A class of power series q-distributions

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Abstract

A class of power series q-distributions, generated by considering a q-Taylor expansion of a parametric function into powers of the parameter, is discussed. The q-Poisson (Heine and Euler), q-binomial, negative q-binomial and q-logarithmic distributions belong in this class. The probability generating functions and q-factorial moments of the power series q-distributions are derived. In particular, the q-mean and the q-variance are deduced.

1 Introduction

Benkherouf and Bather[BB88] derived the Heine and Euler distributions, which constitute q-analogs of the Poisson distribution, as feasible priors in a simple Bayesian model for oil exploration. The probability function of the q-Poisson distributions is given by (Charalambides[Cha16, p. 107])

$$p_x(\lambda; q) = E_q(-\lambda)\frac{\lambda^x}{[x]_q!}, \quad x = 0, 1, \ldots,$$

where $0 < \lambda < 1/(1-q)$ and $0 < q < 1$ (Euler distribution) or $0 < \lambda < \infty$ and $1 < q < \infty$ (Heine distribution). Also, $E_q(t) = \prod_{i=1}^{\infty}(1 + t(1-q)q^{i-1})$ is a q-exponential function. It should be noted that $e_q(t) = \prod_{i=1}^{\infty}(1 - t(1-q)q^{i-1})^{-1}$ is another q-exponential function and that these q-exponential functions are connected by $E_q(t)e_q(-t) = 1$ and $E_{q^{-1}}(t) = e_q(t)$.

Kemp and Kemp [KK91], in their study of the Weldon's classical dice data, introduced a q-binomial distribution. It is the distribution of the number of successes in a sequence of n independent Bernoulli trials, with the odds of success at a trial varying geometrically with the number of trials. Kemp and Newton [KN90] further studied it as stationary distribution of a birth and death process. The probability function of this q-binomial distribution of the first kind is given by

$$p_x(\theta; q) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{\binom{x}{2}}}{\prod_{i=1}^{n}(1 + \theta q^{i-1})}, \quad x = 0, 1, \ldots, n,$$

where $0 < \theta < \infty$, and $0 < q < 1$ or $1 < q < \infty$.

Charalambides [Cha10] in his study of the q-Bernstein polynomials as a q-binomial distribution of the second kind, introduced the negative q-binomial distribution of the second kind. It is the distribution of the number of failures until the occurrence of the nth success in a sequence of independent Bernoulli trials, with the probability

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of success at a trial varying geometrically with the number of successes. The probability function of this negative q-binomial distribution of the second kind is given by

$$p_x(\theta;q) = \begin{bmatrix} n+x-1 \\ x \end{bmatrix}_q \theta^x \prod_{i=1}^{n}(1-\theta q^{i-1}), \quad x = 0,1,\dots,$$

where $0 < \theta < 1$ and $0 < q < 1$.

A q-logarithmic distribution was studied by C. D. Kemp[Kem97] as a group size distribution. Its probability function is given by

$$p_x(\theta;q) = [-l_q(1-\theta)]^{-1}\frac{\theta^x}{[x]_q}, \quad x = 1,2,\dots,$$

where $0 < \theta < 1$, $0 < q < 1$, and

$$-l_q(1-\theta) = \lim_{x\to 0}\left(\prod_{i=1}^{\infty}\frac{1-\theta q^{x+i-1}}{1-\theta q^{i-1}} - 1\right) = \sum_{j=1}^{\infty}\frac{\theta^j}{[j]_q}$$

is a q-logarithmic function.

The class of power series q-distributions, introduced in section 2, provides a unified approach to the study of these distributions. Its probability generating function and q-factorial moments are derived. Demonstrating this approach, the probability generating function and q-factorial moments of the q-Poisson (Heine and Euler), q-binomial, negative q-binomial, and q-logarithmic distributions are obtained.

2 Power series q-distributions

Consider a positive function $g(\theta)$ of a positive parameter θ and assume that it is analytic with a q-Taylor expansion

$$g(\theta) = \sum_{x=0}^{\infty} a_{x,q}\theta^x, \quad 0 < \theta < \rho, \quad \rho > 0, \tag{1}$$

where the coefficient

$$a_{x,q} = \frac{1}{[x]_q!}[D_q^x g(t)]_{t=0} \geq 0, \quad x = 0,1,\dots, \quad 0 < q < 1, \quad \text{or} \quad 1 < q < \infty, \tag{2}$$

with $D_q = d_q/d_qt$ the q-derivative operator,

$$D_q g(t) = \frac{d_q g(t)}{d_q t} = \frac{g(t) - g(qt)}{(1-q)t},$$

does not involve the parameter θ. Clearly, the function

$$p_x(\theta;q) = \frac{a_{x,q}\theta^x}{g(\theta)}, \quad x = 0,1,\dots, \tag{3}$$

with $0 < q < 1$ or $1 < q < \infty$, and $0 < \theta < \rho$, satisfies the properties of a probability (mass) function.

