Role-depth bounded Least Common Subsumer in Prob-$\mathcal{EL}$ with Nominals

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Abstract. Completion-based algorithms can be employed for computing the least common subsumer of two concepts up to a given role-depth, in extensions of the lightweight DL $\mathcal{EL}$. This approach has been applied also to the probabilistic DL Prob-$\mathcal{EL}$, which is variant of $\mathcal{EL}$ with subjective probabilities. In this paper we extend the completion-based lcs-computation algorithm to nominals, yielding a procedure for the DL Prob-$\mathcal{ELO}_c^{01}$.

1 Introduction

The least common subsumer (lcs) is a reasoning service that generalizes a set of input concept descriptions into a single concept description that subsumes all the input concepts and is the least one w.r.t. subsumption. This inference has turned out to be quite useful for a number of applications, like the definition of similarity measures for concept descriptions [5, 7], the bottom-up construction of knowledge bases [4], information retrieval, and more (see [6, 16]).

In particular, for large biomedical ontologies the lcs can be used effectively to aid construction and maintenance. Many of these biomedical ontologies, notably SNOMED CT [12]and the Gene Ontology [1], are written in the $\mathcal{EL}$-family of lightweight description logics.

An interesting extension of $\mathcal{EL}$ that still admits tractable reasoning is the use of nominals. Nominals basically allow to characterize a concept in terms of specific individuals. Nominals are also admitted in the OWL 2 EL profile of OWL [14] and thus are interesting to practical applications. For the $\mathcal{EL}$-family of DLs there exist completion-based classification algorithms [3] that compute all the subsumption relations between all named concepts and nominals in an ontology in polynomial time. Kazakov et al. showed in [11] that the original completion algorithm is indeed incomplete for $\mathcal{ELO}$ and introduced a complete

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consequence-based classification algorithm for this DL. This algorithm is a variant of the completion-based algorithm, with the additional benefit of exhibiting a pay-as-you-go behavior, and thus allows for efficient implementation.

Another extension of $\mathcal{EL}$ is Prob-$\mathcal{EL}$ [13], which allows the modeling of uncertain knowledge by introducing probabilistic constructors. Prob-$\mathcal{EL}$ uses subjective (or Type-2 [10]) probabilities, which correspond to degrees of belief and are interpreted using a multiple-world semantics. For example, in Prob-$\mathcal{EL}$ one can express that obese people are likely to have high pressure, without requiring every obese person to be hypertense, using the GCI

$$\mathsf{Obese} \sqsubseteq P_{\geq 0.9} \exists \mathsf{hasCondition}.\mathsf{HighPressure}.$$

A completion algorithm for classifying TBoxes in the sublanguage Prob-$\mathcal{EL}_c^{01}$ of Prob-$\mathcal{EL}$ was described in [13].

For general $\mathcal{EL}$-TBoxes with cyclic concept definitions, the lcs may not exist, as it may require an infinite nesting of existential restrictions to be expressed. Therefore, an approximation has been introduced in [16], that limits the maximal nesting of quantifiers of the resulting concept descriptions. These approximations, called role-depth bounded lcs (k-lcs), can be computed for $\mathcal{EL}$ with role inclusions by using completion sets produced by the completion-based classification algorithm [8].

In this paper, we give a classification algorithm for the classic DL $\mathcal{ELO}$ introduced in [11] in terms of the completion algorithm in order to extend the k-lcs to this DL. Furthermore, we extend the completion algorithm for (a moderate form of) subjective probabilities to nominals, which results in a classification algorithm for the DL Prob-$\mathcal{ELO}_c^{01}$. In this direction, we correct a small error from the algorithm in [13] that caused it to be incomplete. From this algorithm, we develop an algorithm that computes the k-lcs for Prob-$\mathcal{ELO}_c^{01}$. We show that for both lcs approximations, the resulting concept description subsumes all input concepts, and is the least one w.r.t. subsumption. Thus, if the exact lcs exists for some role-depth bound n, then the k-lcs is exact for $k \geq n$. Recently, necessary and sufficient conditions for the existence of the lcs w.r.t. general $\mathcal{EL}$-TBoxes have been devised [17]. By the use of these conditions the k for which the role-depth bound lcs and the lcs coincide can be determined, if the lcs exists.

The paper is organized as follows. First, we introduce some basic notions. In Section 3 we give a completion algorithm for $\mathcal{EL}$ with nominals and extend the k-lcs algorithm to this DL. The completion-based classification algorithm for Prob-$\mathcal{ELO}_c^{01}$ is devised in Section 4, along with the algorithm for the k-lcs for this probabilistic DL. We end with conclusions and remarks on future work. The main proofs of Section 4 have been deferred to the appendix.

2 Preliminaries

$\mathcal{ELO}$*-concept descriptions* are built from mutually disjoint sets N_C of *concept names*, N_R of *role names* and N_I of *individual names* using the syntax rule:

$$C, D ::= \top \mid A \mid \{a\} \mid C \sqcap D \mid \exists r.C,$$

Table 1. Concept constructors and TBox axioms for $\mathcal{ELO}$.

	Syntax	Semantics
named concept	$A \quad (A \in N_C)$	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
top concept	$\top$	$\Delta^{\mathcal{I}}$
nominal	$\{a\} \quad (a \in N_I)$	$\{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists r.C \quad (r \in N_R)$	$(\exists r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \exists e.(d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\}$
GCI	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

where $A \in N_C$, $r \in N_R$ and $a \in N_I$. As usual the semantics of $\mathcal{ELO}$-concepts are defined by means of interpretations. The semantics of named concepts and roles are extended to concept descriptions as shown in the upper part of Table 1.

An $\mathcal{ELO}$-TBox consists of a finite set of *general concept inclusion axioms* (GCIs) of the form $C \sqsubseteq D$. We use $C \equiv D$ as an abbreviation for $C \sqsubseteq D$ and $D \sqsubseteq C$. With $\mathsf{Sig}(\mathcal{T})$ we denote the *signature* of a TBox $\mathcal{T}$, i.e., the set of all concept, role, and individual names that occur in $\mathcal{T}$. An interpretation is a *model* for a TBox if it satisfies all its GCIs, as shown in the lower part of Table 1.

The central inference discussed in this paper is the *least common subsumer* (lcs) of two concept descriptions, i.e., to compute a concept that subsumes both and is the least one w.r.t. subsumption. Since the lcs does not need to exist for general $\mathcal{EL}$-TBoxes [2], we follow the idea from [16] and compute an approximation of the lcs that limits the *role depth* ($rd(C)$), i.e., the maximal nesting of quantifiers of the resulting concept C:

Definition 1 (Role-depth bounded lcs). *Let $\mathcal{L}$ be a DL, $\mathcal{T}$ be an $\mathcal{L}$-TBox and $k \in \mathbb{N}$. The* role-depth bounded least common subsumer *of two $\mathcal{L}$-concept descriptions C_1, C_2 w.r.t. $\mathcal{T}$ (written: k-lcs$_\mathcal{T}(C_1, C_2)$) is the $\mathcal{L}$-concept description D s.t.:*

1. *$rd(D) \leq k$,*
2. *$C_1 \sqsubseteq_\mathcal{T} D$ and $C_2 \sqsubseteq_\mathcal{T} D$, and*
3. *for each $\mathcal{L}$-concept description E holds: $C_1 \sqsubseteq_\mathcal{T} E$ and $C_2 \sqsubseteq_\mathcal{T} E$ and $rd(E) \leq k$, implies $D \sqsubseteq_\mathcal{T} E$.*

For the DLs considered in this paper, the k-lcs is unique up to equivalence for a given k; thus we speak of *the* k-lcs.

3 The k-lcs in $\mathcal{ELO}$

The algorithms to compute the role-depth bounded lcs are based on completion-based classification algorithms for the corresponding DL. For $\mathcal{ELO}$, a consequence-based classification algorithm is given by Kazakov et al. in [11], building upon the incomplete completion algorithm developed in [3]. The completion algorithm presented next adapts the ideas of the complete algorithm.

3.1 Completion Algorithm for Classification of $\mathcal{ELO}$-TBoxes

The completion algorithms works on normalized TBoxes. We define for $\mathcal{ELO}$ the set of *basic concepts* for a TBox $\mathcal{T}$ as

$$\mathsf{BC}_\mathcal{T} = (\mathsf{Sig}(\mathcal{T}) \cap (N_C \cup N_I)) \cup \{\top\}.$$

An $\mathcal{ELO}$-TBox $\mathcal{T}$ is in *normal form* if every GCI contained in $\mathcal{T}$ is of one of the forms:

$$A \sqsubseteq B, \; A_1 \sqcap A_2 \sqsubseteq B, \; A \sqsubseteq \exists r.B, \quad \text{or } \exists r.A \sqsubseteq B,$$

with $A, A_1, A_2, B \in \mathsf{BC}_\mathcal{T}$. All $\mathcal{ELO}$-TBoxes can be transformed into normal form in linear time by applying a set of normalization rules similar to those given in [3].Before describing the completion algorithm in detail, we introduce the reachability relation $\rightsquigarrow_R$, which plays a fundamental role in the correct treatment of nominals [3, 11].

