Problems concerning parades and poly-Bernoulli numbers

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Abstract

This note records the solution or partial progress made on several open problems related to poly-Bernouilli (pB) numbers, posed by Knuth in [1].

1 Problem 1

Is there a combinatorial proof for the following formula?

$$C_{m,n} = \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{k,l}D_{m-k,n-l}$$

The left part of the formula is the number of parades with $m+1$ girls and n boys starting with a girl. Those are all the parades of the form $\alpha g_1 \beta$. We put the youngest girl g_1 in our parade. For α we choose $m-k$ of the m remaining girls and $n-l$ of the n boys. The remaining k girls, l boys come after the youngest girl and make up β. Now α has to be a parade starting with a girl or be empty, because our full parade has to start with a girl. It has to end with a boy or be empty, because otherwise g_1 is preceded by an older girl, which is not valid in a parade. So the number of possible α is $D_{m-k,n-l}$ by equation (17.1) in [1]. For β any valid parade will do, so there are $B_{k,l}$ possiblilities. And there are $\binom{m}{k}\binom{n}{l}$ ways to divide the girls and boys between α and β. ■

Using variations of the above combinatorial argument, we can dedude new identities between $B_{m,n}$, $C_{m,n}$ and $D_{m,n}$. For example, by using g_m, the oldest girl instead of g_1, we find that

$$C_{m,n} = \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}(C_{k-1,l}+[k=l=0])(C_{m-k,n-l-1}+[m-k=n-l-1=0]))$$

If we start with b_1, the youngest boy instead of g_1, we find (using [17.2] from [1]):

$$C_{m,n} = \sum_{k=0}^{m+1}\sum_{l=0}^{n-1}\binom{m+1}{k}\binom{n-1}{l}B_{k,l}(D_{m-k,n-l-1}+D_{m-k,n-l}), n>0$$

And if we start with b_n we find

$$C_{m,n} = \sum_{k=0}^{m+1}\sum_{l=0}^{n-1}\binom{m+1}{k}\binom{n-1}{l}C_{k-1,l}(C_{m-k,n-l-1}+[m-k+1=n-l-1=0])), n>0$$

Many more identities can be derives using similar constructions on other classes from section 17 in [1].

2 Problem 2

All mxn matrices for $m \le 5, n \le 5$ were enumerated using a computer program. Then for all $\binom{16}{2}$ matrix pairs $\{\phi,\psi\}$ the program counted how many of the mxn matrices were $\{\phi,\psi\}$-free. There were 25 pairs that resulted in the pB-numbers. (See table 1).

Bijections for most of these $\{\phi,\psi\}$-pairs were already discussed in [1]. One slighty new variation are the pairs in the min/max-closed section where one of the matrices is $\begin{pmatrix}1&0\\0&1\end{pmatrix}$. The pair $\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$ corresponds with the 'max-min'-relations, which satisfy the condition

$$x_i \smile y_j \text{ and } x_{i'} \smile y_{i'} \text{ implies } x_{\max\{i,i'\}} \smile y_{\min\{j,j'\}}.$$

And similarly, $\left\{\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$ corresponds with the 'min-max'-relations, which satisfy

$$x_i \smile y_j \text{ and } x_{i'} \smile y_{i'} \text{ implies } x_{\min\{i,i'\}} \smile y_{\max\{j,j'\}}.$$

We can construct bijections to the Γ-free matrices by repeatedly changing 2x2 submatrices of the form $\begin{pmatrix}1&*\\1&1\end{pmatrix}$ to $\begin{pmatrix}1&*\\0&1\end{pmatrix}$ (respectively changing $\begin{pmatrix}1&1\\*&1\end{pmatrix}$ to $\begin{pmatrix}1&0\\*&1\end{pmatrix}$).

3 Problem 5

Investigate the behaviour of the nonuniformity ratio $\max_{0\le r\le n} P_{m,n}(r)/\min_{0\le r\le n} P_{m,n}(r)$ when m and n are large.

Experiments seem to indicate that, for a given n, if m is large enough ($\approx m \ge n/2$), the maximum $P_{m,n}(r)$ is reached when $r=3$, the minimum is reached when $r=0$. Can this behaviour be explained?

