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The solution of fractional order epidemic model by implicit Adams methods

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ABSTRACT

We consider the numerical solution of the fractional order epidemic model on long time intervals of a non-fatal disease in a population. Under real-life initial conditions the problem needs to be treated by means of an implicit numerical scheme. Here we consider the use of implicit fractional linear multistep methods of Adams type. Numerical results are presented.

1. Introduction

The problem of spreading of a non-fatal disease in a population that is assumed to have constant size over the period of the epidemic can be formulated in terms of the following first order model (see [1]):

$$\begin{cases} x'(t) = -\beta x(t)y(t) \\ y'(t) = \beta x(t)y(t) - \gamma y(t) \\ z'(t) = \gamma y(t), \end{cases} \tag{1}$$

with initial conditions

$$x(t_0) = N_1 \geq 0, \quad y(t_0) = N_2 \geq 0, \quad z(t_0) = N_3 \geq 0, \tag{2}$$

where at time $t \geq t_0$, $x(t)$ is the number of susceptible individuals, $y(t)$ is the number of infected individuals, able to spread the disease by contact with susceptible ones, $z(t)$ is the number of isolated individuals, who cannot get or transmit the disease for various reasons. Moreover, $\beta > 0$ is the rate of infection and $\gamma > 0$ is the rate at which current infective population is isolated. The non-fatality of the model (1) depends on the values of this two parameters. In Fig. 1 we show an example of solution of (1) for small initial values.

The fractional order extension of this model have been first studied in [2], where the authors replace the first derivatives in (1) by Caputo's fractional derivative of order $0 < \alpha \leq 1$, defined by (see e.g. [3]),

$$D^{\alpha}_{t_0} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s) ds,$$

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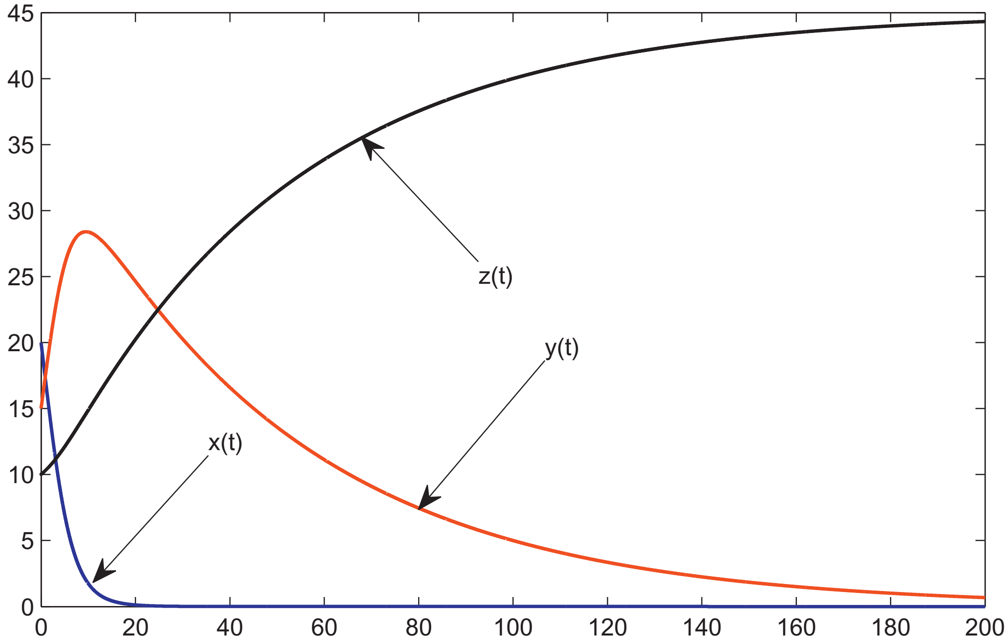

Fig. 1. Plots of numerical solutions for $x(t)$, $y(t)$, $z(t)$ versus time such that $N_1 = 20$, $N_2 = 15$, $N_3 = 10$, $\beta = 0.01$ and $\gamma = 0.02$.

