The Minimization Method of Boolean Functions in Polynomial Set-theoretical Format

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Abstract. A generalized of conjuncterms simplification rules in polynomial set-theoretical format has been considered. These rules are based on the proposed theorems for different initial conditions transform of pair conjuncterms, Hamming distance between them can be arbitrary. These rules may be useful to minimize in polynomial set-theoretical format of arbitrary logic functions with n variables. Advantages of the proposed rules of simplification are illustrated by several examples.

Key words: Boolean function, polynomial set-theoretical format, simplification of conjuncterms, Hamming distance

1 Introduction

Investigations [1–8] have shown that it is economically profitable to build digital devices such as arithmetic units, coding-error detectors as well as devices with programmed logic, etc. on logical elements *AND-EXOR*, which realize polynomial basis $\{\&, \oplus, \mathbf{1}\}$, that is *AND*, *EXCLUSIVE OR* (*EXOR*) logical operations and constant **1**. It is easier to test and diagnose [9–11] digital devices on *AND-EXOR* if compared to the devices built on *AND-OR*. However, in spite of the mentioned advantages it is more difficult to minimize a function in polynomial format, i. e. in *ESOP* (*EXOR Sum-Of-Product*), than in disjunctive format, i. e. in *SOP* (*Sum-Of-Product*). If the merge operation of adjacent conjuncterms (conjunction of literals) is only applied in the SOP minimization, than, in addition, the same operations can be applied in the ESOP minimization [1, 2].

A precise solutions of a minimization problem in ESOP generally are based on analytical [2] or on visual transformations [1–3]. Respectively, such methods are suitable only for functions from small amount of variables [5, 7, 10–13] and only for special classes for functions with up to 10 variables [14]. Heuristic methods have comparatively wider practical application [1, 8, 16–23]. Among them there are minimization method based on a coefficient of generalized canonical Reed-Muller forms using of matrix transformations [1, 8, 11, 16] and the method based on iterative execution of operations with conjuncterms of different ranks of the given function. To the last belongs the algorithm [17], which after transformation of the given function in Positive Polarity Reed-Muller expression minimizes it on the basis of three operations with conjuncterms. Better results have been shown by algorithm based on the procedure of so-called *linked product terms* [18, 19]. Later, on the basis of this procedure, the algorithms have been developed and completed with more perfect operations (that is *primary xlink*, *secondary xlink*, *unlink*, *exorlink*), which can be used for minimization of a system of complete and incomplete functions [20, 21].

However, the mentioned above algorithms have one drawback in common. They involve the procedure of linking in pairs only conjuncterms of the same rank $r \in \{1, 2, \ldots, n\}$, which differs between each other by binary positions. Correspondingly, this limits the use of such algorithms to functions given in SOP or ESOP, which can have triple conjuncterms in the different part. In these cases to conjuncterms that differ in ranks certain procedures of transformation are applied leading to an increase of procedural steps and processing time. Besides, the above mentioned operations of conjuncterms linking and other rules of simplification [24–26] do not have generalized character as to Hamming distance between any two conjuncterms of different ranks of a given function that does not guarantee the final minimized result.

In this paper we consider a new method of minimization of Boolean functions with n variables in *polynomial set-theoretical format* (*PSTF*), based on a procedure of splitting of conjuncterms [27–29] and on usage of generalized set-theoretical rules of conjuncterms simplification [30]. The suggested rules guarantee better (as to costs of realization and number of procedure steps) results of minimization of logic functions proved by the numerous examples that are borrowed from publications of other authors for comparison purposes.

2 Problem Formulation

Boolean function $f(x_1, x_2, \ldots, x_n)$ that undergoes minimization, will be given by a set of binary minterms or in perfect set-theoretical form (STF) as a $Y^1 = \{m_1, m_2, \ldots, m_k\}^1$, or in perfect polynomial set-theoretical form (PSTF) as a $Y^{\oplus} = \{m_1, m_2, \ldots, m_k\}^{\oplus}$ [30, 31].

The generalized set-theoretical rules of simplification [30] of a conjuncterm set of any function f, given in PSTF $Y^{\oplus}$, are based on iterative process of simplification of two conjuncterms $\theta_1^{r_1} = (\sigma_1\sigma_2 \cdots \sigma_n)$ and $\theta_2^{r_2} = (\sigma_1\sigma_2 \cdots \sigma_n)$, $\sigma_i \in \{0, 1, –\}$, $r_1, r_2 \in \{1, 2, \ldots, n\}$, which differ in Hamming difference $d = 1, 2, \ldots, n$ it is number of different in value $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, -\}$ of onename positions. Here the different part $\alpha, \beta, \gamma, \delta, \ldots$ of these conjuncterms may have a different total number of literals k_l. For example, two pairs of conjuncterms $\begin{pmatrix} 11–01 \\ 01–10 \end{pmatrix}$ and $\begin{pmatrix} 11––1 \\ 01–10 \end{pmatrix}$ have $d = 3$, but the first has $k_l = 6$, and the second $k_l = 5$. Therefore, different initial conditions of transformation of two conjuncterms are possible.

We will consider the following conditions:

- when $k_l = 2d$, here two conjuncterms are of the same r-rank θ_1^r and θ_2^r but differ in d onename binary positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1\}$;
- when $k_l = 2d-1$, here one conjuncterm of $(r–1)$-rank θ_1^{r-1} and the second of r-rank θ_2^r differ in d onename positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, –\}$, where dash (–) belongs to θ_1^{r-1};

- when $k_l = 2(d-1)$, here two conjuncterms are of the same $(r-1)$-rank θ_1^{r-1} and θ_2^{r-1} differ in d onename positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, –\}$ and each of them has one dash (–).

As a result of transformation of two conjuncterms in PSTF a *transformed PSTF* $Y^{\oplus}$ will be formed, where power k_Y will depend on distance d. Efficiency of simplification of two different conjuncterms for the mentioned above initial conditions will be estimated on the basis of comparison of interrelation k_θ^*/k_l^*, obtained on the ground of data of transformed PSTF $Y^{\oplus}$, where k_θ^* is number of transformed conjuncterms and k_l^* is number of their literals, with initial interrelation k_θ/k_l, where in this case $k_\theta = 2$.

3 Three Theorems about Conjuncterms Transformation

Theorem 1. *Two conjuncterms of r-rank θ_1^r and θ_2^r, $r \in \{1, 2, \ldots, n\}$, of the function $f(x_1, x_2, \ldots, x_n)$, that differ in values d of onename binary positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1\}$, in polynomial set-theoretical format form a set of transformed PSTF $Y^{\oplus}$ of power $k_Y = d!$, each of them consists of $k_\theta^* = d$ conjuncterms of $(r-1)$-rank and has in different part the total number of literals $k_l^* = d(d-1)$.*

Proof. To determine k_θ^*/k_l^* and k_Y let us consider the transformation of conjuncterms θ_1^r and θ_2^r for $d = 0, 1, 2, 3, 4$. Here it should be mentioned that initial interrelation $k_\theta/k_l = 2/2d$.

- If $d = 0$, than $\theta_1^r = \theta_2^r$. So, transformed PSTF $Y^{\oplus} = \{\theta_1^r, \theta_2^r\}^{\oplus} = \varnothing$, that corresponds to analytical expression $a \oplus a = 0$. In this case $k_\theta^*/k_l^* = 0/0$; $k_Y = 1$.
- Let $d = 1$. Then $\theta_1^r = (\sigma_1 \cdots \bar{\alpha}_i \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \sigma_n)$, $\alpha_i \in \{0, 1\}$. Respectively, for analytical expression $\bar{a} \oplus a = 1$ we can write:

$$Y^{\oplus} = \{(\sigma_1 \cdots \bar{\alpha}_i \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \sigma_n)\}^{\oplus} = (\sigma_1 \cdots -_i \cdots \sigma_n), \tag{1}$$

where the transformed PSTF $Y^{\oplus} = \{(\sigma_1 \cdots -_i \cdots \sigma_n)\}^{\oplus} = \theta^{r-1}$ is a triple conjuncterm of $(r-1)$-rank.

For (1) interrelation $k_\theta^*/k_l^* = 1/0$, and as initial interrelation $k_\theta/k_l = 2/2$, then it indicates on a result of transformation (1) the simplification took place due to the replacement of two conjuncterms of r-rank by one conjuncterm of $(r–1)$-rank; $k_Y = 1$.

To simplify the writing of the conjuncterms of the given and transformed PSTF $Y^{\oplus}$ will be considered only for their different positions which will be written down in a column. In (1) such position is α_i, so, simplified writing down (1) with taking into account $\alpha_i \equiv \alpha \in \{0, 1\}$, will look like:

$$\begin{pmatrix} \bar{\alpha} \\ \alpha \end{pmatrix} \overset{\oplus}{\Rightarrow} (–), \tag{2}$$

where $\overset{\oplus}{\Rightarrow}$ – operator of transftrmation of the conjuncterms θ_1^r and θ_2^r in polynomial format of the function f. In examples of transformation, the same in meaning onename positions of conjuncterms will be rewritten without any change. For example, $\begin{pmatrix} 1–01 \\ 1–11 \end{pmatrix} \overset{\oplus}{\Rightarrow} (1––1)$, that in decimal format corresponds to $\begin{pmatrix} 9, 13 \\ 11, 15 \end{pmatrix} \overset{\oplus}{\Rightarrow}$ $\overset{\oplus}{\Rightarrow} (9, 11, 13, 15)$ and in analytical form is $x_1\bar{x}_3x_4 \oplus x_1x_3x_4 = x_1x_4$.

