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Enumeration of matrices with prohibited bounded sub-windows

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Abstract: Let $\mathscr{A}$ be a finite alphabet, and let $\mathscr{S}$ be a set of of 2-dimensional bounded prohibited patterns over $\mathscr{A}$. We consider the set $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ of matrices over $\mathscr{A}$ that avoid the patterns from $\mathscr{S}$, and attempt to derive (closed or linear recurrence) formulas for the numbers of $m \times n$ matrices in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$. We argue that different sets of prohibited patterns require different types of formulas, with some formulas recurrent in just one of the parameters m,n, some satisfying a two-dimensional linear recurrence relation (depending of both m and n), and some satisfying neither of the two types. We consider characterization of classes that admit a two-dimensional linear recurrence relation, as well as classes that do not allow for such relation. In addition, given $\mathscr{A}$ and $\mathscr{S}$, we address the question of the existence of a constant a such that the number of $m \times n$ matrices in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ is asymptotically equal to $|\mathscr{A}|^{amn}$.

We report on preliminary results for a specific class of boolean matrices with the prohibited set consisting of thirty-two 3×3 matrices for which computational results suggest the non-existence of a two-dimensional linear recurrence relation.

1 Introduction and preliminaries

Many classes of objects are defined via prohibiting specified sub-objects. In our paper, we deal with classes of matrices over finite alphabets that do not contain patterns from a finite set of local prohibited patterns. Such matrices can be viewed as matrices recognizable via a bounded window automaton with a finite memory that can only view a bounded area of the matrix at a time and cannot see (or remember) the matrix in its entirety (while it is allowed to slide through the entire matrix window by window, verifying each window separately). The motivation behind considering these classes of matrices lies in extending the theory of 'one-dimensional' languages of strings avoiding specified substrings to two dimensional arrays. One-dimensional languages that avoid (connected) substrings from a finite set of prohibited substrings have been studied for several decades and their enumeration is well-known to lead to homogeneous linear recurrence relations (see. e.g., [3, 4]), We show that a similar, although more complicated, situation holds in the case of two-dimensional arrays. We stress that when talking of submatrices, we mean connected blocks.

Let $\mathscr{A}$ be a finite alphabet, and let $\mathscr{S}$ be a set of $k \times \ell$ matrices over $\mathscr{A}$, $k,\ell \geq 1$. Let $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ denote the set matrices over $\mathscr{A}$ that do not contain (avoid) sub-matrices from $\mathscr{S}$, i.e., the set of matrices $\mathbf{A} = \| a_{i,j} \|_{m.n}$, $a_{i,j} \in \mathscr{A}$, for $1 \leq i \leq m$, $1 \leq j \leq m$, having the property that none of the $k \times \ell$ submatrices of $\mathbf{A}$ belong to $\mathscr{S}$ (thus, k and ℓ are the dimensions of the viewing window of the automaton recognizing $\mathbf{A}$; it accepts $\mathbf{A}$ if and only if it never finds a matrix from $\mathscr{S}$ in its viewing window).

We illustrate this concept with a specific class of matrices with prohibited patterns that will be used throughout our paper.

Example 1. *Let $\mathscr{A} = \{0,1\}$, and consider the set of boolean matrices over $\mathscr{A}$ not admitting 3×3 crosses of zeroes or ones, i.e., not admitting submatrices of the form*

$$\begin{array}{|c|c|c|}\hline * & 0 & * \\ \hline 0 & 0 & 0 \\ \hline * & 0 & * \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline * & 1 & * \\ \hline 1 & 1 & 1 \\ \hline * & 1 & * \\ \hline \end{array}$$

where the stars stand for arbitrary elements from $\mathscr{A}$ (to avoid using stars, one could think of the set of the 32 prohibited matrices obtained by making all the possible choices). We will call the matrices from this class noise matrices*, and note that they are often considered to be examples of chaotic, structure-less matrices.*