where f is a given function, and $\Gamma(\cdot)$ denotes the gamma function. It is known that $D^{\alpha}_{t_0} f(t) \to f'(t)$ as $\alpha \to 1$. The main reason that leads to this extension (typically with α chosen close to 1) is to reduce the error that may arise from neglected parameters or simplifications in the model (1), as for instance the choice of constant rate of infection β and isolation γ. This seems correct in principle since this two parameters may change accordingly with the experience on the spreading of a certain epidemic, that is, on the history of the process. In this view, the use of $\alpha < 1$ has just the effect of transforming (1) into a model with memory. In general, the epidemic integer model (1) does not carry any information about the memory and learning mechanism of population that affects the spread of disease [4]. It is known that the fractional derivative is a non-local operator, which means that the system response at any time will be affected by all previous responses. Therefore, any dynamical process modeled through fractional order differential equations (FODEs) has a memory effect. The physical meaning of the fractional order is considered in [5,6] to be the index of memory.

Without loss of generality, we assume that the process starts at time $t_0 = 0$. Then, using the simplified notation $f^{(\alpha)}(t) = D^{\alpha}_{0} f(t)$ we consider the numerical solution of the following fractional system of equations:

$$\begin{cases} x^{(\alpha)}(t) = -\beta x(t)y(t) \\ y^{(\alpha)}(t) = \beta x(t)y(t) - \gamma y(t) \\ z^{(\alpha)}(t) = \gamma y(t). \end{cases} \tag{3}$$

Some of the recent analytic methods for solving nonlinear problems like (1) and (3) include the adomian decomposition method (ADM [7]), homotopy-perturbation method (HPM [8]), variational iteration method (VIM [9]) and homotopy analysis method (HAM [10] and [2,11]). They are relatively new approaches to provide an analytical approximate solution to linear and nonlinear problems and they provide immediate and visible symbolic terms of analytic solutions. Anyway, such analytic methods are generally effective only for small time intervals.

In this paper we intend to solve (3) by discrete methods for FODEs, since we prefer to avoid restriction on the time interval under investigation. This may be important to forecast the end of the epidemic. Moreover, using real-life values for the initial conditions (2), problem (3) may become highly stiff and then it will be necessary to employ an implicit time-stepping scheme. In this situation, the use of analytic approximation by polynomials, attainable for instance by homotopy analysis method (HAM [2,11]), seems unreliable. Here, we consider the use of the implicit fractional Adams methods of order 2, which reduces to the classical trapezoidal rule in the case of $\alpha = 1$.

The paper is organized as follows. In Section 2 the stability of the model (3) is investigated. Section 3 is devoted to describe the methods that are used to solve the problem under real-life values for the initial conditions. Numerical results and some implementation details are discussed in Section 4.

2. Stability of the model

As mentioned in the introduction, assuming to work with real-life values for the initial conditions, the model (3) can be quite difficult to solve. Indeed, if we consider the Jacobian of the system,

$$J = \begin{pmatrix} -\beta y & -\beta x & 0 \\ \beta y & \beta x - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}, \tag{4}$$

its nonzero eigenvalues are given by,

$$\lambda_{\pm} = \frac{1}{2}(\beta x - \gamma - \beta y) \pm \frac{1}{2}\sqrt{(\beta x - \gamma - \beta y)^2 - 4\beta\gamma y}.$$

Depending on the values assumed by x and y during the process, there is a negative eigenvalue that may be very large. In particular, the stationary point of the component y of the solution (see Fig. 1 and (3)), it is attained for $\beta x - \gamma = 0$, where $y > y_0$. In this situation,

$$\lambda_{-} = -\frac{\beta y}{2} - \frac{1}{2}\sqrt{\beta^2 y^2 - 4\beta\gamma y}.$$

In most of papers that considers the numerical results of this problem, the initial values and the parameters are taken such that $\beta y \approx 1$ or more generally very small. In this situation the problem can be solved efficiently by an explicit scheme since the eigenvalues remains close to the origin. Using more realistic (large) values for N_i, we clearly have that around the stationary point,

$$\lambda_{-} \approx -\beta y < -\beta N_2.$$

so that the problem needs to be solved by an implicit scheme.