4 Problem 9

This problem asks to investigate the number of lonesum $l\times m\times n$ tensors $L_{l,m,n}$ and also the weakly lonesome $l\times m\times n$ tensors $\hat{L}_{l,m,n}$.

$\{\psi, \phi\}$	In bijection with
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$	Strongly Γ-free
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$	Γ-free and L-free
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$	Min/max-closed relations
$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$	
$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$	Lonesum

Table 1: The 25 pairs $\{\phi, \psi\}$ that result in pB-numbers. The right pair on each line is equal to the left one with the ones and zeros swapped.

For the $2 \times 2 \times n$ weakly lonesum tensors, we only have to make sure that $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ don't occur together in the same tensor. Let's call $s_n(U)$ the number of tensors of size $2 \times 2 \times n$ that don't contain A and B, $s_n(A)$ the ones that contain A and $s_n(B)$ the ones that contain B. This gives the following recurrences:

$$s_0(U) = 1, s_0(A) = s_0(B) = 0$$

$$s_{n+1}(U) = 14 s_n(U)$$
$$s_{n+1}(A) = s_n(U) + 15 s_n(A)$$
$$s_{n+1}(B) = s_n(U) + 15 s_n(B)$$
$$f_n = 1 + 15 s_n(A) + 15 s_n(B) + 16 s_n(U)$$

Solving these recurrences gives the generating function:

$$\hat{G}_{22n}(z) = \frac{13z - 1}{(1 - 15z)(1 - 14z)}$$

.

This results in the following formula:

$$\hat{L}_{2,2,n} = 2 \times 15^n - 14^n$$

Using a similar method, we can also find the generating function and explicit formula for the $2 \times 2 \times n$ lonesum tensors:

$$G_{22n}(z) = \frac{1 - 21z^2 - 28z^3}{(1 - 5z)(1 - 4z)(1 - 3z)(1 - 2z)}$$

.

$$L_{2,2,n} = 8 \times 5^n - 4 \times 4^n - 4 \times 3^n + 2^n$$

5 Problem 10

This problem asks to investigate the number of max-closed $l \times m \times n$ tensors. We will compute the generating function for the $2 \times 2 \times n$ case.

If we start with a $2 \times 2 \times 1$ tensor and try to extend it to size $2 \times 2 \times 2$, there are 14 possible $2 \times 2 \times 1$ tensors that can occur to the right as a successor. Let's write *abcd* for the tensor $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let's represent the successors $S(t)$ by a bit string of length 14, with the bit positions corresponding to the tensors as follows:

$$1111, 1101, 1100, 1011, 1010, 1001, 1000, 0111, 0101, 0100, 0011, 0010, 0001, 0000$$

.

There are only 7 possibilities, shown in table 2.

Tensor t	Possible successors $S(t)$	Count $c(t)$
$*000$	11 1111 1111 1111	14
$*001$	11 0101 0110 1011	9
$*010$	10 0110 0110 1111	9
$*011$	10 0100 0110 1011	7
$*100$	11 1000 0111 1011	9
$*101$	11 0000 0110 1011	7
$*111$	10 0000 0110 1011	6

Table 2: List of possible successors for each $2 \times 2 \times 1$ tensor.

If we have a tensor $t_1 t_2 ... t_{n-1}$ made of $2 \times 2 \times 1$ tensors t_i, we can extend it with a tensor t_n where $t_n \in \bigcap_{i=1...n-1} S(t_i)$.

Let call $s_n(t)$ the number of max-closed $2 \times 2 \times n$ tensors that have the set $S(t)$ as a successor. Then we can setup recurrences for the $s_n(t)$, and based on those count the number of $2 \times 2 \times n$ tensors f_n.

$$s_0(*000) = 1, s_0(t) = 0, t \neq *000$$

$$s_{n+1}(*000) = 2s_n(*000)$$
$$s_{n+1}(*001) = 2s_n(*000) + 3s_n(*001)$$
$$s_{n+1}(*010) = 2s_n(*000) + 3s_n(*010)$$
$$s_{n+1}(*011) = 2s_n(*000) + 2s_n(*001) + 3s_n(*010) + 4s_n(*011)$$
$$s_{n+1}(*100) = 2s_n(*000) + 3s_n(*100)$$
$$s_{n+1}(*101) = 2s_n(*000) + 2s_n(*001) + 3s_n(*100) + 4s_n(*101)$$
$$s_{n+1}(*111) = 2s_n(*000)+2s_n(*001)+3s_n(*010)+3s_n(*011)+3s_n(*100)+3s_n(*101)+6s_n(*111)$$
$$f_n = 1 + \sum_t s_{n-1}(t)c(t)$$

Solving these recurrences gives the generating function

$$F_{22n}(z) = \frac{(1-z)(1-8z^2)}{(1-6z)(1-4z)(1-3z)(1-2z)}$$

.