Given an alphabet $\mathscr{A}$ and a set $\mathscr{S}$ of prohibited $k \times \ell$ submatrices over $\mathscr{A}$, let $N_{\mathscr{A},\mathscr{S}}(m,n)$ denote the number of $m \times n$ matrices in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$. Then clearly $N_{\mathscr{A},\mathscr{S}}(m,n) = |\mathscr{A}|^{mn}$, for all $1 \leq m \leq k$ and $1 \leq n \leq \ell$, with at least one parameter smaller than the upper bound, while

$$0 \leq N_{\mathscr{A},\mathscr{S}}(m,n) \leq |\mathscr{A}|^{mn} \tag{1}$$

in general.

In what follows, we are interested in deriving formulas for $N_{\mathscr{A},\mathscr{S}}(m,n)$ for various alphabets $\mathscr{A}$ and sets of prohibited sub-matrices $\mathscr{S}$.

Example 2. *Considering $\mathscr{A} = \{0,1\}$ again, taking empty $\mathscr{S}_1$ yields $\mathscr{M}_{\mathscr{A},\mathscr{S}_1}$ consisting of all boolean matrices and $N_{\mathscr{A},\mathscr{S}_1}(m,n) = 2^{mn}$, for all m and n.*

Taking $\mathscr{S}_2$ to consist of the single 1×1 matrix with $a_{1,1} = 1$ implies that $\mathscr{M}_{\mathscr{A},\mathscr{S}_2}$ consists of just the $m \times n$ zero-matrices and $N_{\mathscr{A},\mathscr{S}_2}(m,n) = 1$, for all m and n.

Finally, taking $\mathscr{S}_3$ to consist of the 2×2 all-ones matrix $a_{i,j} = 1$, for $1 \le i, j \le 2$, yields $\mathscr{M}_{\mathscr{A},\mathscr{S}_3}$ consisting of all 1×1, 1×2, 2×1 matrices, and $m \times n$ matrices that do not contain a 2×2 sub-matrix of all ones, for $m,n \ge 2$. Thus,

$$N_{\mathscr{A},\mathscr{S}_3}(1,1) = 2,$$

$$N_{\mathscr{A},\mathscr{S}_3}(1,2) = N_{\mathscr{A},\mathscr{S}_3}(2,1) = 2^2 = 4,$$

$$N_{\mathscr{A},\mathscr{S}_3}(2,2) = 2^4 - 1 = 15,$$

and the Inclusion-Exclusion Principle yields that $N_{\mathscr{A},\mathscr{S}_3}(2,3) = 2^6 - 2^2 - 2^2 + 1 = 57$.

One of the main conjectures concerning the asymptotic behavior of the numbers $N_{\mathscr{A},\mathscr{S}}(m,n)$ states the following:

Conjecture 1. *Let $\mathscr{A}$ be a finite alphabet, and let $\mathscr{S}$ be a set of prohibited $k\times \ell$ submatrices over $\mathscr{A}$. Then there exists a constant $0 \le a \le 1$ such that*

$$\lim_{m\to\infty,n\to\infty} \frac{N_{\mathscr{A},\mathscr{S}}(m,n)}{|\mathscr{A}|^{amn}} = 1.$$

If the a from the above conjecture exists for a specific pair $\mathscr{A}$ and $\mathscr{S}$, we say that a is the *critical exponent* for the pair. The sets $\mathscr{S}_1$ and $\mathscr{S}_2$ defined in Example 2 constitute extremal cases with the critical exponents $a_1 = 1$ and $a_2 = 0$, respectively.

The paper [1] contains the following information about the asymptotic behavior of the enumeration function of the noise matrices.

Theorem 1 ([1])**.** *Let $\mathscr{A}$ and $\mathscr{S}$ be those defined in Example 1. For every $m \ge 3$, there exists a constant $0 \le a_m \le 1$ such that*

$$\lim_{n\to\infty} \frac{N_{\mathscr{A},\mathscr{S}}(m,n)}{|\mathscr{A}|^{a_m mn}} = 1.$$

Furthermore, there exist two constants $0 < b_1 < b_2 < 1$ such that

$$2^{b_1 mn} < N_{\mathscr{A},\mathscr{S}}(m,n) < 2^{b_2 mn},$$

for all $m,n \ge 3$.

