# Conformal coupling of gravitational wave field to curvature

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(Received 20 November 1979; accepted for publication 7 December 1979)

Conformal properties of the equations for weak gravitational waves in a curved space–time are investigated. The basic equations are derived in the linear approximation from Einstein's equations. They represent, in fact, the equations for the second-rank tensor field $h_{\alpha\beta}$, restricted by the auxiliary conditions $h_\alpha{}^\beta{}_{;\alpha} = 0$, $h \equiv \gamma_{\alpha\beta} h^{\alpha\beta} = 0$, and embedded into the background space–time with the metric tensor $\gamma_{\alpha\beta}$. It is shown that the equations for $h_{\alpha\beta}$ are not conformally invariant under the transformations $\hat{\gamma}_{\alpha\beta} = e^{2\sigma}\gamma_{\alpha\beta}$ and $\hat{h}_{\alpha\beta} = e^{\sigma} h_{\alpha\beta}$, except for those metric rescalings which transform the Ricci scalar $\hat{R}$ of the original background space–time into $e^{-2\sigma}R$, where $R$ is the Ricci scalar of the conformally related background space–time. The general form of the equations for $h_{\alpha\beta}$ which are conformally invariant have been deduced. It is shown that these equations cannot be derived in the linear approximation from any tensor equations which generalize the Einstein equations.

## I. INTRODUCTION

Conformal symmetry appears to be important in contemporary physics (see, for example, Refs. 1 and 2). Usually, the invariance with respect to the 15-parameter Lie group of conformal transformations $C_0$ which generalizes the Poincaré group is meant by a conformal symmetry.[3,4] A more general kind of conformal transformation is a conformal rescaling of the metric tensor: $\tilde{g}_{\mu\nu} = e^{-2\sigma} g_{\mu\nu}$. Conformal rescalings are mostly applied to the equations that are written down in a generally covariant form. The conformal transformations, viewed as conformal rescaling, correspond to that particular case in which a flat space–time Minkowski metric transforms into another flat space–time metric. Even this simplest kind of conformal symmetry which is important for high-energy physics may have some relevance to gravity as well (not curvature!) since a conformal transformation may be interpreted as a transformation to a constantly accelerated frame of reference.[3] Therefore, a property of a physical system with respect to $C_0$ may describe the behavior of the system in a constant homogeneous gravitational field.

It has been known already for a long time[3,5] that some basic equations of theoretical physics, among them the equations for massless fields, are invariant not only with respect to $C_0$ but also with respect to the group $C_g$ of conformal rescalings. For instance, the field equations for massless fields with integer spins remain unchanged under a replacement of $g_{\mu\nu}$ and field variables $\varphi_{\alpha\beta\cdots\nu}$ according to the rule $\tilde{g}_{\mu\nu} = e^{-2\sigma} g_{\mu\nu}$, $\tilde{\varphi}_{\alpha\beta\cdots\nu} = e^{-\sigma(s-1)}\varphi_{\alpha\beta\cdots\nu}$, where $s$ is the spin of the field. It is important to notice that the field variables transform with different powers of the conformal factor $e^{-2\sigma}$, depending on the spin of the field. For the scalar field it is $\tilde{\varphi} = e^{\sigma}\varphi$, and for the Maxwell equations ($s = 1$) $\tilde{A}_\alpha = A_\alpha$ or $\tilde{F}_{\alpha\beta} = F_{\alpha\beta}$. For the gravitational field ($s = 2$) the conformal invariance is usually referred to the vacuum Bianchi identities with the Weyl tensor transforming as $\tilde{C}_{\alpha\beta\mu\nu} = e^{-\sigma} C_{\alpha\beta\mu\nu}$.

Conformal symmetry of the field equations with respect to $C_g$ is important from the physical point of view since it describes the particular way of coupling of the physical system to the external gravitational field (curvature).[6] The role of conformal invariance in the context of quantum field theory in curved space–time has been emphasized many times.[7,8] It was shown in Ref. 9 that the Einstein linearized equations for weak gravitational waves in nonvacuum conformally flat metrices do not transform into the usual flat space–time wave equations under the conformal transformation of the metric tensor and gravitational-wave variables. Thus, graviton creation in the early Universe is possible[9] while other massless particles such as photons, neutrinos, and gravitinos (spin $s = 3/2$ massless particle) cannot be created. (For the properties of the pure supergravity theory in this context see Ref. 10.) This fact seems to be fundamental enough in order to see to which extent it is inevitable.

The purpose of this article is to investigate the conformal property of the gravitational-wave equations in more detail. In particular, we are trying to find such equations which could be conformally invariant.

It is necessary to clarify the difference between our approach and that which was used in other works, devoted to conformal gravitation on the classical and quantum levels.[11-15] We intend to treat the gravitational-wave variables on the same footing as all other fields embedded in a curved space–time. It means that under conformal rescaling the field variables should transform according to their spin $s = 2$ weight. It might be $\tilde{C}_{\alpha\beta\mu\nu} = e^{-\sigma} C_{\alpha\beta\mu\nu}$ according to Penrose's suggestion or $\tilde{h}_{\alpha\beta} = e^{-\sigma} h_{\alpha\beta}$ in a linearized approximation to the Einstein equations which is considered here.

[a)] Supported in part by the Ministry of Higher Education, USSR, and by the National Science Foundation, USA (AST76-80801 A02).

 0022-2488/80/051168-08$01.00  

(They are obviously consistent in the same way as the conformal transformation rules are consistent for electrodynamical field components $\widehat{F}_{\alpha\beta} = F_{\alpha\beta}$ and electrodynamical potentials $\widehat{A}_\alpha = A_\alpha$.) The linearized version of Einstein's equations provides a natural framework for treating spin $s = 2$ fields in an external gravitational field since these equations describe, in fact, the second-rank symmetric tensor field embedded in a curved space–time. On the other hand, the works[11-15] are concerned with the action and the field equations which are invariant under conformal rescaling of the metric tensor $g_{\alpha\beta}$ and some scalar function. If the components of $g_{\alpha\beta}$ are to be interpreted as spin $s = 2$ field variables, then they transform according to the wrong rule; this rule includes the factor $e^{-2\sigma}$ instead of $e^{-\sigma}$. The same rule is prescribed for the second-rank tensor $h_{\alpha\beta}$ at the linearized level. Although this kind of symmetry may be useful for some purposes, it is certainly not what is meant by conformal invariance for other massless field equations.

In Sec. II from Einstein's equations we derive the basic equations for graviational-wave perturbations $h_{\alpha\beta}$. These equations have the same form both in vacuum space–time and in space–time filled with matter. We introduce also the usual auxiliary conditions $h_\alpha{}^\beta{}_{;\beta} = 0$, $h = 0$ which are similar to that used in a flat space–time for separating spin $s = 2$ states.[16] We investigate the conformal properties of the field equations and show that they are not conformally invariant except for those transformations which transform the Ricci scalar $\widehat{R}$ of the original background space–time into $e^{-2\sigma}R$, where $R$ is the Ricci scalar of the conformally related background space–time. Although we believe that the chosen field equations, the auxiliary conditions, and the transformation law for $h_{\alpha\beta}$ are well motivated, one should not think that the conformal noninvariance is a consequence of these assumptions. The formulas presented in the Appendix show that any other choice of the auxiliary conditions (if any) and of a transformation law cannot improve the situation. Moreover, there is an indication that the prescribed auxiliary conditions and the transformation law emerge in a natural way under an attempt to make the basic equations conformally invariant. Having proved conformal noninvariance of the equations derived from Einstein's equations, we were interested in the formulation of the equations for spin $s = 2$ field which are conformally invariant.

