MATHEMATICAL QUESTIONS

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. VIII.

ARE,

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

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EDITED BY

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CORRIGENDA.

VOL. VII.

p. 69, end of line 12, *omit* "the".
p. 71, lines 3, 4, 5, 6, 12, 13 from bottom, *for* (1) *read* (−1).
p. 89, last line, *for* $\Delta = \&c.$ *read* $\Delta' = \&c.$

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p. 22, line 26, *for* τ *read* π.
p. 35, line 2 from bottom, *insert* "point" *before* (ξ^{-1}) &c.
p. 46, line 3 from bottom, *for* 2*dd* *read* 2*dd'*.
p. 88, line 3 from bottom, *for* 46 *read* 4.6, and *for* (3) *read* (2).
p. 100, line 13, *for* k *read* k^2.
p. 109, last line, *for* Hymer's *read* Hymers'.
p. 110, line 4 from top, *for* DM. DM *read* DM, DM'.
p. 110, line 4 from bottom, *for* CF *read* MH.

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Mathematical Papers, &c.

Solved Questions.

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Unsolved Questions.

No.

2275. (Proposed by the Rev. J. BLISSARD.)—Required to show that if $f(x)=\frac{c_1x}{1^2}-\frac{c_2x(x-1)}{1.2^2}+\frac{c_3x(x-1)(x-2)}{1.2.3^2}-$ &c., then $c_1, c_2, c_3 \ldots$ can be so determined that $\frac{1}{1}+\frac{1}{m+1}+\frac{1}{2m+1}$ &c. $+\frac{1}{(n-1)m+1}$ may be equated to any one of the m functions, viz., $f(mn)$, $f(mn-1)$, $f(mn-2), \ldots . f(mn-m+1)$, which functions therefore are all of equal value.

2276. (Proposed by T. COTTERILL, M.A.)—If tangents to a cubic at the points A B C D meet in a point T on the curve, a conic through T A B C will cut the cubic again in points collinear with D, the intersection of the tangents to the cubic at these points is on the conic, and is collinear with the intersection of the tangents to the conic at the same points and T.

2285. (Proposed by Messrs. HERMITE and SYLVESTER.)—(1.) Prove that the Jacobian of two binary quantics cannot have a less number of real linear factors than the difference between the number of such in the two quantics themselves separately taken.

(2.) Prove that if ϕ, ψ are three binary quantics, the number of real linear factors in F cannot exceed the number of such in

$$(\phi x+\psi y)\left(\phi\frac{d}{dy}-\psi\frac{d}{dx}\right)\text{F}.$$

(3.) From (2) deduce (1), and obtain ROLLE's theorem as a particular case of the latter.

2289. (Proposed by C. TAYLOR, M.A.)—(1.) Interpret the tangential equation $pqr=0$. (2.) Represent by a tangential equation a pair of straight lines regarded as a limit of a conic.

2301. (Proposed by W. K. CLIFFORD, B.A.)—A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B, and if SC, bisecting the angle ASB, cut AB in C, the locus of C is a straight line.

2312. (Proposed by Professor CAYLEY.)—*Trace* the curve defined by the equations given in the solution of Quest. 2110. (*Reprint*, Vol. VII., p. 17.)

2327. (Proposed by the late G. C. DE MORGAN, M.A.)—If $p_0, p_1, p_2, \ldots q_0, q_1, q_2 \ldots$ be whole numbers, such that $\frac{p_n}{q_n}$ has a limit finite or 0, the series $\frac{p_0}{q_0}-\frac{p_1}{q_0q_1}+\frac{p_2}{q_0q_1q_2}-\ldots$ is an incommensurable quantity.

2334. (Proposed by W. S. BURNSIDE, M.A.)—1. Construct a triangle of given species, having its vertices on three concentric circles.

2. The triangle being equilateral, what relation exists between the radii when only one such triangle can be so constructed?

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2337. (Proposed by Professor SYLVESTER.)—By the form probability of a group, understand the probability of a group of a given number of points satisfying a prescribed condition of *form*.

Let ABCD be any quadrilateral; ϕ the form probability of a group of points limited to the area ABCD; p, q, r, s the corresponding probabilities when one of the group is restricted to the lines AB, BC, CD, DA respectively. Show that if four forces act along the successive sides of ABCD, regarded as a rigid figure, with intensities proportional to $\phi-p$, $\phi-q$, $\phi-r$, $\phi-s$ respectively, they will keep the quadrilateral at rest.

2342. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Construct the minimum triangle of given species whose three sides shall touch three given circles.

2371. (Proposed by Professor CAYLEY.)—Consider a triangle ABC, and a point P; and let AP meet BC in M, and BP meet AC in N (if, to fix the ideas, P is within ABC, then M, N are in the sides BC, AC respectively, and the triangles APN, BPM are regarded as positive).

1. Find the locus of the point P, such that the ratio
(Δ APN + Δ BPM) : Δ ABC may have a given value.

2. Drawing from each point P, at right angles to the plane of the triangle, an ordinate PQ of a length proportional to the foregoing ratio (Δ APN + Δ BPM) : Δ ABC, trace the surface which is the locus of the point Q,—a surface which has, it is clear, the loci in (1) for its contour lines; and

3. Find the volume of the portion standing on ABC as base.

4. Deduce the solution of the following case of the four-point problem; viz., taking the points P, Q at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle? [For solutions of part (4) see *Reprint*, Vol. VIII., pp. 36, 74.]

2375. (Proposed by the Rev. W. ROBERTS, M.A.)—Being given a system of confocal spherical conics, in which the distance between the foci is $\frac{1}{2}\pi$, prove that the family of curves cutting them under an angle of 45° is algebraic, and determine its equation.

2381. (Proposed by W. S. BURNSIDE, M.A.)—1. Determine the degree of the locus represented by the equation

$$\frac{\alpha^2}{p^2}+\frac{\beta^2}{p_1^2}+\frac{\gamma^2}{p_2^2}-2\frac{\beta\gamma}{p_1p_2}\cos A-2\frac{\gamma\alpha}{p_2p}\cos B-2\frac{\alpha\beta}{pp_1}\cos C=1,$$

where α, β, γ are the distances of any point from the vertices of the triangle ABC, and p, p_1, p_2 the perpendiculars of the same triangle.

2. Show that the curve given by the above equation is the same as that referred to by Pr CAYLEY in Quest. 2110. (*Reprint*, Vol. I., p. 17.)

2389. (Proposed by Professor CREMONA.)—Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère rencontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.

2390. (Proposed by the late G. C. DE MORGAN, M.A.)—

Prove that $\int_{-\infty}^{+\infty}\frac{1}{x}\phi\left(x-\frac{a}{x}\right)dx=0$, a being anything positive; and

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$$\int_{-\infty}^{+\infty} \frac{1}{x}\phi\left(x+\frac{a}{x}\right)dx = 2\int_{-\infty}^{+\infty}\frac{\phi x}{x}dx,$$ a being infinitely small and positive, and ϕ being such a function that the subject of integration is finite for all finite values of x, however small a may be. In the second case, if $\frac{\phi x}{x}$ be infinite when $x = 0$, the integral on the right must be replaced by $2\int_{-\infty}^{+\infty}\frac{1}{x}\left\{\phi x - \phi(-x)\right\}dx$.

2391. (Proposed by Professor SYLVESTER.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call the *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it.

2393. (Proposed by Dr. SALMON, F.R.S.)—1. What is the degree of the evolute of a circular cubic or bi-circular quartic? 2. Generally, how are the characteristics of the evolute affected when the curve is circular or bi-circular?

2399. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Find a point on a sphere such that the triangle determined by the middle points of the three arcs connecting it with three given points on the sphere shall have two of its sides given.

2403. (Proposed by W. S. BURNSIDE, M.A.)—Prove that confocal circular cubics cut at right angles: taking advantage of Prof. SYLVESTER'S generation of these curves given in Quest. 1990. (*Reprint*, Vol. VI., p. 88.)

2408. (Proposed by S. ROBERTS, M.A.)—Representing by $R(x, y, z)$ the reciprocal of u, explain (1) the nature of the locus $R\left(\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}\right) = 0$, and account for its degree; and (2) give corresponding explanations relative to space of three dimensions.

2412. (Proposed by Professor CREMONA.)—La surface de STEINER ne contient pas de courbes d'ordre impair (ni réelles ni imaginaires); et toute courbe d'ordre $2n$ placée sur cette surface *peut être transformée rationnellement en* (*è punteggiata projettivamente a*) une courbe plane d'ordre n.

2413. (Proposed by Professor MANNHEIM.)—Three planes at right angles to each other, and intersecting in a point A, touch an ellipsoid whose centre is B. Prove that the connector AB and the normals at the three points of contact are generators of a hyperboloid.

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2421. (Proposed by M. W. CROFTON, B.A.)—Two places are taken at random in the northern hemisphere; find the chance of their distance exceeding 90° of a great circle.

2428. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1)\ldots\frac{\Gamma(mx)}{\Gamma(nx)}(x=0)=\frac{n}{m},$$

$$(2)\ldots\frac{\Gamma(x)}{\Gamma(2x)}-\frac{\Gamma(x+2)}{\Gamma(2x+2)}\cdot\frac{\pi^2}{1.2}+\frac{\Gamma(x+4)}{\Gamma(2x+4)}\cdot\frac{\pi^4}{1.2.3.4}-\&c.=0.$$

2432. (Proposed by Captain A. R. CLARKE, R.E., F.R.S.)—Two perfectly flexible and inextensible films, being equal circles in form, are fastened together along the perimeter: what is the greatest amount of air that this could be made to hold (as an *air-cushion*)?

2458. (Proposed by S. WATSON.)—Three points A, B, C are taken at random on the circumference of a given circle; find the chance that a given point P within the circle lies within the triangle ABC.

2462. (Proposed by J. GRIFFITHS, M.A.)—Trace the curve represented by the tangential equation

$$(\sin A.\mu\nu+\sin B.\nu\lambda+\sin C.\lambda\mu)^2=4\lambda\mu\nu(\lambda\cos A+\mu\cos B=\nu\cos C).$$

2474. (Proposed by Professor EVERETT).—

If $\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2=0$, show that when n is any positive integer

$$\left(\cos\theta.\frac{d}{dx}+\sin\theta.\frac{d}{dy}\right)^n=\left(\cos n\theta.\frac{d}{dx}+\sin n\theta.\frac{d}{dy}\right)\left(\frac{d}{dx}\right)^n;$$ and

hence prove the theorem (due to Rankine and Stokes) that at every multiple point in a plane curve which fulfils the condition $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, the branches made equal angles with each other.

2476. (Proposed by EXHUMATUS.)—A point is taken at random in each face of a tetrahedron ABCD; these points form an inscribed tetrahedron A′B′C′D′. Determine the locus of all those points within ABCD which have a given probability p of being within A′B′C′D′.

2482. (Proposed by H. R. GREER, B.A.)—A quadrilateral circumscribes an ellipse, the eccentric angles of the points of contact being $\alpha, \beta, \gamma, \delta$. Prove that the condition that a pair of opposite vertices should lie on a confocal conic may be thus expressed:

$$a^2(\cos A+\cos B+\cos C+\cos D)(\cos A+\cos B-\cos C-\cos D)\times$$
$$(\cos C+\cos A-\cos B-\cos D)(\cos B+\cos C-\cos A-\cos D)=$$
$$b^2(\sin A+\sin B+\sin C+\sin D)(\sin A+\sin B-\sin C-\sin D)\times$$
$$(\sin C+\sin A-\sin B-\sin D)(\sin B+\sin C-\sin A-\sin D);$$

where $A=\frac{1}{4}(-\alpha+\beta+\gamma+\delta)$, $B=\frac{1}{4}(\alpha-\beta+\gamma+\delta)$, $C=\frac{1}{4}(\alpha+\beta-\gamma+\delta)$, $D=\frac{1}{4}(\alpha+\beta+\gamma-\delta)$.

Hence deduce a solution of Mr. C. Taylor's unsolved Question 1819 (*Reprint*, Vol. V., p. 16).

