equality with the two-variable guarded fragment is undecidable in the presence of two transitive relations.*

2.2 Undecidability of $\mathcal{FL}^2$ without equality and with three transitive relations

Suppose the signature contains transitive relations b (black), g (green) and r (red), and additional unary predicates e, e', f, l, $c_{i,j}$ $(0 \leq i \leq 5,\ 0 \leq j \leq 2)$ and $d_{i,j}$ $(0 \leq i \leq 2,\ 0 \leq j \leq 5)$; we refer to the $c_{i,j}$'s and to the $d_{i,j}$'s as *colours*. In this section addition in subscripts of the $c_{i,j}$'s, it is always understood modulo 6 in the first position, and modulo 3 in the second position, i.e. $c_{i+a,j+b}$ denotes $c_{(i+a) mod\ 6,(j+b) mod\ 3}$; and, addition in subscripts of the $d_{i,j}$'s is understood modulo 3 in the first position, and modulo 6 in the second position.

As before we write a formula φ_{grid} that captures several properties of the intended expansion of the $\mathbb{N} \times \mathbb{N}$ grid as shown in Figure 2. There the unary predicates l and f mark the elements in the left column, respectively, bottom row of the grid. The predicate e marks elements on the main diagonal and the predicate e' marks elements with coordinates $(k, k+1)$. The predicates $c_{i,j}$ and $d_{i,j}$ together define a partition of the universe as follows: an element (k, k') with $k' > k$ satisfies $c_{i,j}$ with $i = k \mod 6$, $j = k' \mod 3$, and an element (k, k') with $k \geq k'$ satisfies $d_{i,j}$ with $i = k \mod 3, j = k' \mod 6$. Transitive relations connect elements that are 'close' in the grid; more precisely, paths of the same transitive relation have length at most 7 and they follow one of four designed patterns.

The formula φ_{grid} comprises several conjuncts. The most complex group of conjuncts describes the requirement that each element of a certain colour has a successor of an appropriate colour as shown in Figure 3. Conjuncts from this group have the following form:

$$\forall x\big(colour(x) \wedge diag(x) \wedge border(x) \rightarrow \exists y(t(x,y) \wedge colour'(y))\big), \qquad (1)$$

where *colour* and *colour'* stands for one of the predicates letters $c_{i,j}$ or $d_{i,j}$, $diag(x)$ means one of the literals $e'(x)$, $\neg e'(x)$, $e(x)$, $\neg e(x)$ or $\top$ (i.e. the logical constant true), $border(x)$ means one of the literals $l(x)$, $\neg l(x)$, $f(x)$, $\neg f(x)$ or $\top$, and t stands for one of the transitive predicate letters b, r or g. The precise