Essentially, we look for a conformally invariant second-order differential operator which acts on a symmetric second-rank tensor field restricted by some auxiliary conditions. In other words, we generalize the flat space–time equations in such a way that the coupling of the tensor field to curvature is conformally invariant. To clarify the method used, we start from the simplified problem of finding conformally invariant equations for a scalar field $\varphi$ (see Sec. III). It is known that conformal coupling of the $\varphi$ field to curvature can be represented by the equation

$$\varphi_{;\alpha}{}^{;\alpha} - \frac{R}{6}\varphi = 0. \tag{1}$$

For the sake of generality we take into account some other fields to which the $\varphi$ field can be coupled (other than the curvature) and which transform according to definite rules under a conformal rescaling. It is shown that the most general conformally invariant coupling to curvature is expressed by Eq. (1) while coupling to other fields can also be conformally invariant and then Eq. (1) contains additional terms. The same method of searching for conformally invariant equations was applied to spin $s = 2$ field (Sec. IV). Since in this case the equations are more complicated, we restricted the search to the coupling of this field to curvature. The general form of such conformally invariant equations is deduced. It is seen that these equations could not be derived from Einstein's equations in the linearized approximation. The next step is to look for exact tensorial equations from which conformally invariant equations can follow in the linearized approximation (Sec. V). If such a theory existed it might be interesting to investigate it and compare its predictions with the predictions of Einstein's theory. Quite surprisingly, it turns out that such an exact theory does not exist, at least within those restrictions which were imposed on it. It is also shown that there exists a conformally invariant equation describing the coupling of the second-rank tensor field to curvature and some additional scalar field. The possibility of finding an exact theory which would yield this equation in the linearized limit is not clear. In conclusion (Sec. VI), we give a discussion of the presented results.

## II. CONFORMAL NONINVARIANCE OF THE LINEARIZED EINSTEIN EQUATIONS

First we will derive the equations which we will be working with. Let us start from the vacuum Einstein equations with the cosmological term

$$R_{\mu\nu} = \lambda g_{\mu\nu}.$$

Assume that $g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the metric tensor of a background space–time, and assume that the background field equations $R^{(0)}_{\mu\nu} = \lambda\gamma_{\mu\nu}$ are fulfilled. The linearized equations $-(\gamma_{\mu\alpha}\delta R_\nu{}^\alpha + \gamma_{\nu\alpha}\delta R^\alpha{}_\mu) = 0$ or $-2\delta R_{\mu\nu} + 2\lambda h_{\mu\nu} = 0$ both lead to the same equation

$$h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\mu\alpha\beta\nu}h^{\alpha\beta} - (h_\mu{}^\alpha - \tfrac{1}{2}\delta_\mu{}^\alpha h)_{;\alpha;\nu} - (h_\nu{}^\alpha - \tfrac{1}{2}\delta_\nu{}^\alpha h)_{;\alpha;\mu} = 0, \tag{2}$$

where, as usual, all operations are performed in background space–time. $R_{\mu\alpha\beta\nu}$ denotes the background curvature tensor; here and below we will not especially mark the background quantities. Equation (2) can be regarded as a generally covariant equation for a symmetric second-rank tensor field $h_{\mu\nu}$.

Choose the solutions to Eq. (2) which are subject to the auxiliary conditions

$$h \equiv h_{\alpha\beta}\gamma^{\alpha\beta} = 0, \tag{3}$$

$$h_\mu{}^\alpha{}_{;\alpha} = 0. \tag{4}$$

In analogy to what is known for analogous equations in flat space–time, Eqs. (3) and (4) can be interpreted as the necessary conditions for removing the spin $s = 0$, and $s = 1$ contributions to $h_{\mu\nu}$. For these solutions Eq. (2) takes the form

$$h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\mu\alpha\beta\nu}h^{\alpha\beta} = 0. \tag{5}$$

[Of course, Eq. (2) can be reduced to Eq. (5) under the simpler condition

$$\chi_\mu \equiv (h_\mu{}^\alpha - \tfrac{1}{2}\delta_\mu{}^\alpha h)_{;\alpha} = 0. \tag{4'}$$

We are going to work with Eqs. (3)–(5) but before this let us see which part takes the solutions restricted by Eqs. (3) and (4) (we will call them spin $s = 2$ solutions), among all solutions to Eq. (2).

It can be easily checked using the background field equations that if $h^*_{\mu\nu}$ is a solution to Eq. (2) then

$$h_{\mu\nu} = h^*_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \tag{6}$$

for arbitrary $\xi_\mu$ is also a solution to Eq. (2). This fact is frequently referred to as a gauge freedom.[17–19] For any given solution $h^*_{\mu\nu}$ one can find a vector $\xi_\mu$ which will map this solution into the class of solutions restricted by Eqs. (3) and (4).[20] Therefore, the spin $s = 2$ solutions represent in a sense all the solutions to Eq. (2). Moreover, the spin $s = 2$ solutions map into themselves by the gauge transformations with $\xi_{\mu;\nu}{}^{;\nu} = 0$, $\xi^\nu{}_{;\nu} = 0$. The remaining gauge freedom can be used to impose the initial conditions $(h_{\mu\nu}u^\nu)_{|\Sigma} = 0$, $(h_{\mu\nu}u^\nu)_{;\alpha}n^\alpha{}_{|\Sigma} = 0$ on some hypersurface $\Sigma$ with the normal vector $n^\alpha$, where $u^\alpha$ is a vector field. It was shown in Ref. 19 that the sufficient condition for $h_{\mu\nu}u^\nu$ to be equal to zero not only on $\Sigma$ but also off $\Sigma$ is the existence of $u^\alpha$ obeying the equation

$$u_{\mu;\nu} = u_\mu a_\nu + b\gamma_{\mu\nu}, \tag{7}$$

where $a_\nu$ and $b$ are arbitrary vector and scalar fields, respectively. In flat space–time such a vector $u^\alpha$ does exist and therefore all solutions to Eq. (2) can be mapped into a class of solutions which fulfill Eqs. (3) and (4) and

$$h_{\mu\nu}u^\nu = 0 \tag{8}$$

(TT gauge, according to Ref. 17). We will call this class of solutions the spin $s = 2$ solutions with definite helicity.

As for the curved space–time in general Eq. (7) is not integrable, except for a certain class of background metrics, among them the important case of conformally flat metrics.[21] So in these cases the spin $s = 2$ solutions with definite helicity represent all solutions to the wave equations, similarly to what we have in flat space–time.

Let us turn now to gravitational-wave equations in a nonvacuum space–time. The Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = T_{\mu\nu}$ in the linearized approximation $-\gamma_{\mu\alpha}(\delta R^\alpha_\nu - \frac{1}{2}\delta^\alpha{}_\nu \delta R) - \gamma_{\nu\alpha}(\delta R_\mu{}^\alpha - \frac{1}{2}\delta^\alpha{}_\mu \delta R)$ $= -(\gamma_{\mu\alpha}\delta T^\alpha{}_\nu + \gamma_{\nu\alpha}\delta T^\alpha{}_\mu)$ have the following form:

$$\begin{aligned} h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\mu\alpha\beta\nu}h^{\alpha\beta} - \chi_{\mu;\nu} - \chi_{\nu;\mu} \\ + \gamma_{\mu\nu}(-h_{\alpha\beta}R^{\alpha\beta} + \chi_{\alpha;}{}^\alpha - \tfrac{1}{2}h_{;\alpha}{}^{;\alpha}) \\ = -(\gamma_{\mu\alpha}\delta T^\alpha{}_\nu - \gamma_{\nu\alpha}\delta T^\alpha{}_\mu). \end{aligned} \tag{9}$$

Equation (9), similarly to Eq. (2), is gauge invariant. If $h^*_{\mu\nu}$ and $\delta T^*_{\mu\nu}$ are a solution to Eq. (9), then $h_{\mu\nu}$, defined by Eq. (6), and $\delta T_{\mu\nu}$, defined by

$$\delta T_{\mu\nu} = \delta T^*_{\mu\nu} + T_\mu{}^\alpha \xi_{\alpha;\nu} + T_\nu{}^\alpha \xi_{\alpha;\mu},$$

are also a solution to Eq. (9). For any given solution one can find $\xi_\alpha$ which will map this solution into a class of solutions subjected to Eq. (4') (see Ref. 20). Moreover, there still remains some gauge freedom $\xi_{\mu;}{}^\alpha{}_{;\alpha} + \xi^\alpha R_{\alpha\mu} = 0$ which can be used to impose the zero-initial conditions for $h$, $h_{;\alpha}n^\alpha$, $h_{\mu\nu}u^\nu$, $(h_{\mu\nu}u^\nu)_{;\alpha}n^\alpha$ on some initial hypersurface $\Sigma$. However, in the general case, $h$ and $h_{\mu\nu}u^\nu$ will not vanish off $\Sigma$.