Definition 2 **($\rightsquigarrow_R$).** *Let $\mathcal{T}$ be an $\mathcal{ELO}$-TBox in normal form, $G \in N_C$ a concept name, and $D \in \mathsf{BC}_\mathcal{T}$. $G \rightsquigarrow_R D$ iff there exist roles $r_1, \ldots, r_n$ and basic concepts $A_0, \ldots, A_n, B_0, \ldots, B_n \in \mathsf{BC}_\mathcal{T}$, $n \geq 0$ such that $A_i \sqsubseteq_\mathcal{T} B_i$ for all $0 \leq i \leq n$, $B_{i-1} \sqsubseteq \exists r_i.A_i \in \mathcal{T}$ for all $1 \leq i \leq n$, A_0 is either G or a nominal, and $B_n = D$.*

Informally, the concept name D is reachable from G if there is a chain of existential restrictions leading to D that starts either with G or with a nominal. This implies that, for $G \rightsquigarrow_R D$, if the interpretation of G is not empty, then the interpretation of D cannot be empty either. This, in turn can cause equivalence of concepts, e.g. $|A^\mathcal{I}| > 0$ and $A \sqsubseteq \{b\}$ implies $A \equiv \{b\}$.

The basic idea of the completion algorithm for $\mathcal{EL}$ (without nominals) is to store all basic concepts that subsume a concept $A \in (\mathsf{Sig}(\mathcal{T}) \cap N_C) \cup \{\top\}$ in its subsumer set $S(A)$ and and all basic concepts B for which $\exists r.B$ subsumes A in the subsumer set $S(A, r)$. These completion sets are then extended using a set of completion rules. However, with nominals the algorithm needs to keep track of completion sets of the form $S^G(A)$ and $S^G(A, r)$ for every $G \in (\mathsf{Sig}(\mathcal{T}) \cap N_C) \cup \{\top\}$, since the non-emptiness of an interpretation of a concept G may imply additional subsumption relationships for A. The completion set $S^G(A)$ for $A \in \mathsf{BC}_\mathcal{T}$ therefore stores all basic concepts that subsume A under the assumption that G is not empty. Similarly $S^G(A, r)$ stores all concepts B for which $\exists r.B$ subsumes A under the same assumption. For every $G \in (\mathsf{Sig}(\mathcal{T}) \cap N_C) \cup \{\top\}$, every basic concept A and every role name r, the completion sets are initialized as $S^G(A) = \{A, \top\}$ and $S^G(A, r) = \emptyset$. The completion sets are then extended by applying the completion rules adapted from [11] and shown in Figure 1 exhaustively.

It can be shown that the algorithm terminates in polynomial time, and is sound and complete for classifying the TBox $\mathcal{T}$ [11]. In particular, if no rules are applicable the completion sets have the following properties.

Proposition 1 **([11]).** *Let $\mathcal{T}$ be an $\mathcal{ELO}$-TBox in normal form to which the completion rules have been applied exhaustively, $C, D \in \mathsf{BC}_\mathcal{T}$, $r \in \mathsf{Sig}(\mathcal{T}) \cap N_R$, and $G = C$ if $C \in N_C$ and $G = \top$ otherwise. Then, the following properties hold:*

OR1	If $A_1 \in S^G(A)$, $A_1 \sqsubseteq B \in \mathcal{T}$ and $B \notin S^G(A)$, then $S^G(A) := S^G(A) \cup \{B\}$
OR2	If $A_1, A_2 \in S^G(A)$, $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}$ and $B \notin S^G(A)$, then $S^G(A) := S^G(A) \cup \{B\}$
OR3	If $A_1 \in S^G(A)$, $A_1 \sqsubseteq \exists r.B \in \mathcal{T}$ and $B \notin S^G(A, r)$, then $S^G(A, r) := S^G(A, r) \cup \{B\}$
OR4	If $B \in S^G(A, r)$, $B_1 \in S^G(B)$, $\exists r.B_1 \sqsubseteq C \in \mathcal{T}$ and $C \notin S^G(A)$, then $S^G(A) := S^G(A) \cup \{C\}$
OR5	If $\{a\} \in S^G(A_1) \cap S^G(A_2)$, $G \rightsquigarrow_R A_2$, and $A_2 \notin S^G(A_1)$, then $S^G(A_1) := S^G(A_1) \cup \{A_2\}$

Fig. 1. Completion rules for $\mathcal{ELO}$

$C \sqsubseteq_{\mathcal{T}} D$ *iff* $D \in S^G(C)$, *and*
$C \sqsubseteq_{\mathcal{T}} \exists r.D$ *iff there exists* $E \in \mathsf{BC}_{\mathcal{T}}$ *such that* $E \in S^G(C, r)$ *and* $D \in S^G(E)$.

We now show how to use these completion sets for computing the role-depth bounded lcs for $\mathcal{ELO}$-concept description w.r.t. a general $\mathcal{ELO}$-TBox.

3.2 Computing the Role-depth Bounded $\mathcal{ELO}$-lcs

In order to compute the role-depth bounded lcs of two $\mathcal{ELO}$-concepts C and D, we follow an idea very similar to the one presented for $\mathcal{ELR}$-concepts in [8], where we compute the cross product of the tree unravelings of the completion graph for C and D up to the role-depth k. Clearly, in the presence of nominals, the right completion sets need to be chosen such that they preserve the non-emptiness of the interpretation of concepts derived by $\rightsquigarrow_R$.

An algorithm that computes the role-depth bounded $\mathcal{ELO}$-lcs using completion sets is shown in Figure 2. In the first step, the algorithm introduces two new concept names A and B as abbreviations for the concepts C and D, and the augmented TBox is normalized. The completion sets are then initialized and the completion rules from Figure 1 are applied exhaustively, yielding the saturated completion sets. In the recursive procedure k-lcs-r for concepts A and B, we first obtain all the basic concepts that subsume both A and B from the sets $S^A(A)$ and $S^B(B)$. For every role name r, the algorithm then recursively computes the $(k-1)$-lcs of the concepts A' and B' in the subsumer sets $S^A(A, r)$ and $S^B(B, r)$, i.e. for which $A \sqsubseteq_{\mathcal{T}} \exists r.A'$ and $B \sqsubseteq_{\mathcal{T}} \exists r.B'$. The resulting concepts are conjoined as existential restrictions to the resulting k-lcs.

The algorithm only introduces concept and role names that occur in the original TBox $\mathcal{T}$. Therefore those names introduced by the normalization are not used in the concept description for the k-lcs and an extra denormalization step as in [16, 8] is not necessary.

Notice, that for every pair (A', B') of r-successors of A and B it holds that $A \rightsquigarrow_R A'$ and $B \rightsquigarrow_R B'$. Intuitively, we are assuming that the interpretation of both A and B is not empty. This in turn causes the interpretation of $\exists r.A'$ and $\exists r.B'$ to be not empty, either. Thus, it suffices to consider the completion sets $S^A(\ldots)$

Procedure $\mathsf{k\text{-}lcs}(C, D, \mathcal{T}, k)$
Input: C, D: $\mathcal{ELO}$-concept descriptions; $\mathcal{T}$: $\mathcal{ELO}$-TBox; $k \in \mathbb{N}$
Output: role-depth bounded $\mathcal{ELO}$-lcs of C, D w.r.t. $\mathcal{T}$ and k

$\mathcal{T}' := \mathsf{normalize}(\mathcal{T} \cup \{A \equiv C, B \equiv D\})$
$\mathsf{S}_\mathcal{T} := \mathsf{apply\text{-}completion\text{-}rules}(\mathcal{T}')$
return $\mathsf{k\text{-}lcs\text{-}r}(A, B, \mathsf{S}_\mathcal{T}, k, A, B, \mathsf{Sig}(\mathcal{T}))$

Procedure $\mathsf{k\text{-}lcs\text{-}r}(X, Y, \mathsf{S}_\mathcal{T}, k, A, B, \mathsf{Sig}(\mathcal{T}))$
Input: A, B: concept names, X, Y: basic concepts with $A \rightsquigarrow_R X, B \rightsquigarrow_R Y$; $k \in \mathbb{N}$;
$\mathsf{S}_\mathcal{T}$: set of saturated completion sets; $\mathsf{Sig}(\mathcal{T})$: signature of $\mathcal{T}$
Output: role-depth bounded $\mathcal{ELO}$-lcs of X, Y w.r.t. $\mathcal{T}$ and k