• Let $d=2$. Then $\theta_1^r=(\sigma_1\cdots\bar\alpha_i\cdots\bar\beta_j\cdots\sigma_n)$, $\theta_2^r=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\sigma_n)$, $\alpha_i,\beta_j\in\{0,1\}$, and respectively to analytical expressions $\bar a\bar b\oplus ab=\begin{cases}\bar a\oplus b\\ a\oplus\bar b\end{cases}$ (if $\alpha_i=\beta_i$) and $\bar ab\oplus a\bar b=\begin{cases}\bar a\oplus\bar b\\ a\oplus b\end{cases}$ (if $\alpha_i\neq\beta_i$), in simplified way (for $\alpha_i\equiv\alpha$, $\beta_i\equiv\beta$, $\alpha,\beta\in\{0,1\}$) we will obtain

$$\begin{pmatrix}\bar\alpha\bar\beta\\ \alpha\beta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha-\\ -\beta\end{pmatrix},\begin{pmatrix}\alpha-\\ -\bar\beta\end{pmatrix}\right\}. \tag{3}$$

For (3) we have $k_\theta^*/k_l^*=2/2$, that is indicative of simplification of the given conjuncterms due to reduction of their rank from r to $(r-1)$, as initial interrelation $k_\theta/k_l=2/4$; $k_Y=2$.

• Let $d=3$. Then $\theta_1^r=(\sigma_1\cdots\bar\alpha_i\cdots\bar\beta_j\cdots\bar\gamma_k\cdots\sigma_n)$ and $\theta_2^r=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\gamma_k\cdots\sigma_n)$, $\alpha_i,\beta_j,\gamma_k\in\{0,1\}$. For $\alpha_i\equiv\alpha$, $\beta_i\equiv\beta$, $\gamma_i\equiv\gamma$, $\alpha,\beta,\gamma\in\{0,1\}$ we have

$$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta-\\ \bar\alpha-\gamma\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\\ -\bar\beta\gamma\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\\ -\beta\bar\gamma\\ \alpha\beta-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\\ \bar\alpha\beta-\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta-\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha-\bar\gamma\\ \alpha\beta-\end{pmatrix}\right\}. \tag{4}$$

For (4) $k_\theta^*/k_l^*=3/6$ indicates on an increase of power of each transformed PSTF $Y^\oplus$ and unchangeability of number of literals of their conjuncterms as initial interrelation $k_\theta/k_l=2/6$; $k_Y=6$.

• Let $d=4$. Then $\theta_1^r=(\sigma_1\cdots\bar\alpha_i\cdots\bar\beta_j\cdots\bar\gamma_k\cdots\bar\delta_l\cdots\sigma_n)$ and $\theta_2^r=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\gamma_k\cdots\delta_l\cdots\sigma_n)$, $\alpha_i,\beta_j,\gamma_k,\delta_l\in\{0,1\}$. For $\alpha_i\equiv\alpha$, $\beta_i\equiv\beta$, $\gamma_i\equiv\gamma$, $\delta_i\equiv\delta$, $\alpha,\beta,\gamma,\delta\in\{0,1\}$ we have:

$$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\bar\delta\\ \alpha\beta\gamma\delta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \bar\alpha\bar\beta-\delta\\ \bar\alpha-\gamma\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \bar\alpha\bar\beta-\delta\\ -\bar\beta\gamma\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \bar\alpha-\bar\gamma\delta\\ -\beta\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \bar\alpha-\bar\gamma\delta\\ \bar\alpha\beta-\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ -\bar\beta\bar\gamma\delta\\ \alpha\bar\beta-\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ -\bar\beta\bar\gamma\delta\\ \alpha-\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha-\gamma\bar\delta\\ -\beta\gamma\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\right.$$
$$\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha-\gamma\bar\delta\\ \bar\alpha\beta\gamma-\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ -\bar\beta\gamma\bar\delta\\ \alpha\bar\beta\gamma-\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ -\bar\beta\gamma\bar\delta\\ \alpha-\gamma\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha\bar\beta\gamma-\\ \bar\alpha-\gamma\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha\bar\beta\gamma-\\ -\bar\beta\gamma\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ -\beta\bar\gamma\bar\delta\\ \alpha\beta\bar\gamma-\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ -\beta\bar\gamma\bar\delta\\ \alpha\beta-\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta\bar\gamma-\\ \bar\alpha\beta-\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta\bar\gamma-\\ -\beta\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},$$
$$\left.\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta-\bar\delta\\ -\beta\gamma\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta-\bar\delta\\ \bar\alpha\beta\gamma-\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta\bar\gamma-\\ \alpha\bar\beta-\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta\bar\gamma-\\ \alpha-\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta-\bar\delta\\ \alpha-\gamma\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta-\bar\delta\\ \alpha\bar\beta\gamma-\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha-\bar\gamma\bar\delta\\ \alpha\beta\bar\gamma-\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha-\bar\gamma\bar\delta\\ \alpha\beta-\bar\delta\\ \alpha\beta\gamma-\end{pmatrix}\right\}. \tag{5}$$

So, for (5) $k_\theta^*/k_l^*=4/12$ indicates on an increase of power of transformed PSTF $Y^\oplus$ and the number of literals, as the initial interrelation $k_\theta/k_l=2/8$; $k_Y=24$.

In the case of necessity for any pair of conjuncterms of r-rank of a function f, that have distance $d>4$, one can in similar way form a set of $d!$ of transformed PSTF $Y^\oplus$; $d=1,2,\ldots,n$.

Based on the considered above, one can state that two conjuncterms of r-rank θ_1^r and θ_2^r function f, that differ $d=1,2,\ldots,n$ in different by values onename binary positions $\alpha,\beta,\gamma,\delta,\ldots\in\{0,1\}$, form in polynomial format a set with $k_Y=d!$ of transformed PSTF $Y^{\oplus}$, each of them consisting of different conjuncterms of $(r–1)$-rank with $k^*_\theta/k^*_l = d/d(d–1)$, that is the proof of Theorem 1. $\square$

Theorem 2. *Two conjuncterms of the function $f(x_1, x_2, \ldots, x_n)$, one of which of $(r–1)$-rank θ_1^{r-1} differs from another r-rank θ_2^r in the number of d different in values one-name positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, –\}$, among which the dash $(–)$ belongs to θ_1^{r-1}, $r \in \{1, 2, \ldots, n\}$, in polynomial set-theoretical format create $k_Y = (d-1)!$ of sets of transformed PSTF $Y^{\oplus}$, each of them has power $k^*_\theta = d$ and the total number of literals in different part $k^*_l = d(d-1) - (d-2)$, here:*

- *if* $d = 1$, *then* $\begin{pmatrix} - \\ \tilde{\alpha} \end{pmatrix} \overset{\oplus}{\Rightarrow} (\bar{\tilde{\alpha}});$ (6)

- *if* $d = 2$, *then* $\begin{pmatrix} \bar{\alpha}- \\ \alpha\tilde{\beta} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} -- \\ \alpha\bar{\tilde{\beta}} \end{pmatrix}, \quad \begin{pmatrix} -\bar{\beta} \\ \tilde{\alpha}\beta \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} -- \\ \bar{\tilde{\alpha}}\beta \end{pmatrix};$ (7)

- *if* $d = 3$, *then* $\begin{pmatrix} \bar{\alpha}\bar{\beta}- \\ \alpha\beta\tilde{\gamma} \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}-- \\ -\beta- \\ \alpha\beta\bar{\tilde{\gamma}} \end{pmatrix}, \begin{pmatrix} -\bar{\beta}- \\ \alpha-- \\ \alpha\beta\bar{\tilde{\gamma}} \end{pmatrix} \right\},$ (8)

$$\begin{pmatrix} \bar{\alpha}-\bar{\gamma} \\ \alpha\tilde{\beta}\gamma \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}-- \\ --\gamma \\ \alpha\bar{\tilde{\beta}}\gamma \end{pmatrix}, \begin{pmatrix} --\bar{\gamma} \\ \alpha-- \\ \alpha\bar{\tilde{\beta}}\gamma \end{pmatrix} \right\}, \quad \begin{pmatrix} -\bar{\beta}\bar{\gamma} \\ \tilde{\alpha}\beta\gamma \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} -\bar{\beta}- \\ --\gamma \\ \bar{\tilde{\alpha}}\beta\gamma \end{pmatrix}, \begin{pmatrix} --\bar{\gamma} \\ -\beta- \\ \bar{\tilde{\alpha}}\beta\gamma \end{pmatrix} \right\}, \quad (9),(10)$$