Equation (9) includes metric perturbations as well as perturbations of $T_{\mu\nu}$. It is clear, however, that the source-free gravitational-wave perturbations should be associated in some sense with the perturbations of the gravitational field itself and not the matter. We shall define the spin $s = 2$ solutions in a nonvacuum background as a class of solutions for which Eqs. (3) and (4) are valid together with $\delta T_\alpha{}^\beta = 0$. The last condition reduces Eq. (9) to the form of Eq. (2) and $\delta T_\alpha{}^\beta = 0$ together with Eq. (4') reduces it to the equation

$$h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\mu\alpha\beta\nu}h^{\alpha\beta} = 0, \tag{10}$$

which is exactly the form of Eq. (5) (but $R_{\mu\nu} - \lambda g_{\mu\nu} \neq 0$ now). The fact that Eqs. (5), (10), (3), and (4) formally coincide in a vacuum and in a nonvacuum background corresponds to an intuitive feeling that a free gravitational wave should be "sensitive" to a curvature in the same way, independently of what is the source of that curvature. The other argument in favor of Eqs. (10), (3), and (4) is that for those space–times (for instance, for Robertson–Walker background metrics) for which a unique decomposition of all perturbations into proper modes is possible, the tensorial (gravitational wave) modes obey these equations (cf. Ref. 22).

It is important that we impose the condition $\delta T_\alpha{}^\beta = 0$ (variation of $T_\alpha{}^\beta$ with mixed indices.) Other authors sometimes define the gravitational-wave perturbations as the set of conditions $\delta T_{\mu\nu} = 0$ or $\delta(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) = 0$ together with Eqs. (3) and (4). These alternative equations differ from Eq. (10) and do not lead to tensorial proper modes in symmetric backgrounds, which we regard as an unsatisfactory drawback. As far as conformal invariance is concerned, these alternative equations are not conformally invariant.

Thus, we regard Eqs. (10), (3), and (4) as those whse conformal property should be investigated. We will mark by carets all quantities in an original space–time and apply the conformal transformation

$$\hat{\gamma}_{\mu\nu} = e^{2\sigma}\gamma_{\mu\nu}, \tag{11}$$

$$\hat{h}_{\mu\nu} = e^{\sigma}h_{\mu\nu}, \tag{12}$$

to careted Eqs. (10), (3), and (4). Under the transformation (11) the Ricci scalar $\hat{R}$, Ricci tensor $\hat{R}_{\mu\nu}$, and Weyl tensor $\hat{C}^\alpha{}_{\mu\nu\beta}$ transform as follows:

$$\hat{R} = e^{-2\sigma}[R - 6(\sigma_{\alpha;}{}^\alpha + \sigma_\alpha\sigma^\alpha)], \tag{13}$$

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - 2\sigma_{\mu;\nu} + 2\sigma_\mu\sigma_\nu - (\sigma_{\alpha;}{}^\alpha + 2\sigma_\alpha\sigma^\alpha)\gamma_{\mu\nu}, \tag{14}$$

$$\hat{C}^\alpha{}_{\beta\mu\nu} = C^\alpha{}_{\beta\mu\nu}, \quad \sigma_\alpha \equiv \sigma_{,\alpha}. \tag{15}$$

Recall also the relation between $C^\alpha{}_{\beta\mu\nu}$ and the curvature tensor $R^\alpha{}_{\beta\mu\nu}$:

$$\begin{aligned} C^\alpha{}_{\beta\mu\nu} = R^\alpha{}_{\beta\mu\nu} + \tfrac{1}{2}(\delta^\alpha{}_\nu R_{\beta\mu} - \delta^\alpha{}_\mu R_{\beta\nu} + g_{\beta\mu}R^\alpha{}_\nu \\ - g_{\beta\nu}R^\alpha{}_\mu) - \frac{R}{6}(\delta^\alpha{}_\nu g_{\beta\mu} - \delta^\alpha{}_\mu g_{\beta\nu}), \end{aligned}$$

which helps to restore the transformation rule for $R^\alpha{}_{\beta\mu\nu}$. One can see that $\hat{h} = e^{-\sigma}h$ and $\hat{h}_\mu{}^\nu{}_{;\nu} = e^{-\sigma}(h_\mu{}^\nu{}_{;\nu}$

$+ 3h_\mu{}^\nu\sigma_\nu - h\sigma_\mu$). Thus, Eq. (3) is conformally invariant. To keep Eq. (4) conformally invariant as well one needs

$$h_\mu{}^\nu\sigma_\nu = 0 = \hat{h}_\mu{}^\nu\sigma_\nu. \tag{16}$$

This condition is analogous to one which keeps the electrodynamic Lorentz gauge $A_{\alpha;}{}^\alpha = 0$ conformally invariant.

With Eqs. (3), (4), and (16) valid the left-hand side of Eq. (10) transforms as follows (one can consult formulas in the Appendix with $k = 1$):

$$e^{-\sigma}\left[h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\mu\alpha\beta\nu}h^{\alpha\beta} - (\sigma_{\alpha;}{}^\alpha + \sigma_\alpha\sigma^\alpha)h_{\mu\nu}\right] = 0.$$

It follows that Eq. (10) is not conformally invariant, unless

$$\sigma_{\alpha;}{}^\alpha + \sigma_\alpha\sigma^\alpha = 0. \tag{17}$$

Equation (17) severely restricts conformal transformations with respect to which set of Eqs. (10), (3), and (4) is conformally invariant. For a given $\hat{h}_\mu{}^\nu$, Eq. (16) also restricts $\sigma$; however, the origin and the meaning of the restrictions (17) and (16) are completely different. Equation (17) represents, so to say, the "genuine" noninvariance of the wave equation (10), while Eq. (16) is a necessary condition for keeping the auxiliary condition conformally invariant. Conformal invariance of the wave equations and auxiliary conditions for potentials seems to be a more significant property than just a conformal invariance of the wave equations in terms of field components.

Equation (16), together with Eqs. (3) and (4), selects the spin $s = 2$ solutions with definite helicity as those which could be conformally transformed. In general, for a given $\sigma_\nu$, Eq. (16) restricts $\hat{h}_\mu{}^\nu$; however, in some cases, the conditions (16), (3), and (4) can be achieved at the expense of the gauge freedom and therefore do not, in fact, restrict the transformed solutions. Again, this is true for the Friedmann universes—the case which we are most interested in. For example, in a background metric

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2), \tag{18}$$

the gravitational-wave perturbations obey the auxiliary conditions (3), (4), and (8), where $u^\nu = (1/a,0,0,0)$. The metric (18) transforms into the flat space–time metric by $\sigma = \ln a$ and therefore Eq.(16) is automatically fulfilled.

