Inverted Fuzzy Implications in Backward Reasoning Without Yager Implication

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Abstract. One of the most popular methods of knowledge representation are the fuzzy rules. One of the ways of representation of fuzzy rules is the functional representation. From over eight decades a number of different fuzzy implications have been described, e.g. [5]-[9]. This leads to the following question: how to choose the proper function among basic fuzzy implications. This paper is a continuation of study [15], where we proposed a new method for choosing implications in backward reasoning. Here we presented a way of simplify the analysis by skipping Yager fuzzy implication.

Key words: fuzzy logic, fuzzy implications, inverted fuzzy implications, backward reasoning

1 Introduction

One of the most popular methods of knowledge representation are the fuzzy rules. From imprecise inputs and fuzzy rules imprecise conclusions are obtained. Reasoning is mainly classified into two types: forward reasoning and backward reasoning. The inference mechanism of forward reasoning has a strong forecasting capability, whereas the aim of backward reasoning generally is to find the most possible causes associated with the existing reality. Backward reasoning plays an essential role in fault diagnosis, accident analysis, and defect detection. This kind of reasoning uses fuzzy logic [3] to reason about data in the inference mechanism instead of many other logics, including Boolean logic, (non-fuzzy) many-valued logics, non-monotonic logics, etc.

Paper [4] discusses different representations of rules in a non-fuzzy setting and extends these representations to rules with a fuzzy conclusion part. It introduces the different types of fuzzy rules and put them in the framework of fuzzy sets and possibility theory.

Fuzzy rules are often presented in the form of implications. In [3] a typology of fuzzy rules and the problem of multiple-valued implications are discussed. The paper reviews the problem of representing fuzzy knowledge, and ranges from linguistic variables to conditional if-then rules and qualified statements.

One of the ways of representation of fuzzy rules is the functional representation (e.g.[11],[12],[17]). The definition of fuzzy implications and their mathematical properties can be found e.g. in [1] and [16]. One of basic problems in building an inference system is choosing the relevant fuzzy implication. In [10] authors proposed a method allowing to choose the most suitable fuzzy implication in an inference system application. They introduced an algorithm that calculates the distance between two fuzzy implications and which is based on generalized modus ponens.

In paper [13] we have presented a fuzzy forward reasoning methodology for rule-based systems using the functional representation of fuzzy rules. In [15] we extended this methodology for selecting relevant fuzzy implications for backward reasoning. The proposed methodology takes full advantage of the functional representation of fuzzy implications and the algebraic properties of the family of all fuzzy implications. It allows to compare two fuzzy implications. If the truth value of the conclusion and the truth value of the implication are given, we can easily optimize the truth value of the implication premise. In particular, in [15] we introduced an algorithm of finding the fuzzy implication which has the highest truth value of the antecedent when the truth value of the consequent and the truth value of the implication are given. This methodology can be useful for the design of inference engine based on the rule knowledge for a given rule-based system.

In the solution in [15] we divided the domain of fuzzy implications into areas, in which it was possible to select appropriate fuzzy implication, and to do that we had to use the Lambert W function. Lambert W function is a special function used when solving equations containing unknown to both the base and the exponent power. It is marked $W(z)$ and defined as the inverse of $f(z) = ze^z$, where z belongs to the set of complex numbers. Thus, for each complex number z holds: $z = W(z)e^{W(z)}$. The Lambert W function cannot be expressed in terms of elementary functions.

In this paper we present the way of avoiding this complexity of solution presented in [15].

The rest of this paper is organized as follows. Sect. 2 contains basic information on fuzzy implications. In Sect. 3 the research problem is formulated. Sect. 4 presents the solution of the given research problem. Sect. 5 is devoted to the pseudo-code of an algorithm for determining a basic fuzzy implication which has the highest truth value of the antecedent when the truth value of the consequent and the truth value of the implication are given. Sect. 6 includes summarizing of our research and some remarks.

2 Preliminaries

In this section we recall a definition of a fuzzy implication and we list a few of basic fuzzy implications known from the subject literature [1].