- *if* $d = 4$, *then*

$$\begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ \alpha\beta\gamma\tilde{\delta} \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}\bar{\beta}-- \\ \bar{\alpha}-\gamma- \\ -\beta\gamma- \\ \alpha\beta\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-- \\ -\bar{\beta}\gamma- \\ \alpha-\gamma- \\ \alpha\beta\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}- \\ \bar{\alpha}\beta-- \\ -\beta\gamma- \\ \alpha\beta\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}- \\ -\beta\bar{\gamma}- \\ \alpha\beta-- \\ \alpha\beta\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}- \\ \alpha\bar{\beta}-- \\ \alpha-\gamma- \\ \alpha\beta\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}- \\ \alpha-\bar{\gamma}- \\ \alpha\beta-- \\ \alpha\beta\gamma\bar{\tilde{\delta}} \end{pmatrix} \right\}, \quad (11)$$

$$\begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ \alpha\beta\tilde{\gamma}\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}\bar{\beta}-- \\ \bar{\alpha}--\delta \\ -\beta-\delta \\ \alpha\beta\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-- \\ -\bar{\beta}-\delta \\ \alpha--\delta \\ \alpha\beta\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}--\bar{\delta} \\ \bar{\alpha}\beta-- \\ -\beta-\delta \\ \alpha\beta\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}--\bar{\delta} \\ -\beta-\bar{\delta} \\ \alpha\beta-- \\ \alpha\beta\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}-\bar{\delta} \\ \alpha\bar{\beta}-- \\ \alpha--\delta \\ \alpha\beta\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}-\bar{\delta} \\ \alpha--\bar{\delta} \\ \alpha\beta-- \\ \alpha\beta\bar{\tilde{\gamma}}\delta \end{pmatrix} \right\}, \quad (12)$$

$$\begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ \alpha\tilde{\beta}\gamma\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}-\bar{\gamma}- \\ \bar{\alpha}--\delta \\ --\gamma\delta \\ \alpha\bar{\tilde{\beta}}\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}- \\ --\bar{\gamma}\delta \\ \alpha--\delta \\ \alpha\bar{\tilde{\beta}}\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}--\bar{\delta} \\ --\gamma\bar{\delta} \\ \alpha-\gamma- \\ \alpha\bar{\tilde{\beta}}\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}--\bar{\delta} \\ \bar{\alpha}-\gamma- \\ --\gamma\delta \\ \alpha\bar{\tilde{\beta}}\gamma\delta \end{pmatrix}, \begin{pmatrix} --\bar{\gamma}\bar{\delta} \\ \alpha-\bar{\gamma}- \\ \alpha--\delta \\ \alpha\bar{\tilde{\beta}}\gamma\delta \end{pmatrix}, \begin{pmatrix} --\bar{\gamma}\bar{\delta} \\ \alpha--\bar{\delta} \\ \alpha-\gamma- \\ \alpha\bar{\tilde{\beta}}\gamma\delta \end{pmatrix} \right\}, \quad (13)$$

$$\begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \tilde{\alpha}\beta\gamma\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} -\bar{\beta}\bar{\gamma}- \\ -\bar{\beta}-\bar{\delta} \\ --\bar{\gamma}\bar{\delta} \\ \bar{\tilde{\alpha}}\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}- \\ --\bar{\gamma}\delta \\ -\beta-\delta \\ \bar{\tilde{\alpha}}\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}-\bar{\delta} \\ -\bar{\beta}\gamma- \\ --\gamma\delta \\ \bar{\tilde{\alpha}}\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}-\bar{\delta} \\ --\gamma\bar{\delta} \\ -\beta\gamma- \\ \bar{\tilde{\alpha}}\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} --\bar{\gamma}\bar{\delta} \\ -\beta\bar{\gamma}- \\ -\beta-\delta \\ \bar{\tilde{\alpha}}\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} --\bar{\gamma}\bar{\delta} \\ -\beta-\bar{\delta} \\ -\beta\gamma- \\ \bar{\tilde{\alpha}}\beta\gamma\delta \end{pmatrix} \right\}, \quad (14)$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ are binary positions of any value 0 or 1.

Proof. In this case the given PSTF $Y^{\oplus}$ has interrelation $k_\theta/k_l = 2/(2d-1)$.
• Let $d = 1$. Then $\theta_1^{r-1} = (\sigma_1 \cdots -_i \cdots \sigma_n)$, $\theta_2^r = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \sigma_n)$, $\tilde{\alpha}_i \in \{0, 1\}$, and respectively for the expression $1 \oplus \tilde{a} = \bar{\tilde{a}}$, $\tilde{a} \in \{a, \bar{a}\}$, we can write down such PSTF

$Y^{\oplus}$:

$$Y^{\oplus} = \{(\sigma_1 \cdots -_i \cdots \sigma_n), (\sigma_1 \cdots \tilde{\alpha}_i \cdots \sigma_n)\}^{\oplus} = \{(\sigma_1 \cdots \bar{\tilde{\alpha}}_i \cdots \sigma_n)\}^{\oplus}.$$

As interrelation $k^*_\theta/k^*_l = 1/1$, then compared with $k_\theta/k_l = 2/1$ we have simplification of the given PSTF $Y^{\oplus}$ due to removal of one conjuncterm; $k_Y = 1$.

• Let $d = 2$. Then for $\theta_1^{r-1} = (\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \tilde{\beta}_j \cdots \sigma_n)$ and $\theta_1^{r-1} = (\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \beta_j \cdots \sigma_n)$, $\alpha_i, \beta_j \in \{0, 1\}$, we will obtain $Y^{\oplus} = \{(\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \tilde{\beta}_j \cdots \sigma_n)\}^{\oplus} = \{(\sigma_1 \cdots -_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \bar{\tilde{\beta}}_j \cdots \sigma_n)\}^{\oplus}$ and $Y^{\oplus} = \{(\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots \sigma_n), (\sigma_1 \cdots \tilde{\alpha}_i \cdots \beta_j \cdots \sigma_n)\}^{\oplus} = \{(\sigma_1 \cdots -_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots \bar{\tilde{\alpha}}_i \cdots \beta_j \cdots \sigma_n)\}^{\oplus}$.

Comparing the obtained interrelation $k^*_\theta/k^*_l = 2/2$ with $k_\theta/k_l = 2/3$, we see that transformed PSTF $Y^{\oplus}$ is simpler than the given PSTF $Y^{\oplus}$ for one literal. It should be noted, that a number of transformed PSTF $Y^{\oplus}$ is determined by a number of binary positions in different part θ_1^{r-1} and θ_2^r, that conforms to Theorem 1. Therefore, for $d = 2$ we have $k_Y = 1$.

• Let $d = 3$. Then for $\theta_1^{r-1} = (\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \tilde{\gamma}_k \cdots \sigma_n)$, $\theta_1^{r-1} = (\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \tilde{\beta}_j \cdots \gamma_k \cdots \sigma_n)$, $\theta_1^{r-1} = (\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots \bar{\gamma}_k \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \beta_j \cdots \gamma_k \cdots \sigma_n)$, we have:

$$Y^{\oplus} = \{(\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \tilde{\gamma}_k \cdots \sigma_n)\}^{\oplus} =$$
$$= \left\{ \begin{array}{c} (\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \beta_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \bar{\tilde{\gamma}}_k \cdots \sigma_n) \\ (\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \bar{\tilde{\gamma}}_k \cdots \sigma_n) \end{array} \right\}^{\oplus},$$

$$Y^{\oplus} = \{(\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \tilde{\beta}_j \cdots \gamma_k \cdots \sigma_n)\}^{\oplus} =$$
$$= \left\{ \begin{array}{c} (\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots -_j \cdots \gamma_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \bar{\tilde{\beta}}_j \cdots \gamma_k \cdots \sigma_n) \\ (\sigma_1 \cdots -_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \bar{\tilde{\beta}}_j \cdots \gamma_k \cdots \sigma_n) \end{array} \right\}^{\oplus},$$

$$Y^{\oplus} = \{(\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots \bar{\gamma}_k \cdots \sigma_n), (\sigma_1 \cdots \tilde{\alpha}_i \cdots \beta_j \cdots \gamma_k \cdots \sigma_n)\}^{\oplus} =$$
$$= \left\{ \begin{array}{c} (\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots -_j \cdots \gamma_k \cdots \sigma_n), (\sigma_1 \cdots \bar{\tilde{\alpha}}_i \cdots \beta_j \cdots \gamma_k \cdots \sigma_n) \\ (\sigma_1 \cdots -_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \beta_j \cdots {}_k \cdots \sigma_n), (\sigma_1 \cdots \bar{\tilde{\alpha}}_i \cdots \beta_j \cdots \gamma_k \cdots \sigma_n) \end{array} \right\}^{\oplus}.$$

The obtained $k^*_\theta/k^*_l = 3/5$ indicates on an increase by one conjuncterm, since $k_\theta/k_l = 2/5$; $k_Y = 2$.

• Let $d = 4$. Based on the considered above and taking into account the rule (4) of Theorem 1 for $d = 3$ (three positions are common), one can state that for

$$\theta_1^{r-1} = (\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\beta}_j \cdots \bar{\gamma}_k \cdots -_l \cdots \sigma_n) \text{ and } \theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \gamma_k \cdots \tilde{\delta}_l \cdots \sigma_n),$$
$$\theta_1^{r-1} = (\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\beta}_j \cdots -_k \cdots \bar{\delta}_l \cdots \sigma_n) \text{ and } \theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \tilde{\gamma}_k \cdots \delta_l \cdots \sigma_n),$$
$$\theta_1^{r-1} = (\sigma_1 \cdots \bar{\alpha}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \bar{\delta}_l \cdots \sigma_n) \text{ and } \theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \tilde{\beta}_j \cdots \gamma_k \cdots \delta_l \cdots \sigma_n),$$
$$\theta_1^{r-1} = (\sigma_1 \cdots -_i \cdots \bar{\beta}_j \cdots \bar{\gamma}_k \cdots \bar{\delta} \cdots \sigma_n) \text{ and } \theta_2^r = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \beta_j \cdots \gamma_k \cdots \delta_k \cdots \sigma_n),$$

the interrelation $k^*_\theta/k^*_l = 4/10$ which, compared with $k_\theta/k_l = 2/7$, indicates on an increase of a number of conjuncterms as well as their literals; $k_Y = 6$.