3. Description of the numerical methods

In this section we present the methods used in the paper, which depending on the initial conditions can be used to solve the epidemic model without restrictions on the time length. Consider the general initial value problem,

$$D^{\alpha} y(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad 0 < \alpha \le 1, \quad t_0 < t \le T. \tag{5}$$

In a discrete numerical method the time interval $[t_0, T]$ is replaced by a discrete set of points $t_j = t_0 + jh$, $h = \frac{T - t_0}{N}$, $j = 0, 1, \ldots, N$, so that the solution is approximated by a sequence $\{y_j\}_{j=0,1,\ldots,N}$ such that $y_j \approx y(t_j)$.

The exact solution of (5) can be written in terms of a Volterra integral equation of the second kind with a weakly singular kernel,

$$y(t) = y(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s))ds, \tag{6}$$

so that we can write,

$$y(t_{k+1}) = y(t_0) + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} (t_{k+1}-s)^{\alpha-1} f(s, y(s))ds.$$

In this way we can construct a numerical method by considering the product trapezoidal rule in each sub-interval, obtaining the approximation,

$$\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} (t_{k+1}-s)^{\alpha-1} f(s, y(s))ds \approx \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k+1} a_{j,k+1} f(t_j, y(t_j)), \tag{7}$$

where

$$a_{j,k+1} = \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}, & \text{if } j = 0, \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, & \text{if } 1 \le j \le k, \\ 1, & \text{if } j = k+1. \end{cases}$$

The approximation (7) naturally leads to the implicit formula,

$$y_{k+1} = y_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + f(t_{k+1}, y_{k+1})\right), \tag{8}$$

that generalizes to the fractional case of the standard one-step Adams–Multon method of order 2. In order to compute y_{k+1} from (8) we can replace y_{k+1} by a predictor approximation y^P_{k+1}, and then use a corrector iteration,

$$y^{(i)}_{k+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + f(t_{k+1}, y^{(i-1)}_{k+1})\right), \tag{9}$$

$$y^{(0)}_{k+1} := y^P_{k+1}, \quad i = 1, \ldots, q.$$

The remaining problem is the determination of the predictor formula that we require to calculate the value y^P_{k+1}. Here, we use the generalized one-step Adams–Bashforth method (that is, the fractional Euler method) that consists in replacing the integral on the right-hand side of Eq. (6) by the product rectangle rule,

$$\int_{t_0}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} f(s, y(s)) ds \approx \frac{h^\alpha}{\alpha} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y(t_j)),$$

where now

$$b_{j,k+1} = (k+1-j)^\alpha - (k-j)^\alpha.$$

Thus, the predictor y^P_{k+1} is determined by the explicit recursion,

$$y^P_{k+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \tag{10}$$

The method (9) and (10) has been considered and discussed in [12–14].

Besides the predictor corrector implementation described by (9) and (10), in this work we also consider the fully implicit implementation of (8), in which y_{k+1} is computed by solving with Newton iteration to preserve the good stability properties for the nonlinear system,

$$\Phi(y_{k+1}) - g_k = 0$$

where

$$\Phi(y_{k+1}) := y_{k+1} - \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{k+1}, y_{k+1})$$

$$g_k := y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j)\right)$$

and for implementation, we need to work with the Jacobian (4) of the system.

Finally, we recall the most significant results on the error analysis of the method (8), performed in [15,16]. The first result is based on smoothness assumptions on $D^\alpha y$; the second one on smoothness assumptions on the solution y.