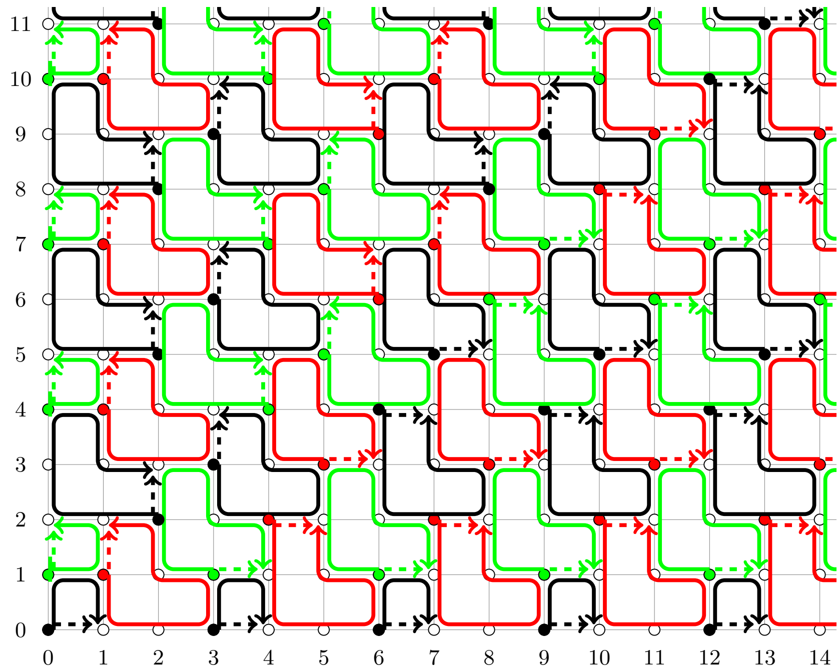

Fig. 2. Intended expansion of the $\mathbb{N} \times \mathbb{N}$ grid with transitive relations b, g and r. Filled nodes depict beginning of a transitive path of the same colour; dotted lines connect the first element with the last element on such path. Paths of length 7 have four distinct patterns; for instance the same pattern is followed by: the red path starting at node (5,3), the green path starting at node (3,1) and the black path starting at node (7,5). Paths starting at nodes (4,2), (8,6) and (6,4) follow another pattern; and similarly, paths starting at nodes (1,1), (5,5) and (3,3); and paths starting at nodes (6,6), (4,4) and (2,2). Two shorter path patterns appear on the border.

combinations of the literals and predicate letters in these conjuncts can be read from Figure 3 and they are defined in Table 1. The remaining conjuncts to φ_{grid} express the following properties:

(2) there is an initial element: $\exists x(d_{00}(x) \wedge e(x) \wedge l(x) \wedge f(x))$.
(3) the predicates $c_{i,j}$ together with $d_{i,j}$ enforce a partition of the universe.
(4) the predicates l, f, e and e' propagate as intended; e.g.

$$\bigwedge_{0 \leq j \leq 5} \forall x\big(c_{0,j}(x) \wedge l(x) \rightarrow \forall y(c_{0,j+1}(y) \wedge (b(x,y) \vee g(x,y)) \rightarrow l(y))\big)$$

$$\bigwedge_{\substack{0 \leq i \leq 2 \\ 0 \leq j \leq 5}} \forall x\big(d_{i,j}(x) \wedge e(x) \rightarrow \forall y((b(x,y) \vee g(x,y) \vee r(x,y)) \wedge d_{i+1,j+1}(y) \rightarrow e(y))\big).$$

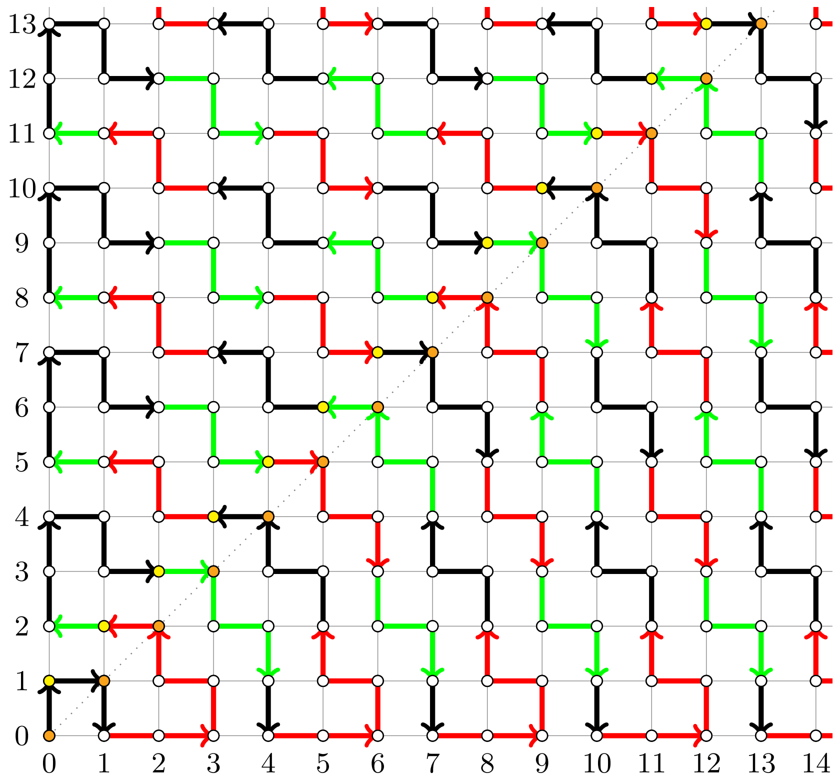

Fig. 3. Construction of the embedding of the $\mathbb{N} \times \mathbb{N}$ grid in a model of φ_{grid}. Nodes on the main diagonal (i.e. satisfying e) are marked orange and nodes satisfying e' are marked yellow. Arrows indicate nodes where the successor relation crosses the diagonal or where its definition switches from one transitive relation to another one.