2492. (Proposed by Professor SYLVESTER.)—Prove that the mean distance between any two points taken arbitrarily within a triangle is a constant fraction of the mean of the mean distances of the angles from

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the several points in the opposite sides, and determine the value of the fraction. Show also that the same is true of the mean distance between two points in any parallelogram, and determine the value of the fraction for that case.

2495. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Equal segments are cut off from two opposite corners of a given rectangle, the lines of section being parallel to the diagonal; determine their magnitude such that if five points be taken at random on the remaining surface, the probability of their forming the apices of a convex pentagon shall be a maximum.

2502. (Proposed by J. GRIFFITHS, M.A.)—If A, B, C, D be functions of x, y, z; and A_1, B_1, C_1, D_1 be certain constants; required the envelope of $A+\lambda B+\mu C+\nu D=0$, when the variables λ, μ, ν are connected by the two equations $A_1+\lambda B_1+\mu C_1+\nu D_1=0$, $\lambda^2+\mu^2+\nu^2-2\lambda\mu\nu=1$.

2503. (Proposed by H. R. GREER, B.A.)—Prove that the number of normals that can be drawn from a point in space of p dimensions to a $(\text{hyper})^{p-3}$ surface (i.e., what corresponds to a *surface* in space of three dimensions) of the nth degree is $\dfrac{n}{n-2}\left\{(n-1)^p-1\right\}$.

2512. (Proposed by Professor CAYLEY.)—If B, C, Q, R are any given points, find the locus of a point A such that

$AQ.AR.BC+BQ.BR.CA+CQ.CR.AB=BC.CA.AB.$

N.B.—The construction of one point on the locus is given by the theorem, Quest. 2010 (*Reprint*, Vol. VII., p. 43).

2515. (Proposed by F. C. MATHEWS, M.A.)—A man has a weight of P lbs. which is broken into n pieces, each being integral in lbs.; find the least value of n such that it may be possible to weigh with the pieces all weights from 1 lb. up to P lbs. Find also the maximum value of P corresponding to any value of n; and find, furthermore, with any value of P and the corresponding minimum of n, the total number of distinct modes of partition which will satisfy the condition.

2520. (Proposed by Captain A. R. CLARKE, F.R.S.)—Three points are taken at random in a given triangle; the probability (p) that they enclose a given point whose triangular coordinates are x, y, z, is

$$p=F(x, y, z)+F(y, z, x)+F(z, x, y),$$

where $F(x, y, z)=2xyz\,(1-6x^2)+(6y^2z^2-24y^3z^3)\log\left(1+\dfrac{x}{yz}\right)$.

2523. (Proposed by T. SAVAGE, M.A.)—1. If A, B, C be three closed areas situated in such a manner that no straight line can be drawn meeting more than two of them, the average area of all triangles formed by joining three points taken at random in A, B, C respectively is equal to the area of the triangle formed by joining the centres of gravity of A, B, C.

2. If A, B, C, D be four closed volumes such that (1) no straight line can meet more than two, and (2) no plane more than three, the average volume of the tetrahedron formed by joining four points taken at random in A, B, C, D respectively is equal to the volume of the tetrahedron formed by joining the centres of gravity of A, B, C, D respectively.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

AT RANDOM: No. IV.

By W. S. B. Woolhouse, F.R.A.S.

The opening paragraph of Mr. Godfray's remarks, No. III., upon the phrase "at random," (*Reprint*, Vol. VII., p. 98), intimating the necessity of strict definitions of random quantities, cannot of course be objected to, since mathematical investigations of every kind ought invariably to proceed upon accurate definitions. But I would here repeat my first statement, that the phrase "at random" has in general only one absolute and unqualified meaning; that the assemblage of quantities *all equally likely*, to which the random one belongs, should be elicited from the true nature of the problem, and not engrafted upon it as an arbitrary hypothesis, however feasible; and that mathematicians perfectly agree as to the precise meaning of "a line drawn at random from a given point," and of "a point taken at random on a given area."

The example relating to the projectile "thrown at random from a point" is not a bad illustration. Looking at the simple phraseology of the question, I unhesitatingly approve of the solution of the minority of candidates who took a hemisphere with the point as centre, and assumed that the chance of the direction of projection meeting any portion of the surface would be proportional to the area of that portion. According to the other solution, it is assumed that all azimuths are equally probable and all altitudes equally probable, so that all directions in space are not equally probable, the probability increasing towards the zenith. The former solution agrees with the supposition that the gun is worked on a pivot so as to move freely in all directions alike; the latter supposes it to be worked separately in azimuth and altitude, and it derives its chief support from the circumstance that the assumption is consistent with certain mechanical conditions, though at variance with the strict wording of the question.

It does not appear that anything more can be said on the subject, without trespassing unnecessarily upon your valuable pages; but if Mr. Godfray should consider it to admit of any further elucidation, I should like to know whether he really supposes the material discrepancies between the results of the different investigations of the "four point problem" have anything whatever to do with any existing misconceptions as to the simple definition of a "random point," or of any want of accord as to what that definition should be. This indeed was the principal theme of Mr. Godfray's first paper "On some Problems in the Theory of Chances," and I am still compelled to withhold my assent from what he had assumed, especially in reference to the "four point problem."

2249. (Proposed by Professor SYLVESTER.)—Prove that the sum of the ternary products of the six quantities

$$\begin{array}{l|l} (a-b)^2(b-c)^2(c-d)^2(d-e)^2(e-a)^2 & (a-d)^2(d-b)^2(b-c)^2(c-e)^2(e-a)^2 \\ (a-b)^2(b-d)^2(d-e)^2(e-c)^2(c-a)^2 & (a-c)^2(c-d)^2(d-b)^2(b-e)^2(e-a)^2 \\ (a-b)^2(b-e)^2(e-c)^2(c-d)^2(d-a)^2 & (a-c)^2(c-b)^2(b-e)^2(e-d)^2(d-a)^2 \end{array}$$

is equal to the sum of the ternary products of the complementary six, viz.,

$$\begin{array}{l|l} (a-c)^2(c-e)^2(e-b)^2(b-d)^2(d-a)^2 & (a-b)^2(b-e)^2(e-d)^2(d-c)^2(c-a)^2 \\ (a-d)^2(d-e)^2(e-b)^2(b-c)^2(c-a)^2 & (a-d)^2(d-e)^2(e-c)^2(c-b)^2(b-a)^2 \\ (a-e)^2(e-b)^2(b-c)^2(c-d)^2(d-a)^2 & (a-b)^2(b-d)^2(d-c)^2(c-e)^2(e-a)^2 \end{array}$$

I. *Solution by the* PROPOSER.

Call ρ_1, ρ_2 the two sums in question, and let $a = 0$; then ρ_1 becomes

$$\Sigma L^2 + \Sigma M^2 + \Sigma N^2,$$

where $\quad L = b^3cde.(c-d)^2(c-e)^2(d-e)^2(b-c)(b-d)(b-e),$

also $\quad M = b^2c^2de.(d-e)^3(b-c)^2(b-d)(b-e)(c-d)(c-e),$

and $\quad N = b^2c^2d^2.(d-e)^2(c-e)^2(b-e)^2(b-c)(c-d)(d-b),$

the summation referring to the interchanges of b, c, d, e with one another.

Thus the effect of an interchange between any two of the letters b, c, d, e is to leave ρ_1 unaltered, and consequently ρ_1 and ρ_2 are identical whenever any one of the quantities a, b, c, d, e vanishes.

But in general $(\rho_1 - \rho_2)^2$ is obviously from its form an invariant of the function (with unity for its leading term) whose roots are a, b, c, d, e, and this invariant, if not zero, from what has been shown above, contains as a factor the last coefficient of such function: but no coefficient of a quantic can ever be contained in any of its invariants; for if so, an arbitrary linear function of all the coefficients would enter into the same. Hence $\rho_1 = \rho_2$.

NOTE.—M. HERMITE appears to have been the first to notice this property of the ternary products of the separate groups, in his memoir on the equation of the 5th Degree in the *Comptes Rendus* for the present year.

The sum of such products (say R) will easily be seen to have the property of vanishing whenever $a = b$ and $c = d$; for in such case 4 out of the 6 terms in either group become zero. It is therefore a linear function of Λ (*i.e.* $2^{11}L - J^3$), and JD; L, J, D having the meanings ascribed to them in my *Trilogy*, in the *Phil. Trans.* for 1865. From M. HERMITE's calculations the actual value of R appears to be a numerical multiple of $\Lambda - \frac{5}{4}JD$, so that by virtue of the rule given in the trilogy ($-\frac{5}{4}$ being comprehended within the limits -2 and 1) R is a good and sufficient third criterion along with J and D the first and second criteria for determining the nature of the roots of a quintic.

II. *Solution by* W. S. BURNSIDE, M.A.

1. Write α^{-1}, β^{-1}, γ^{-1}, δ^{-1} for $a-e$, $b-e$, $c-e$, $d-e$ respectively, and multiply all the terms by $(\alpha\beta\gamma\delta)^4$; then all the denominators disappear, and the first six quantities are thus reduced to

$$\begin{array}{c|c} (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2 & (\alpha-\delta)^2(\delta-\beta)^2(\beta-\gamma)^2 \\ (\alpha-\beta)^2(\beta-\delta)^2(\gamma-\alpha)^2 & (\alpha-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2 \\ (\alpha-\beta)^2(\gamma-\delta)^2(\delta-\alpha)^2 & (\alpha-\gamma)^2(\gamma-\beta)^2(\delta-\alpha)^2 \end{array}$$

and the other six are the quantities complementary to these in

$$(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2.$$

Again write l, m, n for $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$ respectively, and the first six quantities properly arranged become

$$(1)\ldots\ldots\ldots\ldots\ldots\left\{\begin{array}{c|c} l^2(n-l)^2\ (m-n)^2 & m^2n^2(n-l)^2 \\ m^2(l-m)^2\ (n-l)^2 & n^2l^2\ (l-m)^2 \\ n^2(l-m)^2(m-n)^2 & l^2m^2(m-n)^2 \end{array}\right.$$

and the other six are the complementary quantities in

$$l^2m^2n^2(m-n)^2\ (n-l)^2\ (l-m)^2,$$

which may be arranged in the two following ways:—

$$(2)\ldots\ldots\ldots\ldots\left\{\begin{array}{c|c} l^2(m-n)^2\ (l-m)^2 & m^2n^2\ (l-m)^2 \\ m^2(m-n)^2\ (n-l)^2 & n^2l^2(m-n)^2 \\ n^2\ (l-m)^2(n-l)^2 & l^2m^2\ (n-l)^2 \end{array}\right\}$$

$$(3)\ldots\ldots\ldots\ldots\left\{\begin{array}{c|c} m^2\ (n-l)^2\ (m-n)^2 & l^2m^2(n-l)^2 \\ n^2\ (l-m)^2\ (n-l)^2 & m^2n^2\ (l-m)^2 \\ l^2\ (l-m)^2\ (m-n)^2 & l^2n^2\ (m-n)^2 \end{array}\right.$$

Now if any three terms in (1) be combined, the resulting constituent will be obtained from *one or other* of the arrangements (2) and (3) by combining the terms occupying the same positions as those selected in (1); and thus we see that the sum of the ternary combinations of both sets of six quantities are identical term by term.

2. When the 20 ternary combinations expressed in terms of $\alpha, \beta, \gamma, \delta$ are arranged symmetrically, we have

$$\begin{array}{ll} \Sigma\left\{(\alpha-\gamma)^4\ (\alpha-\delta)^4\ (\beta-\gamma)^4\ (\beta-\delta)^4\right\}.(\gamma-\delta)^2 \ldots\ldots\ldots\ldots & \text{six terms} \\ +\Delta\,\Sigma\ (\beta-\gamma)^2\ (\gamma-\alpha)^2(\alpha-\beta)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{four terms} \\ +\Delta\,\Sigma\ (\alpha-\delta)^2\ (\beta-\delta)^2(\gamma-\delta)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{four terms} \\ +\Delta\,\Sigma\ (\alpha-\beta)^2\ (\gamma-\delta)^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \text{six terms} \end{array}$$

where $\Delta = (\alpha-\beta)^2\ (\alpha-\gamma)^2\ (\alpha-\delta)^2\ (\beta-\gamma)^2\ (\beta-\delta)^2\ (\gamma-\delta)^2$.