Theorem 1. *Let $\alpha > 0$ and assume $D^\alpha y \in C^2[t_0, T]$ for some suitable T. Then,*

$$\max_{0\le j\le N} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if} \quad \alpha \ge 1, \\ O(h^{1+\alpha}) & \text{if} \quad \alpha < 1. \end{cases}$$

Theorem 2. *Let $0 < \alpha < 1$ and assume that $y \in C^2[t_0, T]$ for some suitable T. Then, for $1 \le j \le N$ we have,*

$$|y(t_j) - y_j| \le C\, t_j^{\alpha-1} \times \begin{cases} h^{1+\alpha} & \text{if} \quad 0 < \alpha < \frac{1}{2}, \\ h^{2-\alpha} & \text{if} \quad \frac{1}{2} \le \alpha < 1, \end{cases}$$

where C is a constant independent of j and h.

Thus for all choices of $\alpha > 0$ the described method gives a convergence order of at least one if either y or $D^\alpha y$ is at least two times continuous differentiable on $[t_0, T]$.

4. Numerical results and discussion

In this section we present some numerical experiments for the model (3), using different values for the initial conditions. We consider relatively long time intervals and different values of α. In particular we work with $\alpha = 1, 0.99, 0.95$ represented respectively by a solid, dashed and dotted line in each picture. We always take $\beta = 0.01$ and $\gamma = 0.02$. In Fig. 2, for small initial conditions, we consider the Generalized Euler Method (GEM) given by,

$$y(t_{k+1}) = y(t_k) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_k, y(t_k)),$$

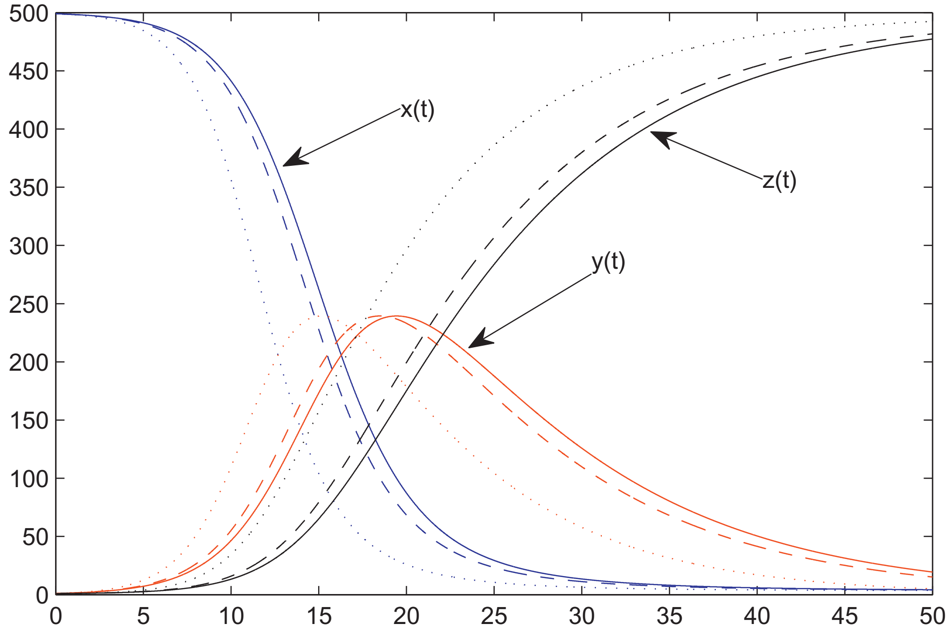

Fig. 2. Numerical solutions by GEM for $N_1 = 499$, $N_2 = 1$ and $N_3 = 1$ (see [10]). The timestep is $h = 0.01$.