Thus, if a conjuncterm of $(r-1)$-rank θ_1^{r-1} differs from a conjuncterm of r-rank θ_2^r in the number d of different by value onename positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, -\}$, here dash (–) belongs to θ_1^{r-1}, $r \in \{1, 2, \ldots, n\}$, then $(d-1)!$ of sets transformed PSTF $Y^{\oplus}$ will be formed, each of which having the interrelation $k^*_\theta/k^*_l = d/(d(d-1) - (d-2))$, that is the proof of Theorem 2. □

Theorem 3. *Two conjuncterms of* $(r-1)$*-rank* θ_1^{r-1} *and* θ_2^{r-1}, $r \in \{1, 2, \ldots, n\}$, *of the function* $f(x_1, x_2, \ldots, x_n)$ *differ in* d *onename binary positions*

$\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, -\}$, where each conjuncterm has one (–), in polynomial set-theoretical format starting with $d = 2$, create $k_Y = (d-2)!$ of sets transformed PSTF $Y^{\oplus}$, each of which has power $k^*_\theta = d$ and the total number of literals in the different part $k^*_l = d(d-1) - 2(d-2)$ and :

- *if* $d=2$, *then* $\begin{pmatrix} \tilde{\alpha}- \\ -\tilde{\beta} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \bar{\tilde{\alpha}}- \\ -\bar{\tilde{\beta}} \end{pmatrix}$, (15)

- *if* $d=3$, *then* $\begin{pmatrix} \tilde{\alpha}\bar{\beta}- \\ -\beta\tilde{\gamma} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} --- \\ \bar{\tilde{\alpha}}\bar{\beta}- \\ -\beta\bar{\tilde{\gamma}} \end{pmatrix}$, $\begin{pmatrix} \bar{\alpha}\tilde{\beta}- \\ \alpha-\tilde{\gamma} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} --- \\ \bar{\alpha}\bar{\tilde{\beta}}- \\ \alpha-\bar{\tilde{\gamma}} \end{pmatrix}$, $\begin{pmatrix} \tilde{\alpha}-\bar{\gamma} \\ -\tilde{\beta}\gamma \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} --- \\ \bar{\tilde{\alpha}}-\bar{\gamma} \\ -\bar{\tilde{\beta}}\gamma \end{pmatrix}$,

(16),(17),(18)

- *if* $d=4$, *then* $\begin{pmatrix} \tilde{\alpha}\bar{\beta}\bar{\gamma}- \\ -\beta\gamma\tilde{\delta} \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} -\bar{\beta}-- \\ --\gamma- \\ \bar{\tilde{\alpha}}\bar{\beta}\bar{\gamma}- \\ -\beta\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} -\beta-- \\ --\bar{\gamma}- \\ \bar{\tilde{\alpha}}\bar{\beta}\bar{\gamma}- \\ -\beta\gamma\bar{\tilde{\delta}} \end{pmatrix} \right\}$, $\begin{pmatrix} \bar{\alpha}\tilde{\beta}\bar{\gamma}- \\ \alpha-\gamma\tilde{\delta} \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}--- \\ --\gamma- \\ \bar{\alpha}\bar{\tilde{\beta}}\bar{\gamma}- \\ \alpha-\gamma\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} \alpha--- \\ --\bar{\gamma}- \\ \bar{\alpha}\bar{\tilde{\beta}}\bar{\gamma}- \\ \alpha-\gamma\bar{\tilde{\delta}} \end{pmatrix} \right\}$,

(19),(20)

$$\begin{pmatrix} \bar{\alpha}\bar{\beta}\tilde{\gamma}- \\ \alpha\beta-\tilde{\delta} \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}--- \\ -\beta-- \\ \bar{\alpha}\bar{\beta}\bar{\tilde{\gamma}}- \\ \alpha\beta-\bar{\tilde{\delta}} \end{pmatrix}, \begin{pmatrix} \alpha--- \\ -\bar{\beta}-- \\ \bar{\alpha}\bar{\beta}\bar{\tilde{\gamma}}- \\ \alpha\beta-\bar{\tilde{\delta}} \end{pmatrix} \right\}, \quad \begin{pmatrix} \tilde{\alpha}\bar{\beta}-\bar{\delta} \\ -\beta\bar{\gamma}\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} -\bar{\beta}-- \\ ---\delta \\ \bar{\tilde{\alpha}}\bar{\beta}-\bar{\delta} \\ -\beta\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} -\beta-- \\ ---\bar{\delta} \\ \bar{\tilde{\alpha}}\bar{\beta}-\bar{\delta} \\ -\beta\bar{\tilde{\gamma}}\delta \end{pmatrix} \right\}, \quad (21),(22)$$

$$\begin{pmatrix} \bar{\alpha}\tilde{\beta}-\bar{\delta} \\ \alpha-\tilde{\gamma}\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}--- \\ ---\delta \\ \bar{\alpha}\bar{\tilde{\beta}}-\bar{\delta} \\ \alpha-\bar{\tilde{\gamma}}\delta \end{pmatrix}, \begin{pmatrix} \alpha--- \\ ---\bar{\delta} \\ \bar{\alpha}\bar{\tilde{\beta}}-\bar{\delta} \\ \alpha-\bar{\tilde{\gamma}}\delta \end{pmatrix} \right\}, \quad \begin{pmatrix} \tilde{\alpha}-\bar{\gamma}\bar{\delta} \\ -\tilde{\beta}\gamma\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} --\bar{\gamma}- \\ ---\delta \\ \bar{\tilde{\alpha}}-\bar{\gamma}\bar{\delta} \\ -\bar{\tilde{\beta}}\gamma\delta \end{pmatrix}, \begin{pmatrix} --\gamma- \\ ---\bar{\delta} \\ \bar{\tilde{\alpha}}-\bar{\gamma}\bar{\delta} \\ -\bar{\tilde{\beta}}\gamma\delta \end{pmatrix} \right\}, \quad (23),(24)$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ are binary positions of any value 0 or 1.

Proof. Given PSTF $Y^{\oplus}$ has the initial interrelation $k_\theta/k_l = 2/2(d-1)$.
- Let $d = 2$. Then $\theta^r_1 = (\sigma_1 \cdots \tilde{\alpha}_i \cdots -_j \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots -_i \cdots \tilde{\beta}_j \cdots \sigma_n)$, $\tilde{\alpha}_i, \tilde{\beta}_j \in \{0, 1\}$. For $f(a, b)$ respectively to (15) we have $\tilde{a} \oplus \tilde{b} = (\bar{\tilde{a}} \oplus 1) \oplus (\bar{\tilde{b}} \oplus 1) = = \bar{\tilde{a}} \oplus \bar{\tilde{b}}$, that corresponds to PSTF

$$Y^{\oplus} = \{(\sigma_1 \cdots \tilde{\alpha}_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \tilde{\beta}_j \cdots \sigma_n)\}^{\oplus} = \{(\sigma_1 \cdots \bar{\tilde{\alpha}}_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \bar{\tilde{\beta}}_j \cdots \sigma_n)\}^{\oplus}.$$

Here the interrelation $k_\theta/k_l = k^*_\theta/k^*_l = 2/2$ that indicates on unchangeability of parameters of the transformed PSTF $Y^{\oplus}$, in which only inversion of different positions took place; $k_Y = 1$.
- Let $d = 3$. Then $\theta^r_1 = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots -_i \cdots \beta_j \cdots \tilde{\gamma}_k \cdots \sigma_n)$, $\theta^r_1 = (\sigma_1 \cdots \bar{\alpha}_i \cdots \tilde{\beta}_j \cdots -_k \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \tilde{\gamma}_k \cdots \sigma_n)$, and $\theta^r_1 = (\sigma_1 \cdots \tilde{\alpha}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots -_i \cdots \tilde{\beta}_j \cdots \gamma_k \cdots \sigma_n)$, $\tilde{\alpha}_i, \tilde{\beta}_j, \tilde{\gamma}_k \in \{0, 1\}$.

So, transformed PSTF $Y^{\oplus}$ will look like:

$$Y^{\oplus} = \{(\sigma_1 \cdots \tilde{\alpha}_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \beta_j \cdots \tilde{\gamma}_k \cdots \sigma_n)\}^{\oplus} =$$

$$= \{(\sigma_1 \cdots -_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \bar{\tilde{\alpha}}_i \cdots \bar{\beta}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \beta_j \cdots \bar{\tilde{\gamma}}_k \cdots \sigma_n)\}^{\oplus},$$

$$Y^{\oplus} = \{(\sigma_1 \cdots \bar{\alpha}_i \cdots \tilde{\beta}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \tilde{\gamma}_k \cdots \sigma_n)\}^{\oplus} =$$

$$= \{(\sigma_1 \cdots -_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\bar{\tilde{\beta}}}_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \bar{\bar{\tilde{\gamma}}}_k \cdots \sigma_n)\}^{\oplus},$$

$$Y^{\oplus} = \{(\sigma_1 \cdots \tilde{\alpha}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \tilde{\beta}_j \cdots \gamma_k \cdots \sigma_n)\}^{\oplus} =$$

$$= \{(\sigma_1 \cdots -_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \bar{\bar{\tilde{\alpha}}}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \bar{\bar{\tilde{\beta}}}_j \cdots \gamma_k \cdots \sigma_n)\}^{\oplus}.$$

Compared with $k_\theta/k_l = 2/4$ here $k^*_\theta/k^*_l = 3/4$ means that the transformed PSTF $Y^{\oplus}$ has one more conjuncterm, here its rank is $(r-3)$; $k_Y = 1$.