3. Since the function we have been discussing is an invariant of the quintic having a, b, c, d, e for its roots, and is of the 12th degree in the coefficients, and of the 30th in the roots, it must necessarily be expressible in terms of the three fundamental invariants of the quintic, and must therefore be of the form $p\,.\,A^3+q\,.\,AB+r\,.\,C$, (Salmon's *Higher Algebra*, Art. 136); and p, q, r are such that this function vanishes for a pair of double roots, viz. $a = b$, and $c = d$.

2406. (Proposed by Rev. J. BLISSARD.)—Show that $\dfrac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)}$

$$(1)\ldots = 2\left\{\frac{B_{n+1}}{n+1} - \frac{n}{1}\cdot\frac{B_{n+2}}{n+2} + \frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3} - \&c.\right\}$$

$$(2)\ldots = 4\left\{(2^{n+2}-1)\frac{B_{n+2}}{(n+1)(n+2)} - \frac{n}{1}(2^{n+3}-1)\frac{B_{n+3}}{(n+2)(n+3)} + \&c.\right\},$$

where B is the Representative of Bernoulli's numbers.

Solution by the PROPOSER.

By use of what I have called *Representative Notation*, it is easy to arrive at the following general formula, viz.,

$$\frac{(1+x)^{n+1}}{n+1} - \frac{m}{1}\cdot\frac{(1+x)^{n+2}}{n+2} + \frac{m(m-1)}{1.2}\cdot\frac{(1+x)^{n+3}}{n+3} - \&c.$$

$$+(-1)^{m+1}\left\{\frac{x^{m+1}}{m+1} + \frac{n}{1}\cdot\frac{x^{m+2}}{m+2} + \frac{n(n-1)}{1.2}\cdot\frac{x^{m+3}}{m+3} + \&c.\right\}$$

$$= \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}.$$

Let $m = n$, then $\dfrac{(1+x)^{n+1}}{n+1} - \dfrac{n}{1}\dfrac{(1+x)^{n+2}}{n+2} + \&c.\ldots$

$$+(-1)^{n+1}\left(\frac{x^{n+1}}{n+1} + \frac{n}{1}\cdot\frac{x^{n+2}}{n+2} + \&c.\right) = \frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} \quad \ldots\ldots(1).$$

Now (1) for x put B, then since $(B+1)^r = B_r$ $(r>1)$ and $=0$ (r odd), we have (See Theory of Generic Equations in the *Quarterly Journal of Mathematics*, Vol. IV.)

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = \frac{B_{n+1}}{n+1} - \frac{n}{1}\frac{B_{n+2}}{n+2} + \frac{n(n-1)}{1.2}\frac{B_{n+3}}{n+3} - \&c.\ldots$$

$$+(-1)^{n+1}\left\{\frac{B_{n+1}}{n+1} + \frac{n}{1}\frac{B_{n+2}}{n+2} + \&c.\right\}$$

$$= 2\left\{\frac{B_{n+1}}{n+1} + \frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3} + \&c.\right\} \; n \text{ odd, and}$$

$$= -2\left\{\frac{n}{1}\frac{B_{n+2}}{n+2} + \frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{B_{n+3}}{n+3} + \&c.\right\} \; n \text{ even.}$$

But B_{n+1}, B_{n+3}, B_{n+5} &c. all vanish (n even) and B_{n+2}, B_{n+4}, &c. all vanish n odd. Hence whether n be even or odd

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = 2\left\{\frac{B_{n+1}}{n+1} - \frac{n}{1}\cdot\frac{B_{n+2}}{n+2} + \frac{n(n-1)}{1.2}\cdot\frac{B_{n+3}}{n+3} - \&c.\right\}$$

Ex. Let $n = 3$, then $\dfrac{\{\Gamma(4)\}^2}{\Gamma(8)}\left(=\dfrac{1}{140}\right) = 2\left(\dfrac{B_4}{4} - 3\dfrac{B_5}{5} + 3\dfrac{B_6}{6} - \dfrac{B_7}{7}\right);$

but $B_4 = -\frac{1}{30}$, $B_5 = 0 = B_7$, $B_6 = \frac{1}{42}$, $\therefore \frac{1}{140} = 2\left(-\frac{1}{120} + \frac{1}{84}\right)$, which holds.

Again (2), in (I) for x put A, where A belongs to the Generic Equation $(A+1)^r + A^r = 0\,(r>0)$, and $\therefore (A+1)^r = -A_r$ which $= \frac{2^{r+2}-2}{r+1} B_{r+1}$.

$$\text{Hence } \frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = (2^{n+3}-2)\frac{B_{n+2}}{(n+1)(n+2)} - \frac{n}{1}(2^{n+4}-2)\frac{B_{n+3}}{(n+2)(n+3)}$$
$$+ \frac{n(n+1)}{1.2}.(2^{n+5}-2)\frac{B_{n+4}}{(n+3)(n+4)} - \&c.$$
$$+(-1)^{n+1}\left\{(2^{n+3}-2)\frac{B_{n+2}}{(n+1)(n+2)} + \frac{n}{1}(2^{n+4}-2)\frac{B_{n+3}}{(n+2)(n+3)} + \&c.\right\}$$

therefore, as in the preceding case, whether n be even or odd,

$$\frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+2)} = 4\left\{(2^{n+2}-1)\frac{B_{n+2}}{(n+1)(n+2)} - \frac{n}{1}(2^{n+3}-1)\frac{B_{n+3}}{(n+2)(n+3)}\right.$$
$$\left. + \frac{n(n-1)}{1.2}(2^{n+4}-1)\frac{B_{n+4}}{(n+3)(n+4)} - \&c.\right\}$$

Ex. (Let $n = 3$), then $\frac{1}{140} = 4\left(\frac{31B_5}{4.5} - \frac{3.63B_6}{5.6} + \frac{3.127B_7}{6.7} + \frac{255B_8}{7.8}\right)$

$$= 4\left(-\frac{3.63}{5.6}\cdot\frac{1}{42} + \frac{255}{7.8}\cdot\frac{1}{30}\right), \text{ which holds.}$$

2370. (Proposed by M. W. CROFTON, B.A.)—If three points be taken at random on the surface of a sphere, find the chance (1) that some one of the angles of the triangle they determine will be greater than the sum of the other two, (2) that the area will exceed a great circle of the sphere, (3) that the area will exceed a given area.

Solution by EXHUMATUS.

1. Let A and B be two points taken at random on the surface of a sphere; then if the third point C be taken within the small circle which is described on AB as a diameter, it is clear (on joining C with the middle point of AB) that the angle at C is greater than the sum of the angles at A and B. Hence the chance that the angle C is greater than (A + B) is

$$p = \frac{1}{16\pi^2}\int_0^\pi (2\pi \sin c\,.\,dc)\; 2\pi\left(1-\cos\frac{c}{2}\right)$$
$$= -\int_0^\pi \left(\cos\frac{c}{2} - \cos^2\frac{c}{2}\right) d.\cos\frac{c}{2} = \frac{1}{6}.$$

Consequently, ***the chance that some one of the angles will exceed the sum of the other two is*** $\frac{1}{2}$.

2. It is easily proved by the ordinary formulæ of Spherical Trigonometry that if the area of a spherical triangle on a given base AB be constant, the locus of the vertex is a small circle. If the given area be a great circle of the sphere, the locus is a small circle described on a diameter equal and opposite to AB. If a third point C be taken within this small circle, the area of the triangle ABC will exceed a great circle of the sphere; and the probability of this by the foregoing case is $\frac{1}{8}$, which is therefore ***the chance that the area will exceed a great circle of the sphere.***

3. Generally, to find the chance that the area will be less than a given area σ; let ρ be the length of an arc of a great circle joining C to the middle of AB; then it may be shown that

$$\tan\frac{\sigma}{2} = \frac{\sin\frac{1}{2}c \sin\rho \sin\phi}{\cos\frac{1}{2}c + \cos\rho},$$

where ϕ is the angle ρ makes with AB at their intersection. This equation is that of a small circle of the sphere whose centre is in the great circle which bisects AB at right angles, and of which the radius is determined by the equation $\tan r = \tan\frac{1}{2}c \operatorname{cosec}\frac{1}{2}\sigma$. This small circle cuts the great circle AB in two points A′, B′ directly opposite to A and B. It is clear that the point C must not fall within the circumference of this small circle, nor within the circumference of *another* small circle equal to it and similarly situated on the other side of AB, cutting AB also in the same two points A′, B′. The total area then to which C is restricted is $4\int_0^{\frac{1}{2}\pi}(1-\cos\rho)\,d\phi$, which may be integrated by expressing $\cos\rho$ in terms of ϕ, from the above expression for $\tan\frac{1}{2}\sigma$. Or thus; if M be the middle point of AB, and P the pole of the small circle, the area of the spherical triangle MPB′ is $\frac{1}{2}\pi + (\pi-\zeta) + (\frac{1}{2}\pi+\frac{1}{2}\sigma)-\pi$, where ζ is half the angle A′PB′, and $\cos r \tan\zeta = \tan\frac{1}{2}\sigma$. That the angle A′B′P is equal to $\frac{1}{2}\pi-\frac{1}{2}\sigma$ is evident from the expression $\tan r = \tan\frac{1}{2}c \operatorname{cosec}\frac{1}{2}\sigma$. From the spherical triangle MB′P we have to deduct that part of the small circle which falls within it; the area of this is $(1-\cos r)(\pi-\zeta)$; the remainder multiplied by 4 is the area to which the third point C is restricted. Hence the chance required is

$$p = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\left\{\frac{\sigma}{2} + \cos r\,(\pi-\zeta)\right\}\frac{\sin c}{2}\,dc.$$

Substitute here ζ for c by means of the equation $\cos\zeta = \cos\frac{1}{2}c\cos\frac{1}{2}\sigma$, and the expression is transformed to

$$p = \frac{2}{\pi\cos^2\frac{1}{2}\sigma}\int\left\{\frac{\sigma}{2}\cos\zeta\sin\zeta + (\pi-\zeta)\tan\frac{\sigma}{2}\cos^2\zeta\right\}d\zeta,$$

the limits being from $\zeta = \frac{1}{2}\sigma$, to $\frac{1}{2}\pi$. The result is

$$p = \frac{1}{\pi\cos^2\frac{1}{2}\sigma}\left\{\frac{\sigma}{2} + \frac{1}{2}\sin\frac{\sigma}{2}\cos\frac{\sigma}{2} - \pi\sin^2\frac{\sigma}{2} + \left(\frac{\pi}{2}-\frac{\sigma}{2}\right)\left(\frac{3}{4}\pi-\frac{\sigma}{4}\right)\tan\frac{\sigma}{2}\right\},$$

which is *the chance that the area will be less than* σ.

4. When the given area is that of a great circle of the sphere, we have $\sigma = \pi$ in (3), and p takes the apparent form of infinity, but is really a fraction of the form $\frac{0}{0}$ whose value is readily determined if we put $\sigma = \pi - 2\theta$.

For this value of σ we get

$$p = \frac{1}{\pi \sin^2 \theta}\left\{\pi \sin^2 \theta - \pi + \frac{\pi}{2} - \theta + \frac{1}{2}\sin\theta\cos\theta + \frac{\theta\cos\theta}{\sin\theta}\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right\}$$

$$= 1 - \frac{\sin\theta - \theta\cos\theta}{2\sin^3\theta} - \frac{\theta}{2\pi}\left\{\frac{\theta - \sin\theta\cos\theta}{\theta\sin^2\theta} + \frac{\sin\theta-\theta\cos\theta}{\sin^3\theta}\right\}.$$

Now, when θ is very small, it is easily seen that ultimately

$$\frac{\sin\theta - \theta\cos\theta}{\sin^3\theta} = \frac{1}{3}, \text{ and } \frac{\theta - \sin\theta\cos\theta}{\theta\sin^2\theta} = \frac{2}{3};$$

so that when $\theta = 0$, that is when $\sigma = \pi$, the required probability is $\frac{5}{6}$. *The chance that the area will exceed a great circle of the sphere is consequently* $\frac{1}{6}$.