(5) some pairs of elements connected by one transitive relation are also connected by another one as intended in Figure 2; for instance:

$$\forall x(c_{11}(x) \rightarrow \forall y(b(x,y) \wedge c_{10}(y) \rightarrow r(x,y)))$$
$$\forall x(c_{20}(x) \rightarrow \forall y(g(x,y) \wedge c_{30}(y) \rightarrow r(x,y)))$$

The structure depicted in Figure 2 is a model of φ_{grid}. We show the following

Claim. Every model of φ_{grid} is grid-like.

Proof. Let $\mathfrak{A} \models \varphi_{grid}$. We define an embedding h of the standard grid $\mathfrak{G}_{\mathbb{N}}$ on $\mathbb{N} \times \mathbb{N}$ into $\mathfrak{A}$ as follows. Let $next : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ be a successor function defined on $\mathfrak{G}_{\mathbb{N}}$ as depicted by the thick path in Figure 3 starting in $(0,0)$ (ignoring any colours). Let $a \in A$ be an element such that $\mathfrak{A} \models d_{00}(a) \wedge e(a) \wedge l(a) \wedge f(a)$ that exists by condition (2). Define $h(0,0) = a$. Now, we proceed inductively: suppose

$colour$	$diag(x)$	$border(x)$	$t(x,y)$	$colour'$	$colour$	$diag(x)$	$border(x)$	$t(x,y)$	$colour'$
d_{00}	e	l	b	c_{01}					
d_{00}	e	$\neg l$	g	c_{50}	c_{01}	e'		b	d_{11}
d_{14}	e		b	c_{31}	c_{20}	e'		g	d_{03}
d_{22}	e		r	c_{12}	c_{42}	e'		r	d_{25}
d_{00}	$\neg e$		r	d_{01}	c_{01}	$\neg e'$		b	c_{11}
d_{01}			r	d_{21}	c_{11}			b	c_{10}
d_{21}			r	d_{22}	c_{10}			b	c_{20}
d_{22}	$\neg e$		b	d_{23}	c_{20}	$\neg e'$		g	c_{30}
d_{23}			b	d_{13}	c_{30}			g	c_{32}
d_{13}			b	d_{14}	c_{32}			g	c_{42}
d_{14}	$\neg e$		g	d_{15}	c_{42}	$\neg e'$		r	c_{52}
d_{15}			g	d_{05}	c_{52}			r	c_{51}
d_{05}			g	d_{00}	c_{51}			r	c_{01}
d_{11}			b	d_{10}	c_{12}			g	c_{02}
d_{10}		$\neg f$	b	d_{20}	c_{02}		$\neg l$	g	c_{00}
d_{20}		$\neg f$	b	d_{25}	c_{00}		$\neg l$	g	c_{50}
d_{25}			r	d_{24}	c_{50}			b	c_{40}
d_{24}			r	d_{04}	c_{40}			b	c_{41}
d_{04}			r	d_{03}	c_{41}			b	c_{31}
d_{03}			g	d_{02}	c_{31}			r	c_{21}
d_{02}			g	d_{12}	c_{21}			r	c_{22}
d_{12}			g	d_{11}	c_{22}			r	c_{12}
d_{20}		f	r	d_{00}	c_{02}		l	b	c_{00}
d_{10}		f	r	d_{20}	c_{00}		l	b	c_{01}

Table 1. Various combinations of the literals in conjuncts of the form (1); empty entries in columns $diag(x)$ or $border(x)$ mean $\top$.

$h(k,k')$ has already be defined and $h(k,k') = a$. Let a' be the witness of a for the appropriate conjunct from the group (1), i.e. where the monadic literals for x agree with the monadic literals satisfied by a in $\mathfrak{A}$. Define $h(next(k,k')) = a'$.