Notice, that gravitational-wave equations and auxiliary conditions in Minkowski space–time are conformally invariant with respect to $C_0$. Really, a solution to Eq. (17) in Minkowski space–time is

$$\sigma = -\ln(1 + 2a_\alpha x^\alpha + a^2x_\alpha x^\alpha), \tag{19}$$

where $a^2 = a_\alpha a^\alpha$, and $a_\alpha$ are constants. Conformal rescaling with the $\sigma$ factor (19) corresponds to a group of conformal transformations $C_0$. This rescaling transforms the Minkowski line element $ds^2{}_m$ into the line element $ds^2 = e^{2\sigma}ds_m^2$:

$$ds^2 = \frac{1}{(1 + 2a_\alpha x^\alpha + a^2x_\alpha x^\alpha)^2}(c^2dt^2 - dx^2 - dy^2 - dz^2). \tag{20}$$

Due to the gauge freedom in Minkowski space–time one can introduce Eqs. (3) and (4) and reduce the field equations to $h_{\mu\nu,\alpha}{}^{,\alpha} = 0$. The gauge freedom which remains is described by $\xi_{\mu,\alpha}{}^{,\alpha} = 0$, $\xi^\alpha{}_{,\alpha} = 0$. For $\sigma$, given by Eq. (19), we introduce $u^\nu = a^\nu + a^2x^\nu \equiv -\frac{1}{2}e^{-\sigma}\sigma^\nu$. Note that $(h_{\mu\nu}u^\nu)_{,\alpha}{}^{,\alpha} = 0$. The remaining gauge freedom can be used to impose $(h_{\mu\nu}u^\nu)_{|\Sigma} = 0$ $(h_{\mu\nu}u^\nu)_{,\alpha}n^\alpha{}_{|\Sigma} = 0$, and hence $h_{\mu\nu}u^\nu = 0 = h_{\mu\nu}\sigma^\nu$. Thus, the field equations, the auxiliary conditions, and the spin $s = 2$ solutions with definite helicity in the conformally related space–times (20) transform into each other.

## III. COUPLING OF A SCALAR FIELD TO CURVATURE AND OTHER FIELDS

The aim of this section is to find the covariant and conformally invariant, second order, linear, homogeneous differential equations for a scalar field $\varphi$. As far as this equation will be formulated in a curved space–time it will describe the coupling of $\varphi$ to curvature, but, for the sake of generality, we also allow for coupling of $\varphi$ to other (nongravitational) scalar, vector, and tensor fields, which could be present in the given space–time.

The general form of the equation is

$$\hat{C}^{\alpha\beta}\hat{\varphi}_{\alpha\hat{;}\beta} + \hat{B}^\alpha\hat{\varphi}_\alpha + \hat{A}\hat{\varphi} = 0, \tag{21}$$

where $\varphi_\alpha \equiv \varphi_{,\alpha}$; $A, B^\alpha$, and $C^{\alpha\beta}$ are some scalar, vector, and tensor fields, respectively, $C^{\alpha\beta} = C^{\beta\alpha}$, since $\varphi_{\alpha;\beta} = \varphi_{\beta;\alpha}$. We make two additional assumptions also. Firstly, we assume that the coupling to the external gravitational field can be realized only through the metric tensor, the curvature tensor, and their different algebraic combinations, so that the gravitational part of coefficients, $A$, $B^\alpha$, and $C^{\alpha\beta}$ should be constructed from them. Secondly, we note that the first term in Eq.(21) contains a piece $\hat{\varphi}_{\alpha\hat{;}}{}^\alpha$ among all other possible contributions. We want this piece to be present in the original and in the transformed equation.

Under a conformal rescaling of the metric tensor the gravitational part of the coefficients $A$, $B^\alpha$, and $C^{\alpha\beta}$ transform according to the law which is basically determined by Eqs. (13) and (15). As for the transformation laws for the nongravitational contributions to $A$, $B^\alpha$, and $C^{\alpha\beta}$ we will derive them from the condition of conformal invariance of Eq. (21).

First we will transform the $\varphi$ field and its derivatives in Eq. (21). Under the transformation rules

$$\hat{\gamma}_{\mu\nu} = \Omega^2\gamma_{\mu\nu}, \quad \hat{\varphi} = \Omega^{-1}\varphi, \tag{22}$$

Eq. (21) takes the form

$$\begin{aligned}\Omega^{-1}\Big\{\hat{C}^{\alpha\beta}[&\varphi_{\alpha;\beta} - 2\Omega^{-1}(\varphi_\alpha\Omega_\beta + \Omega_\alpha\varphi_\beta) \\ &+ \Omega^{-1}\gamma_{\alpha\beta}\varphi_\sigma\Omega^\sigma - \Omega^{-1}\Omega_{\alpha;\beta}\varphi \\ &+ 4\Omega^{-2}\Omega_\alpha\Omega_\beta\varphi - \Omega^{-2}\gamma_{\alpha\beta}\Omega_\sigma\Omega^\sigma\varphi\,] \\ &+ \hat{B}^\alpha\left(\varphi_\alpha - \frac{\Omega_\alpha}{\Omega}\varphi\right) + \hat{A}\varphi\Big\} = 0.\end{aligned} \tag{23}$$

For Eq. (21) to be conformally invariant, we need the left-hand side of Eq. (23) to be equal to the left-hand side of Eq. (21) (without "carets") multiplied by $\Omega$ in some power: $\Omega^n$. Since $\hat{C}^{\alpha\beta}\hat{\varphi}_{\alpha\hat{;}\beta}$ includes $\hat{\gamma}^{\alpha\beta}\hat{\varphi}_{\alpha\hat{;}\beta}$ and this term transforms as $1/\Omega^3(\gamma^{\alpha\beta}\varphi_{\alpha;\beta} + \cdots)$, we want $n = -3$. Comparing the coefficients in front of $\varphi_{\alpha;\beta}$, $\varphi_{;\alpha}$, and $\varphi$ in between Eqs. (23) and (21) (without "carets") multiplied by $\Omega^{-3}$, one can obtain the transformation laws

$$\hat{C}_{\alpha\beta} = \Omega^2 C_{\alpha\beta}, \quad \hat{\gamma}^{\alpha\beta}\hat{C}_{\alpha\beta} \equiv \hat{C} = C \equiv \gamma^{\alpha\beta}C_{\alpha\beta}, \tag{24}$$

$$\hat{B}_\alpha = B_\alpha + 4\Omega^{-1}(\Omega_\beta C_\alpha{}^\beta - \tfrac{1}{4}\Omega_\alpha C), \tag{25}$$

$$\hat{A} = \Omega^{-2}A + \Omega^{-3}(\Omega^\alpha B_\alpha + \Omega_{\alpha;\beta}C^{\alpha\beta}). \tag{26}$$

Let us see what the gravitational contributions to $\hat{C}_{\alpha\beta}$ could be. This coefficient could contain $\hat{\gamma}_{\alpha\beta}$, $\hat{R}_{\alpha\beta}$, $\hat{R}_\alpha{}^\sigma \hat{R}_{\sigma\beta}$, etc. However, only $\hat{\gamma}_{\alpha\beta}$ can meet the transformation law (24). Hence,

$$\hat{C}_{\alpha\beta} = \hat{\gamma}_{\alpha\beta} + \hat{c}_{\alpha\beta}, \tag{27}$$

where $\hat{c}_{\alpha\beta}$ is some tensor field which does not depend on the metric, but is connected by a relation

$$\hat{c}_{\alpha\beta} = \Omega^2 c_{\alpha\beta} \tag{28}$$

in the conformally related space–time. Since the coefficient $\hat{B}_\alpha$ has an odd number of indices, it cannot have any gravitational contribution, so $\hat{B}_\alpha = \hat{b}_\alpha$, where $\hat{b}_\alpha$ is a vector field, independent of metric. Substituting $C_{\alpha\beta} = \gamma_{\alpha\beta} + c_{\alpha\beta}$ into Eq. (25), one obtains the transformation law for $\hat{b}_\alpha$:

$$\hat{b}_\alpha = b_\alpha + 4\Omega^{-1}\,\Omega_\beta(c^\beta{}_\alpha - \tfrac{1}{4}c\delta^\beta{}_\alpha). \tag{29}$$