$\text{CN} := S^A(X) \cap S^B(Y) \cap \mathsf{BC}_\mathcal{T}$
if $k = 0$ **then**
  return $\displaystyle\bigsqcap_{P \in \text{CN}} P$
else
  return
  $\displaystyle\bigsqcap_{P \in \text{CN}} P \sqcap \bigsqcap_{r \in \mathsf{Sig}(\mathcal{T}) \cap N_R} \Big(\bigsqcap_{\substack{E \in S^A(X,r),\\ F \in S^B(Y,r)}} \exists r.\mathsf{k\text{-}lcs\text{-}r}\big(E, F, \mathsf{S}_\mathcal{T}, k-1, A, B, \mathsf{Sig}(\mathcal{T})\big) \Big)$

Fig. 2. Computation algorithm for role-depth bounded $\mathcal{ELO}$-lcs.

and $S^B(\ldots)$, without the need to additionally compute $S^{A'}(\ldots)$ and $S^{B'}(\ldots)$, or the completion sets $S^E(\ldots)$ for any other basic concept E encountered during the recursive computation of the k-lcs. This allows for a goal-oriented optimization in cases where there is no need to classify the full TBox.

4 The k-lcs in Prob-$\mathcal{ELO}_c^{01}$

So far, we have discussed only classic DLs that can be used to represent and reason with certain knowledge. However, it is not uncommon to encounter situations where a degree of uncertainty is unavoidable. This is often the case in the medical and biological domains, where knowledge is obtained through clinical testing, and there might exist hidden, or not completely understood, factors affecting the outcome. For instance, we would like to be able to express that obese people are *likely* to have high pressure, without asserting that every obese person *must* have high pressure.

The probabilistic logic Prob-$\mathcal{EL}$ was introduced by [13] as an extension of $\mathcal{EL}$ that allows to express uncertain knowledge through probabilistic concepts and roles. Here, we extend these ideas to cover also nominals. Formally, Prob-$\mathcal{ELO}$ concepts extend classical $\mathcal{ELO}$ concepts with the constructors

$$P_{\triangleright q} C \quad \text{and} \quad \exists P_{\triangleright q} r.C,$$

where $r \in N_R$, $\triangleright \in \{>, <, \geq, \leq, =\}$, and $q \in [0, 1]$. Intuitively, a concept of the form $P_{\triangleright q} C$ denotes the class of all objects that belong to C with a probability $\triangleright q$.

For the above example, we can use the concept $P_{\geq 0.9}\exists\mathsf{hasCondition}.\mathsf{HighPressure}$ to represent the class of all individuals that are likely to have high pressure.

The semantics of this logic generalizes the semantics of classical $\mathcal{ELO}$ by considering a set of possible worlds, corresponding to a formalization of subjective (or Type 2) probabilities [10]. Formally, the semantics of Prob-$\mathcal{ELO}$ are based on *probabilistic interpretations* of the form $\mathcal{I} = (\Delta^\mathcal{I}, W, (\mathcal{I}_w)_{w\in W}, \mu)$, where $\Delta^\mathcal{I}$ is a (non-empty) domain, W is a non-empty set of (possible) *worlds*, μ is a discrete probability distribution over W, and for every $w \in W$, $\mathcal{I}_w$ is a classical $\mathcal{ELO}$ interpretation with domain $\Delta^\mathcal{I}$. Additionally, for every $a \in N_I$ and every two worlds $w, w' \in W$, it holds that $a^{\mathcal{I}_w} = a^{\mathcal{I}_{w'}}$.

From a probabilistic interpretation, we can compute the probability that a given element of the domain $d \in \Delta^\mathcal{I}$ belongs to the interpretation of a named concept A, and respectively, the probability that a pair of individuals are related via a role r as follows:

$$p^\mathcal{I}_d(A) := \mu(\{w \in W \mid d \in A^{\mathcal{I}_w}\}),$$
$$p^\mathcal{I}_{d,e}(r) := \mu(\{w \in W \mid (d,e) \in r^{\mathcal{I}_w}\}).$$

The functions $\mathcal{I}_w$ and $p^\mathcal{I}_d$ are extended to general concepts through the following mutual recursion.

$$\begin{aligned}
\top^{\mathcal{I}_w} &= \Delta^\mathcal{I}, & (\exists r.C)^{\mathcal{I}_w} &= \{d \in \Delta^\mathcal{I} \mid \exists e \in C^{\mathcal{I}_w}.(d,e) \in r^{\mathcal{I}_w}\},\\
(C \sqcap D)^{\mathcal{I}_w} &= C^{\mathcal{I}_w} \cap D^{\mathcal{I}_w}, & (P_{\triangleright q} C)^{\mathcal{I}_w} &= \{d \in \Delta^\mathcal{I} \mid p^\mathcal{I}_d(C) \triangleright q\},\\
(\{o\})^{\mathcal{I}_w} &= \{o^{\mathcal{I}_w}\}, & (\exists P_{\triangleright q} r.C)^{\mathcal{I}_w} &= \{d \in \Delta^\mathcal{I} \mid \exists e \in C^{\mathcal{I}_w}.p^\mathcal{I}_{d,e}(r) \triangleright q\},
\end{aligned}$$
$$p^\mathcal{I}_d(C) = \mu(\{w \in W \mid d \in C^{\mathcal{I}_w}\}).$$

We say that the probabilistic interpretation $\mathcal{I} = (\Delta^\mathcal{I}, W, (\mathcal{I}_w)_{w\in W}, \mu)$ *satisfies* the GCI $C \sqsubseteq D$ if for every world $w \in W$ it holds that $C^{\mathcal{I}_w} \subseteq D^{\mathcal{I}_w}$. $\mathcal{I}$ is a *model* of the TBox $\mathcal{T}$ if $\mathcal{I}$ satisfies all GCIs in $\mathcal{T}$. A concept C is subsumed by concept D w.r.t. the TBox $\mathcal{T}$ ($C \sqsubseteq_\mathcal{T} D$), if every model $\mathcal{I}$ of $\mathcal{T}$ satisfies $C \sqsubseteq D$.

Unfortunately, the probabilistic constructors increase the complexity of reasoning, and deciding subsumption becomes intractable. In fact, as shown in [9], the problem is ExpTime-complete, even if only one constructor of the form $P_{\triangleright q}$ with $q \in (0,1)$ is allowed. Moreover, the problem becomes PSpace-hard if probabilistic existential restrictions of the form $\exists P_{>0} r$ or $\exists P_{=1} r$ are used. To regain tractability Prob-$\mathcal{EL}$ was restricted in [13] to probabilistic concepts of the form $P_{>0}C$ or $P_{=1}C$ and no probabilistic existential restrictions or roles, yielding the DL Prob-$\mathcal{EL}^{01}_c$. The extension of this DL by nominals is Prob-$\mathcal{ELO}^{01}_c$, which is the DL we consider for the remainder of the paper.

4.1 Completion Algorithm for Prob-$\mathcal{ELO}^{01}_c$

The basic idea of the completion algorithm for Prob-$\mathcal{ELO}^{01}_c$ is the same as in the crisp case: to construct a canonical model of the given TBox. In order to introduce it, we need to extend the notion of basic concepts to the new constructors

$$\mathsf{BC}_{\mathcal{T}} = (\mathsf{Sig}(\mathcal{T}) \cap (N_C \cup N_I)) \cup \{\top\} \cup \{P_{>0}A, P_{=1}A \mid A \in \mathsf{Sig}(\mathcal{T}) \cap N_C\}.$$

Since probabilistic interpretations contain a set of worlds, the completion algorithm has to work on sets of completion sets: one for each world and, since Prob-$\mathcal{ELO}^{01}_c$ also contains nominals, for each basic concept. Let $\mathcal{P}_0^{\mathcal{T}}$ denote the set of all probabilistic concepts of the form $P_{>0}A$ appearing in $\mathcal{T}$. The completion algorithm uses a set of worlds $V := \{0, 1, \varepsilon\} \cup \mathcal{P}_0^{\mathcal{T}}$; this way, for each GCI $B \sqsubseteq P_{>0}A \in \mathcal{T}$ the world $v = P_{>0}A$ serves as witness for this subsumption. Therefore, the Prob-$\mathcal{ELO}^{01}_c$ completion sets are now of the form $S^G_*(X, v)$ or $S^G_*(X, r, v)$ where X is a concept name, $\top$, or a nominal, $v \in V$, G is a concept name or $\top$, $* \in \{0, \varepsilon\}$ and r is a role name. The completion sets contain *basic concepts* from $\mathsf{BC}_{\mathcal{T}}$. The normal form for Prob-$\mathcal{ELO}^{01}_c$-TBoxes is the same as the normal form as for $\mathcal{ELO}$-TBoxes, i.e. each axiom is of the form $C \sqsubseteq D$, $C_1 \sqcap C_2 \sqsubseteq D$, $C \sqsubseteq \exists r.A$, or $\exists r.A \sqsubseteq D$, with $C, C_1, C_2, D \in \mathsf{BC}_{\mathcal{T}}$ and $A \in (\mathsf{Sig}(\mathcal{T}) \cap N_C) \cup \{\top\}$.