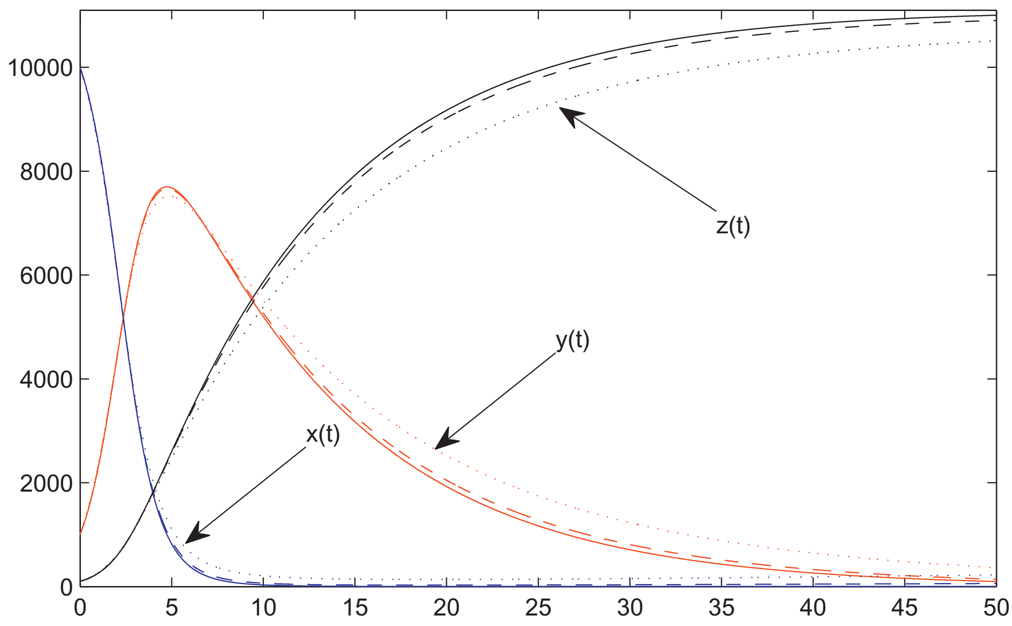

Fig. 3. Numerical solutions by the predictor corrector formula (9) and (10) with initial conditions $N_1 = 10^4$, $N_2 = 10^3$ and $N_3 = 10$.The timestep is $h = 0.01$.

that of course is an explicit code. This method has been used in [17] for fractional order models of HTLV infection, in [18] to study the HIV during the primary infection and in [19] for the problem of the population dynamics of the human immunodeficiency type 1 virus (HIV-1). More details on this approach can be found in [20]. In Fig. 3 we plot the numerical solution given by the predictor corrector formula (9) and (10) for higher values of the initial conditions $N_1 = 10^4$, $N_2 = 10^3$ and $N_3 = 10$. We take $h = 0.01$ and apply only one corrector iteration. For this problem GEM produces unstable solutions unless we take $h = 0.001$. Using the same timestep $h = 0.01$, the predictor corrector formula shows a certain instability at the beginning of the process when $N_2 \geq 10^4$, as shown in Fig. 4. This phenomenon seems independent of the number of corrector iterations. The fully implicit implementation of (8) has been used to solve the problem with $N_1 = 10^5$, $N_2 = 10^4$ and $N_3 = 10^3$ (timestep $h = 0.01$). The results are reported in Fig. 5.