The gravitational contributions to $\hat{A}$ could be of the form, $\hat{R}$, $\hat{R}^2$, $\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta}$, etc. However, only the first term can meet the condition $\hat{R} = \Omega^{-2}R + \cdots$ which is dictated by Eq. (26). Hence, the general form of $\hat{A}$ is

$$\hat{A} = a\hat{R} + \hat{m}, \tag{30}$$

where $a$ is some function and $\hat{m}$ is a scalar field. Substituting Eq. (30) and $B_\alpha = b_\alpha$ and $C_{\alpha\beta} = \gamma_{\alpha\beta} + c_{\alpha\beta}$ into Eq. (26), we obtain the equation

$$a\Omega^{-2}(R - 6\Omega^{-1}\,\Omega_{\alpha;}{}^\alpha) + \hat{m}$$
$$= \Omega^{-2}[aR + m + \Omega^{-1}(\Omega^\alpha b_\alpha + \Omega_{\alpha;}{}^\alpha + \Omega_{\alpha;\beta}c^{\alpha\beta})],$$

which gives rise to the relations

$$-6a = 1, \quad \hat{m} = \Omega^{-2}m + \Omega^{-3}(\Omega^\alpha b_\alpha + \Omega_{\alpha;\beta}c^{\alpha\beta}). \tag{31}$$

Thus, the most general conformally invariant equation of the form of Eq. (21) is

$$\varphi_{;\alpha}{}^{;\alpha} - \tfrac{1}{6}R\varphi + c^{\alpha\beta}\varphi_{;\alpha;\beta} + b^\alpha\varphi_\alpha + m\varphi = 0, \tag{32}$$

where $c_{\alpha\beta}$, $b_\alpha$, and $m$ transform according to Eqs. (28), (29), and (31), respectively, unless all of them or some of them are equal to zero. Notice that if there exists any other conformally invariant equation for the $\varphi$ field, it cannot contain the operator $\varphi_{;\alpha}{}^{;\alpha}$. The first two terms in Eq. (32) give the familiar equation for a scalar field in a curved space–time.

## IV. CONFORMALLY INVARIANT EQUATIONS FOR A SECOND-RANK SYMMETRIC TENSOR FIELD

Having proved that the equations which follow from the Einstein equations for the second-rank tensor field $h_{\alpha\beta}$ are not conformally invariant, we will try now to find certain equations which are conformally invariant. For simplicitly, we will consider only the coupling of $h_{\alpha\beta}$ to the external gravitational field and not to other fields. The general strategy will be similar to the one used in the previous section. However, in the case of $h_{\alpha\beta}$ there is a complication related to the fact that $h_{\alpha\beta}$ should obey not only the field equations but also the auxiliary conditions.

We are looking for conformally invariant equations within the following class of equations. They should be covariant, second-order, homogeneous differential equations. Coefficients in these equations can contain the metric tensor $\gamma_{\alpha\beta}$, the curvature tensor, and their different algebraic combinations.

The general form of these equations can be written as follows;

$$\hat{F}_{\mu\nu}{}^{\sigma\alpha\beta\rho}\hat{h}_{\alpha\beta\hat{;}\sigma\hat{;}\rho} + \hat{P}_{\mu\nu}{}^{\alpha\beta\gamma}\hat{h}_{\alpha\beta\hat{;}\gamma} + \hat{U}^\alpha{}_{\mu\nu}{}^\beta\hat{h}_{\alpha\beta} = 0. \tag{33}$$

We also assume the validity of the auxiliary conditions

$$\hat{h} = 0, \tag{34}$$

$$\hat{h}_\mu{}^\nu{}_{\hat{;}\nu} = 0, \tag{35}$$

and will demand their conformal invariance. In fact, we should put $\hat{P}_{\mu\nu}{}^{\alpha\beta\gamma} = 0$ since a tensor with an odd number of indices cannot be constructed as an algebraic combination of metric tensor and curvature tensor.

Some properties of the symmetry of tensors $\hat{F}$ and $\hat{U}$ follow from the fact that $\hat{h}_{\alpha\beta}$ is a symmetric tensor and Eq. (33) is assumed to be symmetric with respect to the free indices $\mu$ and $\nu$. To the same end, since

$$h_{\alpha\beta;\sigma;\rho} - h_{\alpha\beta;\rho;\sigma} = h_{\alpha\epsilon}R^\epsilon{}_{\beta\sigma\rho} + h_{\epsilon\beta}R^\epsilon{}_{\alpha\sigma\rho},$$

we may assume that the first term in Eq. (33) contains only the symmetric (with respect to $\sigma,\rho$) part of $h_{\alpha\beta\hat{;}\sigma\hat{;}\rho}$ while the antisymmetric part is included in the last term in Eq. (33). This assumption also determines, in part, the symmetry properties of the tensor $\hat{F}$.

As we know from Sec. II, under the transformations (11) and (12) the gauge conditions (34) and (35) transform into Eqs. (3) and (4), respectively, if Eq. (16) is satisfied. With the use of Eq. (16) one obtains the following transformation rules for $\hat{h}_{\mu\nu\hat{;}\alpha}$ and $\hat{h}_{\mu\nu\hat{;}\alpha\hat{;}\beta}$:

$$\hat{h}_{\mu\nu\hat{;}\alpha} = e^\sigma(h_{\mu\nu;\alpha} - h_{\mu\alpha}\sigma_\nu - h_{\nu\alpha}\sigma_\mu - h_{\mu\nu}\sigma_\alpha), \tag{36}$$

$$\hat{h}_{\mu\nu\hat{;}\alpha\hat{;}\beta} = e^\sigma(h_{\mu\nu;\alpha;\beta} + h_{\mu\gamma;\delta}Y_{\nu\alpha\beta}{}^{\gamma\delta} + h_{\nu\gamma;\delta}Y_{\mu\alpha\beta}{}^{\gamma\delta}$$
$$+ h_{\mu\gamma}Z_{\nu\alpha\beta}{}^\gamma + h_{\nu\gamma}Z_{\nu\alpha\beta}{}^\gamma + h_{\gamma\delta}V_{\mu\nu\alpha\beta}{}^{\gamma\delta}), \tag{37}$$

where

$$Y_{\nu\alpha\beta}{}^{\gamma\delta} \equiv \tfrac{1}{2}\gamma_{\alpha\beta}\delta_\nu{}^\gamma\sigma^\delta - \sigma_\nu(\delta_\alpha{}^\gamma\delta_\beta{}^\delta + \delta_\beta{}^\gamma\delta_\alpha{}^\delta)$$
$$- \delta_\nu{}^\gamma(\sigma_\alpha\delta_\beta{}^\delta + \sigma_\beta\delta_\alpha{}^\delta), \tag{38}$$

$$Z_{\nu\alpha\beta}{}^\gamma = \tfrac{1}{2}\delta_\nu{}^\gamma(-\sigma_{\alpha;\beta} + 3\sigma_\alpha\sigma_\beta - \gamma_{\alpha\beta}\sigma_\delta\sigma^\delta)$$
$$+ \delta_\alpha{}^\gamma(-\sigma_{\nu;\beta} + 3\sigma_\nu\sigma_\beta - \gamma_{\beta\nu}\sigma_\delta\sigma^\delta)$$
$$- \gamma_{\beta\nu}\sigma^\gamma{}_{;\alpha} + 2\sigma_\nu\sigma_\alpha\delta_\beta{}^\gamma, \tag{39}$$

$$V_{\mu\nu\alpha\beta}{}^{\gamma\delta} = 2\sigma_\mu\sigma_\nu\delta_\alpha{}^\gamma\delta_\beta{}^\delta. \tag{40}$$

The conformal invariance of Eq. (33) implies that after substituting Eqs. (12) and (37) and transformation laws for $\hat{F}$ and $\hat{U}$ into Eq. (33), the left-hand side of Eq. (33) transforms into the same expression (without "carets") multiplied by some power of $e^\sigma$. The power is determined from considerations similar to the ones used in Sec. III. The first term in Eq. (33) contains a piece $\hat{h}_{\mu\nu\hat{;}\sigma}{}^{\hat{;}\sigma}$. This arises from the contribution to $\hat{F}_{\mu\nu}{}^{\sigma\alpha\beta\rho}$ of the following form:

$$\hat{F}_{\mu\nu}{}^{\sigma\alpha\beta\rho} = \delta_\mu{}^\alpha\delta^\beta{}_\nu\hat{\gamma}^{\delta\rho}. \tag{41}$$