The reachability relation for Prob-$\mathcal{ELO}^{01}_c$-concepts extends the one for $\mathcal{ELO}$ in distinguishing between concept names X and probabilistic concepts $P_{>0}X$. For example, non-emptiness of G does not imply non-emptiness of $P_{>0}X$, even if $G \rightsquigarrow_R X$, e.g. in worlds with probability 0. Similarly, non-emptiness of G does not imply non-emptiness of X for $G \rightsquigarrow_R P_{>0}X$. Therefore we introduce two kinds of reachability relation, $G \overset{0}{\rightsquigarrow}_R X$ for $G \rightsquigarrow_R X$ and $G \overset{\varepsilon}{\rightsquigarrow}_R X$ for $G \rightsquigarrow_R P_{>0}X$:

Definition 3 (Reachability of Prob-$\mathcal{ELO}^{01}_c$-concept descriptions). *Let $\mathcal{T}$ be a Prob-$\mathcal{ELO}^{01}_c$-TBox in normal form, $G \in N_C \cup \{\top\}$, and $D \in N_C \cup N_I$.*

- $G \overset{0}{\rightsquigarrow}_R D$ *iff there exist $r_1, \dots, r_n \in N_R$ and basic concepts $A_0, \dots, A_n$ where $A_i \in S^G_0(A_{i-1}, r_i, 0)$ for all $1 \le i \le n$, such that A_0 is either G or a nominal and $A_n = D$.*
- $G \overset{\varepsilon}{\rightsquigarrow}_R D$ *iff $G \overset{0}{\rightsquigarrow}_R X$, $P_{>0}Y \in S^G_0(X, 0)$ and there are $r_1, \dots, r_n \in N_R$ and $A_0, \dots, A_n \in N_C$ with $A_i \in S^G_\varepsilon(A_{i-1}, r_i, \varepsilon)$ for all $1 \le i \le n$ such that $A_0 = Y$ and $A_n = D$.*

When it is clear from the context which relation $\overset{0}{\rightsquigarrow}_R$ or $\overset{\varepsilon}{\rightsquigarrow}_R$ we are using, we will sometimes denote it simply as $\rightsquigarrow_R$.

Additionally, nominals interact with the set of possible worlds in a different way than normal concepts. In particular, the concepts $P_{>0}\{a\}$ and $P_{=1}\{a\}$ are indeed equivalent to $\{a\}$, since $\{a\}$ is interpreted as the singleton domain element $a^{\mathcal{I}}$ in each world. This implies that also $X \sqsubseteq P_{>0}\{a\}$, $X \sqsubseteq P_{=1}\{a\}$ and $X \sqsubseteq \{a\}$ are equivalent. In other words, whenever $\{a\} \in S^G_*(X, v)$, then $\{a\}$ must be in $S^G_*(X, w)$ for all $w \in V$.

The completions sets $S^G_*(X, v)$ and $S^G_*(X, r, v)$ are initialized as follows:

- $S^G_0(X, 0) = \{\top, X\}$ and $S^G_0(X, v) = \{\top\}$ for all $v \in V \setminus \{0\}$,
- $S^G_\varepsilon(X, \varepsilon) = \{\top, X\}$ and $S^G_\varepsilon(X, v) = \{\top\}$ for all $v \in V \setminus \{\varepsilon\}$,
- $S^G_0(X, r, v) = S^G_\varepsilon(X, r, v) = \emptyset$ for all $v \in V$.

PR1	If $C' \in S^G_*(X, v)$ and $C' \sqsubseteq D \in \mathcal{T}$, then $S^G_*(X, v) := S^G_*(X, v) \cup \{D\}$
PR2	If $C_1, C_2 \in S^G_*(X, v)$ and $C_1 \sqcap C_2 \sqsubseteq D \in \mathcal{T}$, then $S^G_*(X, v) := S^G_*(X, v) \cup \{D\}$
PR3	If $C' \in S^G_*(X, v)$ and $C' \sqsubseteq \exists r.D \in \mathcal{T}$, then $S^G_*(X, r, v) := S^G_*(X, r, v) \cup \{D\}$
PR4	If $D \in S^G_*(X, r, v)$, $D' \in S^G_{\gamma(v)}(D, \gamma(v))$ and $\exists r.D' \sqsubseteq E \in \mathcal{T}$, then $S^G_*(X, v) := S^G_*(X, v) \cup \{E\}$
PR5	If $\{a\} \in S^G_{*_1}(X, *_1) \cap S^G_{*_2}(D, *_2)$ and $G \overset{*_2}{\leadsto}_R D$, then $S^G_{*_1}(X, *_1) := S^G_{*_1}(X, *_1) \cup \{P_{*_2} D\}$
PR6	If $P_{>0}A \in S^G_*(X, v)$, then $S^G_*(X, P_{>0}A) := S^G_*(X, P_{>0}A) \cup \{A\}$
PR7	If $P_{=1}A \in S^G_*(X, 0)$, then $S^G_*(X, 1) := S^G_*(X, 1) \cup \{A\}$
PR8	If $P_{=1}A \in S^G_*(X, v)$, $v \neq 0$, then $S^G_*(X, v) := S^G_*(X, v) \cup \{A\}$
PR9	If $A \in S^G_*(X, v)$ and $v \neq 0$, $P_{>0}A \in \mathcal{P}^{\mathcal{T}}_0$, then $S^G_*(X, v') := S^G_*(X, v') \cup \{P_{>0}A\}$
PR10	If $A \in S^G_*(X, 1)$ and $P_{=1}A$ occurs in $\mathcal{T}$, then $S^G_*(X, v) := S^G_*(X, v) \cup \{P_{=1}A\}$
PR11	If $\{a\} \in S^G_*(X, v)$ then $S^G_*(X, v') := S^G_*(X, v') \cup \{\{a\}\}$

Fig. 3. Completion rules for Prob-$\mathcal{ELO}^{01}_c$

These completion sets are then extended by applying the completion rules from Figure 3 exhaustively. The function $\gamma : V \to \{0, \varepsilon\}$ used in rule **PR4** is defined by $\gamma(0) = 0$, and $\gamma(v) = \varepsilon$ for all $v \in V \setminus \{0\}$. The completion rules can be divided into two groups. The rules **PR1** to **PR5** form the first group and are basically the same as the rules **OR1** to **OR5** for $\mathcal{ELO}$—they are used to compute all subsumption relationships between concepts inside each world. The next five rules **PR6** to **PR11** handle probabilistic concepts and therefore propagate derived facts between the different worlds. For example, whenever we have $P_{>0}A$ in the subsumer set of B, then rule **PR6** will push A into the subsumer set of B for the world $v = P_{>0}A$, i.e., the world v is a witness of the subsumption. Similarly, whenever $P_{=1}A$ is in the subsumer set of B for some world v, then rule **PR7** and **PR10** will push $P_{=1}A$ into the subsumer sets of B for all other worlds w and rule **PR8** will finally put A into the subsumer set of B for each world with non-zero probability (i.e. all worlds except world 0). Rule **PR11** distributes nominals in subsumer sets between the worlds in V as explained earlier.

This set of completion rules extends the rules given in [13] for Prob-$\mathcal{EL}^{01}_c$ in two ways. First, by the rules for nominals (written as completion rules). Second, by rule **PR7**, which is actually necessary to achieve completeness of the completion algorithm for Prob-$\mathcal{EL}^{01}_c$ (without nominals). To see this, consider the following TBox:

$$\mathcal{T}_{ex} = \{A \sqsubseteq P_{=1}B, \quad B \sqsubseteq C, \quad P_{=1}C \sqsubseteq D\}$$

Clearly, we have $A \sqsubseteq_{\mathcal{T}_{ex}} D$, however, without rule **PR7**, the completion algorithm is stuck with $P_{=1}B \in S_0(A, 0)$ and will never derive $B \in S_0(A, 1)$, $C \in S_0(A, 1)$, $P_{=1}C \in S_0(A, 0)$ and finally $D \in S_0(A, 0)$.