• Let $d = 4$. Then $\theta^r_1 = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \bar{\beta}_j \cdots \bar{\gamma}_k \cdots -_l \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots -_i \cdots \beta_j \cdots \gamma_k \cdots \tilde{\delta}_l \cdots \sigma_n)$,

$\theta^r_1 = (\sigma_1 \cdots \bar{\alpha}_i \cdots \tilde{\beta}_j \cdots \bar{\gamma}_k \cdots -_l \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \gamma_k \cdots \tilde{\delta}_l \cdots \sigma_n)$,

$\theta^r_1 = (\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\beta}_j \cdots \tilde{\gamma}_k \cdots -_l \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots -_k \cdots \tilde{\delta}_l \cdots \sigma_n)$,

$\theta^r_1 = (\sigma_1 \cdots \tilde{\alpha}_i \cdots \bar{\beta}_j \cdots -_k \cdots \bar{\delta}_l \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots -_i \cdots \beta_j \cdots \tilde{\gamma}_k \cdots \delta_l \cdots \sigma_n)$,

$\theta^r_1 = (\sigma_1 \cdots \bar{\alpha}_i \cdots \tilde{\beta}_j \cdots -_k \cdots \bar{\delta}_l \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \tilde{\gamma}_k \cdots \delta_l \cdots \sigma_n)$,

$\theta^r_1 = (\sigma_1 \cdots \tilde{\alpha}_i \cdots -_j \cdots \bar{\gamma}_k \cdots \bar{\delta}_l \cdots \sigma_n)$ and $\theta^r_2 = (\sigma_1 \cdots -_i \cdots \tilde{\beta}_j \cdots \gamma_k \cdots \delta_l \cdots \sigma_n)$.

Here, for $d = 4$ the interrelation $k^*_\theta/k^*_l = 4/8$ is greater than the initial one $k_\theta/k_l = 2/6$; $k_Y = 2$.

So, if two conjuncterms of $(r-1)$-rank θ^{r-1}_1 and θ^{r-1}_2, $r \in \{1, 2, \ldots, n\}$, of the function $f(x_1, x_2, \ldots, x_n)$ differ in d different by values onename positions $\alpha, \beta, \gamma, \delta, \ldots \in \{0, 1, -\}$, among which each of these conjuncterms has one dash (–), then in polynomial set-theoretical format, starting with $d = 2$, they form $k_Y = (d-2)!$ of the sets transformed PSTF $Y^{\oplus}$, each of them has interrelation $k^*_\theta/k^*_l = d/(d(d-1) - 2(d-2))$, that is the proof of Theorem 3. □

Example 1. To apply Theorems 1, 2 and 3 to the function $f(x_1, x_2, x_3, x_4)$, that is given by perfect STF $Y^1 = \{0, 3, 5, 6, 7, 8, 9, 10, 12, 15\}^1$, which is minimized in polynomial format by *K*-maps method to the expression $f = x_1 \oplus x_2x_3 \oplus x_2x_4 \oplus x_3x_4 \oplus x_1x_2x_3x_4 \oplus \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$ [32, p. 97].

Solution. This function has PSTF $Y^{\oplus} = \{$(1–––),(–11–),(–1–1),(––11),(1111),(0000)$\}^{\oplus}$. To the pair (1111) and (0000), that has $d = 4$, we will apply, for example, the fourth PSTF from the rule (5):

$$Y^{\oplus} = \left\{ (1\text{–––}), (\text{–}11\text{–}), (\underline{\text{–}1\text{–}1}), (\underline{\underline{\text{––}11}}), \begin{pmatrix} 000\text{–} \\ 0\text{–}01 \\ \underline{01\text{–}1} \\ \underline{\underline{\text{–}111}} \end{pmatrix} \right\}^{\oplus}.$$

Applying the rule (6) of Theorem 2 to the underlined pairs that have $d = 1$, namely $\begin{pmatrix} \text{–}1\text{–}1 \\ 01\text{–}1 \end{pmatrix} \overset{\oplus}{\Rightarrow} (11\text{–}1)$, $\begin{pmatrix} \text{––}11 \\ \text{–}111 \end{pmatrix} \overset{\oplus}{\Rightarrow} (\text{–}011)$, and the rule (7), in the formed set namely $\begin{pmatrix} \text{–}011 \\ \text{–}11\text{–} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \text{––}1\text{–} \\ \text{–}010 \end{pmatrix}$, we will obtain PSTF $Y^{\oplus} = \{(1\text{–––}), (\text{––}1\text{–}), (\underline{\text{–}010}), (\underline{000\text{–}}), (11\text{–}1), (0\text{–}01)\}^{\oplus}$. Doing further transformations and according to the rules (16) and (17) of Theorem 3, namely $\begin{pmatrix} \text{–}010 \\ 000\text{–} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \text{–}0\text{––} \\ 100\text{–} \\ \text{–}011 \end{pmatrix}$, $\begin{pmatrix} 100\text{–} \\ 0\text{–}01 \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \text{––}0\text{–} \\ 0\text{–}00 \\ 110\text{–} \end{pmatrix}$, we'll obtain the final minimal PSTF $Y^{\oplus} = \{(1\text{–––}), (\text{––}1\text{–}), (\text{–}0\text{––}), (\text{––}0\text{–}), (\text{–}011), (0\text{–}00), (110\text{–}), (11\text{–}1)\}^{\oplus} \Rightarrow$

$\Rightarrow$ {(1−−−),(−1−−),(−011),(0−00),(110−),(11−1)}$^{\oplus}$. Here the cost of realization of the minimized function $f = x_1 \oplus x_2 \oplus \bar{x}_2 x_3 x_4 \oplus \bar{x}_1 \bar{x}_3 \bar{x}_4 \oplus x_1 x_2 \bar{x}_3 \oplus x_1 x_2 x_4$ is equal to $k^*_\theta / k^*_l = 6/14$ that is a better result if compared to [32], where $k_\theta / k_l = 6/15$.

4 Minimization of Complete and Incomplete Functions

The proposed method of Boolean functions minimization in the polynomial set-theoretical format is based on the idea of minterms splitting of a given function $f(x_1, x_2, \ldots, x_n)$ in the disjunctive format [27–29].

The algorithm of minimization of a function f in the polynomial set-theoretical format is realized on two stages:

1-st stage: the procedure of splitting of minterms of a given function f and the obtaining of a set of covering of a matrix of splitting;

2-nd stage: the procedure of iterative simplification of conjuncterms of a set of covering (obtained on the 1-st stage) on the basis of generalized rules of Theorems 1, 2 and 3 and formation of a minimal PSTF $Y^{\oplus}$ of a given function f.

The 1-st stage is realized by sequence of such steps:

Step 1: the given binary minterms $m_1, m_2, \ldots m_k$ of the perfect PSTF $Y^{\oplus} = \{m_1, m_2, \ldots m_k\}^{\oplus}$ of the function f are split (operator $\overset{S}{\Rightarrow}$) by using the matrix-column of the masks of literals of rank $r \geq n - \log_2 k$, $r = 1, 2, \ldots, n$, as a result of this a matrix of splitting M^r_n of $C^r_n \times k$ dimension is formed, where $C^r_n = \frac{n!}{(n-r)!r!}$; for example, let $n = 5$; if the number k of minterms is $8 \leq k < 16$, then we use the matrix of masks of rank $r = 2$, and as a result the matrix M^2_5 of the dimension $C^2_5 \times k$ is formed;

Step 2: in the matrix M^r_n (in our example M^2_5) for execution of the procedure of covering (operator $\overset{C}{\Rightarrow}$) the conjuncterms-copies of r-rank, the number of which $2^{n-r-1} < k_r \leq 2^{n-r}$ ($4 < k_r \leq 8$) are highlighted by underlining; priority is given to conjuncterms-copies and their number is $k_r = 2^{n-r}$ ($k_r = 8$); if $k = k_r$, then the matrix is covered with a conjuncterms-copy of r-rank; if $k_r < 2^{n-r}$ ($k_r < 8$), then the covering of the matrix will be made of the conjuncterms-copies the number of which $2^{n-r-1} < k_r < 2^{n-r}$, and if there are not enough of them, then together with generating minterms of the matrix M^r_n; if $k_r < 2^{n-r-1}$, then the transition to step 1 is done for realization of similar procedures with application of the matrix of masks of the rank $r = 3$ and etc. until to getting in the covering of the matrix M^r_n of the minterms splitting of which provides its covering, if such minterms > 2, then the transition to step 1 is done.

The 1-st stage of algorithm is completed when there are not only minterms in the set of covering of the matrix M^r_n or when the split elements do not provide its covering.

The 2-nd stage of the minimization algorithm is the procedure of iterative simplification. It is done with the conjuncterms of the set of the covering in sequence of the following steps:

Step 1: for every pair with $d = 1$ (pairs with $d = 0$ are not taken into account) either the rule (2) of Theorem 1, or the rule (6) of Theorem 2; are applied; after respective replacement the transition to the 1-st is done, if there are not such pairs, then go to the step 2;

Step 2: for every pair with $d = 2$ we apply either one from the sets of the rule (3) of Theorem 1, or the rule (7) of Theorem 2; after respective replacement the transition to the 1-st step is done and if there are not such pairs, then to the 3-rd step;

Step 3: for every pair with $d = 3$ we apply either one from the sets of the rule (4) of Theorem 1, or one from the sets of the rules (8), (9) or (10) of Theorem 2, or one from the rules (15), (16) or (17) of Theorem 3; after respective replacement the transition to the 1-st step is done and if there are not such pairs, then go to the 4-th step;

Step 4: for every pair with $d = 4$ we apply one from the sets of the rule (5) of Theorem 1, or one from the rules (8), (9) or (10) of Theorem 2, or one from the sets of the rules (18)–(23) of Theorem 3; after respective replacement the transition to the 1-st step is done and if there are not such pairs, then go to the 5-th step;

Step 5: if further transformation does not lead to the simplification of the set of conjuncterms, then this set is the found minimal PSTF $Y^{\oplus}$ of the function f, the cost of realization of which is determined by the interrelation k_{θ}^{*}/k_{l}^{*}.