2417. (Proposed by the Rev. W. ROBERTS, M.A.)—Denoting the roots of the cubic equation $x^3 + px + q = 0$ by α, β, γ, it is required to form the equation the roots of which are $\alpha\beta + \gamma$, $\beta\gamma + \alpha$, $\alpha\gamma + \beta$.

I. *Solution by the* REV. ROBERT HARLEY, F.R.S.

More generally, let α, β, γ be the roots of the complete cubic equation $x^3 + bx^2 + cx + d = 0$; then the equation $y^3 + By^2 + Cy + D = 0$, whose roots y_1, y_2, y_3, are $\alpha\beta+\gamma$, $\alpha\gamma+\beta$, $\beta\gamma+\alpha$, is readily calculated. For we have

$$-B = \Sigma y = \Sigma\alpha\beta + \Sigma\alpha = c - b,$$

$$C = \Sigma y_1 y_2 = \alpha\beta\gamma\Sigma\alpha + \Sigma\alpha^2\beta + \Sigma\alpha\beta$$

$$= b(d-c) + 3d + c,$$

$$-D = y_1 y_2 y_3 = \alpha^2\beta^2\gamma^2 + \alpha\beta\gamma\Sigma\alpha^2 + \Sigma\alpha^2\beta^2 + \alpha\beta\gamma$$

$$= (d+c)^2 - bd(b+2) - d.$$

Therefore the required equation is

$$y^3 - (c-b)y^2 + \left\{b(d-c) + 3d + c\right\}y - (d+c)^2 + bd(b+2) + d = 0.$$

In the particular case proposed, we have $b = 0$, and writing p and q in of c and d respectively, the equation becomes

$$y^3 - py^2 + (3q+p)y + q - (q+p)^2 = 0.$$

II. *Solution by* J. MCDOWELL, M.A., F.R.A.S.; J. DALE; *and others.*

We have $\qquad x^3 + px + q = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1);$

and since $\qquad \alpha\beta + \gamma = \dfrac{\alpha\beta\gamma}{\gamma} + \gamma = \dfrac{-q}{\gamma} + \gamma,$

if we put $\qquad y = \dfrac{-q}{x} + x$, or $x^2 - yx - q = 0 \;\ldots\ldots\ldots\ldots\ldots\;(2),$

and eliminate x between (1) and (2), the resulting equation in y will be the required one.

(1) $- x$ (2) gives $\qquad yx^2 + (p+q)x + q = 0 \;\ldots\ldots\ldots\ldots\ldots\;(3);$

$(3) - y\,(2)$ gives $(y^2+p+q)\,x+q\,(y+1)=0$, or $x=-\dfrac{q\,(y+1)}{y^2+p+q}$.
This value of x substituted in (2) gives, after a few simple reductions,

$$y^3-py^2+(p+3q)\,y+q-(p+q)^2=0,$$

the required equation.

2234. (Proposed by Professor EVERETT.)— Find a curve such that the area $\int y dx$ bounded by two ordinates, whose mutual distance is constant, varies directly as the ordinate midway between them.

Solution by THOMAS SAVAGE, M.A.

Let $y=f'(x)$ be the equation to the curve, $2a$ the given mutual distance of the two ordinates.

Then $$f(x+a)-f(x-a)=mf'(x)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or $$2\left\{f'(x)\,a+f'''(x)\frac{a^3}{\Pi(3)}+f^v(x)\frac{a^5}{\Pi(5)}+\ldots\right\}=mf'x,$$

or $$\frac{f'''(x)}{f'(x)}\frac{a^3}{\Pi(3)}+\frac{f^v(x)}{f'(x)}\frac{a^5}{\Pi(5)}+\ldots=\frac{m}{2}-a.$$

If $\frac{1}{2}m=a$, this equation is satisfied by $f'''(x)=0$, which gives us any straight line. If not, and k^2 be determined by either of the equations

$$\pm k^2\frac{a^3}{\Pi(3)}+k^4\frac{a^5}{\Pi(5)}\pm k^6\frac{a^7}{\Pi(7)}+\ldots=\frac{m}{2}-a$$

the equation (1) will be satisfied by $\dfrac{f'''(x)}{f'(x)}=\pm k^2$, which gives

$$y=A\epsilon^{kx}+B\epsilon^{-kx},\ \text{or}\ y=C\sin(kx+D),$$

where A, B, C, D are any arbitrary constants. The linear form of equation (1) shows that the sum or difference of any particular solution is also a solution.

2252. (Proposed by Professor EVERETT.)—Find the forms of ϕ and ψ which make the mean value of $\phi(x)$ from $x=a-c$ to $x=a+c$ take the form $\phi(a)\,.\,\psi(c)$.

Solution by the PROPOSER.

The mean value in question is $\dfrac{1}{2c}\displaystyle\int_{a-c}^{a+c}\phi(x)\,dx$, which, if F denote the primitive of ϕ, is equal to $\dfrac{1}{2c}\left\{F(a+c)-F(a-c)\right\}$; and this by Tay'or's theorem is equal to the series $\phi(a) + \phi''(a)\frac{c^2}{\Pi(3)} + \phi''''(a\frac{c^4}{\Pi(5)} + \ldots\ldots$

Equating $\phi(a).\psi(c)$ to this series, differentiating twice with respect to c and then putting $c=0$, we have $\phi(a).\psi''(0) = \frac{1}{3}\phi''(a)$. Hence the ratio of $\phi''(a)$ to $\phi(a)$ is to be independent of the value of a. The only functious which fulfil this condition are

$\phi(x) = k\sin(mx+l)$ and $\phi(x) = k\epsilon^{mx} + l\epsilon^{-mx}$, k, l, m being any constants. Substituting these values in the series which is to be equal to $\phi(a).\psi(c)$, we find that

when $\phi(x) = k\sin(mx+l)$, we have $\psi(c) = \frac{\sin mc}{mc}$;

when $\phi(x) = k\epsilon^{mx} + l\epsilon^{-mx}$, we have $\psi(c) = \frac{1}{2mc}(\epsilon^{mc} - \epsilon^{-mc})$.

[The solution of Quest. 2234 follows at once from the above by putting $a-c$ and $a+c$ for the extreme values of x. The equation of the curve is $y = \phi(x)$, where $\phi(x)$ has either of the values found above.]

2388. (Proposed by E. PROUHET.)—Si l'on designe par a, b, c, a', b', c', les côtés d'un hexagone inscrit dans un cercle; par A la diagonale qui n'a aucune extrémité commune avec les côtés opposés a et a'; par B, C les deux autres diagonales analogues; on aura les deux équations suivantes:—

$$B^2Ca + B^2b'c - Ba'(a^2 - b'^2 - c'^2) + a'^2b'c' = C^2Ba' + C^2b'c' - Ca(a'^2 - b'^2 - c'^2) + a^2b'c',$$

$$B^2Ca' + B^2bc - Ba(a'^2 - b^2 - c^2) + a^2bc = C^2Ba + C^2bc - Ca'(a^2 - b^2 - c^2) + a'^2bc,$$

qui permettent d'établir une équation entre les côtés de l'hexagone et une des grandes diagonales. (J'appelle grande diagonale toute diagonale qui joint deux sommets opposés.)

Solution by S. WATSON; G. A. OGILVIE; *and others.*

Let PQRSTU be the hexagon, and join PT; then $\angle$ PQT = PST = π − PUT = α (suppose);

$$\therefore\ PT^2 = a^2 + C^2 - 2aC\cos\alpha = a'^2 + B^2 - 2a'B\cos\alpha = b'^2 + c'^2 + 2b'c'\cos\alpha;$$

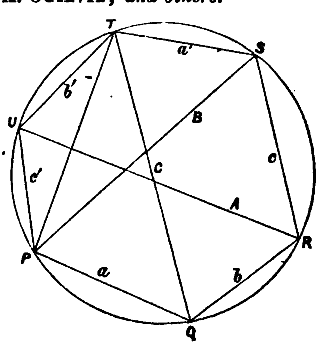
therefore, by eliminating $\cos\alpha$,

$$(b'c' + aC)(b'^2 + c'^2 - a'^2 - B^2) = (b'c' + a'B)(b'^2 + c'^2 - a^2 - C^2)\ldots(1).$$

Similarly, by joining QS, we get

$$(bc + a'C)(b^2 + c^2 - a^2 - B^2) = (bc + aB)(b^2 + c^2 - a'^2 - C^2)\ldots(2).$$

Equations (1) and (2) reduce to the forms given in the question.



2321. (Proposed by Professor CAYLEY.)—Given a conic, to find four points such that all the conics through the four points may have their centres in the given conic.

Solution by SAMUEL ROBERTS, M.A.

If we take any three points forming a triangle of reference, and (l, m, n) are the coordinates of any other point, a quadrangle, of which the three points first taken are the intersections of opposite sides, is determined by

$$(l, m, n), (l, m, -n), (l, -m, n), (-l, m, n).$$

This convenient result gives for any conic circumscribing the quadrangle

$$A\alpha^2 + B\beta^2 + c\gamma^2 = 0, \quad Al^2 + Bm^2 + Cn^2 = 0.$$

Now the centre of this conic is given by

$$x : y : z = \frac{a}{A} : \frac{b}{B} : \frac{c}{C} \text{ or } \frac{a}{x} : \frac{b}{y} : \frac{c}{z} = A : B : C,$$

a, b, c representing the sides of the triangle of reference.

Hence the locus of centres is

$$\frac{al^2}{x} + \frac{bm^2}{y} + \frac{cn^2}{z} = 0.$$

If then we refer the given conic to any inscribed triangle and its equation is

$$\frac{A'}{x} + \frac{B'}{y} + \frac{C}{z} = 0,$$

four points fulfilling the condition imposed are determined by

$$\pm\sqrt{\left(\frac{A'}{a}\right)} : \pm\sqrt{\left(\frac{B'}{b}\right)} : \pm\sqrt{\left(\frac{C'}{c}\right)}.$$

2368. (Proposed by X. U. J.)—Show that the four straight lines represented by the equations $u=0$, $v=0$, $lu^2 + 2muv + nv^2 = 0$, will form a harmonic pencil if $8m^2 = 9ln$.

I. *Solution by* J. WALMSLEY.

Putting $k = \dfrac{2m}{3l}$ and $u' \equiv u + kv$, we have $2m = 3lk$ and $n = 2lk^2$; and the four straight lines are easily found to be, in another order,

$$u'=0, \quad u'-kv=0, \quad v=0, \quad u'+kv=0.$$

II. *Solution by* JAMES DALE.

The anharmonic ratio of the pencil formed by the four straight lines

$$u + k_1v = 0, \quad u + k_2v = 0, \quad u + k_3v = 0, \quad u + k_4v = 0$$

is

$$\frac{(k_1-k_2).(k_3-k_4)}{(k_1-k_4).(k_3-k_2)},$$

Putting $k_1 = 0$ and $k_2 = \frac{1}{0}$, the anharmonic ratio of the pencil

$$u = 0,\ v = 0,\ u + k_3 v = 0,\ u + k_4 v = 0$$

is $\frac{k_3 - k_4}{-k_4}$, and if this be a harmonic ratio, then $k_3 = 2k_4$.

Now from the equation $lu^2 + 2muv + nv^2 = 0$, we get

$$k_3 = \frac{1}{l}\left\{m + \sqrt{(m^2 - ln)}\right\},\quad k_4 = \frac{1}{l}\left\{m - \sqrt{(m^2 - ln)}\right\};$$

and substituting these values in $k_3 = 2k_4$, and freeing from radicals, we get

$$8m^2 = 9ln.$$

2346. (Proposed by the Rev. R. H. WRIGHT, M.A.)—Through the angular points of the triangle of reference ABC, draw B′C′, C′A′, A′B′ respectively perpendicular to BC, CA, AB, and find the trilinear equation of the circle drawn round the triangle A′B′C′.