While computer experimentation appears to support Conjecture 1, in principle, it cannot be used to prove the claim for any specific pair $\mathscr{A}$ and $\mathscr{S}$. However, it usually fairly quickly provides for estimates for the value of a. In particular, finding the numbers $N_{\mathscr{A},\mathscr{S}}(m,n)$ for a large range of pairs m,n allows one to calculate $\frac{\log_{|\mathscr{A}|}(N_{\mathscr{A},\mathscr{S}}(m,n))}{mn}$ for each such pair. The actual values for large pairs often match for a considerable number of decimal places. For example, calculations concerning the enumeration of noise matrices reported in [6] yield that the corresponding a (if it exists!) lies in the range:

$$0.9068 \le a \le 0.947564.$$

2 One-dimensional linear recurrence relations

Let $\mathscr{A}$ and $\mathscr{S}$ be a finite alphabet and a set of $k\times \ell$ prohibited matrices over $\mathscr{A}$. In this section, we prove that for any given $m \ge k$ there exists a linear recurrence formula tying together the numbers $N_{\mathscr{A},\mathscr{S}}(m,n)$, $n \ge 1$. We use a generalization of a technique used in [1] for noise matrices.

Example 3. *To illustrate the basic idea of this approach, suppose we extend a $3\times n$ matrix ending in a specific triple of columns by adding a specific new column which results in a $3\times(n+1)$ matrix ending in a new triple of columns (but sharing two columns with the original triple). We may encounter two different situations:*

$$\cdots \begin{array}{|c|c|c|}\hline 0&1&1\\\hline 1&1&1\\\hline 0&0&1\\\hline\end{array} + \begin{array}{|c|}\hline 0\\\hline 0\\\hline 1\\\hline\end{array} = \cdots \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 1&1&0\\\hline 0&1&1\\\hline\end{array}$$

or

$$\cdots \begin{array}{|c|c|c|}\hline 0&1&1\\\hline 1&1&1\\\hline 0&0&1\\\hline\end{array} + \begin{array}{|c|}\hline 0\\\hline 1\\\hline 1\\\hline\end{array} = \cdots \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 1&1&1\\\hline 0&1&1\\\hline\end{array}$$

When considering the noise matrices defined in Example 1, these two situations differ as follows. Any noise matrix ending in the first triple remains *a noise matrix after adding the specified column, while the matrix formed from a noise matrix by adding the second specified column ceases being a noise matrix.*

With regard to the above example, it is important to point out that the entire situation only depends of the last three columns, and the actual number of columns of the matrices is irrelevant with regard to the above claims.

In view of these observations, let $\mathscr{A}$ be a finite alphabet and $\mathscr{S}$ be a set of $k \times \ell$ prohibited matrices over $\mathscr{A}$, and let us fix the number $m \geq k$ of rows. Let $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_{|\mathscr{A}|^{m\ell}}\}$ be the set of all $m \times \ell$ matrices over $\mathscr{A}$ (listed in an arbitrary but fixed order). For each $n \geq \ell$, divide the $m \times n$ matrices in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ with regard to their last ℓ columns, and let α_i^n denote the number of $m \times n$ matrices in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ ending in the matrix $\mathbf{M}_i$, $1 \leq i \leq |\mathscr{A}|^{m\ell}$. Denote $\alpha(n) = (\alpha_1^n, \alpha_2^n, \ldots, \alpha_{|\mathscr{A}|^{m\ell}}^n)$, and note that

$$\alpha(n) \cdot \mathbf{1}^T = \sum_{i=1}^{|\mathscr{A}|^{m\ell}} \alpha_i^n = N_{\mathscr{A},\mathscr{S}}(m,n), \qquad (2)$$

(where $\mathbf{1}^T$ stands for the column of all ones).