Example 2. To minimize the function $f(x_1, x_2, x_3, x_4)$ in the polynomial format by using the splitting method. This function has perfect STF $Y^1 = \{0, 6, 14, 15\}^1$ (*this function is borrowed from* [21, p. 28]).

Solution.

$$Y^{\oplus} = \{(0000),(0110),(1110),(1111)\}^{\oplus} \overset{S}{\Rightarrow} \begin{bmatrix} ll{-}{-} \\ l{-}l{-} \\ l{-}{-}l \\ {-}ll{-} \\ {-}l{-}l \\ {-}{-}ll \end{bmatrix} = \begin{bmatrix} 00{-}{-} & 01{-}{-} & 11{-}{-} & 11{-}{-} \\ 0{-}0{-} & 0{-}1{-} & 1{-}1{-} & 1{-}1{-} \\ 0{-}{-}0 & 0{-}{-}0 & 1{-}{-}0 & 1{-}{-}1 \\ {-}00{-} & \underline{{-}11{-}} & \underline{{-}11{-}} & \underline{{-}11{-}} \\ {-}0{-}0 & {-}1{-}0 & {-}1{-}0 & {-}1{-}1 \\ {-}{-}00 & {-}{-}10 & {-}{-}10 & {-}{-}11 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow} \{{-}ll{-}\} = \big\{\big(({-}11{-}),(0111)\big),(0000)\big\}^{\oplus}.$$

We apply the rule (4) of Theorem 1 to the minterms (0000) and (0111):

$$\begin{pmatrix} 0000 \\ 0111 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 000{-} \\ 00{-}1 \\ 0{-}11 \end{pmatrix}, \begin{pmatrix} 000{-} \\ 01{-}1 \\ 0{-}01 \end{pmatrix}, \begin{pmatrix} 001{-} \\ 00{-}0 \\ 0{-}11 \end{pmatrix}, \underline{\begin{pmatrix} 011{-} \\ 00{-}0 \\ 0{-}10 \end{pmatrix}}, \begin{pmatrix} 010{-} \\ 01{-}1 \\ 0{-}00 \end{pmatrix}, \underline{\begin{pmatrix} 011{-} \\ 01{-}0 \\ 0{-}00 \end{pmatrix}} \right\}^{\oplus}.$$

After replacement of minterms (0000) and (0111) by the underlined sets in the set of covering, we obtain two equal as to the realization cost of solutions of minimization of the given function which is reflected by the minimal PSTF:

$$Y^{\oplus} = \{({-}11{-}),(0000),(0111)\}^{\oplus} \Rightarrow \left\{ (\underline{{-}11{-}}),(\underline{011{-}}), \begin{cases} 1.\ (00{-}0),(0{-}10) \\ 2.\ (01{-}0),(0{-}00) \end{cases} \right\}^{\oplus} \Rightarrow$$

$$\Rightarrow \left\{ (111{-}), \begin{cases} 1.\ (00{-}0),(0{-}10) \\ 2.\ (01{-}0),(0{-}00) \end{cases} \right\}^{\oplus}.$$

Answer. The cost of realization of the minimized function determines the interrelation $k_{\theta}^{*}/k_{l}^{*} = 3/9$ that is a better result than in [21], where the PSTF $Y^{\oplus} = \{(-11-), (0000), (0111)\}^{\oplus}$ that is equal to 3/10.

In [27, 28], the incomplete function $f(x_1, x_2, \ldots, x_n)$ can be given by the perfect STF $\{Y^1, Y^{\sim}\}$, where Y^1 and $Y^{\sim}$ there are subsets of the complete set $\mathbf{E}_2^n$, on which the function f takes the value respectively 1 and $\sim$ (so-called “don’t-care”). In the polynomial set-theoretical format the sets $Y^{\oplus}$ and $Y^{\tilde{\oplus}}$ correspond to the sets Y^1 and $Y^{\sim}$, the elements of which are binary minterms. Thus, incomplete function f can be given by the perfect PSTF $\{Y^{\oplus}, Y^{\tilde{\oplus}}\}$.

Similarly, [27, 29] the procedure of splitting of conjuncterms of incomplete function f is realized by the matrix of splitting M_n^r, which is designated as $Mr^{\oplus}\vdots Mr^{\tilde{\oplus}}$, where $Mr^{\oplus}$ that is basic submatrix, $Mr^{\tilde{\oplus}}$ that is additional submatrix, and $\vdots$ that is a symbol of separation of the matrix M_n^r. As a result of covering of the matrix M_n^r a set of the splitting conjuncterms $Y^{\oplus}\vdots Y^{\tilde{\oplus}}$ is be obtained.

An algorithm of minimization of an incomplete function in the polynomial set-theoretical format is realized in the same way as for a complete function in two stages. On the 1-st stage the minterms of the perfect PSTF $\{Y^{\oplus}, Y^{\tilde{\oplus}}\}$ are split by using of the matrix M_n^r, where the main role in its cover is played by the conjuncterms-copies of the basic submatrix $Mr^{\oplus}$. Whereas the 2-nd stage of the algorithm of minimization of an incomplete function is realized in similar way as for a complete function.

Example 3. To minimize incomplete function $f(x_1, x_2, x_3, x_4)$ in the polynomial format by using splitting method. This function is has perfect STF $\begin{cases} Y^1 = \{3, 5, 6, 9, 12, 15\}^1 \\ Y^{\sim} = \{1, 2, 8, 11\}^{\sim} \end{cases}$ *(this function is borrowed from* [33, p. 460]).

Solution.

$$\begin{cases} Y^{\oplus} = \{(0011), (0101), (0110), (1001), (1100), (1111)\}^{\oplus} \\ Y^{\tilde{\oplus}} = \{(0001), (0010), (1000), (1011)\}^{\tilde{\oplus}} \end{cases} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow} \begin{bmatrix} l\,l\text{––} \\ l\text{–}l\text{–} \\ l\text{––}l \\ \text{–}ll\text{–} \\ \text{–}l\text{–}l \\ \text{––}ll \end{bmatrix} = \begin{bmatrix} \text{00––} & \text{01––} & \text{01––} & \text{10––} & \text{11––} & \text{11––} & \text{00––} & \text{00––} & \text{10––} & \text{10––} \\ \underline{\mathbf{0–1–}} & \text{0–0–} & \underline{\mathbf{0–1–}} & \underline{\mathbf{1–0–}} & \underline{\mathbf{1–0–}} & \text{1–1–} & \text{0–0–} & \underline{\mathbf{0–1–}} & \underline{\mathbf{1–0–}} & \text{1–1–} \\ \underline{\text{0––1}} & \underline{\text{0––1}} & \text{0––0} & \underline{\text{1––1}} & \text{1––0} & \underline{\text{1––1}} & \underline{\text{0––1}} & \text{0––0} & \text{1––0} & \underline{\text{1––1}} \\ \text{–01–} & \underline{\text{–00–}} & \text{–11–} & \underline{\text{–00–}} & \text{–10–} & \text{–11–} & \underline{\text{–00–}} & \text{–01–} & \underline{\text{–00–}} & \text{–01–} \\ \underline{\text{–0–1}} & \text{–1–1} & \text{–1–0} & \underline{\text{–0–1}} & \text{–1–0} & \text{–1–1} & \underline{\text{–0–1}} & \text{–0–0} & \text{–0–0} & \underline{\text{–0–1}} \\ \underline{\underline{\text{––11}}} & \underline{\underline{\text{––01}}} & \text{––10} & \underline{\underline{\text{––01}}} & \text{––00} & \underline{\underline{\text{––11}}} & \underline{\underline{\text{––01}}} & \text{––10} & \text{––00} & \underline{\underline{\text{––11}}} \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow} \{l\text{–}l\text{–}\} = \Big\{\big((\text{0–1–}), (\underline{0111})\big), \big((\text{1–0–}), (\underline{\underline{1101}})\big), (\underline{\underline{0101}}), (\underline{1111}) \vdots (0001), (1011)\Big\}^{\oplus} \Rightarrow$$

$$\Rightarrow \{(\text{0–1–}), (\underline{\text{–111}}), (\text{1–0–}), (\underline{\text{–101}})\}^{\oplus} \Rightarrow \{(\underline{\text{0–1–}}), (\text{–1–1}), (\underline{\text{1–0–}})\}^{\oplus}.$$

After the transformation of the pair of highlighted conjunñterms by the rule (3) of the Theorem 1, i. e. $\begin{pmatrix} \text{0–1–} \\ \text{1–0–} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \text{––1–} \\ \text{1–––} \end{pmatrix}$, we will obtain the final minimal PSTF $Y^{\oplus} = \{(\text{––1–}), (\text{1–––}), (\text{–1–1})\}^{\oplus}$, to which coresponds the minimal PSTF $Y^{\oplus} = \{(2, 3, 6, 7, 10, 11, 14, 15), (8, 9, 10, 11, 12, 13, 14, 15), (5, 7, 13, 15)\}^{\oplus} = \{\mathbf{2}, 3, 5, 6, \mathbf{8}, 9, 12, 15\}^{\oplus}$, where the highlighted in bold font elements belong to set $Y^{\sim}$.

Answer. The cost of realization of the given function is equal to k_θ^*/k_l^* = 3/4. If compared with [33] it is a better result, where $Y^{\oplus} = \{(\text{–11–}), (\text{11––}), (\text{–––1})\}^{\oplus}$ and the cost of realization is equal to $k_\theta^*/k_l^* = 3/5$.