A function $I : [0,1]^2 \to [0,1]$ is called *a fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

- if $x_1 \leq x_2$, then $I(x_1, y) \geq I(x_2, y)$, i.e., $I(., y)$ is decreasing;

Name	Year	Formula of basic fuzzy implication
Łukasiewicz	1923, [9]	$I_{LK}(x,y) = min(1, 1-x+y)$
Gödel	1932, [4]	$I_{GD}(x,y) = \begin{cases} 1 \ if \ x \leq y \\ y \ if \ x > y \end{cases}$
Reichenbach	1935, [11]	$I_{RC}(x,y) = 1 - x + xy$
Kleene-Dienes	1938, [8]; 1949, [2]	$I_{KD}(x,y) = max(1-x, y)$
Goguen	1969, [7]	$I_{GG}(x,y) = \begin{cases} 1 \ \ if \ x \leq y \\ \frac{y}{x} \ if \ x > y \end{cases}$
Rescher	1969, [12]	$I_{RS}(x,y) = \begin{cases} 1 \ if \ x \leq y \\ 0 \ if \ x > y \end{cases}$
Yager	1980, [18]	$I_{YG}(x,y) = \begin{cases} 1 \quad if \ x = 0 \ and \ y = 0 \\ y^x \ if \ x > 0 \ or \ y > 0 \end{cases}$
Weber	1983, [17]	$I_{WB}(x,y) = \begin{cases} 1 \ if \ x < 1 \\ y \ if \ x = 1 \end{cases}$
Fodor	1993, [3]	$I_{FD}(x,y) = \begin{cases} 1 & if \ x \leq y \\ max(1-x, y) & if \ x > y \end{cases}$

Table 1. Examples of basic fuzzy implications

- if $y_1 \leq y_2$, then $I(x, y_1) \leq I(x, y_2)$, i.e., $I(x, .)$ is increasing;
- $I(0,0) = 1; I(1,1) = 1; I(1,0) = 0$.

There exist uncountably many fuzzy implications. The following Table 1 contains a few examples of basic fuzzy implications. One of the fuzzy implication in the table is Yager implication. As we noted in Sect. 1 we skip this implication in our analysis in this paper to avoid complexity of solution presented in [15].

Figure 1 gives us some plots of these functions.

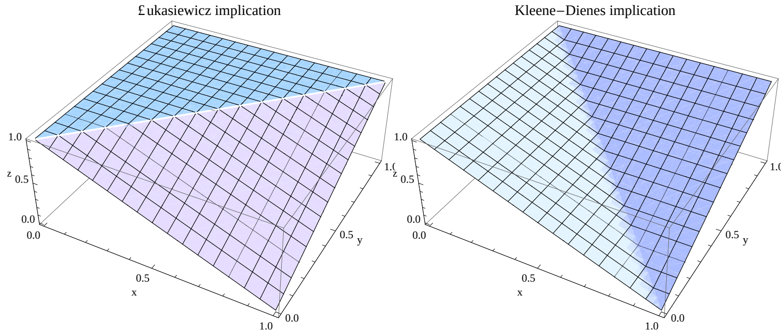
3 Problem Statement

Our goal is to design an algorithm to find a method of selecting fuzzy implication in view of the value of the implication antecedent.

Assume that there is given a basic fuzzy implication $z = I(x, y)$, where x, y belong to [0,1]. y is the truth value of the consequent and is known. z is the truth value of the implication and is also known. In order to determine the value of the truth of the implication antecedent x it is needed to compute the inverse function $InvI(y, z)$. In other words, the inverse function $InvI(y, z)$ has to be determined. Not every of basic implications can be inverted. The function can be inverted only when it is injective.

4 Results

Table 2 lists inverse fuzzy implications and their domains and in Figure 2 there are some plots of them.

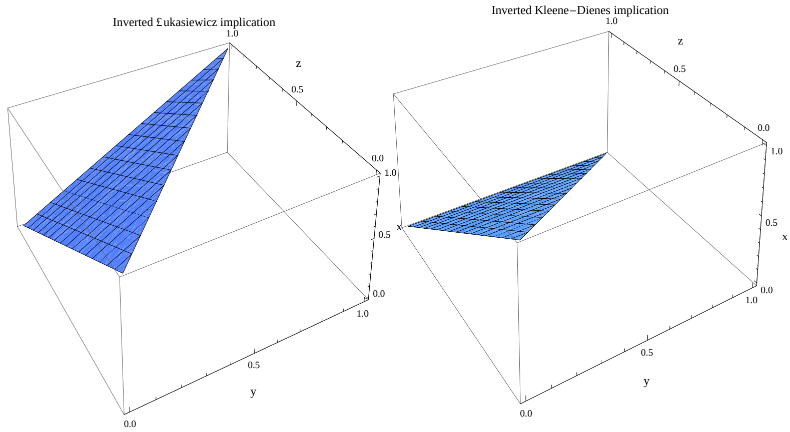


(a) (b)

Fig. 1. Plots of I_{LK} and I_{KD} fuzzy implications

Formula of inverted fuzzy implication	Domain of inverted fuzzy implication
$InvI_{LK}(y,z) = 1 - z + y$	$y \leq z < 1, y \in [0,1)$
$InvI_{RC}(y,z) = \frac{1-z}{1-y}$	$y \leq z \leq 1, y \in [0,1)$
$InvI_{KD}(y,z) = 1 - z$	$y < z \leq 1, y \in [0,1)$
$InvI_{GG}(y,z) = \frac{y}{z}$	$y \leq z < 1, y \in (0,1)$
$InvI_{FD}(y,z) = 1 - z$	$y < z < 1 - y, y \in [0,1)$