We want to save the term $h_{\mu\nu;\sigma}{}^{;\sigma}$ in the transformed equation. So we will sacrifice all other contributions to $\hat{F}_{\mu\nu}{}^{\sigma\alpha\beta\rho}$ if they do not fit into the transformation law for $\hat{h}_{\mu\nu;\hat{\sigma}}{}^{;\sigma}$. Since

$$\hat{h}_{\mu\nu;\hat{\sigma}}{}^{;\sigma} = e^{-\sigma}(h_{\mu\nu;\sigma}{}^{;\sigma} + \cdots),$$

we will demand that Eq. (33) be transformed into

$$e^{-\sigma}(F_{\mu\nu}{}^{\sigma\alpha\beta\rho} h_{\alpha\beta;\sigma;\rho} + U^{\alpha}{}_{\mu\nu}{}^{\beta} h_{\alpha\beta}) = 0. \quad (42)$$

The right-hand side of Eqs. (36) and (37) already contains the factor $e^{\sigma}$; therefore, Eq. (42) implies that the transformation law for the tensors $\hat{F}$ and $\hat{U}$ must have the following form:

$$\hat{F}_{\mu\nu}{}^{\sigma\alpha\beta\rho} = e^{-2\sigma}(F_{\mu\nu}{}^{\sigma\alpha\beta\rho} + \cdots), \quad (43)$$

$$\hat{U}^{\alpha}{}_{\mu\nu}{}^{\beta} = e^{-2\sigma}(U^{\alpha}{}_{\mu\nu}{}^{\beta} + \cdots). \quad (44)$$

Unlike what was done in Sec. III, here, we could not obtain the transformation rules for $\hat{F}$ and $\hat{U}$ directly as a result of comparing the coefficients in front of $h_{\alpha\beta;\sigma;\rho}$, $h_{\alpha\beta;\sigma}$, and $h_{\alpha\beta}$ in both Eqs. (33) and (42). This is because there may be additional terms in these transformation rules which after multiplying them by $h_{\alpha\beta;\sigma;\rho}$ and $h_{\alpha\beta}$ can vanish due to the auxiliary conditions (3) and (4).

Let us see what are the possible contributions to $\hat{F}$ and $\hat{U}$ and whether they can satisfy the conditions (43) and (44). As far as the tensor $\hat{F}$ is concerned, the only contribution which meets the condition (43) is Eq. (41). All other contributions which can contain different combinations of Kronecker symbols, metric tensor, Ricci tensor, and curvature are not appropriate. Some of them, $\delta^{\sigma}{}_{\mu}\delta_{\nu}{}^{\rho}\hat{\gamma}^{\alpha\beta}$ or $\delta^{\sigma}{}_{\mu}\delta^{\alpha}{}_{\gamma}\hat{\gamma}^{\beta\rho}$, though they have the correct transformation property, do not play any role because they disappear due to eqs. (34) and (35). The other terms, like $\delta^{\sigma}{}_{\mu}\delta_{\nu}{}^{\rho}\hat{R}^{\alpha\beta}$ or $\gamma_{\mu\nu}\hat{R}^{\alpha\beta}\hat{R}^{\sigma\rho}$ or $\hat{R}_{\mu\nu}\hat{R}^{\sigma\alpha}\hat{R}^{\beta\rho}$, which do not disappear due to Eqs. (34) and (35), transform with the wrong dependence on $e^{-2\sigma}$; they acquire coefficients $e^{-4\sigma}$, $e^{-6\sigma}$, or even $e^{-8\sigma}$.

Substituting Eqs. (12), (37), and (41) into Eq. (33) and comparing the result with Eq. (42), one can derive the transformation rule for $\hat{U}$. In the course of the calculation it is important to notice that because of Eq. (4) the following relaton is valid:

$$\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\gamma^{\sigma\rho}(h_{\alpha\gamma;\delta}Y_{\beta\sigma\rho}{}^{\gamma\delta} + h_{\beta\gamma;\delta}Y_{\alpha\sigma\rho}{}^{\gamma\delta}) = 0.$$

The other terms give the equation

$$h_{\mu\gamma}Z_{\nu\alpha}{}^{\alpha\gamma} + h_{\nu\gamma}Z_{\mu\alpha}{}^{\alpha\gamma} + h_{\gamma\delta}V_{\mu\nu\alpha}{}^{\alpha\gamma\delta} + \hat{U}^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta}e^{2\sigma}$$
$$= U^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta},$$

which in more detail reads as

$$\hat{U}^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} = e^{-2\sigma}[U^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} + 2h_{\mu\alpha}\sigma_{\nu;}{}^{\alpha}$$
$$+ 2h_{\nu\alpha}\sigma_{\mu;}{}^{\alpha} + h_{\mu\nu}(\sigma_{\alpha;}{}^{\alpha} + 3\sigma_{\alpha}\sigma^{\alpha})]. \quad (45)$$

The general form of $\hat{U}^{\alpha}{}_{\mu\nu}{}^{\beta}$ which might be consistent with Eq. (44) is

$$\hat{U}^{\alpha}{}_{\mu\nu}{}^{\beta} = a\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\hat{R} + b(\delta^{\alpha}{}_{\mu}\hat{R}_{\nu}{}^{\beta} + \delta^{\alpha}{}_{\nu}\hat{R}_{\mu}{}^{\beta})$$
$$+ c\hat{\gamma}_{\mu\nu}\hat{R}^{\alpha\beta} + d\hat{C}^{\alpha}{}_{\mu\nu}{}^{\beta}, \quad (46)$$

where $a$, $b$, $c$, and $d$, are arbitrary functions of the space–time variables. Since $\hat{C}^{\alpha}{}_{\mu\nu}{}^{\beta} = e^{-2\sigma}C^{\alpha}{}_{\mu\nu}{}^{\beta}$ and $\hat{\gamma}_{\mu\nu}\hat{R}^{\alpha\beta}h_{\alpha\beta} = e^{-2\sigma}\gamma_{\mu\nu}(R^{\alpha\beta} - 2\sigma^{\alpha;\beta} + 2\sigma^{\alpha}\sigma^{\beta} - \sigma_{\rho;}{}^{\rho}\gamma^{\alpha\beta} - 2\sigma_{\rho}\sigma^{\rho}\gamma^{\alpha\beta})h_{\alpha\beta} = e^{-2\sigma}\gamma_{\mu\nu}R^{\alpha\beta}h_{\alpha\beta}$, the functions $c$ and $d$ are not restricted by Eq. (45) and they can remain arbitrary. As for the functions $a$ and $b$, they are determined after substitution of Eq. (46) into Eq. (45) and they must be $a = 1/6$, $b = -1$.

Thus, the general form for conformally invariant equations for a second-rank symmetric tensor field $h_{\mu\nu}$ is

$$h_{\mu\nu;\alpha}{}^{;\alpha} + \frac{R}{6}h_{\mu\nu} - h_{\mu\alpha}R^{\alpha}{}_{\nu} - h_{\nu\alpha}R^{\alpha}{}_{\mu} + c\gamma_{\mu\nu}R^{\alpha\beta}h_{\alpha\beta}$$
$$+ dC^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} = 0. \quad (47)$$

These equations and the auxiliary conditions (3) and (4) are conformally invariant if Eq. (16) is satisfied.