I. *Solution by the* PROPOSER; E. MCCORMICK; *and others.*

The coordinates of the vertices of the triangle A′B′C′ are

$$\begin{array}{lll} A' \ldots \alpha_1 = c \cot A \cos B, & \beta_1 = c \operatorname{cosec} A, & \gamma_1 = -c \cot A; \\ B' \ldots \alpha_2 = -a \cot B, & \beta_2 = a \cot B \cos C, & \gamma_2 = a \operatorname{cosec} B; \\ C' \ldots \alpha_3 = b \operatorname{cosec} C; & \beta_3 = -b \cot C; & \gamma_3 = \cot C \cos A. \end{array}$$

Now any circle may (Salmon's *Conics*, Art. 128) be represented by

$$(l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C) + \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0;$$

and substituting for (α, β, γ) their respective values at A′, B′, C′ we obtain three equations in terms of l, m, n from which we get

$$l = -\frac{\cos C}{\sin A \sin C},\quad m = -\frac{\cos A}{\sin A \sin B},\quad n = -\frac{\cos B}{\sin B \sin C};$$

and thus the required equation to the circle round A′B′C′ becomes

$$\alpha^2 \cot C + \beta^2 \cot A + \gamma^2 \cot B + \left(\cot A \frac{\sin C}{\sin B} + \sin A + \frac{\cos B}{\sin C}\right)\beta\gamma$$
$$+ \left(\cot B \frac{\sin A}{\sin C} + \sin B + \frac{\cos C}{\sin A}\right)\gamma\alpha + \left(\cot C \frac{\sin B}{\sin A} + \sin C + \frac{\cos A}{\sin B}\right)\alpha\beta = 0.$$

II. *Solution by* S. WATSON; *the* REV. J. L. KITCHIN, M.A.; *and many others.*

The trilinear equations of B′C′, C′A′, A′B′, are

$$\alpha \cos C + \beta = 0,\ \beta \cos A + \gamma = 0,\ \gamma \cos B + \alpha = 0;$$

and the perpendiculars from any point $(\alpha', \beta', \gamma')$ on these lines, are respectively (Salmon's *Conics*, Art. 61),

$$x = \frac{\alpha' \cos C + \beta'}{\sin C}, \quad y = \frac{\beta' \cos A + \gamma'}{\sin A}, \quad z = \frac{\gamma' \cos \beta + \alpha'}{\sin B}.$$

Now, since the angles A′, B′, C′ are respectively equal to A, B, C, the trilinear equation of the circle about A′B′C′ in terms of (x, y, z) is

$$yz \sin A + zx \sin B + xy \sin C = 0;$$

hence, substituting and dropping the accents, the required equation is found to be

$$\frac{\text{cosec } A}{\alpha + \gamma \cos B} + \frac{\text{cosec } B}{\beta + \alpha \cos C} + \frac{\text{cosec } C}{\gamma + \beta \cos A} = 0.$$

2005. (Proposed by MATTHEW COLLINS, B.A.)—Find the whole number x so that $3x^7 - 29$ may be divisible by 557.

I. *Solution by* SAMUEL BILLS.

I am not aware that this question admits of any *direct* solution. It is well known that if the expression $3x^7 - 29$ is divisible by 557 at all, it will be so for some value of x, positive or negative, not greater than $\frac{557}{2}$, or 278. The only method that I know of is to proceed by *trials*. Now by using various artifices and contractions I have tried *all* the numbers, both *positive* and *negative*, as far as 278; and I find that either $x = -35$, or $x = 522$ will answer the condition. If we take m = any whole number either positive or negative we shall have, generally, $x = 557m - 35$; from which any number of answers may be found.

I may observe that -35 and 522 are the *only* numbers *less* than 557 which will satisfy the condition. If $m = 2$, $x = 1079$; if $m = -1$, $x = -592$, &c. &c.

II. *Solution by* JAMES DALE.

If $3x^7 - 29 = M(557)$ be solvable in whole numbers, M must be a term of the series $2, 5, 8, \ldots. 2 + (r-1)3$, and the equation may be reduced to the form

$$x^7 - 381 = N(557),$$

M and N being connected by the equation $M = 3N + 2$.

To solve the equation $x^7 - 381 = N(557)$, let $7p = q(557-1) + 1$; then the least values of p and q in positive integers are $p = 159$, $q = 2$.

Put $x = 381^{159}$, then $x^7 = 381^{1113}$, and

$$x^7 - 381 = 381(381^{1112} - 1) = 381(381^{556} + 1).(381^{556} - 1).$$

Now by Fermat's Theorem $(381^{556} - 1) = (381^{557-1} - 1)$ is divisible by 557; therefore $x = 381^{159}$ is a root of the equation $x^7 - 381 = N(557)$, and hence also of the given equation $3x^7 - 29 = M(557)$.

Other values may be obtained by taking suitable values for p and q, such as $x = 381^{1.71}$, &c.

2324. (Proposed by W. S. BURNSIDE, M.A.)—If four roots of a quintic be in harmonic progression, prove that the following relation between the invariants holds, viz., $J^3 - 2^7 . 3^2 JK + 2^{13} . 3^3 L \equiv 0$, (where J, K, L are the fundamental invariants as given in Salmon's *Higher Algebra*, 2nd edition, pp. 184—187.)

Solution by the PROPOSER.

In this case the quintic may be reduced to the form

$$5 \left\{ lx^5 + mx^4y + lxy^4 + my^5 \right\};$$

then $L \equiv (5^4 + 18 . 5^2 - 3^3)(lm)^6 - 4 . 5^3 l^2m^2 (l^8 + m^8)$;
also $K \equiv (5^4 + 3^3)(lm)^4 - 2 . 5^3 (l^8 + m^8)$, and $J = 2^7 . 3 \, l^2m^2$.
Eliminating l and m we obtain the relation in the question.

1947. (Proposed by W. B. DAVIS, B.A.)—Prove in any way that

$$41^{41} \equiv 1 \text{ (mod. 83)}, \text{ and } 82^{41} \equiv 82 \text{ (mod. 83)}.$$

N.B.—Mod. is short for *modulus*, the first expression meaning that $41^{41} - 1$ is exactly divisible by 83.

I. *Solution by* M. JENKINS, B.A.; W. H. LAVERTY; *and others.*

Since 41 is odd, $82^{41} + 1$ is divisible by $82 + 1$,
therefore $82^{41} \equiv -1 \equiv 82$ (mod. 83), which proves the 2nd congruence.
Again; $(2n)^n - (-1)^n$ is divisible by $2n - (-1)$, i.e. by $2n + 1$ ......(1);
and, by a well known theorem, if $2n + 1$ be a prime number,

$$2^n \equiv 1 \left\{\text{mod. } (2n+1)\right\}, \text{ if } 2n+1 \text{ be of the form } 8m \pm 1$$
$$\equiv -1 \text{ .... if .... } 8m \pm 3;$$

that is, $2^n - (-1)^{\frac{1}{2}n(n+1)}$ is divisible by $(2n+1)$ ....(2).
Multiplying (1) by $(-1)^n$, (2) by $(-1)^{\frac{1}{2}n(n+1)}$ and subtracting,

$$(-1)^n (2n)^n - (-1)^{\frac{1}{2}n(n+1)} 2^n \text{ is divisible by } (2n+1);$$

and dividing by $(-1)^n 2^n$, $n^n - (-1)^{\frac{1}{2}n(n-1)}$ is divisible by $(2n+1)$.
This shows that, if n be of the form $4m$ or $4m + 1$,

$$n^n \equiv 1 \,[\text{mod. } (2n+1)];$$

therefore, since 41 is of the latter form, and 83 a prime number,

$$41^{41} \equiv 1 \text{ (mod. 83)}.$$

II. *Solution by* J. DALE; S. W. BROMFIELD; *and others.*

1. By Fermat's Theorem, we have

$$41^{82} - 1 = (41^{41} + 1)(41^{41} - 1) = M(83),$$

or $$\left\{(83-1)^{41} + 2^{41}\right\} \left\{(83-1)^{41} - 2^{41}\right\} = 2^{41} . M(83).$$

Expanding $(83-1)^{41}$, every term except the last is a multiple of 83, and the above equation may be written

$$\left\{M(83)+(2.2^{40}-1)\right\}\left\{M(83)-(2.2^{40}+1)\right\}=2^{41}.M.(83),$$

therefore either $(2.2^{40}-1)$ or $(2.2^{40}+1)$ is divisible by 83; but $(2.2^{40}-1)$ is not divisible by 83, (see Legendre, sect. 145); therefore $2.2^{40}+1$ is divisible by 83, and consequently $41^{41}-1$, and not $41^{41}+1$, is a multiple of 83.

2. We have $82^{41}-82=82(82^{40}-1)$

$$=82(82^{20}+1)(82^{20}-1)$$
$$=82(82^{20}+1)(82^{10}+1)(82^{10}-1)$$
$$=82(82^{20}+1)(82^{10}+1)(82^{5}-1)(82^{5}+1)$$
$$=82.83(82^{20}+1)(82^{10}+1)(82^{5}-1)(82^{4}-82^{3}+82^{2}-82+1)$$
$$=M(83),$$

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153 Primes of the 10th Million:

by W. B. Davis, B.A.

9019573	9091391	9091879	9092387	9093017
9042409	9091429	9091891	9092437	9093061
9090919	9091441	9091909	9092449	9093089
9090947	9091451	9091997	9092471	9093121
9090953	9091477	9092003	9092527	9093127
9090973	9091483	9092011	9092533	9093143
9090989	9091513	9092053	9092543	9093151
9091009	9091541	9092063	9092557	9093163
9091021	9091543	9092071	9092561	9093169
9091051	9091553	9092081	9092563	9093197
9091057	9091561	9092087	9092581	9093239
9091067	9091591	9092113	9092591	9151643
9091079	9091597	9092117	9092593	9173471
9091099	9091609	9092131	9092617	9219139
9091109	9091613	9092137	9092647	9214043
9091127	9091631	9092143	9092653	9234463
9091151	9091637	9092177	9092711	9260117
9091157	9091657	9092179	9092717	9357163
9091207	9001679	9092183	9092723	9422407
9091213	9091703	9092191	9092789	9524717
9091219	9091729	9092219	9092791	9705911
9091223	9091741	9092227	9092833	9745639
9091259	9091759	9092231	9092851	9768487
9091267	9091769	9092257	9092873	9820289
9091273	9091777	9092261	9092891	9826079
9091289	9091783	9092299	9092893	9880447
9091331	9091807	9092333	9092911	9890219
9091333	9091813	9092339	9092933	9949259
9091343	9091829	9092353	9092947	9963137
9091373	9091843	9092371	9092971	
9091387	9091871	9092381	9092989	

Note on the Number of Primes in the 7th Million; by W. B. Davis, B.A.

It is well known to readers of the Theory of Numbers that Legendre counted the number of Primes, in order to verify the log-integral formula, and I have since found, that Bessel in his *Briefwechsel* with Olbers, has done the same thing, verifying the general accuracy of Legendre's computation.

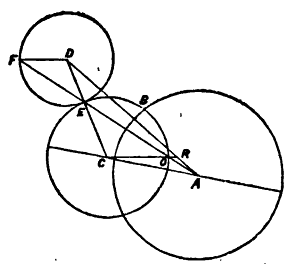
The following are the number of Primes in the successive 100,000, of the 7th Million. I suppress details.

1st	6386
2nd....	6401
3rd	6418
4th	6340
5th	6340
6th	6398
7th	6323
8th	6366
9th	6368
10th	6368

63708 Total number of primes in the 7th million; counted from Dase's *Factoren Tafeln.*

2330. (Proposed by H. Tomlinson.)—Describe a circle whose centre shall be on a given diameter of a given circle, to cut that circle orthogonally and to touch another given circle.

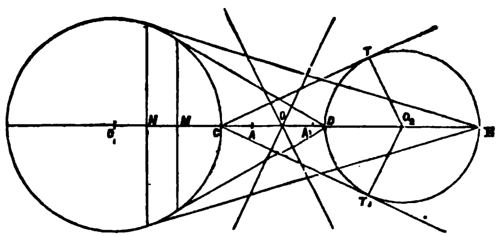
I. *Solution by* Hugh Murphy.