From this generating function, we can find a general formula for f_{22n}:

$$f_{22n} = 35 \times 6^{n-1} - 24 \times 4^{n-1} + 2 \times 3^{n-1} + 2^{n-1}$$

Using a similar method, it is possible to find F_{lmn} and f_{lmn}, for a few other $l \times m \times n$:

$$F_{23n}(z) = \frac{(1-z)(1+2z-309z^2+2846z^3-9208z^4+8516z^5+3408z^6)}{(1-10z)(1-8z)(1-7z)(1-6z)(1-5z)(1-4z)(1-3z)(1-2z)}$$

.

$$f_{23n}(z) = 1547156\times10^{n-1}-\frac{203}{30}\times8^n-44\times7^n+\frac{105}{8}\times6^n+32\times5^{n-1}+7\times4^n-\frac{74}{35}\times3^n-\frac{13}{48}\times2^n$$

.

$$F_{24n}(z) = (1-z)\frac{\begin{array}{c}1+42z-4211z^2+108914z^3-1291187z^4+6343786z^5\\ +13861607z^6-344488734z^7+1832401390z^8\\ -4550005128z^9+5074183440z^{10}-1696749984z^{11}\end{array}}{\begin{array}{c}(1-15z)(1-13z)(1-12z)(1-11z)(1-10z)(1-9z)\\ (1-8z)(1-7z)(1-6z)(1-5z)(1-4z)(1-3z)(1-2z)\end{array}}$$

.

From the above, we can conjecture that F_{2mn} has the form

$$\frac{(1-z)(\text{some polynomial in } z \text{ of order} \binom{m+2}{2}-4)}{\prod_{k=1,\ldots,\binom{m+2}{2}-2,\binom{m+2}{2}} 1-kz}$$

.

$$F_{33n}(z) = (1-z)\frac{\begin{array}{c}1+90z-10157z^2+347670z^3-5577997z^4+34044314z^5\\ +272507281z^6-7049450706z^7+62690000360z^8-307963368792z^9\\ +869031680800z^{10}-1285812875072z^{11}+701635229056z^{12}-111688739328z^{13}\end{array}}{\begin{array}{c}(1-20z)(1-17z)(1-14z)(1-13z)(1-12z)(1-11z)(1-10z)\\ (1-9z)(1-8z)(1-7z)(1-6z)(1-5z)(1-4z)(1-3z)(1-2z)\end{array}}$$

.

6 Future work

Here are some additional open problems (some of which may be answered in a future version of this note):

1. Are there any other interesting numbers and corresponding bijections hiding in the $120-25=95$ pairs of matrices that don't give pB-numbers in Problem 2?

We can compute the number $X_{m,n}$ of $\{\phi,\psi\}$-free matrices for all 95 pairs $\{\phi,\psi\}$, and then group them in classes with the same $X_{m,n}$ for all m and n. Then there are (1, 5, 2, 1) classes of size (1, 4, 8, 16) where $X_{m,n}=X_{n,m}$, and (2, 6, 2, 0) classes where $X_{m,n}\neq X_{n,m}$.

The symmetric class of size 16 seems a good starting point for further investigations.

2. Is there an algorithm to generate all parades of m girls and n boys by swapping 2 persons at each step?

References

[1] Donald E. Knuth, Parades and poly-Bernoulli numbers, retrieved from `https://www-cs-faculty.stanford.edu/~knuth/papers/poly-Bernoulli.pdf`.

[2] Donald E. Knuth, *Sorting and Searching*; Volume 3 of *The Art of Computer Programming*.

[3] `https://oeis.org`, The On-Line Encyclopedia of Integer Sequencer ®, N. J. Sloane.