Table 2. Inverted fuzzy implications



(a) (b)

Fig. 2. Plots of $InvI_{LK}$ and $InvI_{KD}$ fuzzy implications

The domains of every considered inverted fuzzy implications are included in a half of the unit square, where $y \leq z < 1$ and $y \in (0, 1)$. Only one inverted fuzzy implication has a domain which is smaller than this area. This is inverted Fodor implication and in the whole its domain ($y \leq z < 1 - y, y \in [0, 1]$) this function is equal to inverted Kleene-Dienes implication.

For $y \leq z < 1 - y$ there are the following inequalities: $InvI_{FD} = InvI_{KD} < InvI_{RC} < InvI_{LK}$, $InvI_{RC} < InvI_{LK}$, $InvI_{GG} < InvI_{LK}$. A graphical representation of the ordering of inverted basic fuzzy implications is given in Figure 3.

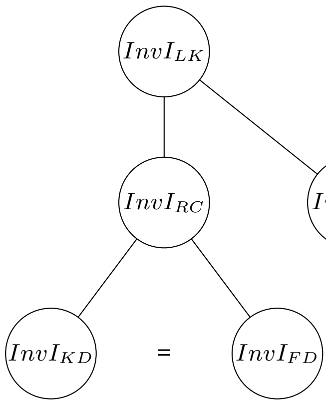


Fig. 3. A graphical representation of the ordering of inverted basic fuzzy implications for $y \leq z < 1 - y$

For $1 - y \leq z < 1$ and $y \leq z$ there are the same inequalities, but without inverted Fodor implication, because this function does not exist in this area.

The resulting inverse functions can be compared with each other so that it is possible to order them. However, some of those functions are incomparable in the whole domain. By taking into account six inverted fuzzy implications (including inverted Yager implication) and by dividing their domain into separable areas, we obtained 19 inequalities between inverted fuzzy implications for any $y \leq z < 1$ and $y \in (0, 1)$ described in [15].

To simplify that solution and avoid Lambert W function in this paper we skip Yager fuzzy implication in our analysis. With this assumption there is only five different area and inequalities instead of nineteen. The areas are shown in the Figure 4 and the inequalities are given in Table 3.

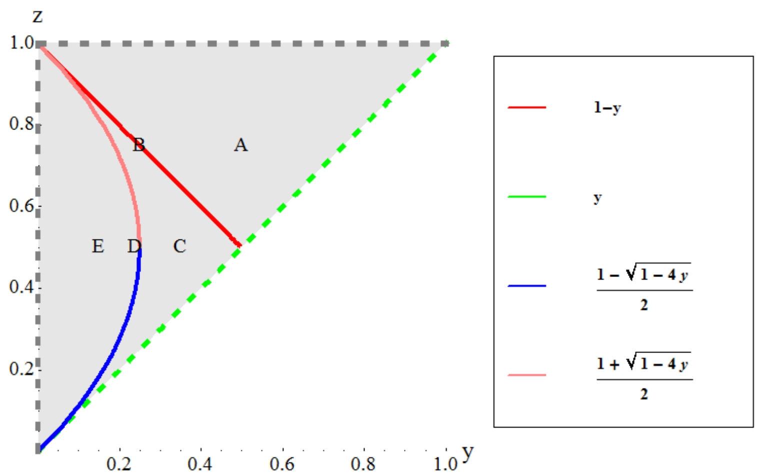


Fig. 4. The unit square $[0,1]^2$ divided into five separable areas

Table 3: Table of inequalities

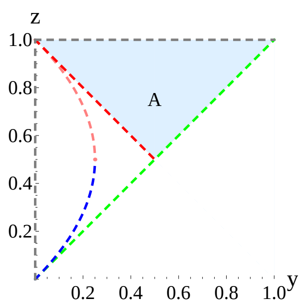
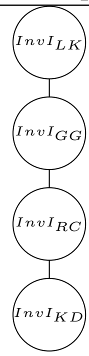
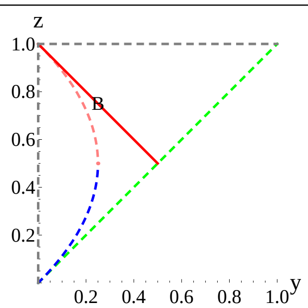
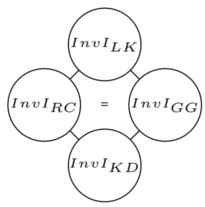
No	Area and inequality	Chart of area	Graph of inequalities
A.	For $z > 1-y$ $InvI_{KD} < InvI_{RC} <$ $InvI_{GG} < InvI_{LK}$		
B.	For $z = 1-y$ $InvI_{KD} < InvI_{RC} =$ $InvI_{GG} < InvI_{LK}$		