5 Minimization of System of Complete and Incomplete Functions

In general, the system $F(X)$, $X = \{x_1, x_2, \ldots, x_n\}$, of Boolean functions, $f_i(X)$, $i = 1, 2, \ldots, s$ can be represented in the polynomial set-theoretical format as a perfect PSTF $\{Y_i^{\oplus}, Y_i^{\oplus *}\}$ [27, 29]:

$$F(X) = \begin{cases} Y_1^{\oplus}=\{m_{11}, m_{12}, \ldots, m_{1k_1}\}^{\oplus}, Y_1^{\oplus *}=\{m_{k_1+1}, m_{k_1+2}, \ldots, m_{2^n-k_1-\nu_1}\}^{\oplus *}, \\ Y_2^{\oplus}=\{m_{21}, m_{22}, \ldots, m_{2k_2}\}^{\oplus}, Y_2^{\oplus *}=\{m_{k_2+1}, m_{k_2+2}, \ldots, m_{2^n-k_2-\nu_2}\}^{\oplus *}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_s^{\oplus}=\{m_{s1}, m_{s2}, \ldots, m_{sk_s}\}^{\oplus}, Y_s^{\oplus *}=\{m_{k_s+1}, m_{k_s+2}, \ldots, m_{2^n-k_s-\nu_s}\}^{\oplus *}, \end{cases} \tag{25}$$

where $\nu_i < 2^n - k_i$; m_{ij} are binary minterms of functions $f_i(X)$, $j = 1, 2, \ldots, k_i$; while if $F(X)$ it is a system of the completely specified functions, then $Y_i^{\oplus *} \equiv \varnothing$ and we have perfect PSTF $\{Y_i^{\oplus}\}$, and if $F(X)$ is a system of the incompletely specified functions, then $Y_i^{\oplus *} \equiv Y_i^{\tilde{\oplus}}$ and we have perfect PSTF, where symbol $\sim$ represents incomplete ("don't care") values of functions f_i of system $F(X)$.

Similarly as in SOP form [27, 29] compatible minimization of the system of PSTF $\{Y_i^{\oplus}, Y_i^{\oplus *}\}$ (25) is performed by splitting method with the *system minterms* $(m)_{1,2,\ldots,s'}$, $s' \in \{1, 2, \ldots, s\}$, formed from minterms of the system $F(X)$.

The algorithm of compatible minimization of the system $F(X)$ of complete functions given by the perfect PSTF $\{Y_i^{\oplus}\}$, $Y_i^* \equiv \varnothing$, is realized in the following way. On the first stage the system minterms $(m)_{1,2,\ldots,s'}$ of the set $\{Y_I^{\oplus}\}$, $I \in \{1, 2, \ldots, s\}$, are split by using the matrix M_n^r creating a system conjuncterms of $(n-1)$-rank $(\theta_i^{n-1})_{1,2,\ldots,s'}$. The minimal covering of the matrix M_n^r is done in similar way [27, 29] by the identical system conjuncterms-copies of (n–1)-rank. But among them, a decisive role for realization of compatible minimization of the given system will be played by those ones the indices of which contain the greatest quantity of numbers with the set $\{1, 2, \ldots, s\}$. Herewith, if $(\theta_i^{r-1})_{1,2,\ldots,s'}$ and $(\theta_j^{r-1})_{1,2,\ldots,s''}$, $s', s'' \in \{1, 2, \ldots, s\}$, these are identical system conjuncterms-copies of $(r-1)$-rank of the matrix M_n^r, $r = 1, 2, \ldots, n-1$, then they can be elements of its covering if the indices of their generative form $(\theta_i^r)_{1,2,\ldots,s'}$ and $(\theta_j^r)_{1,2,\ldots,s''}$ will form a not empty intersection, i. e. $\{1, 2, \ldots, s'\} \bigcap \{1, 2, \ldots, s''\} \neq \varnothing$. For example, let the system minterms $(100)_{1,2,4}$, $(110)_{1,3}$, $(010)_{1,2,3}$ be generative elements of the matrix M_n^{n-1}. For the mask {l–l} the identical system conjuncterms-copies will be $(1\text{–}0)_1$ and $(1\text{–}0)_1$, the index of which determines the intersection $\{1, 2, 4\} \bigcap \{1, 3\} = \{1\}$, and for the mask {–$ll$} will be $(\text{–}10)_{1,3}$ and $(-10)_{1,3}$ because $\{1, 3\} \bigcap \{1, 2, 3\} = \{1, 3\}$. So, in this case for covering of the matrix M_n^{n-1} it is recommended to choose the pair of the mask {–ll} because the power $|\{1, 3\}| > |\{1\}|$.

Example 4. In the polynomial set-theoretical format to minimize the system $F(X)$ of complete functions $f_i(x_1, x_2, x_3)$, $i = 1, 2, 3$, by the splitting method. This has perfect STF $\begin{cases} Y_1^1 = \{(000), (010), (101), (110)\}^1 \\ Y_2^1 = \{(001), (011), (101)\}^1 \\ Y_3^1 = \{(000), (001), (010), (011)\}^1 \end{cases}$. This example is borrowed from [21, p. 35] where the author illustrates efficiency of *xlinking* method.

Solution. Having transformed the given system of the perfect STF $\{Y^1_{1,2,3}\}$ into the system of the perfect PSTF $\{Y^{\oplus}_{1,2,3}\}$ and, having formed from it a set of system minterms, we will execute the splitting procedure by using the matrix M^1_3 and its covering procedure:

$$Y^{\oplus}_{1,2,3} = \left\{(000)_{1,3}, (001)_{2,3}, (010)_{1,3}, (011)_{2,3}, (101)_{1,2}, (110)_1\right\}^{\oplus} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} l{-}{-} \\ {-}l{-} \\ {-}{-}l \end{bmatrix} = \begin{bmatrix} \underline{0{-}{-}}_{1,3} & \underline{0{-}{-}}_{2,3} & \underline{0{-}{-}}_{1,3} & \underline{0{-}{-}}_{2,3} & 1{-}{-}_{1,2} & 1{-}{-}_1 \\ {-}0{-} & {-}0{-} & {-}1{-} & {-}1{-} & {-}0{-} & {-}1{-} \\ {-}{-}0 & {-}{-}1 & {-}{-}0 & {-}{-}1 & {-}{-}1 & {-}{-}0 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\Big\{\big((0{-}{-})_{1,3}, (001)_{1,3}, (011)_{1,3}\big), \big((0{-}{-})_{2,3}, (000)_{2,3}, (010)_{2,3}\big), (101)_{1,2}, (110)_1\Big\}^{\oplus}.$$

We will do the splitting procedure with system minterms of the formed set by using the matrix M^2_3 and its covering procedure:

$$(001)_{1,3}, (011)_{1,3}, (000)_{2,3}, (010)_{2,3}, (101)_{1,2}, (110)_1 \overset{S}{\Rightarrow} \begin{bmatrix} ll{-} \\ l{-}l \\ {-}ll \end{bmatrix} =$$

$$= \begin{bmatrix} 00{-} & 01{-} & 00{-} & 01{-} & 10{-} & 11{-} \\ \underline{0{-}1}_{1,3} & \underline{0{-}1}_{1,3} & \underline{0{-}0}_{2,3} & \underline{0{-}0}_{2,3} & 1{-}1 & 1{-}0 \\ {-}01 & {-}11 & {-}00 & {-}10 & {-}01 & {-}10 \end{bmatrix} \overset{C}{\Rightarrow} \{(0{-}1)_{1,3}, (0{-}0)_{2,3}, (101)_{1,2}, (110)_1\}^{\oplus}.$$

Having distributed system conjuncterms in the functions we obtain the system of the PSTF $\{Y^{\oplus}_{1,2,3}\}$, with the underlined elements of which we will do step by step the transformations according to the rules (2), (3) and (7) of the theorems:

$$\begin{cases} Y^{\oplus}_1 = \{(\underline{0{-}{-}}), (\underline{0{-}1}), (\underline{\underline{101}}), (\underline{\underline{110}})\}^{\oplus} = \{(\underline{0{-}0}), (\underline{1{-}1}), (11{-})\}^{\oplus} = \{(0{-}{-}), ({-}{-}1), (11{-})\}^{\oplus} \\ Y^{\oplus}_2 = \{(\underline{0{-}{-}}), (\underline{0{-}0}), (101)\}^{\oplus} = \{(\underline{0{-}1}), (\underline{101})\}^{\oplus} = \{({-}{-}1), (111)\}^{\oplus} \\ Y^{\oplus}_3 = \{(\underline{0{-}1}), (\underline{0{-}0})\}^{\oplus} = \{(0{-}{-})\}^{\oplus} \end{cases}.$$

Answer. Minimal system of the PSTF $\{Y^{\oplus}_{1,2,3}\}$: $\begin{cases} Y^{\oplus}_1 = \{(0{-}{-}), ({-}{-}1), (11{-})\}^{\oplus} \\ Y^{\oplus}_2 = \{({-}{-}1), (111)\}^{\oplus} \\ Y^{\oplus}_3 = \{(0{-}{-})\}^{\oplus} \end{cases}$. Cost of its realization reflects the interrelation k^*_{θ}/k^*_l = 4/7. If compared with [21] it is a better result, where this system is compatibly minimized by *xlinking method* with k^*_{θ}/k^*_l = 4/9, namely $\begin{cases} Y^{\oplus}_1 = \{(0{-}{-}), (0{-}1), (110), (101)\}^{\oplus} \\ Y^{\oplus}_2 = \{(0{-}1), (101)\}^{\oplus} \\ Y^{\oplus}_3 = \{(0{-}{-})\}^{\oplus} \end{cases}$.