Correctness of the embedding can be shown by induction considering various cases depending on the colours of the elements. Interested readers are invited to do the proof by drawing: print Figure 3 on a sheet of paper, use appropriate colours to draw the edges induced by transitivity of b, g and r, apply universal conjuncts from the group (5) and compare the resulting picture with Figure 2 (when they agree the proof is finished!).

We have one more obstacle to overcome. Defining the embedding we moved through the grid in four directions, so in order to appropriately assign tiles to nodes in models of φ_{grid} we need to respect the four directions as we have fixed order of variables. The idea is to define four binary relations $\mathsf{rt}(x,y)$, $\mathsf{lt}(x,y)$, $\mathsf{up}(x,y)$ and $\mathsf{dn}(x,y)$ with the intention that rt and the inverse of lt together give the expected horizontal grid successor, and up and the inverse of dn together give the expected vertical grid successor. For instance, the relation $\mathsf{rt}(x,y)$ is defined as follows:

$$\begin{aligned}\mathsf{rt}(x,y) :=&(b(x,y) \vee g(x,y) \vee r(x,y)) \quad \wedge \\ &(c_{50}(x) \wedge d_{00}(y)) \vee (c_{31}(x) \wedge d_{14}(y)) \vee (c_{12}(x) \wedge d_{22}(y)) \quad \vee \\ &\bigvee_{(i,j)\notin\{(0,2),(1,2),(2,1),(3,1),(4,0),(5,0)\}} (c_{ij}(x) \wedge c_{i+1,j}(y)) \quad \vee \bigvee_{(i,j)\notin\{(2,1),(1,3),(0,5)\}} (d_{ij}(x) \wedge d_{i+1,j}(y))\end{aligned}$$

The formula above says that the relation rt connects elements that are connected by (at least) one of the transitive relation and satisfy one of the possible combination of colours: in the second line the combinations for crossing the diagonal from left to right are listed, in the third line the left big disjunction describes combinations when both elements are located above the diagonal, and in the right big disjunction—when both elements are located on and below the diagonal. The definition of $\mathsf{lt}(x,y)$ is written complementing the definition of rt:

$$\begin{aligned}\mathsf{lt}(x,y) :=&(b(x,y) \vee g(x,y) \vee r(x,y)) \quad \wedge \\ &(d_{00}(x) \wedge c_{50}(y)) \vee (d_{14}(x) \wedge c_{31}(y)) \vee (d_{22}(x) \wedge c_{12}(y)) \quad \vee \\ &\bigvee_{(i,j)\in\{(1,2),(2,2),(3,1),(4,1),(5,0),(0,0)\}} (c_{ij}(x) \wedge c_{i-1,j}(y)) \quad \vee \bigvee_{(i,j)\in\{(0,1),(2,3),(1,5)\}} (d_{ij}(x) \wedge d_{i-1,j}(y))\end{aligned}$$

Analogously one defines the relations up and dn. Now the counterpart of the formula (9a) from the previous subsection can be written as follows:

$$\bigwedge_{C\in\mathcal{C}} \forall x(Cx \to \forall y(\mathsf{rt}(x,y) \to \bigvee_{C':(C,C')\in\mathcal{C}_H} C'y))$$
$$\bigwedge_{C\in\mathcal{C}} \forall x(Cx \to \forall y(\mathsf{lt}(x,y) \to \bigvee_{C':(C',C)\in\mathcal{C}_H} C'y))$$

and similarly for $\mathcal{C}_V$. As before, the formulas can be further rewritten as fluted thanks to the fact that in the definitions of $\mathsf{rt}(x,y)$, $\mathsf{lt}(x,y)$, $\mathsf{up}(x,y)$ and $\mathsf{dn}(x,y)$ the binary literals have the form $t(x,y)$ (and not $t(y,x)$).

As in the previous section all formulas used for the reduction are either guarded or can easily be rewritten as guarded. Hence we also get the following

Corollary 2. *The satisfiability problem for the intersection of the fluted fragment without equality with the two-variable guarded fragment is undecidable in the presence of three transitive relations.*

It is not obvious if the formula φ_{grid} can be modified to define embeddings of finite grids. Hence decidability of the corresponding finite satisfiability problem remains open. It also is open whether the full fluted fragment, $\mathcal{FL}$, with one or two transitive relations (without equality) is decidable.

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