Theorem 1. *The completion algorithm for Prob-$\mathcal{ELO}^{01}_c$ is sound and complete.*

A detailed proof is given in the appendix (see Lemmas 1 and 2). Also note that the completion algorithm for Prob-$\mathcal{ELO}_c^{01}$ still runs in polynomial time, since $|\mathsf{BC}_\mathcal{T}|$, $|\mathsf{Sig}(\mathcal{T}) \cap N_C|$, $|\mathsf{Sig}(\mathcal{T}) \cap N_I|$, $|V|$, are all linear in the size of $|\mathcal{T}| = n$.

4.2 Computing the Role-depth Bounded Prob-$\mathcal{ELO}_c^{01}$-lcs

The approach for computing the role-depth bounded Prob-$\mathcal{ELO}_c^{01}$-lcs is similar to the classical case, where we first introduce new concept names for the input concepts, normalize the TBox, apply the completion rules, then we intersect the direct subsumers stored in the completion sets and add the cross-product of the existential restrictions of both concepts. However, in the presence of probabilistic concepts, we need to compute also the probabilistic direct subsumers and probabilistic existential restrictions. Therefore, this algorithm computes the cross-product of the existential restrictions three times: for the unconditional concepts, for those concepts with probability 1, and for concepts with non-zero probability. In contrast, we need to compute the intersection of the basic concepts only once, since whenever $X \in S_0^G(A, v)$ with $v \neq 0$, then by completion rule **PR9** we have also $P_{>0}X \in S_0^G(A, 0)$ and similarly, whenever $X \in S_0^G(A, 1)$ then we also have $P_{=1}X \in S_0^G(A, 0)$ by completion rule **PR10**. The algorithm to compute the role-depth bounded lcs in Prob-$\mathcal{ELO}_c^{01}$ is described in Figure 4.

Theorem 2. *Let $\mathcal{T}$ be a Prob-$\mathcal{ELO}_c^{01}$-TBox, C and D be Prob-$\mathcal{ELO}_c^{01}$-concepts and k be a natural number. Then $\mathsf{k\text{-}lcs}(C, D, \mathcal{T}, k)$ is the role-depth bounded least common subsumer of C and D w.r.t. $\mathcal{T}$ and the role-depth k.*

Correctness of this algorithm for computing the role-depth bounded Prob-$\mathcal{ELO}_c^{01}$-lcs follows from soundness and completeness of the completion rules which are used to generate the underlying completion sets. The full proof can be found in the appendix. As in the crisp case, the resulting k-lcs can have a size exponential in k if computed for n input concepts, but it is still polynomial in the size of the input TBox $\mathcal{T}$.

5 Conclusions

In this paper we have studied extensions of the light-weight description logic $\mathcal{EL}$, that include nominals and are capable of handling uncertainty. For the DL Prob-$\mathcal{ELO}_c^{01}$, we have introduced a completion algorithm that generalizes the previously known algorithm for Prob-$\mathcal{EL}_c^{01}$, with correct rules for handling nominals, while still retaining the polynomial time complexity of classification.

Second, we described how the completion sets saturated by the completion algorithm can be combined to compute (approximations of) the lcs of two concepts in DLs with nominals—both for the crisp case in $\mathcal{ELO}$ and the probabilistic case in Prob-$\mathcal{ELO}_c^{01}$. In cases where the exact lcs exists, the algorithms compute the exact lcs for a sufficiently large k.

The extension of $\mathcal{EL}^+$ by nominals, covers (most of) the OWL 2 EL profile, thus combining the algorithms for computing the role-depth bounded lcs in $\mathcal{EL}^+$

Procedure k-lcs$(C, D, \mathcal{T}, k)$
Input: C, D: Prob-$\mathcal{ELO}_c^{01}$-concept descriptions; $\mathcal{T}$: Prob-$\mathcal{ELO}_c^{01}$-TBox; $k \in \mathbb{N}$
Output: k-lcs(C, D): role-depth bounded Prob-$\mathcal{ELO}_c^{01}$-lcs of C, D w.r.t. $\mathcal{T}$ and k

$\mathcal{T}' := \mathsf{normalize}(\mathcal{T} \cup \{A \equiv C, B \equiv D\})$
$\mathsf{S}_\mathcal{T} := \mathsf{apply\text{-}completion\text{-}rules}(\mathcal{T}')$
$L := \mathsf{k\text{-}lcs\text{-}r}(A, B, \mathsf{S}_\mathcal{T}, k, A, B)$
return L

Procedure k-lcs-r$(X, Y, \mathsf{S}, k, A, B)$
Input: A, B: concept names, X, Y: basic concepts with $A \rightsquigarrow_R X$, $B \rightsquigarrow_R Y$;
S: set of saturated completion sets; k: natural number
Output: k-lcs(A, B): role-depth bounded Prob-$\mathcal{ELO}_c^{01}$-lcs of X, Y w.r.t. $\mathcal{T}$ and k

$\mathrm{CN} := \bigsqcap_{E \in S_0^A(X,0) \cap S_0^B(Y,0) \cap \mathsf{BC}_\mathcal{T}} E$
if $k = 0$ **then**
  return CN
else
  return

$$\mathrm{CN} \sqcap \bigsqcap_{r \in \mathsf{Sig}(\mathcal{T}) \cap N_R} \Bigg(\bigsqcap_{(E,F) \in S_0^A(X,r,0) \times S_0^B(Y,r,0)} \exists r.\mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k-1, A, B) \sqcap \bigsqcap_{(E,F) \in S_0^A(X,r,1) \times S_0^B(Y,r,1)} P_{=1}\Big(\exists r.\mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k-1, A, B)\Big) \sqcap \bigsqcap_{(E,F) \in \mathsf{PR}^A(X,r) \times \mathsf{PR}^B(Y,r)} P_{>0}\Big(\exists r.\mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k-1, A, B)\Big)\Bigg),$$

where $\mathsf{PR}^G(X, r) = \bigcup_{v \in V \setminus \{0\}} S_0^G(X, r, v)$

Fig. 4. Computation algorithm for role-depth bounded Prob-$\mathcal{ELO}_c^{01}$-lcs.

[8] and in $\mathcal{ELO}$ presented here allows to compute generalizations in this profile. Similarly, as in case of the lcs, an approximation of the most specific concept (msc) can be computed based on a completion algorithm, see [15]. For DLs with nominals, the completion algorithm given in this paper can be directly used for this, since an ABox can always be absorbed into the TBox in a preprocessing step using these nominals. However, since the msc in the presence of nominals is trivial ($msc(a) = a$), another target DL should be considered in order to yield an informative version of the msc.

Besides the msc, there exist several other non-standard inferences that have been studied for classical DLs and would be of interest in the context of subjective probabilities. One of them is *axiom pinpointing*, i.e. the task of discovering the precise axioms from a knowledge base that are responsible for a consequence to follow. The use of probabilities introduces a new challenge as seemingly innocuous axioms may interact to produce unexpected (and possibly unwanted) consequences. A further study of this problem will be a matter of future work.

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A Omitted Proofs

A.1 Correctness of the Classification Algorithm for Prob-$\mathcal{ELO}_c^{01}$

Theorem 1. *The completion algorithm for Prob-$\mathcal{ELO}_c^{01}$ is sound and complete.*

We prove each of the claims in the following. Please note that the consequence-driven algorithms for $\mathcal{ELO}$ from [11] and also the completion algorithm presented in Section 3 compute subsumption relationships *under the assumption* that certain concepts have a non-empty interpretation. To indicate the assumption that, say G is not empty, when considering the subsumption relationship between C and D, we write $G : C \sqsubseteq_{\mathcal{T}} D$.

Lemma 1. *The completion algorithm is sound, i.e.*

$$C \in S_*^G(X, v) \text{ implies } G : P_* X \sqsubseteq_{\mathcal{T}} P_v C \tag{1}$$

$$C \in S_*^G(X, r, v) \text{ implies } G : P_* X \sqsubseteq_{\mathcal{T}} P_v \exists r.C \tag{2}$$

Proof. We show this by induction on the number of rule applications. It is easy to see that the initial subsumer sets satisfy (1) and (2). Also, after each rule application (1) and (2) will still be satisfied. We will only show (1) for the new completion rules for Prob-$\mathcal{ELO}_c^{01}$, which are not in the original completion algorithm in [13]. For property (2), note that none of these new rules changes the subsumer sets $S_*^G(X, r, v)$.