Let the circle whose centre is A be that which is to be cut orthogonally, and the circle whose centre is D be that which is to be touched. Draw the lines as in the diagram: then AF . AE [= (tan)^2 from A to circle (D)], and AO . AE [= AB^2] are both given; therefore AF : AO or DF : RO is given, COR and DF being parallel; therefore RO is given; moreover the point R is given by the proportion AF : AO = AD : AR. Hence the question is reduced to the following one: given two points D and R to find a point C in a given line AC, so that (CD−CR) may be given. (See Mulcahy's *Modern Geometry*, p. 63, Note.)

Otherwise: as the envelope of the circle touching one fixed circle and cutting another orthogonally is a given circle, its centre and the centre of the touched circle are given points; hence the problem is reduced to finding a point in the given diameter, such that the difference of the lines drawn therefrom to these two given points may be equal to the difference of the radii.

II. *Solution by* JAMES DALE.



1. To find the locus of the centre of a circle which shall cut one given circle orthogonally, and touch another given circle. Taking the origin of rectangular coordinates at the centre of the orthogonally cut circle, the line of centres being the axis of x, let r_1, r_2 be the radii, and d the distance of centres; then, if (x, y) be any point on the locus we have

$$(x^2+y^2-r_1^2)^{\frac{1}{2}} = \left\{(d-x)^2+y^2\right\}^{\frac{1}{2}} \pm r_2.$$

Clearing of radicals and arranging, this equation becomes

$$4(d^2-r_2^2)x^2-4r_2^2y^2-4d(r_1^2-r_2^2+d^2)x + \left\{r_1^2+(r_2+d)^2\right\}\left\{r_1^2+(r_2-d)^2\right\}=0,$$

which represents an hyperbola, the equation of which, referred to the principal axes, is

$$4(d^2-r_2^2)x^2-4r_2^2y^2 = \frac{r_2^2\left\{d^2-(r_1^2+r_2^2)\right\}^2}{d^2-r_2^2}.$$

As a diameter of the circle (r_1) will in general cut the hyperbola in two points, two circles, having their centres respectively at the points of intersection, can be drawn cutting the one circle (r_1) orthogonally, and touching the other (r_2), the one externally, and the other internally.

The vertices and asymptotes of the hyperbola can be readily constructed as follows:—Let O_1, O_2 be the centres of the two circles, D, E the points in which the line O_1O_2 cuts the circle (O_2), then the vertices are the points of bisection of the distances between D, E and their polars with regard to the circle (O_1). Again, from C, the point of (O_1) nearest to O_2, draw tangents, CT, CT_1 to O_2, join O_2T and O_2T', then the asymptotes are parallel to these lines O_2T, O_2T'.

When $d^2=2r_2^2$, the hyperbola becomes equilateral; and when $d^2=r_1^2+r_2^2$, that is, when the circles O_1, O_2 cut orthogonally, the hyperbola becomes the two straight lines $r_1x+r_2y=0$ and $r_1x-r_2y=0$.

2227. (Proposed by W. LEA.)—Form 2^n-1 symbols into triads, so that every pair of the symbols shall appear once in the triads.

Solution by SAMUEL BILLS.

The following is perhaps the simplest method of solution to this question. I shall give examples when there are 7, 15, and 31 symbols, and it will be seen that the method will be equally applicable for any value of n.

Take 7 symbols a, b, c, d, e, f, g, and form all the pairs in order as follows, *ab, ac, ad, ae, af, ag, bc, bd, be, bf, bg, cd, ce, cf, cg, de, df, dg, ef, eg, fg.* Now each triad will comprise three pairs; take the letters in the first two pairs for the first triad, viz., *abc*, and mark off the pairs of which it is composed with a dash underneath; then take the next pair in order, not marked off, and join it with the next succeeding one which will not comprise a pair that has been previously marked off, for the second triad, which will be *ade*, and mark off the three pairs as before. By proceeding in this manner till *all* the pairs are exhausted, we obtain the following seven triads,

abc, ade, afg, bdf, beg, cdg, cef.

Next take 15 symbols, *a, b, c, d, e, f, g, h, i, j, k, l, m, n, o,* and form the pairs in order thus:—

ab, ac, ad, ae, af, ag, ah, ai, aj, ak, al, am, an, ao, bc, bd, be, bf, bg, bh, bi, bj, bk, bl, bm, bn, bo, cd, ce, cf, cg, ch, ci, cj, ck, cl, cm, cn, co, de, df, dg, dh, di, dj, dk, dl, dm, dn, do, ef, eg, eh, ei, ej, ek, el, em, en, eo, fg, fh, fi, fj, fk, fl, fm, fn, fo, gh, gi, gj, gk, gl, gm, gn, go, hi, hj, hk, hl, hm, hn, ho, ij, ik, il, im, in, io, jk, jl, jm, jn, jo, kl, km, kn, ko, lm, ln, lo, mn, mo, no.

The number of pairs is 105, and by forming the triads from them as above described we obtain the following 35, which are put down exactly in the order in which they were obtained; *abc, ade, afg, ahi, ajk, alm, ano, bdf, beg, bhj, bik, bln, bmo, cdg, cef, chk, cij, clo, cmn, dhl, dim, djn, dko, ehn, eil, ejo, ekn, fhn, fio, fjl, fkm, gho, gin, gjm, gkl.*

In the above 35 triads *all* the pairs will be found. In precisely the same manner we may proceed for 31, 63, &c., symbols.

For the case of 31 symbols, let the symbols be $a, b, c, \ldots y, z, a_1, b_1, c_1, d_1, e_1$; and form all the pairs in order as follows:—

$ab, ac, ad, \ldots ae_1$; $bc, bd, \ldots be_1$; $cd, ce_1 \ldots ce_1$; $de, df, \ldots de_1$; $ef, eg, \ldots ee_1$; $fg, fh, \ldots fe_1$; $gh, gi, \ldots ge_1$; $hi, hj \ldots he_1$; $ij, ik, \ldots ie_1$, $jk, jl, \ldots je_1$; $kl, km, \ldots ke_1$; $lm, ln, \ldots le_1$; $mn, mo, \ldots me_1$; $no, np, \ldots ne_1$; $op, oq, \ldots oe_1$; $pq, pr, \ldots pe_1$; $qr, qs, \ldots qe_1$; $rs, rt, \ldots re_1$; $st, su, \ldots se_1$; $tu, tv, \ldots te_1$; $nv, nw, \ldots ne_1$; $vw, vx, \ldots ve_1$; $wx, wg, \ldots we_1$; $xy, xz, \ldots xe_1$; $yz, ya_1, \ldots ye_1$; $za_1, zb_1, \ldots ze_1$; $a_1b_1, a_1c_1, \ldots a_1e_1$; $b_1c_1, b_1d_1, b_1e_1, c_1d_1, c_1e_1, d_1e_1$.

The complete list of pairs may be very readily written out as required.

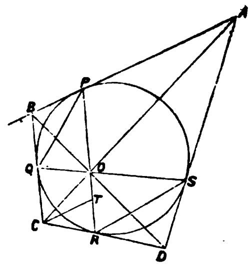
Now by forming the triads from the pairs, as previously directed, we readily find the following 155 triads, which are put down in exactly the same order as they were obtained, and which will be found to exhaust *all* the pairs.

abc, ade, afg, ahi, ajk, alm, ano, apq, ars, atu, avw, axy, aza_1, ab_1c_1, ad_1e_1, bdf, beg, bhj, bik, bln, bmo, bpr, bqs, btv, buw, bxz, bya_1, bb_1d_1, bc_1e_1, cdg, cef, chk, cij, clo, cmn, cps, cqr, ctw, cuv, cxa_1, cyz, cb_1e_1, cc_1d_1, dhl, dim, djn, dko, dht, dqu, drv, dsw, dxb_1, dyc_1, dzd_1, da_1e_1, ehn, eil, ejo, ekn, epu, eqt, erw, esv, exc_1, eyb_1, eze_1, ea_1d_1, fhn, fio, fjl, fkm, fpv, fqw, frt, fsu, fxd_1, fye_1, fzb_1, fa_1c_1, gho, gin, gjm, gkl, gpw, gqv, gru, gst, gxe_1, gyd_1, gzc_1, ga_1b_1, hpx, hgy, hrz, hsa_1, htb_1, huc_1, hvd_1, hwe_1, ipy, iqx, iva_1, isz, itc_1, iub_1, ive_1, iwd_1, jpz, jqa_1, jrx, jsy, jtd_1, jue_1, jvb_1, jwc_1, kpa_1, kqz, kry, ksx, kte_1, kud_1, kvc_1, kwb_1, lpb_1, lqc_1, lrd_1, lse_1, ltx, luy, lvz, lwa_1, mpc_1, mqb_1, mre_1, msd_1, mty, mux, mva_1, mwz, npd_1, nqe_1, nrb_1, nsc_1, ntz, nua_1, nvx, nwy, ope_1, oqd_1, orc_1, osb_1, ota_1, ouz, ovy, owx.

2310. (Proposed by J. M. Wilson, M.A., F.G.S.)—Prove (1) that if ABCD be a quadrilateral circumscribed about a circle, and P, Q, R, S the points of contact, then AC, BD, RP, SQ pass through one point.

Hence show (2) that the locus of the intersection of the lines joining the ends of chords in a circle which pass through a given point is the polar of that point.

I. *Solution by the* Proposer.



1. Consider the segments into which PR divides the diagonal AC.

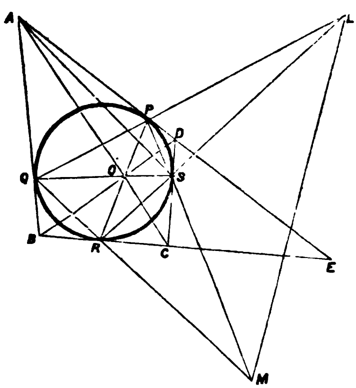
Through C draw a line CT parallel to AP, then CT = CR and AO : OC = AP : CR.

Similarly QS divides AC into segments which have the ratio of AS to QC, that is of AP to CR. Hence PR and QS intersect on the diagonal AC, and therefore also on BD.

2. To prove the second part, join PQ, RS. Then B is the pole of PQ, D of SR, and therefore the intersection of PQ, SR is the pole of BD which passes through O, and therefore PQ and SR intersect on the polar of O.

Note.—These well-known properties are proved in McDowell's *Exercises*, by the harmonic properties of transversals to a triangle, and of chords of a circle.

II. *General Solution by the* Rev. R. H. Wright, M.A.; *the* Rev. J. L. Kitchin, M.A.; H. Tomlinson; W. Chadwick; *and others*.



Let ABE be a triangle of reference where BE, EA, AB are respectively represented by $\alpha = 0$, $\beta = 0$, $\gamma = 0$; also let PQR be an inscribed conic; and draw any tangent DC touching the conic in S. Now if the equation of the inscribed conic be $(l\alpha)^{\frac12} + (m\beta)^{\frac12} + (n\gamma)^{\frac12} = 0$, and if AS be represented by $\beta - h\gamma = 0$, the equation of CD is (Wright's *Examples on Trilinears*, Art. 74.)

$$lh^{\frac12}\alpha - (m^{\frac12}h^{\frac12} + n^{\frac12})\,m^{\frac12}\beta - (m^{\frac12}h^{\frac12} + n^{\frac12})\,n^{\frac12}h^{\frac12}\gamma = 0 \quad \ldots\ldots\ldots\ldots (1).$$

Making $\beta = 0$ and $\alpha = 0$ respectively in (1), we have

for BD and AC.... $(m^{\frac12}h^{\frac12} + n^{\frac12})\,n^{\frac12}\gamma - l\alpha = 0$, and $m^{\frac12}\beta + n^{\frac12}h^{\frac12}\gamma = 0$.. (2, 3),

and we readily get

for PR.................. $l\alpha + m\beta - n\gamma = 0$ .... (4),

and QS.... $lm^{\frac12}\alpha - (m^{\frac12}h^{\frac12} + n^{\frac12})\,m^{\frac12}n^{\frac12}\beta - (m^{\frac12}h^{\frac12} + n^{\frac12})\,(n^{\frac12}h^{\frac12} + m^{\frac12})\,n^{\frac12}\gamma = 0$ .. (5).