For an easier comparison of Eq. (47) with Eq. (10) we can rewrite the former one in the form

$$h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\alpha\mu\nu\beta}h^{\alpha\beta} - \tfrac{1}{6}Rh_{\mu\nu} - \tfrac{1}{2}k\gamma_{\mu\nu}R^{\alpha\beta}h_{\alpha\beta}$$
$$+ lC_{\alpha\mu\nu\beta}h^{\alpha\beta} = 0, \quad (48)$$

where $k$ and $l$ are arbitrary functions. Multiplying this equation by $\gamma_{\mu\nu}$ we obtain its consequence

$$2(1-k)h^{\alpha\beta}R_{\alpha\beta} = 0,$$

which says that either $k = 1$ and then $R_{\alpha\beta}h^{\alpha\beta}$ is not necessarily equal to zero, or $h^{\alpha\beta}R_{\alpha\beta} = 0$ (what has been true for the linearized Einstein equations) and then we can put $k = 0$. In any case, it is seen from Eqs. (48) and (10) that the most important difference between them is the term $\frac{1}{6}Rh_{\mu\nu}$. The lack of this term was the cause of conformal noninvariance of Eq. (10).[23]

One should remember that the conditions (43) and (44) were obtained as a consequence of a desire to keep the operator $h_{\mu\nu;\alpha}{}^{;\alpha}$ in the equations. So if there exist any other conformally invariant equations, different from Eq. (48), it does not include this operator.

We have considered the conformally invariant coupling of $h_{\alpha\beta}$ to curvature. There must exist conformally invariant equations which describe the coupling of $h_{\alpha\beta}$ to curvature and other fields. Derivation of the general form of such equations is a complicated problem, so we shall restrict ourselves to a specific example. This is provided by the equation

$$\hat{h}_{\mu\nu;\hat{\alpha}}{}^{;\alpha} - 2\hat{R}_{\alpha\mu\nu\beta}\hat{h}^{\alpha\beta} - \frac{\hat{\varphi}_{;\alpha}{}^{;\alpha}}{\hat{\varphi}}\hat{h}_{\mu\nu} = 0, \quad (49)$$

where $\varphi$ is a scalar field. This equation transforms into

$$e^{-\sigma}\left(h_{\mu\nu;\alpha}{}^{;\alpha} - 2R_{\alpha\mu\nu\beta}h^{\alpha\beta} - \frac{\varphi_{;\alpha}{}^{;\alpha}}{\varphi}h_{\mu\nu}\right) = 0, \quad (50)$$

under the transformation rules $\hat{\varphi} = e^{-\sigma}\varphi$ and Eqs. (11) and (12), and the conditions (16), (34), and (35). Since the scalar field $\varphi$ transforms with the correct dependence on the conformal factor, it may obey the conformally invariant equation as well. In that case Eq. (50) can be represented in terms of the background variables only, since $(\varphi_{;\alpha}{}^{;\alpha})/\varphi = R/6$.

It is interesting to note that Eq. (49) can be obtained from Eq. (10) as a result of applying the conformal transformation $h_{\mu\nu} = \hat{\varphi}\hat{h}_{\mu\nu}$, $\gamma_{\mu\nu} = \hat{\varphi}^{2}\hat{\gamma}_{\mu\nu}$.

## V. NONEXISTENCE OF A GRAVITATIONAL THEORY WITH CONFORMALLY INVARIANT LINEARIZED WAVE EQUATIONS

Equation (10) was derived from the Einstein equations in the linear approximation. Let us see if Eq. (48) can be derived in a similar way from some exact equations which generalize Einstein's equations.

Suppose that the generalized equations have the following form:

$$N_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} + F_{\mu\nu} = T_{\mu\nu}, \tag{51}$$

where $F_{\mu\nu}$ is a symmetric tensor, constructed in an arbitrary way from the exact metric $g_{\mu\nu}$ ($g_{\mu\nu} \simeq \gamma_{\mu\nu} + h_{\mu\nu}$) and its derivatives. In principle, among possible contributions of $F_{\mu\nu}$ could be terms like $RR_{\mu\nu}$, $R_{\alpha\beta} R^{\alpha}{}_{\mu\nu}{}^{\beta}$, etc. We do not assume that $N_{\mu}{}^{\nu}{}_{;\nu} \equiv 0$ should hold necessarily, at least for the time being.

In analogy to the way in which Eq. (10) was derived from the first variation of the Einstein equations, Eq. (48) should follow from the equations

$$-(\gamma_{\mu\alpha} \delta N_{\nu}{}^{\alpha} + \gamma_{\nu\alpha} \delta N_{\mu}{}^{\alpha}) = 0 \tag{52}$$

and the auxiliary conditions (3) and (4). Since the first two terms of Eq. (48) follow from the expression $-\gamma_{\mu\alpha}(\delta R_{\nu}{}^{\alpha} + \lambda \delta_{\nu}{}^{\alpha}) - \gamma_{\nu\alpha}(\delta R_{\mu}{}^{\alpha} + \lambda \delta_{\mu}{}^{\alpha})$, Eq. (52) can be reduced to

$$2\delta F_{\mu\nu} - h^{\alpha}{}_{\mu} F_{\alpha\nu} - h_{\nu}{}^{\alpha} F_{\alpha\mu} = \tfrac{1}{6} R h_{\mu\nu} + (\tfrac{1}{2}k - 1)\gamma_{\mu\nu} h_{\alpha\beta} R^{\alpha\beta} - l C_{\alpha\mu\nu\beta} h^{\alpha\beta}. \tag{53}$$

[Obviously, the background, or "unperturbed," values of the curvature enter the right-hand side of Eq. (53) and background values of $F_{\mu\nu}$ enter the last two terms on the left-hand side of this equation.] The question is whether there exists a tensor $F_{\mu\nu}$ which is a solution to Eq. (53). First of all, one can notice that since the right-hand side of Eq. (53) is linear in the background curvature, the tensor $F_{\mu\nu}$ can only consist of terms which are not higher than quadratic order in curvature, or otherwise the variation of $F_{\mu\nu}$ would give rise to quadratic and higher order terms, which are not present at the right-hand side of Eq. (53). Secondly, since the right-hand side of Eq. (53) does not contain derivatives of $h_{\mu\nu}$, we should exclude the contributions to $F_{\mu\nu}$ which could lead to them, unless they disappear due to Eq. (4).

Then, the general form of $F_{\mu\nu}$ which could meet these restrictions is $F_{\mu\nu} = a g_{\mu\nu} + bRg_{\mu\nu} + cR_{\mu\nu} + mRR_{\mu\nu} + nR_{\mu}{}^{\alpha} R_{\alpha\nu} + pR_{\alpha\beta} R^{\alpha}{}_{\mu\nu}{}^{\beta} + q\Psi(R^2) g_{\mu\nu} + rR_{;\mu;\nu}$, where all the coefficients are arbitrary functions of space–time and $\Psi(R^2)$ symbolizes any quadratic function of scalars constructed out of the curvature tensor. A more detailed analysis shows that, in fact, none of the terms with coefficients, $c$, $m$, $n$, $p$, $q$ and $r$ is useful because the variation of each of them gives either (i) the second (or higher) derivatives of $h_{\mu\nu}$, which cannot be cancelled out, or (ii) the terms which are quadratic in the background curvature. Both these cases contradict the form of the right-hand side of Eq. (53). Thus, we should seek among the terms with coefficients $a$ and $b$. The term with coefficient $a$ (like the cosmological term in the Einstein equations) does not play any role because the left-hand side of Eq. (53) calculated from this term is identically equal to zero. The term with coefficient $b$, for $b = \frac{1}{2} - \frac{1}{4}k$, can give rise to the second term on the right-hand side of Eq. (53). However, this term cannot explain the appearance of the first and the third term. Since $l$ is an arbitrary coefficient, we can choose $l = 0$. However, the presence of the term $\frac{1}{6} R h_{\mu\nu}$ is a real obstacle.