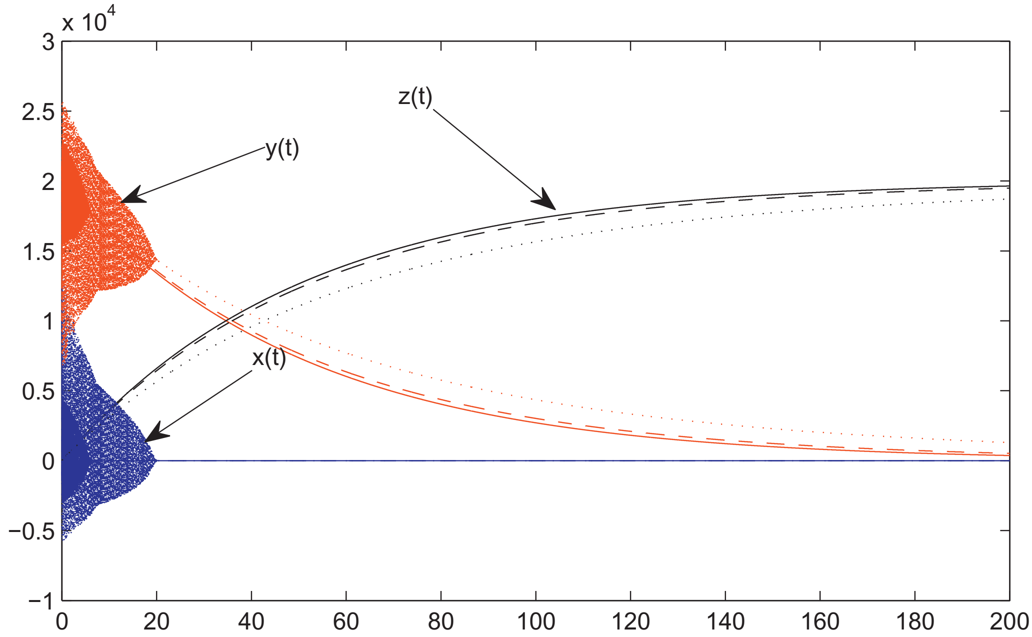

Fig. 4. Numerical solutions by the predictor corrector formula (9) and (10) with initial conditions $N_1 = 10^4$, $N_2 = 10^4$, $N_3 = 10^2$, timestep $h = 0.01$ and the number of corrector iterations = 2.

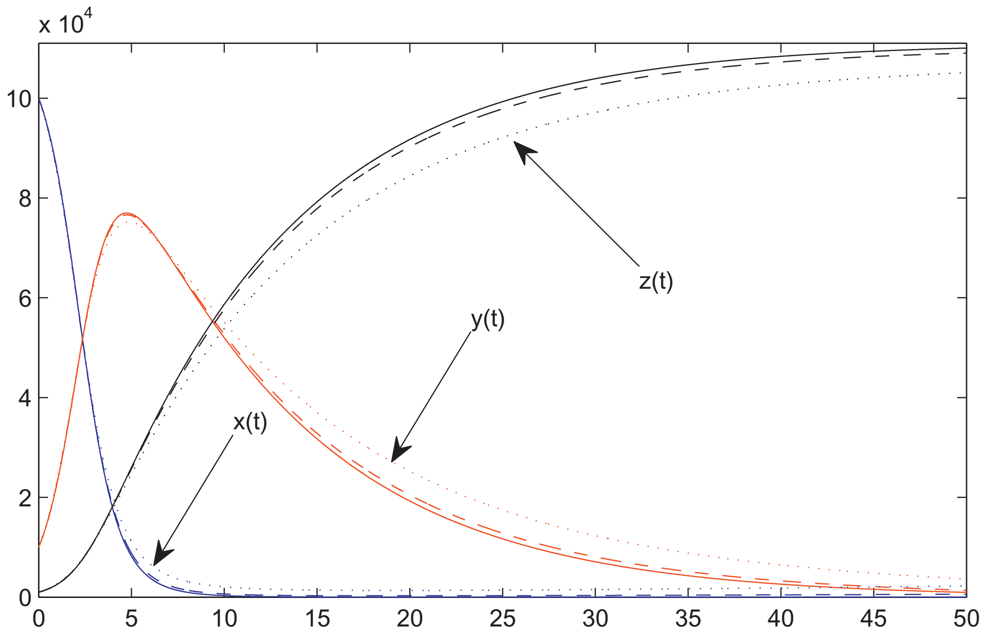

Fig. 5. Numerical solutions by (8) with Newton iteration for $x(t)$, $y(t)$, $z(t)$ versus time such that $N_1 = 10^5$, $N_2 = 10^4$ and $N_3 = 10^3$.

5. Conclusion

In this paper we discussed numerical methods to obtain the solution of fractional epidemic model (3) over a long time period where HAM [2,11] is not effective. Increasing the initial conditions the problem becomes difficult to solve. In particular, under realistic values of the initial conditions $N_1, N_2 \geq 10^4$ with $\beta = 0.01$, $\gamma = 0.02$, only the fully implicit implementation of formula (8) is able to provide accurate solution with a reasonable choice of the timestep.

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