Now $\quad (n^{\frac12}h^{\frac12} + m^{\frac12})\,(2) + n\,(3) + n^{\frac12}h^{\frac12}\,(4) + (5) \equiv 0$,

which shows that AC, BD, RP, SQ pass through the same point.

2. If we refer to the *Mathematician*, (Vol. I., p. 133), we find that if the sides PS and QR of the quadrilateral PQRS inscribed in the conic be produced to meet in M, and QP and RS to meet in L, and if the diagonals meet in O; then ML is the polar of O, and OL is the polar of M.

Note on a locus connected with the Geometry of the Triangle. By J. Griffiths, M.A.

The locus referred to is that of a point (ξ, η, ζ) such that its pedal circle* touches the nine-point circle of the triangle of reference. Its equation is,

$$\Sigma\left\{\xi \cos^2 A\,(\eta \cos B-\zeta \cos C)(\eta \cos C-\zeta \cos B)\right\}^{\frac{1}{2}}=0,$$

and the object of the present note is to show that the curve, though apparently a proper one of the sixth degree, consists in fact of two coincident *cubics* given by the common equation

$$\xi \cos A\,(\eta^2-\zeta^2)+\eta \cos B\,(\zeta^2-\xi^2)+\zeta \cos C\,(\xi^2-\eta^2)=0.$$

For if we write

$$\cos A\,(\eta \cos C-\zeta \cos B)=x,\quad \cos B\,(\zeta \cos A-\xi \cos C)=y,$$
$$\cos C\,(\xi \cos B-\eta \cos A)=z;\quad \xi \cos A\,(\eta \cos B-\zeta \cos C)=x',$$
$$\eta \cos B\,(\zeta \cos C-\xi \cos A)=y',\quad \zeta \cos C\,(\xi \cos A-\eta \cos B)=z',$$

the above equation becomes $(xx')^{\frac{1}{2}}+(yy')^{\frac{1}{2}}+(zz')^{\frac{1}{2}}=0$,
where $x+y+z=0$, and $x'+y'+z'=0$.

Hence, eliminating z and z', for instance, we easily find $(xy'-x'y)^2=0$, or

$$\cos A\,(\eta \cos C-\zeta \cos B)\,\eta \cos B\,(\zeta \cos C-\xi \cos A)$$
$$=\xi \cos A\,(\eta \cos B-\zeta \cos C)\cos B\,(\zeta \cos A-\xi \cos C),$$

that is, $$\Sigma\,\xi \cos A\,(\eta^2-\zeta^2)=0.$$

Hence we see that the pedal of (ξ, η, ζ) touches the nine-point circle, provided that the line joining (ξ, η, ζ) to its inverse $(\xi^{-1}, \eta^{-1}, \zeta^{-1})$ passes through the centre of the circumscribing circle.

AT RANDOM: No. V. By Professor Whitworth.

As Mr. Woolhouse still adheres to his assertion, that the phrase "at random," has in general only one absolute and unqualified meaning, perhaps I may be allowed to illustrate the question by reference to Mr. Woolhouse's own language in the statement of the last two problems which he has published in the *Educational Times*.

His Question (2420) in the June number, commences as follows :—"Three numbers, each of them not exceeding a number n, being arbitrarily taken, &c." This may mean, either that a *group* is selected arbitrarily (any *group* being equally likely), or else that the numbers are severally selected arbitrarily, in which case the group 2, 3, 4 will be *six* times as likely to occur, and the group 3, 3, 4 will be *three* times as likely to occur as the group 3, 3, 3. It seems to me, that Mr. Woolhouse's language, in the statement of his question, is more suggestive of the latter interpretation; but the result which he arrives at shows that he intends the former. If he had used an absolute verb instead of a participle, the distinction might easily have been expressed. The first case would have been indicated by the phrase, "A group of three numbers *is* arbitrarily taken;" the second by the phrase, "Three numbers are arbitrarily taken." In the one case, we have to make *one* selection of *three* numbers; in the other, *three* selections of *one* number.

[* That is, the circle through the feet of the perpendiculars from the point (ξ, η, ζ) on the sides of the triangle of reference. This circle also passes through the feet of the perpendiculars from the $(\xi^{-1}, \eta^{-1}, \zeta^{-1})$ on the sides, see solutions of Quests. 1815 and 1975, *Reprint*, Vol. V. p. 19, and Vol. VI. p. 43.]

In the one case, all *combinations* (repetitions admissible) are equally likely: in the other, all *permutations* (repetitions admissible) are equally likely.

A precisely similar ambiguity occurs in Mr. Woolhouse's Question (2433) in the July Number. Does he mean, that five points *are* selected at random, or that a selection of five points *is* made at random?

But as to Mr. Woolhouse's general assertion, that the phrase "at random" must be unambiguous, having only one proper meaning in any enunciation, I can only say that I, for one, agree with him if he means that by due care we can generally so express ourselves as to preclude any doubt about the meaning of our most unqualified statements; but if he means that such loose language as we are in the habit of meeting with, his own Question 2420 affording an instance, is never open to a double interpretation, his appreciation of the niceties of language must be much more precise than mine.

2371. (Proposed by Professor CAYLEY.)—(4). If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

Solution by PROFESSOR SYLVESTER.

1. If the quadrangle ABPQ is non-convex, ABQ encloses P or ABP encloses Q, and therefore the chance of a reentrant figure is double the chance of ABP containing Q.

Divide the triangle into infinitesimal bands parallel to AB; then taking any element of given magnitude in this band, the chance of P being upon it and Q inside the triangle APB is the same all along the band and is proportonal to the distance of the band from the base. Hence if the distance of the apex from the base is h, and from the band x (remembering that the length of the band is proportional to x, and its distance from the base to $h-x$) the total chance of APB enclosing Q is

$$\int_0^h dx\,(hx-x^2) \div \int_0^h dx\,.\,hx = \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}.$$

Hence the chance of the figure APQB being reentrant is $\frac{2}{3}$, and of being convex is $\frac{1}{3}$, being independent of the form of the triangle.

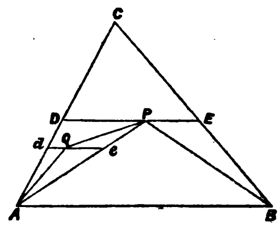
2. In like manner it may easily be shown that for a parallelogram on AB the chance of the quandrangle APQB being reentrant is

$$2\int_0^h dx\,.\,\frac{x}{2} \div \int_0^h dx\,.\,h = \frac{1}{2}.$$

Hence it is an even chance whether the quadrangle is convex or reentrant. The principle of duplicity above employed is one of frequent use in the calculus of local probability; by aid of it in Professor CAYLEY's question we have been enabled to elude the calculation of the troublesome integral $\int dm$ (CMPN), M and N being the points where AP and BP meet the opposite sides, as we know à priori that this must be equal to $\int dm\,.\,$APB.

3. The above method applies with still more conspicuous advantage to an extension of Professor Cayley's problem to *five* points. Suppose we want to

calculate the probability of ABPQR forming a convex figure ; P, Q, R being any three points inside the triangle ABC.



If the pentagon is not convex, either two of the points P, Q, R will be within the triangle formed by the third with AB, or one of them will lie inside the quadrangle formed by the remaining two with AB.

The chance of the former is three times the chance of two specified points Q, R lying within the triangle ABP.

The chance of the latter is three times the chance of one specified point R lying within the quadrangle ABPQ ; or if we please so to say, is six times the chance of this taking place coupled with the new condition of Q being nearer than P is to the base AB.

If then we draw a band parallel to AB through P, the latter chance will be twelve times the chance of Q lying in APD whilst R lies in APB or AQP, the chances relating to Q placed in APD and to Q placed in BPE being of course the same.

If then the chance of Q, R being each of them inside APB be called ϕ,
that of Q being in APD and R in APB be called ψ,
that of Q being in APD and R in AQP be called ω,
the chance of the figure APQRB being convex will be

$$1-3\phi-12\psi-12\omega,$$

It is obvious *a priori* that the form of the triangle does not enter into the result. Call then the perpendicular distance of C from the base and the length of the latter each unity, so that the area of ABC $= \frac{1}{2}$, and let the triangle APD be divided into bands de parallel to AB or DE.

Let x be the distance of DE from AB, ξ that of de from AB, and put DP $= y$. Then since the cube of the given triangle is $\frac{1}{8}$, we have

$$\tfrac{1}{8}\phi = \int_0^1 dx\,(1-x)\left(\frac{x}{2}\right)^2 = \tfrac{1}{4}\left(\tfrac{1}{3}-\tfrac{1}{4}\right) = \tfrac{1}{48};$$

$$\tfrac{1}{8}\psi = \int_0^1 dx \int_0^{1-x} dy\,.\,y\left(\frac{x}{2}\right)^2 = \int_0^1 dx\,(1-x)^2\,\frac{x^2}{8} = \tfrac{1}{240};$$

$$\tfrac{1}{8}\omega = \int_0^1 dx \int_0^{1-x} dy \int_0^x d\xi\,\left(\frac{\xi y}{x}\right)^2 \frac{x}{4} = \int_0^1 dx\,(1-x)^3\,\frac{x^2}{36} = \tfrac{1}{2160}.$$

Hence the required chance of the figure APQRB being convex is

$$1-\tfrac{1}{2}-\tfrac{2}{5}-\tfrac{2}{45}, \text{ or } \tfrac{1}{18}.$$

In analysis, as in statecraft, the great rule applies, "Divide et impera."

4. The same method gives a simple solution of the four-point problem when the four points are taken anywhere inside a given triangle.

The chance of their forming a reentrant figure is four times that of a specified one of them (as S) lying within the triangle whose apices are the other three (P, Q, R). Now the mean value of the triangle PQR is one-twelfth of the limiting triangle, as is shown in the solutions of Quest. 1229, (*Reprint*, Vol. II., p. 95, and Vol. IV., p. 101 ; in the *first* line of the latter solution, however, *m* is printed in error for M). Hence the chance of S being within

PQR is $\frac{1}{12}$; and the whole chance of the figure PQRS being reentrant is therefore $\frac{1}{3}$; so that the complementary chance of the four points forming a convex quadrangle is $\frac{2}{3}$.

5. It is worth while to show here how the result of Art. 4 may be obtained more directly by another mode of applying the theorem (Ω) in the passage from the *Reprint*, Vol. IV., last referred to.

We may make n (the number of points) equal to 4, i (the order of the *affection*) equal to zero; we have then

$$M = \frac{3.4.3.2}{8.7.6}m + \frac{4.4.3}{8.7.6}(3\mu) = \frac{4m+3\mu}{7}$$

where $m = \frac{1}{4}$, and μ = the double of $\frac{2}{9} = \frac{4}{9}$. Hence $M = \frac{3+4}{21} = \frac{1}{3}$, the probability, as previously found, of four points inside a given triangle forming the apices of a reentrant quadrangle.

6. Let us apply the same method to determining the probability of five points taken anywhere at random within the triangle ABC forming or not a convex figure. It may easily be seen *à priori* that the relative lengths of the sides will not affect the result, and I shall regard them all as *units*. Call m the probability of *reentrance* when three of the five points lie respectively along three sides, μ the same when two of them lie respectively at a vertex and on an opposite side of the limiting triangle. Then, if the required chance of reentrance is M, we have by (Ω)

$$M = \frac{3.5.4.3\,m + 4.5.4(3\mu)}{10.9.8} = \frac{2m+\mu}{3}.$$

(α). To find μ; calling B the fixed vertex, Q the point moveable in AC, we have

$$\mu = \frac{17}{18}\mu_1 + \frac{2}{3}\mu_2,$$

where μ_1 = mean value of $\frac{(ABQ)^3+(CBQ)^3}{(ABC)^3}$,

and μ_2 = mean value of $\frac{3(ABQ)^2(CBQ)+3(ABQ)(CBQ)^2}{(ABC)^3}$.