- **PR5** If $\{a\} \in S_{*_1}^G(X, *_1) \cap S_{*_2}^G(D, *_2)$, then by induction hypothesis we have $G : P_{*_1} X \sqsubseteq_{\mathcal{T}} \{a\}$ and $G : P_{*_2} D \sqsubseteq_{\mathcal{T}} \{a\}$. Additionally, by definition of $\leadsto_R$, $G \overset{*_2}{\leadsto}_R D$ implies that if G is not empty, then $P_{*_2} D$ must be not empty as well. Thus we have $G : P_{*_2} D \equiv_{\mathcal{T}} \{a\}$ and therefore $G : P_{*_1} X \sqsubseteq P_{*_2} D$. This means, that the addition of $P_{*_2} D$ to $S_{*_1}^G(X, *_1)$ still satisfies (1).
- **PR7** If $P_{=1} A \in S_*^G(X, 0)$, then by induction hypothesis $G : P_* X \sqsubseteq_{\mathcal{T}} P_{=1} A$. Thus the implication $A \in S_*^G(X, 1) \Rightarrow G : P_* X \sqsubseteq_{\mathcal{T}} P_{=1} A$ is obviously correct, and we can add A to $S_*^G(x, 1)$.
- **PR11** If $\{a\} \in S_*^G(X, v)$, then by induction hypothesis $G : P_* X \sqsubseteq_{\mathcal{T}} P_v \{a\}$, i.e. for all models $\mathcal{I} = (\Delta^{\mathcal{I}}, W, (\mathcal{I}_w)_{w \in W}, \mu)$ of $\mathcal{T}$ and all worlds $w \in W$ we have: if G is not empty, then $(P_* X)^{\mathcal{I}, w} \subseteq (P_v \{a\})^{\mathcal{I}, w}$. Together with

  $$\begin{aligned}(P_{>0}\{a\})^{\mathcal{I},w} &= \{d \mid \exists v \in W : \mu(v) > 0 \wedge d \in \{a\}^{\mathcal{I},v}\} \\ &= \{d \mid \exists v \in W : \mu(v) > 0 \wedge d = a^{\mathcal{I}}\} = \{a^{\mathcal{I}}\} = \{a\}^{\mathcal{I},w}\end{aligned}$$

  and similarly $(P_{=1}\{a\})^{\mathcal{I},w} = \{a\}^{\mathcal{I},w}$, this yields: if G is not empty, then $(P_* X)^{\mathcal{I},w} \subseteq \{a\}^{\mathcal{I},w} = \{a^{\mathcal{I}}\} = \{a\}^{\mathcal{I},v'}$ for all $v' \in W$. Hence it holds that $G : P_* X \sqsubseteq_{\mathcal{T}} P_{v'} \{a\}$. This means, that the addition of $\{a\}$ to $S_*^G(X, v')$ still satisfies (1).

Lemma 2. *The completion algorithm is complete, i.e. for a normalized TBox $\mathcal{T}$, $G \in N_C$, $B \in \mathsf{BC}\mathcal{T}$, and $r \in N_R$ we have*

$$G \sqsubseteq_{\mathcal{T}} B \text{ implies } B \in S_0^G(G, 0)$$

$$G \sqsubseteq_{\mathcal{T}} \exists r.B \text{ implies } \exists A \text{ with } A \in S_0^G(G, r, 0) \text{ and } B \in S_0^G(A, 0)$$

Proof. We assume that $B \notin S_0^G(G,0)$ (resp. there is no A with $A \in S_0^G(G,r,0)$ and $B \in S_0^G(A,0)$) and construct a model $\mathcal{I}_G$ of $\mathcal{T}$ which shows that $G \not\sqsubseteq_{\mathcal{T}} B$ (resp. $G \not\sqsubseteq_{\mathcal{T}} \exists r.B$). To construct this model, we need the classes of equivalent nominals: $[a] = \{b \in \mathsf{Sig}(\mathcal{T}) \cap N_I \mid \{a\} \in S_0^G(\{b\},0)\}$. The domain of the interpretation will contain all nominals (modulo equivalence) and for each world $w \in V$ all concepts that are not subsumed by a nominal and can be reached from G or a nominal using the relation $\rightsquigarrow_R$.

Let $\mathcal{I}_G = (\Delta^{\mathcal{I}_G}, W, (\mathcal{I}_{G,w})_{w\in W}, \mu)$ be the following interpretation:

$$\begin{aligned}
\Delta^{\mathcal{I}_G} &:= \{[a] \mid a \in \mathsf{Sig}(\mathcal{T}) \cap N_I\} \cup \\
&\qquad \{(A,v) \in \mathsf{Sig}(\mathcal{T}) \cap N_C \times V \mid G \overset{\gamma(v)}{\rightsquigarrow}_R A, \{a\} \notin S_{\gamma(v)}^G(A,\gamma(v))\} \\
W &:= V \\
\mu(0) &:= 0 \\
\mu(w) &:= \frac{1}{|W \setminus \{0\}|} \qquad \text{for all } w \in W \setminus \{0\} \\
a^{\mathcal{I}_G} &= [a] \qquad \text{for all } a \in \mathsf{Sig}(\mathcal{T}) \cap N_I
\end{aligned}$$

To interpret concept and role names, we also need a bijection $\pi_v(w) : W \to W$ for each $v \in W \setminus \{0\}$ with $\pi_v(v) = \varepsilon$ and $\pi_v(0) = 0$. Moreover, π_0 is the identity mapping on W. Then:

$$\begin{aligned}
A^{\mathcal{I}_G,w} = &\{[a] \mid A \in S_0^G(\{a\},w)\} \\
&\cup \{(B,v) \in \Delta^{\mathcal{I}_G} \mid A \in S_{\gamma(v)}^G(B,\pi_v(w))\} \\
r^{\mathcal{I}_G,w} = &\{([a],[b]) \in \Delta^{\mathcal{I}_G} \times \Delta^{\mathcal{I}_G} \mid \exists A : A \in S_0^G(\{a\},r,w) \wedge \{b\} \in S_{\gamma(w)}^G(A,\gamma(w))\} \\
&\cup \{([a],(A,w)) \in \Delta^{\mathcal{I}_G} \times \Delta^{\mathcal{I}_G} \mid A \in S_0^G(\{a\},r,w)\} \\
&\cup \{((B,v),[b]) \in \Delta^{\mathcal{I}_G} \times \Delta^{\mathcal{I}_G} \mid \exists A : A \in S_{\gamma(v)}^G(B,r,\pi_v(w)) \\
&\qquad \wedge \{b\} \in S_{\gamma(w)}^G(A,\gamma(w))\} \\
&\cup \{((B,v),(A,w)) \in \Delta^{\mathcal{I}_G} \times \Delta^{\mathcal{I}_G} \mid A \in S_{\gamma(v)}^G(B,r,\pi_v(w))\}
\end{aligned}$$

Before proving that $\mathcal{I}_G$ is indeed a model of $\mathcal{T}$, we generalize the definition of $A^{\mathcal{I}_G,w}$ to probabilistic concepts:

$$X \in S_{\gamma(v)}^G(B,\pi_v(w)) \text{ iff } (B,v) \in X^{\mathcal{I}_G,w} \qquad \text{for } X \in \mathsf{BC}_{\mathcal{T}}, (B,v) \in \Delta^{\mathcal{I}_G} \tag{3}$$

$$X \in S_0^G(\{a\},0) \text{ iff } [a] \in X^{\mathcal{I}_G,w} \qquad \text{for } X \in \mathsf{BC}_{\mathcal{T}}, [a] \in \Delta^{\mathcal{I}_G} \tag{4}$$

To show this, we make a case distinction according to the kind of concept X is. First notice that in (3) X cannot be a nominal, since otherwise B would be subsumed by one and hence not be in the domain $\Delta^{\mathcal{I}_G}$ as we assumed.

- If $X = \top$, then (3) and (4) are true by definition of $\top^{\mathcal{I}_G,w} = \Delta^{\mathcal{I}_G}$ and the fact that $\top$ is in each subsumer set.
- If $X = A \in N_C$, then (3) and (4) are true by definition of $A^{\mathcal{I}_G,w}$.