As observed above, if we expand an $m \times n$ matrix ending in $\mathbf{M}_i$ by adding a column, we obtain an $m \times (n+1)$ matrix ending in $\mathbf{M}_j$ having the property that the first $\ell - 1$ columns of $\mathbf{M}_j$ match the last $\ell - 1$ columns of $\mathbf{M}_i$. If this is the case, we will say that $\mathbf{M}_j$ is a *successor* of $\mathbf{M}_i$. Moreover, if $n \geq \ell$, the question whether an $m \times n$ matrix in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ ending in $\mathbf{M}_i$ remains in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ after a column is added to it so that it ends in $\mathbf{M}_j$ depends of $\mathbf{M}_i$ and $\mathbf{M}_j$ only and it is *independent* of the number of columns n. Therefore, for $1 \leq i, j \leq |\mathscr{A}|^{m\ell}$, let $a_{i,j} = 1$ if $\mathbf{M}_j$ is a successor of $\mathbf{M}_i$ having the property that if an $m \times n$ matrix ending in $\mathbf{M}_i$ belongs to $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ then so does the $m \times (n+1)$ matrix ending in $\mathbf{M}_j$ (constructed from the smaller matrix by adding a column). Let $a_{i,j} = 0$ otherwise, and denote $\mathbb{A} = \| a_{i,j} \|$. Since every $m \times (n+1)$ matrix in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$ is obtained from a specific $m \times n$ matrix in $\mathscr{M}_{\mathscr{A},\mathscr{S}}$, it follows that

$$\alpha(n+1) = \alpha(n)\mathbb{A}, \qquad (3)$$

for all $n \geq \ell$.

Suppose now that the square matrix $\mathbb{A}$ is a root of a monic polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{s-1}x^{s-1} + x^s$ in $\mathbb{Z}[x]$, i.e.,

$$a_0\mathbb{I} + a_1\mathbb{A} + a_2\mathbb{A}^2 + \ldots + a_{s-1}\mathbb{A}^{s-1} + \mathbb{A}^s = \mathbb{O},$$

where $\mathbb{I}$ stands for the identity matrix and $\mathbb{O}$ for the all-zeroes matrix. Thus,

$$\mathbb{A}^s = -a_0\mathbb{I} - a_1\mathbb{A} - a_2\mathbb{A}^2 - \ldots - a_{s-1}\mathbb{A}^{s-1}, \qquad (4)$$

and after multiplying by $\alpha(n)$ on the left and by $\mathbf{1}^T$ on the right we obtain

$$\alpha(n)\mathbb{A}^s\mathbf{1}^T = -a_0\alpha(n)\mathbf{1}^T - a_1\alpha(n)\mathbb{A}\mathbf{1}^T - a_2\alpha(n)\mathbb{A}^2\mathbf{1}^T - \ldots - a_{s-1}\alpha(n)\mathbb{A}^{s-1}\mathbf{1}^T.$$

Applying equations (2) and (3) finally yields

$$N_{\mathscr{A},\mathscr{S}}(m,n+s) = -a_0N_{\mathscr{A},\mathscr{S}}(m,n) - \ldots - a_{s-1}N_{\mathscr{A},\mathscr{S}}(m,n+s-1), \qquad (5)$$

which is a linear recurrence relation.

The above arguments allow us to prove the following generalization of Theorem 2 to *all* sets of matrices over finite alphabets with prohibited bounded patterns.