Definition 2.1. A family of discrete q-distributions $p_x(\theta;q)$, $\theta \in \Theta$, $q \in Q$, is said to be a class of power series q-distributions, with parameters θ, q and series function $g(\theta)$ if it has the representation (3), with series function satisfying condition (1).

Remark 2.2. The range of x in (3), as in the case of the (usual) power series distributions), may be reduced. Thus, we may have $a_{x,q} > 0$ for $x \in T$, with

$$T = \{x_0, x_0+1, \dots, x_0+x_1-1\}, \quad x_0 \geq 0, \quad x_1 \geq 1.$$

Moreover, note that the truncated versions of the a power series q-distribution are also power series q-distributions in their own right.

The probability generating function $P(t) = \sum_{x=0}^{\infty} p_x(\theta;q)t^x$, on using (1) and (3), is readily deduced as

$$P(t) = \frac{g(\theta t)}{g(\theta)}. \tag{4}$$

Clearly, the mth q-derivative, with respect to t, of the probability generating function is

$$\frac{d_q^m P(t)}{d_q t^m} = \sum_{x=m}^{\infty} p_x(\theta;q)[x]_{m,q} t^{x-m}.$$

Thus, the mth q-factorial moment of the power series q-distribution, on using (4), is obtained as

$$E([X]_{m,q}) = \frac{1}{g(\theta)} \cdot \left[\frac{d_q^m g(\theta t)}{d_q t^m}\right]_{t=1} = \frac{\theta^m}{g(\theta)} \cdot \frac{d_q^m g(\theta)}{d_q \theta^m}, \quad m = 1, 2, \dots . \tag{5}$$

In particular the q-mean is given by

$$E([X]_q) = \frac{\theta}{g(\theta)} \cdot \frac{d_q g(\theta)}{d_q \theta}. \tag{6}$$

Also, on using the expression

$$V([X]_q) = qE([X]_{2,q}) - E([X]_q)\big(E([X]_q) - 1\big), \tag{7}$$

the q-variance is obtained as

$$V([X]_q) = \frac{q\theta^2}{g(\theta)} \cdot \frac{d_q^2 g(\theta)}{d_q \theta^2} - \frac{\theta}{g(\theta)} \cdot \frac{d_q g(\theta)}{d_q \theta}\left(\frac{\theta}{g(\theta)} \cdot \frac{d_q g(\theta)}{d_q \theta} - 1\right). \tag{8}$$

Example 2.3. *q-Poisson distributions.* These are power series q-distributions, with series function $g(\lambda) = e_q(\lambda) = 1/E_q(-\lambda)$, where $0 < \lambda < 1/(1-q)$ and $0 < q < 1$ or $0 < \lambda < \infty$ and $1 < q < \infty$. Since $D_q e_q(t) = e_q(t)$ and $e_q(0) = 1$, it follows from (2) that

$$a_{x,q} = \frac{1}{[x]_q!}[D_q^x e_q(t)]_{t=0} = \frac{1}{[x]_q!}, \quad x = 0, 1, \dots,$$

Also, the probability generating function of the q-Poisson distributions, on using (4), is deduced as

$$P(t) = \frac{e_q(\lambda t)}{e_q(\lambda)} = E_q(-\lambda)e_q(\lambda t).$$

The q-factorial moments, by (5) and since $D_q^m e_q(\lambda) = e_q(\lambda)$, are readily deduced as

$$E([X]_{m,q}) = \lambda^m, \quad m = 1, 2, \dots .$$

In particular, the q-mean is given by

$$E([X]_q) = \lambda.$$

Also, using (7), the q-variance is obtained as

$$V([X]_q) = q\lambda^2 - \lambda(\lambda - 1) = \lambda(1 + (q-1)\lambda).$$

Example 2.4. *q-Binomial distribution of the first kind.* The series function of this distribution is $g(\theta) = \prod_{i=1}^{n}(1 + \theta q^{i-1})$, where $0 < \theta < \infty$ and $0 < q < 1$ or $1 < q < \infty$. Since

$$\begin{aligned} D_q g(\theta) &= \frac{\prod_{i=1}^{n}(1+\theta q^{i-1}) - \prod_{i=1}^{n}(1+\theta q^i)}{(1-q)\theta} \\ &= \frac{[(1+\theta) - (1+\theta q^n)]\prod_{i=1}^{n-1}(1+\theta q^i)}{(1-q)\theta} = [n]_q \prod_{i=1}^{n-1}(1 + (\theta q)q^{i-1}), \end{aligned}$$