- If $X = P_{>0}A$. For the "$\Rightarrow$" direction, let $P_{>0}A \in S^G_{\gamma(v)}(B, \pi_v(w))$. By rule PR6, we have $A \in S^G_{\gamma(v)}(B, P_{>0}A)$ and by definition of $\mathcal{I}_G$ $(B, v) \in A^{\mathcal{I}_G,u}$ with $\pi_v(u) = P_{>0}A$. By definition of π_v and $\mathcal{I}_G$ this means $\mu(u) > 0$ and thus $(B, v) \in (P_{>0}A)^{\mathcal{I}_G,w}$. For the "$\Leftarrow$" direction, let $(B, v) \in (P_{>0}A)^{\mathcal{I}_G,w}$, i.e. there is $u \in W \setminus \{0\}$ with $(B, v) \in A^{\mathcal{I}_G,u}$. The definition of $\mathcal{I}_G$ yields $A \in S^G_{\gamma(v)}(B, \pi_v(u))$ with $\pi_v(u) \neq 0$ by definition of π_v. Then by rule **PR9** $P_{>0}A \in S^G_{\gamma(v)}(B, \pi_v(w))$.
- If $X = P_{=1}A$. For the "$\Rightarrow$" direction, let $P_{=1}A \in S^G_{\gamma(v)}(B, \pi_v(w))$. By rules **PR7** and **PR10** we have $P_{=1}A \in S^G_{\gamma(v)}(B, u)$ for all $u \in W$ and by rule **PR8** $A \in S^G_{\gamma(v)}(B, u)$ for all $u \in W \setminus \{0\}$. Since π_v is a bijection on W with $\pi_v(0) = 0$, this also means $A \in S^G_{\gamma(v)}(B, \pi_v(u'))$ for all $u' \in W \setminus \{0\}$ and hence by definition of $\mathcal{I}_G$, $(B, v) \in A^{\mathcal{I}_G,u'}$ for all $u' \in W \setminus \{0\}$. Finally, the definition of μ then yields $(B, v) \in (P_{=1}A)^{\mathcal{I}_G,w}$.
  For the "$\Leftarrow$" direction, let $(B, v) \in (P_{=1}A)^{\mathcal{I}_G,w}$, i.e. for all $u \in W \setminus \{0\}$ we have $(B, v) \in A^{\mathcal{I}_G,u}$, especially for u' with $\pi_v(u') = 1$. The definition of $\mathcal{I}_G$ yields $A \in S^G_{\gamma(v)}(B, \pi_v(u') = 1)$ and by rule **PR10** $P_{=1}A \in S^G_{\gamma(v)}(B, w')$ for all $w' \in W$, especially $P_{=1}A \in S^G_{\gamma(v)}(B, \pi_v(w))$.

This interpretation $\mathcal{I}_G$ is indeed a model of $\mathcal{T}$, which we will show using a case distinction on the types of GCIs in $\mathcal{T}$.

- $C \sqsubseteq D \in \mathcal{T}$. Let $(B, v) \in C^{\mathcal{I}_G,w}$, then (3) yields $C \in S^G_{\gamma(v)}(B, \pi_v(w))$ and by rule **PR1** also $D \in S^G_{\gamma(v)}(B, \pi_v(w))$. (3) then yields $(B, v) \in D^{\mathcal{I}_G,w}$.
  Let $[a] \in C^{\mathcal{I}_G,w}$, then $C \in S^G_0(\{a\}, 0)$ by (4) and by rule **PR1** it also holds that $D \in S^G_0(\{a\}, 0)$. (4) then yields $[a] \in D^{\mathcal{I}_G,w}$.
- $C_1 \sqcap C_2 \sqsubseteq D \in \mathcal{T}$. Let $(B, v) \in (C_1 \sqcap C_2)^{\mathcal{I}_G,w}$, i.e. by the semantics of $\sqcap$ $(B, v) \in C_1^{\mathcal{I}_G,w}$ and $(B, v) \in C_2^{\mathcal{I}_G,w}$. Then (3) yields $C_1, C_2 \in S^G_{\gamma(v)}(B, \pi_v(w))$, and by rule **PR2** $D \in S^G_{\gamma(v)}(B, \pi_v(w))$. (3) then yields $(B, v) \in D^{\mathcal{I}_G,w}$.
  Let $[a] \in (C_1 \sqcap C_2)^{\mathcal{I}_G,w}$, i.e. $[a] \in C_1^{\mathcal{I}_G,w}$ and $[a] \in C_2^{\mathcal{I}_G,w}$. We then have $C_1, C_2 \in S^G_0(\{a\}, 0)$ by (4) and by rule **PR2** also $D \in S^G_0(\{a\}, 0)$. (4) then yields $[a] \in D^{\mathcal{I}_G,w}$.
- $C \sqsubseteq \exists r.A$. Let $(B, v) \in C^{\mathcal{I}_G,w}$, then (3) yields $C \in S^G_{\gamma(v)}(B, \pi_v(w))$ and by rule **PR3** $A \in S^G_{\gamma(v)}(B, r, \pi_v(w))$. Then, there are two cases: If $(A, w) \in \Delta^{\mathcal{I}_G}$, i.e. there is no nominal $\{b\} \in S^G_{\gamma(w)}(A, \gamma(w))$, then the definition of $r^{\mathcal{I}_G,w}$ yields $((B, v), (A, w)) \in r^{\mathcal{I}_G,w}$. By the initialization of the completion sets we also have $A \in S^G_{\gamma(w)}(A, \pi_w(w))$ as $\gamma(w) = \pi_w(w)$ by definition, and thus $(A, w) \in A^{\mathcal{I}_G,w}$. Together with $((B, v), (A, w)) \in r^{\mathcal{I}_G,w}$, this yields $(B, v) \in (\exists r.A)^{\mathcal{I}_G,w}$.
  If $(A, w) \notin \Delta^{\mathcal{I}_G}$, then there is a nominal $\{b\} \in S^G_{\gamma(w)}(A, \gamma(w))$ and the definition of $r^{\mathcal{I}_G,w}$ yields $((B, v), [b]) \in r^{\mathcal{I}_G,w}$. On the other hand, rule **PR11** with $\{b\} \in S^G_{\gamma(w)}(A, \gamma(w))$ yields also $\{b\} \in S^G_{\gamma(w)}(A, 0)$ and then rule **PR5** with $\{b\} \in S^G_0(\{b\}, 0) \cap S^G_{\gamma(w)}(A, 0)$ and $G \overset{\gamma(w)}{\rightsquigarrow}_R A$ shows that $P_{\gamma(w)}A \in S^G_0(\{b\}, 0)$ and thus $A \in S^G_0(\{b\}, w)$, i.e. $[b] \in A^{\mathcal{I}_G,w}$.

Similarly, let $[a] \in C^{\mathcal{I}_G,w}$, then (4) yields $C \in S_0^G(\{a\},0)$. By rule **PR3** it holds that $A \in S_0^G(\{a\},r,0)$. Again, we have the two cases as before, which can be shown analogously.

– $\exists r.A \sqsubseteq D$. Let $(B,v) \in (\exists r.A)^{\mathcal{I}_G,w}$, i.e. there is an $\alpha \in \Delta^{\mathcal{I}_G,w}$ such that $((B,v),\alpha) \in r^{\mathcal{I}_G,w}$ and $\alpha \in A^{\mathcal{I}_G,w}$. By definition of $\mathcal{I}_G$, there are two cases. If $\alpha = (C,w) \in \Delta^{\mathcal{I}_G}$, then the definitions of $A^{\mathcal{I}_G,w}$ and $r^{\mathcal{I}_G,w}$ yield $A \in S_{\gamma(w)}^G(C,\pi_w(w))$ and $C \in S_{\gamma(v)}^G(B,r,\pi_v(w))$. Since $\pi_w(w) = \gamma(w)$, and $\gamma(\pi_v(w)) = \gamma(w)$ for all $v \in V$, by rule **PR4** we get $D \in S_{\gamma(v)}^G(B,\pi_v(w))$ and thus by (3) $(B,v) \in D^{\mathcal{I}_G,w}$.
  If $\alpha = [b] \in \Delta^{\mathcal{I}_G}$, the definitions of $A^{\mathcal{I}_G,w}$ and $r^{\mathcal{I}_G,w}$ yield $A \in S_0^G(\{b\},w)$ and there exists C such that $C \in S_{\gamma(v)}^G(B,r,\pi_v(w))$ and $\{b\} \in S_{\gamma(w)}^G(C,\gamma(w))$. Because of $\{b\} \in S_{\gamma(w)}^G(C,\gamma(w))$, we have $S_{\gamma(w)}^G(C,\gamma(w)) \supseteq S_0^G(\{b\},w)$ (which can be shown by induction on the number of rule applications to the latter), and hence also $A \in S_{\gamma(w)}^G(C,\gamma(w))$. Since $C \in S_{\gamma(v)}^G(B,r,\pi_v(w))$, $\pi_w(w) = \gamma(w)$, and $\gamma(\pi_v(w)) = \gamma(w)$ for all $v \in V$ hold, rule **PR4** finally yields $D \in S_{\gamma(v)}^G(B,\pi_v(w))$ and thus by (3) $(B,v) \in D^{\mathcal{I}_G,w}$.
  Similarly, let $[a] \in (\exists r.A)^{\mathcal{I}_G,w}$, i.e. there is an $\alpha \in \Delta^{\mathcal{I}_G,w}$ with $([a],\alpha) \in r^{\mathcal{I}_G,w}$ and $\alpha \in A^{\mathcal{I}_G,w}$. Again, we have the two cases as before, which can be shown analogously.