Theorem 2 ([1]). *Let $\mathscr{A}$ be a finite alphabet and $\mathscr{S}$ be a set of $k \times \ell$ prohibited matrices over $\mathscr{A}$. For every $m \geq k$, there exists a linear recurrence relation such that*

$$N_{\mathscr{A},\mathscr{S}}(m,n+s) = -a_0N_{\mathscr{A},\mathscr{S}}(m,n) - \ldots - a_{s-1}N_{\mathscr{A},\mathscr{S}}(m,n+s-1),$$

as well as a constant $0 \leq c_m \leq 1$ such that

$$\lim_{n\to\infty} \frac{N_{\mathscr{A},\mathscr{S}}(m,n)}{|\mathscr{A}|^{c_m mn}} = 1.$$

Proof. Let $\mathscr{A}$ and $\mathscr{S}$ be as stated, and suppose that $m \geq k$. The matrix $\mathbb{A}$ defined in the discussion preceding the statement of our theorem is an $|\mathscr{A}|^{m\ell} \times |\mathscr{A}|^{m\ell}$ boolean matrix which (by the Cayley-Hamilton theorem) is the root of its characteristic polynomial $char_{\mathbb{A}}(x)$, which belongs to $\mathbb{Z}[x]$, and is either monic when $|\mathscr{A}|^{m\ell}$ is even or can be made monic by multiplying by -1 when $|\mathscr{A}|^{m\ell}$ is odd. This yields a linear recurrence relation of order $|\mathscr{A}|^{m\ell}$ for the numbers $N_{\mathscr{A},\mathscr{S}}(m,n)$, $n \geq \ell$. Since $\mathbb{A}$ is a boolean (i.e., non-negative) matrix, using the Perron-Frobenius theorem yields that its spectral radius $\rho(\mathbb{A})$ is its eigenvalue of the largest modulus. Consequently, $\rho(\mathbb{A})$ determines the magnitude of any sequence satisfying the recurrence relation determined by $char_{\mathbb{A}}(x)$ [2, 5], therefore $N_{\mathscr{A},\mathscr{S}}(m,n) = \theta(\rho(\mathbb{A})^n)$, and the second claim of our theorem follows. □

3 One-dimensional linear recurrence relations of smallest order

Even though we have proved the existence of an one-dimensional linear recurrence relation for each $\mathscr{A}, \mathscr{S}$, and $m \geq k$, the orders $|\mathscr{A}|^{m\ell}$ of these relations are rather large. The key problem when using such relations lies in the need to find the first $|\mathscr{A}|^{m\ell}$ elements of the corresponding sequence by brute force. Thus, in order to start using the above described recurrence relation for the numbers $N_{\mathscr{A},\mathscr{S}}(3,n)$ in the case of noise matrices (for which the prohibited matrices are of dimension 3×3), one first needs to find the numbers

$$N_{\mathscr{A},\mathscr{S}}(3,3), N_{\mathscr{A},\mathscr{S}}(3,4), \ldots, N_{\mathscr{A},\mathscr{S}}(3,2^9),$$

which turns out to be a computationally demanding task simply because of the sheer size of the search spaces and the corresponding frequency numbers. For example, $N_{\mathscr{A},\mathscr{S}}(3,55000) \approx 2^{0.970956\cdot 3\cdot 55000} \approx 2^{160207}$, and 640 MB of memory space were needed to store the first 55,000 members of the sequence $N_{\mathscr{A},\mathscr{S}}(3,n)$ [6]. (Clearly, in order to obtain the correct recurrence relation, one needs to calculate and store the exact numbers.)

Finding recurrence relations of smaller degrees is therefore of utter importance. The first obvious choice for reducing the degree of the obtained recurrence relation is to use the minimal polynomial for $\mathbb{A}$ over $\mathbb{C}$ instead of its characteristic polynomial. However, while calculating the characteristic polynomial for $\mathbb{A}$ is a computationally demanding but simple determinant calculation, finding the minimal polynomial for $\mathbb{A}$ requires finding the roots for $char_{\mathbb{A}}(x)$ or its irreducible divisors. Moreover, the minimal polynomial over $\mathbb{C}$ most likely does not belong to $\mathbb{Z}[x]$, making the exact calculation of the coefficients of the corresponding recurrence relation impossible. While this problem can be remedied by considering the minimal polynomial over $\mathbb{Q}$ (which does belong to $\mathbb{Z}[x]$), in general, this would be of higher degree than the minimal polynomial over $\mathbb{C}$, and still hard to find.