it follows successively that

$$D_q^x g(\theta) = [n]_{x,q} q^{1+2+\cdots+(x-1)} \prod_{i=1}^{n-x} (1 + (\theta q^x) q^{i-1}) = [n]_{x,q} q^{\binom{x}{2}} \prod_{i=1}^{n-x} (1 + (\theta q^x) q^{i-1}),$$

for $x = 1, 2, \ldots, n$. Thus, by (2),

$$a_{x,q} = \frac{1}{[x]_q!} [D_q^x g(t)]_{t=0} = \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\binom{x}{2}}, \quad x = 0, 1, \ldots, n.$$

Also, the probability generating function of the q-binomial distribution of the first kind, on using (4), is deduced as

$$P(t) = \frac{\prod_{i=1}^{n} (1 + \theta t q^{i-1})}{\prod_{i=1}^{n} (1 + \theta q^{i-1})}.$$

The q-factorial moments, by (5) and since

$$D_q^m g(\theta) = [n]_{m,q} q^{\binom{m}{2}} \prod_{i=1}^{n-m} (1 + (\theta q^m) q^{i-1}) = [n]_{m,q} q^{\binom{m}{2}} \prod_{i=m+1}^{n} (1 + \theta q^{i-1}),$$

are obtained as

$$E([X]_{m,q}) = \frac{[n]_{m,q} \theta^m q^{\binom{m}{2}}}{\prod_{i=1}^{m} (1 + \theta q^{i-1})}, \quad m = 1, 2, \ldots .$$

In particular, the q-mean is

$$E([X]_q) = \frac{[n]_q \theta}{(1 + \theta)}.$$

Also, using (7) and, subsequently, the expression $q[n-1]_q = [n]_q - 1$, the q-variance is obtained as

$$\begin{aligned} V([X]_q) &= \frac{[n]_q [n-1]_q \theta^2 q^2}{(1+\theta)(1+\theta q)} + \frac{[n]_q \theta}{1+\theta} \left(1 - \frac{[n]_q \theta}{1+\theta}\right) \\ &= \frac{[n]_q \theta}{(1+\theta)(1+\theta q)} \left(1 + \frac{[n]_q \theta (q-1)}{1+\theta}\right). \end{aligned}$$

Example 2.5. *Negative q-binomial distribution of the second kind.* It is a power series q-distribution, with series function $g(\theta) = \prod_{i=1}^{n} (1 - \theta q^{i-1})^{-1}$, where $0 < \theta < 1$ and $0 < q < 1$. Since

$$\begin{aligned} D_q g(\theta) &= \frac{\prod_{i=1}^{n} (1 - \theta q^{i-1})^{-1} - \prod_{i=1}^{n} (1 - \theta q^{i})^{-1}}{(1-q)\theta} \\ &= \frac{[(1 - \theta q^n) - (1 - \theta)] \prod_{i=1}^{n+1} (1 - \theta q^{i-1})}{(1-q)\theta} = [n]_q \prod_{i=1}^{n+1} (1 - \theta q^{i-1}), \end{aligned}$$

it follows successively that

$$D_q^x g(\theta) = [n]_q [n+1]_q \cdots [n+x-1]_q \prod_{i=1}^{n+x} (1 - \theta q^{i-1}) = [n+x-1]_{x,q} \prod_{i=1}^{n+x} (1 - \theta q^{i-1}),$$

for $x = 1, 2, \ldots$. Thus, by (2),

$$a_{x,q} = \frac{1}{[x]_q!} [D_q^x g(t)]_{t=0} = \begin{bmatrix} n+x-1 \\ x \end{bmatrix}_q, \quad x = 0, 1, \ldots .$$

Also, the probability generating function of the negative q-binomial distribution of the second kind, on using (4), is deduced as

$$P(t) = \frac{\prod_{i=1}^{n} (1 - \theta t q^{i-1})^{-1}}{\prod_{i=1}^{n} (1 - \theta q^{i-1})^{-1}}.$$

The q-factorial moments, by (5) and since

$$D_q^m g(\theta) = [n+m-1]_{m,q} \prod_{i=1}^{n+m} (1-\theta q^{i-1})^{-1}$$
$$= [n+m-1]_{m,q} \prod_{i=1}^{n} (1-\theta q^{i-1})^{-1} \prod_{i=1}^{m} (1-\theta q^{n+i-1})^{-1},$$

are obtained as

$$E([X]_{m,q}) = [n+m-1]_{m,q}\theta^m \prod_{i=1}^{m} (1-\theta q^{n+i-1})^{-1}, \quad m = 1, 2, \ldots .$$