Finally, by the assumption $B \notin S_0^G(G,0)$ and the definition of $\mathcal{I}_G$ we have $(G,0) \notin B^{\mathcal{I}_G,0}$, whereas $G \in S_0^G(G,0)$ yields $(G,0) \in G^{\mathcal{I}_G,0}$. Since $\mathcal{I}_G$ is a model of $\mathcal{T}$, this proves $G \not\sqsubseteq_{\mathcal{T}} B$.

The second case is similar. If we assume that there exists no concept A with $A \in S_0^G(G,r,0)$ and $B \in S_0^G(A,0)$, then by definition of the interpretation $\mathcal{I}_G$, there is no element $\alpha \in \Delta^{\mathcal{I}_G}$ with $((G,0),\alpha) \in r^{\mathcal{I}_G,0}$ and $\alpha \in B^{\mathcal{I}_G,0}$. Since $\mathcal{I}_G$ is a model of $\mathcal{T}$, this shows that $G \not\sqsubseteq_{\mathcal{T}} \exists r.B$.

A.2 Correctness of the *k*-lcs Algorithm for Prob-$\mathcal{ELO}_c^{01}$

Lemma 3. *Let $\mathcal{T}$ be a Prob-$\mathcal{ELO}_c^{01}$-TBox, $\mathcal{T}'$ be the TBox obtained from $\mathcal{T}$ by applying the normalization rules,* S *be the set of completion sets obtained from $\mathcal{T}'$, A, B be concept names, X, Y be basic concepts with $A \rightsquigarrow_R X, B \rightsquigarrow_R Y$, k be a natural number and $L =$ k-lcs-r(X,Y,S,k,A,B). Then $X \sqsubseteq_{\mathcal{T}'} L$ and $Y \sqsubseteq_{\mathcal{T}'} L$.*

Proof. Similar to the crisp case, this lemma can be shown by induction on k for the recursive procedure k-lcs-r. For the case $k = 0$, the result

$$L = \bigsqcap_{E \in S_0^A(X,0) \cap S_0^B(Y,0) \cap \mathsf{BC}_{\mathcal{T}}} E$$

of k-lcs-r is a conjunction of (possibly probabilistic) concept names, but no existential restrictions. By soundness of the completion rules, we know that $E \in S_0^A(X,0) \cap S_0^B(Y,0)$ implies $X \sqsubseteq_{\mathcal{T}'} E$ and $Y \sqsubseteq_{\mathcal{T}'} E$. Since L contains exactly those conjuncts, we also have $X \sqsubseteq_{\mathcal{T}'} L$ and $Y \sqsubseteq_{\mathcal{T}'} L$.

For the case $k > 0$, L is a conjunction of (possibly probabilistic) concept names and existential restrictions $\exists r.E$, $P_{=1}\exists r.E$, and $P_{>0}\exists r.E$. For the concept names, the same argument as for the case $k = 0$ applies. For existential restrictions of the form $\exists r.\mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k{-}1, A, B)$ with

$$(E, F) \in S_0^A(X, r, 0) \times S_0^B(Y, r, 0),$$

we know that $E \in S_0^A(X, r, 0)$ implies $X \sqsubseteq_{\mathcal{T}'} \exists r.E$ by soundness of the completion algorithm, and similarly $Y \sqsubseteq_{\mathcal{T}'} \exists r.F$. Then the induction hypothesis yields $E \sqsubseteq_{\mathcal{T}'} L'$ and $F \sqsubseteq_{\mathcal{T}'} L'$ for $L' = \mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k{-}1, A, B)$ and thus also $X \sqsubseteq_{\mathcal{T}'} \exists r.L'$ and $Y \sqsubseteq_{\mathcal{T}'} \exists r.L'$.

Similarly, by soundness we get that $E \in S_0^A(X, r, 1)$ implies $X \sqsubseteq_{\mathcal{T}'} P_{=1}\exists r.E$ respectively $E \in \mathsf{PR}^A(X, r)$ implies $X \sqsubseteq_{\mathcal{T}'} P_{>0}\exists r.E$ and by induction hypothesis $E \sqsubseteq_{\mathcal{T}'} \mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k{-}1, A, B)$, thus $X \sqsubseteq_{\mathcal{T}'} P_{=1}\exists r.\mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k{-}1, A, B)$, respectively $X \sqsubseteq_{\mathcal{T}'} P_{>0}\exists r.\mathsf{k\text{-}lcs\text{-}r}(E, F, \mathsf{S}, k{-}1, A, B)$. All together, this means $X \sqsubseteq_{\mathcal{T}'} L$. The case for $Y \sqsubseteq_{\mathcal{T}'} L$ is analogous.

Lemma 4. *Let $\mathcal{T}$ be a Prob-$\mathcal{ELO}_c^{01}$-TBox, $\mathcal{T}'$ be the TBox obtained from $\mathcal{T}$ by applying the normalization rules, S be the set of completion sets obtained from $\mathcal{T}'$, A, B be concept names, X, Y be basic concepts with $A \rightsquigarrow_R X, B \rightsquigarrow_R Y$, k be a natural number and $L = \mathsf{k\text{-}lcs\text{-}r}(X, Y, \mathsf{S}, k, A, B)$. Then for each Prob-$\mathcal{ELO}_c^{01}$-concept F with $\mathsf{Sig}(F) \subseteq \mathsf{Sig}(\mathcal{T})$ and $rd(F) \leq k$, $X \sqsubseteq_{\mathcal{T}'} F$ and $Y \sqsubseteq_{\mathcal{T}'} F$ imply $L \sqsubseteq_{\mathcal{T}'} F$.*

Proof. By induction on the role-depth $rd(F)$. Let $rd(F) = 0$, i.e. $F = \bigsqcap E$ contains no existential restrictions. Since $X \sqsubseteq_{\mathcal{T}'} F$ and $Y \sqsubseteq_{\mathcal{T}'} F$, we also have $X \sqsubseteq_{\mathcal{T}'} E$ and $Y \sqsubseteq_{\mathcal{T}'} E$ for all conjuncts E of F. Then, completeness of the algorithm yields that $E \in S_0^X(X, 0)$ and since $A \rightsquigarrow_R X$ also $E \in S_0^A(X, 0)$. Similarly, we have $E \in S_0^B(Y, 0)$ for all conjuncts E of F and thus

$$L = \bigsqcap_{E \in S_0^A(X,0) \cap S_0^B(Y,0) \cap \mathsf{BC}_{\mathcal{T}}} E \sqsubseteq_{\mathcal{T}'} F.$$

If $rd(F) > 0$, F may contain two kinds of conjuncts: basic concepts and (possibly probabilistic) existential restrictions. The basic concepts in F must appear in L as well by an argument analog to the case $rd(F) = 0$. Let $\exists r.F'$ be a top-level conjunct of F. Since $X \sqsubseteq_{\mathcal{T}'} F$ and $Y \sqsubseteq_{\mathcal{T}'} F$, completeness yields that there exists an $E \in S_0^A(X, r, 0)$ such that $F' \in S_0^A(E, 0)$ (i.e. $E \sqsubseteq_{\mathcal{T}'} F'$), and an $E' \in S_0^B(Y, r, 0)$ such that $F' \in S_0^B(E', 0)$ (i.e. $E' \sqsubseteq_{\mathcal{T}'} F'$). By induction hypothesis, it follows that $\mathsf{k\text{-}lcs\text{-}r}(E, E', \mathsf{S}, k{-}1, A, B) \sqsubseteq_{\mathcal{T}'} F'$, thus $L \sqsubseteq_{\mathcal{T}'} \exists r.\mathsf{k\text{-}lcs\text{-}r}(E, E', \mathsf{S}, k{-}1, A, B) \sqsubseteq_{\mathcal{T}'} \exists r.F'$. The other two cases of probabilistic existential conjuncts $P_{=1}\exists r.F'$ and $P_{>0}\exists r.F'$ of F are similar. Together, this implies $L \sqsubseteq_{\mathcal{T}'} F$.

Together, Lemmata 3 and 4 fulfill all requirements of the role-depth bounded least common subsumer. Thus, the following theorem is a direct consequence of both lemmas and the fact that the $\mathsf{k\text{-}lcs}$ procedure introduces new concept names

A and B for the concepts C and D and then calls the procedure k-lcs-r for these new concept names A and B, using the completion sets of the extended and normalized TBox.

Theorem 2. *Let $\mathcal{T}$ be a Prob-$\mathcal{ELO}_c^{01}$-TBox, C and D be Prob-$\mathcal{ELO}_c^{01}$-concepts and k be a natural number. Then* k-lcs$(C, D, \mathcal{T}, k)$ *is the role-depth bounded least common subsumer of C and D w.r.t. $\mathcal{T}$ and the role-depth k.*