In [6], the third author under the supervision of the second author of this article considered the noise matrices and chose a much simpler computational approach. Using essentially brute force, he found the numbers of noise matrices $N_{\mathscr{A},\mathscr{S}}(3,n)$ for $1 \leq n \leq 55000$. Having the numbers from this list, he created a list consisting of the numbers $\frac{\log_2(N(3,n))}{3n}$, looking for a pattern. An easy inspection reveals that $\frac{\log_2(N(3,1500))}{3\cdot 1500} \approx 0.970992$, while $\frac{\log_2(N(3,55000))}{3\cdot 55000} \approx 0.970956$; the critical exponent for $m=3$ becomes exact up to the first four decimal digits fairly quickly.

Similarly, calculating the numbers $N_{\mathscr{A},\mathscr{S}}(4,n)$ for $1 \leq n \leq 35000$ determined the critical exponent for $m=4$ equal to 0.959452; the numbers $N_{\mathscr{A},\mathscr{S}}(5,n)$ for $1 \leq n \leq 50000$ determined the critical exponent for $m=5$ equal to 0.952307; and finally calculating the numbers $N_{\mathscr{A},\mathscr{S}}(6,n)$ for $1 \leq n \leq 25000$ determined the critical exponent for $m=6$ equal to 0.9475645.

As for the recurrence relation of minimal degree, having the actual values of the corresponding sequence allows one to find the minimal degree experimentally. Specifically, let $k \geq 2$, and suppose an equivalence relation of degree k exists. If that were the case, the solution $a_0, a_1, a_2, \ldots, a_{k-1}$ of the $k \times k$ system of linear equations

$$\begin{aligned}
a_0 N_{\mathscr{A},\mathscr{S}}(m,\ell) + \ldots + a_{k-1} N_{\mathscr{A},\mathscr{S}}(m,\ell+k-1) \\
= N_{\mathscr{A},\mathscr{S}}(m,\ell+k) \\
a_0 N_{\mathscr{A},\mathscr{S}}(m,\ell+1) + \ldots + a_{k-1} N_{\mathscr{A},\mathscr{S}}(m,\ell+k) \\
= N_{\mathscr{A},\mathscr{S}}(m,\ell+k+1) \\
\ldots \\
a_0 N_{\mathscr{A},\mathscr{S}}(m,\ell+k-1) + \ldots + a_{k-1} N_{\mathscr{A},\mathscr{S}}(m,\ell+2k-2) \\
= N_{\mathscr{A},\mathscr{S}}(m,\ell+2k-1)
\end{aligned}$$

would have to satisfy all the 'latter' systems, $i \geq 1$,

$$\begin{aligned}
a_0 N_{\mathscr{A},\mathscr{S}}(m,\ell+i) + \ldots + a_{k-1} N_{\mathscr{A},\mathscr{S}}(m,\ell+k-1+i) \\
= N_{\mathscr{A},\mathscr{S}}(m,\ell+k+i) \\
a_0 N_{\mathscr{A},\mathscr{S}}(m,\ell+1+i) + \ldots + a_{k-1} N_{\mathscr{A},\mathscr{S}}(m,\ell+k+i) \\
= N_{\mathscr{A},\mathscr{S}}(m,\ell+k+1+i) \\
\ldots \\
a_0 N_{\mathscr{A},\mathscr{S}}(m,\ell+k-1+i) + \ldots + a_{k-1} N_{\mathscr{A},\mathscr{S}}(m,\ell+2k-2+i) \\
= N_{\mathscr{A},\mathscr{S}}(m,\ell+2k-1+i).
\end{aligned}$$

This can be experimentally tested starting from $k=2$, and looking for the first k that satisfies these requirements (which will necessary be the smallest degree of a linear recurrence relation for the considered sequence).