To demonstrate this in a more straightforward way we will choose $l = k = 0$ in Eq. (48). Then, a consequence of this equation and Eq. (3) is $h^{\alpha\beta} R_{\alpha\beta} = 0$, and hence $\delta R$ is equal to zero, since $\delta R = -h_{\alpha\beta} R^{\alpha\beta} - h_{;\alpha}{}^{;\alpha} + h_{\alpha}{}^{\beta}{}_{;\beta}{}^{;\alpha}$. Equation (53) can now be rewritten in the following form:

$$\gamma_{\mu\alpha} \delta F^{\alpha}{}_{\nu} + \gamma_{\nu\alpha} \delta F^{\alpha}{}_{\mu} = \delta(\tfrac{1}{6} R g_{\mu\nu}). \tag{54}$$

One can solve Eq. (54) with respect to quantities $\delta F_{\beta}{}^{\alpha}$ which, by assumption, should be variations of some tensor. However, from the very way of constructing the solution to Eq. (54) it is clear that this solution is not a variation of a tensor. Thus, conformally invariant Eqs. (48) cannot follow in a linear approximation from any tensor equations of the form of Eq. (51).

## VI. CONCLUSIONS

It seems that one is left with two options, though each of them looks interesting. One of them is to agree that the equations which govern weak gravitational waves in a curved space–time are not conformally invariant. Then, it means that on both levels—classical and quantum—gravitons behave drastically different from other massless particles. Classical gravitational waves can be amplified and gravitons can be created (contrary to other massless fields and particles) in a nonstationary isotropic gravitational field, particularly in the strong gravitational field of the early Universe. The other option is to try to endow gravitons with the same kind of coupling to the external gravitational field that other massless particles have. Then one has to find some nontrivial generalization of the Einstein equations.

## ACKNOWLEDGMENT

Thanks are due to B. Mashhoon for reading the manuscript.

## APPENDIX

We will give here the transformation rules for different terms which could enter the linearized Einstein equations. The background metric tensor $\hat{\gamma}_{\alpha\beta}$ and the field variables $\hat{h}_{\alpha\beta}$ transform as follows:

$$\hat{\gamma}_{\alpha\beta} = e^{2\sigma} \gamma_{\alpha\beta}, \quad \hat{h}_{\alpha\beta} = e^{k\sigma} h_{\alpha\beta},$$

where $k$ is an arbitrary constant. We denote $\sigma_{\alpha} \equiv \sigma_{,\alpha}$; notice also that $\sigma_{\mu;\nu} = \sigma_{\nu;\mu}$. Then,

$$e^{-(k-2)\sigma} \hat{h}_{\mu\nu\hat{;}\alpha}{}^{\hat{;}\alpha} = h_{\mu\nu;\alpha}{}^{;\alpha} + 2(k-1) h_{\mu\nu;\alpha} \sigma^{\alpha} + (k^2 - 2k - 2) h_{\mu\nu} \sigma_{\alpha} \sigma^{\alpha} + (k-2) h_{\mu\nu} \sigma_{\alpha;}{}^{\alpha} + 2\sigma_{\alpha}(h_{\mu}{}^{\alpha}{}_{;\nu} + h_{\nu}{}^{\alpha}{}_{;\mu}) - 2(h_{\mu}{}^{\alpha}{}_{;\alpha} \sigma_{\nu} + h_{\nu}{}^{\alpha}{}_{;\alpha} \sigma_{\mu}) - 4\sigma_{\alpha}(h_{\mu}{}^{\alpha} \sigma_{\nu} + h_{\nu}{}^{\alpha} \sigma_{\mu}) + 2\gamma_{\mu\nu} h_{\alpha\beta} \sigma^{\alpha} \sigma^{\beta} + 2h \sigma_{\mu} \sigma_{\nu},$$

$$
\begin{aligned}
-e^{-(k-2)\sigma}\widehat{R}_{\alpha\mu\nu\beta}\hat{h}^{\alpha\beta} = {} & -R_{\alpha\mu\nu\beta}h^{\alpha\beta} + h^{\alpha\beta}(\sigma_\alpha\sigma_\beta - \sigma_{\alpha;\beta})\gamma_{\mu\nu} + h_{\mu\nu}\sigma_\alpha\sigma^\alpha + h_\mu{}^\alpha\sigma_{\nu;\alpha} \\
& + h_\nu{}^\alpha\sigma_{\mu;\alpha} - \sigma_\alpha(h_\mu{}^\alpha\sigma_\nu + h_\nu{}^\alpha\sigma_\mu) + h(\sigma_\mu\sigma_\nu - \sigma_{\mu;\nu} - \gamma_{\mu\nu}\sigma_\alpha\sigma^\alpha),
\end{aligned}
$$

$$
\begin{aligned}
-e^{-(k-2)\sigma}\hat{h}^{\alpha}{}_{\mu\hat{;}\alpha\hat{;}\nu} \\
= {} & -h^\alpha{}_{\mu;\alpha;\nu} - (k-3)h^\alpha{}_{\mu;\alpha}\sigma_\nu + h^\alpha{}_{\nu;\alpha}\sigma_\mu - h^{\alpha\beta}{}_{;\beta}\sigma_\alpha\gamma_{\mu\nu} - (k+2)h^\alpha{}_{\mu;\nu}\sigma_\alpha - (k^2-k-6)h^\alpha{}_\mu\sigma_\alpha\sigma_\nu \\
& -(k+2)h^\alpha{}_\mu\sigma_{\alpha;\nu} + (k+2)h^\alpha{}_\nu\sigma_\alpha\sigma_\mu - (k+2)h^{\alpha\beta}\sigma_\alpha\sigma_\beta\gamma_{\mu\nu} + h_{,\nu}\sigma_\mu + h\,[\sigma_{\mu;\nu} + \sigma_\alpha\sigma^\alpha\gamma_{\mu\nu} + (k-4)\sigma_\mu\sigma_\nu],
\end{aligned}
$$

$$
e^{-(k-2)\sigma}\hat{h}_{\hat{;}\mu\hat{;}\nu} = h_{;\mu;\nu} + (k-3)(h_{,\mu}\sigma_\nu + h_{,\nu}\sigma_\mu) + \gamma_{\mu\nu}h_{,\alpha}\sigma^\alpha + (k-2)h\,[(k-4)\sigma_\mu\sigma_\nu + \sigma_{\mu;\nu} + \sigma_\alpha\sigma^\alpha\gamma_{\mu\nu}],
$$

$$
\begin{aligned}
e^{-(k-2)\sigma}(\hat{h}_\mu{}^\alpha\widehat{R}_{\alpha\nu} + \hat{h}_\nu{}^\alpha\widehat{R}_{\alpha\mu}) = {} & -(h_\mu{}^\alpha R_{\alpha\nu} + h_\nu{}^\alpha R_{\alpha\mu}) + 2(h^\alpha{}_\mu\sigma_{\nu;\alpha} + h^\alpha{}_\nu\sigma_{\mu;\alpha}) \\
& -2\sigma_\alpha(h_\mu{}^\alpha\sigma_\nu + h_\nu{}^\alpha\sigma_\mu) + 2h_{\mu\nu}(\sigma_{\alpha;}{}^\alpha + 2\sigma_\alpha\sigma^\alpha).
\end{aligned}
$$

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[20] To find $\xi_\mu$ one has to solve a set of differential equations. Strictly speaking, the existence of $\xi_\mu$ is only guaranteed locally, in a vicinity of some initial hypersurface.

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[23] As a curious fact, we should point out that Eq. (9) reduces to Eq. (48) (with $k = l = 0$) under the conditions $\chi_{\mu;\nu} + \chi_{\nu;\mu} - \gamma_{\mu\nu}(-h_{\alpha\beta}R^{\alpha\beta} + \chi_{\alpha;}{}^\alpha - \frac{1}{2}h_{;\alpha}{}^{;\alpha}) - \frac{1}{6}Rh_{\mu\nu} - (\gamma_{\mu\alpha}\delta T^\alpha{}_\nu + \gamma_{\nu\alpha}\delta T^\alpha{}_\mu) = 0$. However, this equation cannot be used as a gauge condition for $\chi_\mu$ since it is not, in general, integrable. On the other hand, if this equation is to be interpreted as a definition of $\delta T_\mu{}^\nu$ (together with $\chi_\mu = 0$), then it picks up a mixture of tensor, scalar, and vector harmonics in the case of Robertson–Walker background metrics, which does also seem to be unsatisfactory.

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