In particular, the q-expected value is

$$E([X]_q) = \frac{[n]_q\theta}{1-\theta q^n}.$$

Also, using (7) and, subsequently, the expression $[n+1]_q = [n]_q + q^n$, the q-variance is successively obtained as

$$V([X]_q) = \frac{[n]_q[n+1]_q\theta^2 q}{(1-\theta q^n)(1-\theta q^{n+1})} + \frac{[n]_q\theta}{1-\theta q^n}\left(1 - \frac{[n]_q\theta}{1-\theta q^n}\right)$$
$$= \frac{[n]_q\theta}{(1-\theta q^n)(1-\theta q^{n+1})}\left(1 + \frac{[n]_q\theta(q-1)}{1-\theta q^n}\right).$$

Example 2.6. *q-Logarithmic distribution.* The series function of this distribution is

$$g(\theta) = -l_q(1-\theta) = \sum_{j=1}^{\infty} \frac{\theta^j}{[j]_q}, \quad 0 < \theta < 1, \quad 0 < q < 1.$$

Taking successively its q-derivatives,

$$D_q^x g(\theta) = \sum_{j=x}^{\infty} [j-1]_{x-1,q}\theta^{j-x} = [x-1]_q! \sum_{j=x}^{\infty} \begin{bmatrix} j-1 \\ j-x \end{bmatrix}_q \theta^{j-x},$$

and using the negative q-binomial formula

$$\sum_{k=0}^{\infty} \begin{bmatrix} x+k-1 \\ k \end{bmatrix}_q \theta^k = \prod_{i=1}^{x} (1-\theta q^{i-1})^{-1},$$

we find

$$D_q^x g(\theta) = [x-1]_q! \prod_{i=1}^{x} (1-\theta q^{i-1})^{-1}.$$

Thus, by (2),

$$a_{x,q} = \frac{1}{[x]_q!}[D_q^x g(t)]_{t=0} = \frac{1}{[x]_q}, \quad x = 1, 2, \ldots .$$

Also, the probability generating function of the q-logarithmic distribution, on using (4), is deduced as

$$P(t) = \frac{-l_q(1-\theta t)}{-l_q(1-\theta)}.$$

The q-factorial moments, by (5) and since

$$D_q^m g(\theta) = [m-1]_q! \prod_{i=1}^{m} (1-\theta q^{i-1})^{-1},$$

are obtained as

$$E([X]_{m,q}) = \frac{[-l_q(1-\theta)]^{-1}[m-1]_q!\theta^m}{\prod_{i=1}^{m}(1-\theta q^{i-1})}, \quad m = 1, 2, \ldots .$$

In particular, the q-mean value is

$$E([X]_q) = \frac{[-l_q(1-\theta)]^{-1}\theta}{1-\theta}.$$

Also, using (7), the q-variance is obtained as

$$\begin{aligned} V([X]_q) &= \frac{[-l_q(1-\theta)]^{-1}\theta^2 q}{(1-\theta)(1-\theta q)} + \frac{[-l_q(1-\theta)]^{-1}\theta}{1-\theta}\left(1 - \frac{[-l_q(1-\theta)]^{-1}\theta}{1-\theta}\right) \\ &= \frac{[-l_q(1-\theta)]^{-1}\theta}{1-\theta}\left(\frac{1}{1-\theta q} - \frac{[-l_q(1-\theta)]^{-1}\theta}{1-\theta}\right). \end{aligned}$$

References

[BB88] L. Benkherouf, J. A. Bather. Oil exploration: sequential decisions in the face of uncertainty. *Journal of Applied Probability*, 25:529-543, 1988.

[Cha10] Ch. A. Charalambides The q-Bernstein basis as a q-binomial distribution. *Journal of Statististical Planning and Inference*, 140:2184-2190, 2010.

[Cha16] Ch. A. Charalambides. *Discrete q-Distributions.* John Wiley & Sons, Hoboken, New Jersey, 2016.

[Kem97] C. D. Kemp. A q-logarithmic distribution. In Balakrishnan, N. (Ed.) *Advances in Combinatorial Methods and Applications to Probability and Statistics*, Birkhäuser, Boston, MA, pp. 465-470, 1997.

[KK91] A. Kemp, C D. Kemp. Weldon's dice data revisited. *American Statistician*, 45:216-222, 1991.

[KN90] A. Kemp, J. Newton. Certain state-dependent processes for dichotomized parasite population. *Journal of Applied Probability*, 27:251-258, 1990.