Relying on [6] again reveals the following. The minimal degree of a linear recurrence relation for $N_{\mathscr{A},\mathscr{S}}(3,n)$ is 2, the minimal degree of a linear recurrence relation for $N_{\mathscr{A},\mathscr{S}}(4,n)$ is 4, the minimal degree of a linear recurrence relation for

$N_{\mathscr{A},\mathscr{S}}(5,n)$ is 8, and the minimal degree of a linear recurrence relation for $N_{\mathscr{A},\mathscr{S}}(6,n)$ is 20.

In particular,

$$N_{\mathscr{A},\mathscr{S}}(3,n+2) = 4N_{\mathscr{A},\mathscr{S}}(3,n) + 7N_{\mathscr{A},\mathscr{S}}(3,n+1),$$

for all $n \geq 3$.

4 Two-dimensional linear recurrence relations

The results obtained for the noise matrices mentioned in the previous section suggest that the degree of the minimal linear recurrence relation increases with increasing number of rows. This is, however, not a universal fact concerning all matrices with prohibited bounded patterns. For example, all the numbers $N_{\mathscr{A},\mathscr{S}_2}(m,n)$ for the matrices from Example 2 are equal to 1, and hence satisfy the recurrence relation $N_{\mathscr{A},\mathscr{S}_2}(m,n+1) = N_{\mathscr{A},\mathscr{S}_2}(m,n)$. Nevertheless, we feel that the following conjecture might turn out to be true.

Conjecture 2. *Let $\mathscr{A}$ be a finite alphabet and $\mathscr{S}$ be a set of $k \times \ell$ prohibited matrices over $\mathscr{A}$ with $k,\ell \geq 2$. Then the minimal degree of a linear recurrence relation for the sequence*

$$N_{\mathscr{A},\mathscr{S}}(m,n), N_{\mathscr{A},\mathscr{S}}(m,n+1), N_{\mathscr{A},\mathscr{S}}(m,n+2), \ldots$$

$m \geq k$, increases with increasing m.

In view of Conjecture 2, instead of looking for one-dimensional linear recurrence relations, we propose to search for two-dimensional recurrence relations.

Specifically, let $r_{m,n}$ be a two dimensional sequence of reals (i.e., a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$). We say that a two-dimensional sequence $r_{m,n}$ *satisfies a two-dimensional linear recurrence relation* provided there exist coefficients $a_{i,j}$, $0 \leq i \leq t$, $0 \leq j \leq s$, with $a_{t,s} = 0$, such that

$$\begin{aligned} r_{m+t,n+s} &= \\ a_{0,0}r_{m,n} + a_{0,1}r_{m,n+1} + \ldots + a_{0,s}r_{m,n+s} &+ \\ a_{1,0}r_{m+1,n} + a_{1,1}r_{m+1,n+1} + \ldots + a_{1,s}r_{m+1,n+s} &+ \\ \ldots &+ \\ a_{t,0}r_{m+t,n} + a_{t,1}r_{m+t,n+1} + \ldots + a_{t,s}r_{m+t,n+s}&, \end{aligned}$$

for all $m,n \in \mathbb{N}$.

Our preliminary results suggest the following two conjectures.

Conjecture 3. *Let $\mathscr{A}$ be a finite alphabet and $\mathscr{S}$ be a set of $k \times \ell$ prohibited matrices over $\mathscr{A}$ with at least one of the numbers k,ℓ equal to 1. Then the two-dimensional sequence $N_{\mathscr{A},\mathscr{S}}(m,n)$, $m,n \geq 1$, satisfies a two-dimensional recurrence relation.*

Conjecture 4. *Let $\mathscr{A}$ be a finite alphabet and $\mathscr{S}$ be a set of $k \times \ell$ prohibited matrices over $\mathscr{A}$ with $k,\ell \geq 2$. Then the two-dimensional sequence $N_{\mathscr{A},\mathscr{S}}(m,n)$, $m,n \geq 1$, does not satisfy a two-dimensional recurrence relation.*

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