In case of the system of incomplete functions (25) the set of system minterms will consist of two subsets separated by the symbol $\vdots$ and reflected as $\{Y^{\oplus}_I \vdots Y^{\tilde{\oplus}}_I\}$, $I \in \{1, 2, \ldots, s\}$. Here the system minterms undergo the splitting procedure by using the matrix M^r_n, the elements of which are the conjuncterms of r-rank. It should be noted, that in the course of covering the matrix M^r_n two procedures are realized at a time: making the matrix compatible as its elements are used to maximum extent with higher capacity of the set I, and making it more predetermined in which the elements of the submatrix $Y^{\tilde{\oplus}}_I$ are used. After distribution of the last ones in the functions of the system we obtain $\{Y_I^{\oplus}\vdots Y_I^{\tilde{\oplus}}\}$, the elements of which for every function further undergo the simplification procedure according to the rules of the respective theorems in Section 3, selecting out of possible variants of transformation those which will provide the compatible minimization of the given system $F(X)$ in the best way.

Example 5. [37, p. 228, example 5.1] To minimize the system $F(X)$ of incomplete functions $f_1(a,b,c)$ and $f_2(a,b,c)$, given by the truth table (see right) with the help of splitting method in the polynomial set-theoretical format.

I	a	b	c	f_1	f_2
0	0	0	0	1	0
1	0	0	1	$\sim$	$\sim$
2	0	1	0	0	1
3	0	1	1	1	$\sim$
4	1	0	0	$\sim$	$\sim$
5	1	0	1	$\sim$	0
6	1	1	0	1	0
7	1	1	1	0	1

Solution. The given system $F(X)$ has the perfect PSTF

$$\begin{cases} Y_1^{\oplus}=\{(000),(011),(110)\}^{\oplus},\ Y_1^{\tilde{\oplus}}=\{(001),(100),(101)\}^{\tilde{\oplus}} \\ Y_2^{\oplus}=\{(010),(111)\}^{\oplus},\ Y_2^{\tilde{\oplus}}=\{(001),(011),(100)\}^{\tilde{\oplus}} \end{cases}$$

We will form a set of system minterms $\left(\text{i.e. } \{Y_{1,2}^{\oplus}\vdots Y_{1,2}^{\tilde{\oplus}}\}\right)$, with which we will do the splitting procedure by using the matrix M_3^1 and the procedure of its covering, for example, for the mask $\{–l–\}$:

$$Y_{1,2}^{\oplus}\vdots Y_{1,2}^{\tilde{\oplus}}=\{(000)_1,(010)_2,(011)_1,(110)_1,(111)_2\vdots(001)_{1,2},(011)_2,(100)_{1,2},(101)_1\}^{\oplus}\overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} l–– \\ –l– \\ ––l \end{bmatrix}=\begin{bmatrix} 0–– & 0–– & 0–– & 1–– & 1–– & 0–– & 0–– & 1–– & 1–– \\ –0–_1 & \underline{\mathbf{–1–}}_2 & \underline{\mathbf{–1–}}_1 & \underline{\mathbf{–1–}}_1 & \underline{\mathbf{–1–}}_2 & –0–_{1,2} & \underline{\mathbf{–1–}}_2 & –0–_{1,2} & –0–_1 \\ ––0 & ––0 & ––1 & ––0 & ––1 & ––1 & ––1 & ––0 & ––1 \end{bmatrix}\overset{C}{\Rightarrow}\{–l–\}=$$

$$=\Big\{(000)_1,\big((–1–)_2,(011)_2,(110)_2\big),\big((–1–)_1,(010)_1,(111)_1\big)\vdots(001)_{1,2},(011)_2,(100)_{1,2},(101)_1\Big\}^{\oplus}.$$

After removal of the system minterm $(011)_2$ the set of covering will look like

$$Y_{1,2}^{\oplus}\vdots Y_{1,2}^{\tilde{\oplus}}=\Big\{(000)_1,\big((–1–)_2,(110)_2\big),\big((–1–)_1,(010)_1,(111)_1\big)\vdots(001)_{1,2},(011)_2,(100)_{1,2},(101)_1\Big\}^{\oplus}.$$

Having distributed the system conjuncterms on the functions, we obtain the system

$$\text{PSTF}\begin{cases} Y_1^{\oplus}\vdots Y_1^{\tilde{\oplus}}=\{(–1–),(000),(010),(111)\vdots(001),(100),(101)\}^{\oplus} \\ Y_2^{\oplus}\vdots Y_2^{\tilde{\oplus}}=\{(–1–),(110)\vdots(001),(011),(100)\}^{\oplus} \end{cases}.$$

We will do the splitting procedure with the minterms of the PSTF of the function f_1 by using the matrixM_3^1 and the procedure of its covering:

$$\{(000),(010),(111)\vdots(001),(100),(101)\}^{\oplus}\overset{S}{\Rightarrow}\begin{bmatrix} l–– \\ –l– \\ ––l \end{bmatrix}=\begin{bmatrix} \underline{0––} & \underline{0––} & 1–– & \underline{0––} & 1–– & 1–– \\ –0– & –1– & –1– & –0– & –0– & –0– \\ ––0 & ––0 & \underline{––1} & \underline{––1} & ––0 & \underline{––1} \end{bmatrix}\overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\Big\{\big((0––),(011)\big),\big((––1),(011)\big)\Big\}^{\oplus}\Rightarrow\{(0––),(––1)\}^{\oplus}.$$

Having taken into account the rule (3) $\begin{pmatrix} 0–– \\ ––1 \end{pmatrix}^{\oplus}\equiv\begin{pmatrix} 1–– \\ ––0 \end{pmatrix}^{\oplus}$ we will obtain two solutions of the minimal PSTF $Y_1^{\oplus}=\left\{(–1–),\begin{Bmatrix} 1.\ (0––),(––1) \\ 2.\ (1––),(––0) \end{Bmatrix}\right\}^{\oplus}.$

We will do similar procedures for the minterms of the PSTF of the function f_2 by applying the matrix M_3^2 for their splitting:

$$\{(110)\vdots(001),(011),(100)\}^{\oplus} \overset{S}{\Rightarrow} \begin{bmatrix} ll- \\ l\text{–}l \\ -ll \end{bmatrix} = \begin{bmatrix} 11- & 00- & 01- & 10- \\ \underline{1\text{–}0} & \underline{\underline{0\text{–}1}} & \underline{\underline{0\text{–}1}} & \underline{1\text{–}0} \\ -10 & -01 & -11 & -00 \end{bmatrix} \overset{C}{\Rightarrow} \{(1\text{–}0),(0\text{–}1)\}^{\oplus}.$$

After the transformation according to the rule (3) $\begin{pmatrix} 1\text{–}0 \\ 0\text{–}1 \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{\begin{pmatrix} 1-- \\ --1 \end{pmatrix},\begin{pmatrix} --0 \\ 0-- \end{pmatrix}\right\}$, we obtain two solutions of the minimal PSTF $Y_2^{\oplus} = \left\{(\text{–}1\text{–}),\left\{\begin{matrix} 1.\ (1--),(--1) \\ 2.\ (--0),(0--) \end{matrix}\right\}\right\}^{\oplus}$.

Answer. The given system of functions has two solutions of minimization which reflect the PSTF

$$1.\begin{cases} Y_1^{\oplus} = \{(\text{–}1\text{–}),(--1),(0--)\}^{\oplus} \\ Y_2^{\oplus} = \{(\text{–}1\text{–}),(--1),(1--)\}^{\oplus} \end{cases}; \qquad 2.\begin{cases} Y_1^{\oplus} = \{(\text{–}1\text{–}),(--0),(1--)\}^{\oplus} \\ Y_2^{\oplus} = \{(\text{–}1\text{–}),(--0),(0--)\}^{\oplus} \end{cases}.$$

The analytical expressions correspond to these solutions:

$$1.\begin{cases} f_1(a,b,c) = b \oplus c \oplus \bar{a} \\ f_2(a,b,c) = b \oplus c \oplus a \end{cases}; \qquad 2.\begin{cases} f_1(a,b,c) = b \oplus \bar{c} \oplus a \\ f_2(a,b,c) = b \oplus \bar{c} \oplus \bar{a} \end{cases}.$$

Cost of realization of the system for the both solutions is equal to $k_\theta^*/k_l^* = 4/4$. If compared with [37] it is a better result, where cost of realization is equal to $k_\theta^*/k_l^* = 4/7$, namely: $\begin{cases} f_1(a,b,c) = b \oplus \bar{c} \oplus ab \\ f_2(a,b,c) = b \oplus ab\bar{c} \end{cases}$.

6 Conclusions

A new minimization method in the polynomial set-theoretical format of complete and incomplete logic functions with n variables and their system has been presented. It consists in the splitting procedure of given minterms and iterative simplification of conjuncterms on the based set-theoretical rules. The method's efficiency has been proved by numerous examples borrowed from well-known publications (see References) related to different minimization methods. The vast majority of functions and their systems minimized by the proposed method showed better results. This is due to the fact that in the process of transformation are involved the conjuncterms with Hamming distance $d \geq 3$, the transformed PSTF of which may have elements for which in the given set there will be a pair with smaller d. The search procedure of such elements has s combinative character: after each replacement of a chosen pair of conjuncterms of the given PSTF $Y^{\oplus}$ for certain set of the transformed PSTF $Y^{\oplus}$ we obtain a new set where it is necessary to determine distance d between new pairs and, having chosen from them the elements with minimal d, to apply the rules of respective theorem and build again a new set and so on. As a result, the probability of effective simplification of conjuncterms set increases through the use of appropriate transformation rules that reduce the implementation cost of realization of minimized function.

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