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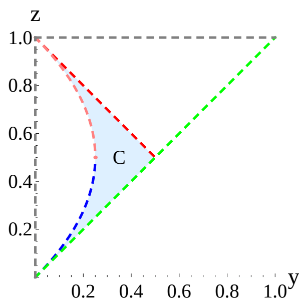
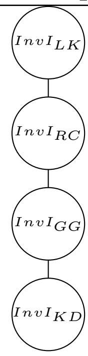
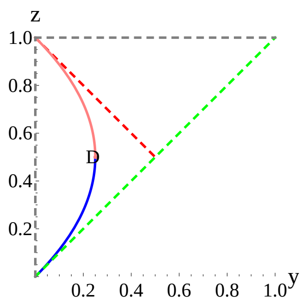
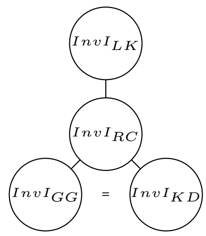
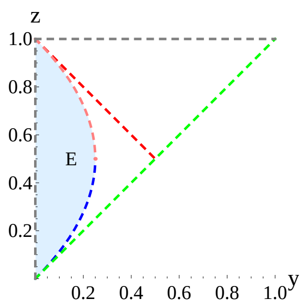
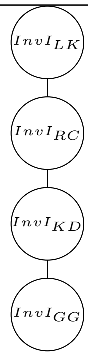
Table 3 – *Continued from previous page*

No	Area and inequality	Chart of area	Graph of inequalities
C.	For ($z > \frac{1+\sqrt{1-4y}}{2}$ or $z < \frac{1-\sqrt{1-4y}}{2}$ or $z \in (0.25, 0.5)$) and $z < 1 - y$ $InvI_{KD} < InvI_{GG} < InvI_{RC} < InvI_{LK}$		
D.	For $z = \frac{1+\sqrt{1-4y}}{2}$ or $z = \frac{1-\sqrt{1-4y}}{2}$ $InvI_{GG} = InvI_{KD} < InvI_{RC} < InvI_{LK}$		
E.	For $z > \frac{1-\sqrt{1-4y}}{2}$ and $z < \frac{1+\sqrt{1-4y}}{2}$ $InvI_{GG} < InvI_{KD} < InvI_{RC} < InvI_{LK}$		

All inequalities given in Table 3 can be proven in a similar way. As examples, we will consider one of inequalities. Let $y \in (0,1)$ and $z \in (y,1)$. $y < z$, so obviously $y^2 < yz$. By adding and subtracting $1 - z + y$ to the equation we obtained $1 - z < 1 - z + y - y + yz - y^2$. And therefore, $\frac{1-z}{1-y} < 1 - z + y$. This completes the proof of the inequality: $InvI_{RC} < InvI_{LK}$ in domains of these functions.

5 Algorithm

Below we present the pseudo-code of the algorithm (*DetermineImplicationGTVA*) for determining a basic fuzzy implication which has the highest truth value of the antecedent whereas the truth value of the consequent and the truth value of the implication are given.

The algorithm uses the results of our research presented in Table 3. The first step in the algorithm determines to which area $(A) - (E)$ from Table 3 point (y, z) belongs to.

Algorithm *DetermineImplicationGTVA*

Input: W - a given subset of the basic fuzzy implications;
y - the truth value of the consequent;
z - the truth value of the implication
Output: $I \in W$ - fuzzy implication(s) which has (have) the highest truth value of the antecedent

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1. $a \leftarrow area(y, z)$ //determines the area from $(A) - (E)$ to which a point (y, z) belongs to;
2. **order** the set W with respect to the graph G_a of inequalities from the area a;
3. $I \leftarrow$ the maximal element(s) from the ordered set W;
4. **return** I;

6 Concluding Remarks

In the paper, we introduced an algorithm for finding the fuzzy implication which has the highest truth value of the antecedent from a given subset of the basic fuzzy implications, when the truth value of the consequent and the truth value of the implication are given. In order to simplify the solution we skipped Yager fuzzy implication in the presented analysis.

We considered a set of basic implications mentioned in Table 1, because they are well known and widely used. But considering only these basic implications implied the solution which does not cover the whole unit square as in the case with the forward reasoning [13], only one of its halves. It raises the question how to find such a set of implications that could give a solution for a backward reasoning in the whole unit square. Our future works will focus on answering the question whether such implications could exist and how they could be defined.

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