Hence $\mu_1 = 2\int_0^1 dx\,.\,x^3 = \frac{1}{2}$; $\mu_2 = 6\int_0^1 dx\,(x^2-x^3) = \frac{1}{2}$;

therefore $$\mu = \frac{17}{36} + \frac{1}{3} = \frac{29}{36}.$$

(β). Again, P, Q, R being points taken at random in BC, CA, AB respectively, we have $m = m_1 - m_2 + \frac{2}{3}m_3$,

where m_1 = mean value of $2\frac{PQR}{ABC}$,

also m_2 = mean value of $\frac{(PQR)^2}{(ABC)^2}$,

and m_3 = mean value of $\Sigma\frac{(PBR)^2}{(ABC)^2}$, that is of $3\frac{(PBR)^2}{(ABC)^2}$, where it is to be noticed that $\frac{2}{3}$ is the coefficient of reentrance given in Art. (1), and that the mean values of such products as $\frac{(PBR)\,(QCP)}{(ABC)^2}$ do not enter into m because the probabilities due to the *distribution* of the two outstanding points *between* PBR, QCB is *nil.*

The mean value of PQR is easily seen to be the triangle whose apices are the mid-points of the sides of ABC.

Hence we have $\quad \frac{m_1}{2} = \frac{1}{4}$ or $m_1 = \frac{1}{2}$.

To find m_2; for the moment suppose P and R fixed whilst Q travels along AC; make $AQ = x$, $ARP = \alpha$, and $CPR = \beta$.
Then the mean value of $(PQR)^2$ becomes

$$\int_0^1 dx \left\{(1-x)\,\alpha + x\beta\right\}^2 = \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2).$$

The mean value of α^2 (as of β^2) is $\left(\int_0^1 dx\,.\,x^2\right)^2 (ABC)^2$, i.e. $\frac{(ABC)^2}{9}$.

The mean value of $\alpha\beta$ is $\left(\int_0^1 dx\,(x-x^2)\right)^2 (ABC)^2$, i.e. $\frac{(ABC)^2}{36}$;

where in explanation of the *latter* equation it may be observed that when R remains fixed α varies as BP, say x, whilst β varies as CP or $(1-x)$, so that the mean value of $\alpha\beta$ on this preliminary supposition varies as

$$\int_0^1 dx\,.\,x\,(1-x)\,(ACR\,.\,BCR).$$

Hence we have $\quad m_2 = \frac{1}{3}\left(\frac{1}{9} + \frac{1}{36} + \frac{1}{9}\right) = \frac{1}{12}$.

Finally the mean value of $\frac{(PBR)^2}{(ABC)^2} = \left(\int_0^1 dx\,.\,x^2\right)^2 = \frac{1}{9}$, so that $m_3 = \frac{2}{9}$.

Thus $m = \frac{1}{2} - \frac{1}{12} + \frac{2}{9} = \frac{23}{36}$ and $M = \frac{1}{3}\left(\frac{46+29}{36}\right) = \frac{25}{36}$, which fraction is the chance of reentrance for 5 points taken at random in *any* given triangle, thus giving $\frac{11}{36}$ as the chance of convexity.

7. The reentrance referred to above may be single or double. The calculation for the case of double reentrance isolated from that of single reentrance is very easy. Following the notation in Art. (1), the chance of three points in any triangle ABC forming with A and B the apices of a doubly reentrant pentagon is three times

$$\int_0^h x\,(h-x)^2\,.\,dx \div \int_0^h xh^2\,.\,dx, \text{ that is } \frac{1}{2}.$$

Then following the notation of Art. 6, so that μ_1, μ_2, m_1, m_2, m_3 remain the same as before, we have now

$$\mu = \frac{1}{2}\mu_1 + 0 . \mu_2 = \frac{1}{4}, \text{ and } m = m_2 = \frac{1}{12};$$

therefore $$M = \frac{1}{3}\left(\frac{2}{12} + \frac{1}{4}\right) = \frac{5}{36}.$$

Mr. Woolhouse's formula (Quest. 2433) gives for this case

$$M = \frac{5}{12} \cdot \frac{(2+\cos \frac{2}{3}\pi)^2}{(3 \sin \frac{2}{3}\pi)^2} = \frac{5}{12} \cdot \frac{1}{3} = \frac{5}{36}, \text{ as found above.}$$

Thus the probabilities of the pentagon cornered by five points taken at random on any triangle being doubly concave, simply concave, and totally convex, will be as the numbers 5, 20, 11.

8. The value of the *total* chances of concavity calculated in Art 3, by aid of the diagram therein given may easily be seen to be

$$\int_0^1 dx \int_0^{1-x} dy . 2\left(3x^2 + 12x^2y + 6 . \frac{2}{3}x^2y^2\right).$$

The first term $6x^2$ within the brackets gives as the final result of the integration $\frac{1}{2}$, the chance calculated in Art. 2.

The second term $24x^2y$ gives $12\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{2}{5}$.

The third term $8x^2y^2$ gives $\frac{8}{3}\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) = \frac{2}{45}$.

Now $\frac{2}{5} + \frac{2}{45} = \frac{4}{9}$, which is the chance of simple reentrance for the pentagon whose angles are A, B, and three points taken at random within ABC; and $\frac{1}{2} + \frac{4}{9} = \frac{17}{18}$ which confirms the correctness of the result found in Art. 3.

9. There is no difficulty in finding the chance that of $\nu+2$ points, taken at random in a triangle, $\nu-1$ of them shall be within the remaining three. This however it should be observed when $\nu>3$ is not the case of a simple "morph", but comprises various distinguishable species of configuration. Beginning with the supposition of two of the points lying on given angles of the triangle, this collective chance will easily be found to be

$$2\nu\int_0^1 dx\, x^{\nu-1}(1-x) = \frac{2}{\nu+1}.$$

Then passing to the case of all $\nu+1$ points being free, we obtain

$$\mu = \frac{4}{\nu+1}\int_0^1 dx . x^{\nu} = \frac{4}{(\nu+1)^2},$$

$$m = \text{mean value of } \left(\frac{PQR}{ABC}\right)^{\nu}$$

$$= \text{mean value of } \frac{1}{\nu . (ABC)^{\nu}}(\alpha^{\nu-1} + \alpha^{\nu-2}\beta + \alpha^{\nu-3}\beta^2 + \dots\dots + \beta^{n-1})$$

$$=\frac{1}{\nu^3}\left\{1+(\nu-1)^{-2}+\left[(\nu-1)\left(\frac{\nu-2}{2}\right)\right]^{-2}+\left[(\nu-1)\left(\frac{\nu-2}{2}\right)\left(\frac{\nu-3}{3}\right)\right]^{-2}+.+1\right\}$$

Finally $$\mathrm{M} = \frac{8\nu(\nu+1)(\nu+2)m+12(\nu+1)(\nu+2)\mu}{(2\nu+2)(2\nu+3)(2\nu+4)}$$

$$=\frac{2}{2\nu+3}\left\{\frac{s}{\nu^2}+\frac{6}{(\nu+1)^2}\right\},$$

where s is the sum of the squared reciprocals of the coefficients in the development of $(1+t)^{\nu-1}$ according to the powers of t.*

1917. (Proposed by W. S. B. Woolhouse, F.R.A.S.) — Suppose the surface of a sphere to be made up of an indefinite number of points, and straight lines to be drawn through every two of those points, and determine the law of density of this mass of lines as depending on the distance from the centre of the sphere.

Solution by the Proposer.

Let a plane be drawn through the centre of the sphere, and first consider the system of parallel chords that are normal to this plane. To simplify our notions, suppose the plane to be horizontal and the chords under consideration to be therefore vertical.

Let a denote the radius of the sphere;
- z the singular zenith distance of any point on the surface;
- n the number of points, supposed to be indefinitely great and uniformly distributed over the surface;
- ρ the radius of a sphere concentric with the given one;
- V its volume;
- L the quantity of line imparted to the volume of this concentric sphere by the mass of chords joining every pair of the n points;
- D the density of the mass of lines at the distance ρ from the centre;

$k=\frac{\rho}{a}$, and $\lambda=\frac{n^2}{8\pi a}$.

Then, conceiving two consecutive horizontal parallel circles whose angular zenith distances are z, $z+dz$, the spherical radii of these circles are az, $a(z+dz)$, and the area of the spherical surface contained between them is $2\pi a \sin z \times a dz = 2\pi a^2 dz \sin z$. And as the n points are uniformly distributed over the total spherical area $4\pi a^2 \int dz \sin z = 4\pi a^2$, the proportionate number of points contained between these parallel circles is $\frac{2\pi a^2 dz \sin z}{4\pi a^2} n$ $= \frac{1}{2} n dz \sin z$. The vertical chords which connect these points with the corresponding points in the lower hemisphere generate a concentric cylindrical annulus, radii $a \sin z$, $a \sin(z+dz)$ and altitude $2a \cos z$, and the portion of each of these vertical chords which lies within the concentric sphere,

* Thus, making ν successively 1, 2, 3, M assumes the values 1, $\frac{1}{3}$, $\frac{5}{36}$, which is right.

radius ρ, volume V, is evidently $2\sqrt{(\rho^2-a^2\sin^2 z)}$. Hence, differentiating with respect to ρ, we get the sum of the two equal portions which fall within the differential volume dV contained between two consecutive concentric spheres, radii ρ, $\rho+d\rho$, equal to $\dfrac{2\rho d\rho}{\sqrt{(\rho^2-a^2\sin^2 z)}}$.

Now if the sphere be moveable round its centre, and each of the n points on the surface be made a zenith, and a set of vertical parallel chords drawn, we shall evidently obtain, in the aggregate, the whole of the chords which connect every two points on the surface; and as these n sets of parallel chords are precisely alike, they will all be included by simply multiplying by n. From these considerations we find that the quantity of line thrown into the differential volume dV by the whole mass of chords is

$$dL = n\times\int\frac{n}{2}\,dz\sin z\,.\,\frac{2\rho d\rho}{\sqrt{(\rho^2-a^2\sin^2 z)}} = n^2\rho d\rho\int\frac{dz\sin z}{\sqrt{(\rho^2-a^2\sin^2 z)}}$$

$$= \frac{n^2\rho d\rho}{a}\log\frac{a\cos z-\sqrt{(\rho^2-a^2\sin^2 z)}}{a-\rho}\quad(\text{up to } a\sin z=\rho)$$

$$= \frac{n^2\rho d\rho}{2a}\log\frac{a+\rho}{a-\rho}.$$

And as this quantity of line is obviously equally distributed round the differential volume, $dV = 4\pi\rho^2 d\rho$, the density of the mass of lines round this spherical annulus at the distance ρ from the centre is

$$D = \frac{dL}{dV} = \frac{n^2}{8\pi a\rho}\log\frac{a+\rho}{a-\rho} = \frac{\lambda}{\rho}\log\frac{a+\rho}{a-\rho},$$

which is the required law of density.

When developed in a series it is

$$D = \frac{\lambda}{\rho}\int\frac{2a\,d\rho}{a^2-\rho^2} = \frac{2\lambda}{a\rho}\int\frac{d\rho}{1-\frac{\rho^2}{a^2}} = \frac{2\lambda}{a}\left(1+\frac{\rho^2}{3a^2}+\frac{\rho^4}{5a^4}+\&c.\right)$$

$$= \frac{n^2}{4\pi a^2}\left(1+\frac{\rho^2}{3a^2}+\frac{\rho^4}{5a^4}+\&c.\right)$$

And at the centre $\quad D_0 = \dfrac{2\lambda}{a} = \dfrac{n^2}{4\pi a^2}.$

So that, putting $\dfrac{\rho}{a}=k$, the law of density may be stated thus:

$$\frac{D}{D_0} = \frac{1}{2k}\log\frac{1+k}{1-k}.$$

A small marginal table is here annexed to show the numerical character of this function, which expresses the relative density at the distance k from the centre when the radius is taken as the linear unity.

k	$\frac{D}{D_0}$
·0	1·0000
·1	1·0033
·2	1·0137
·3	1·0317
·4	1·0591
·5	1·0986
·6	1·1552
·7	1·2390
·8	1·3733
·9	1·6358
1·0	∞

The quantity of line within the sphere (ρ) is found by integrating

$$dL = \frac{n^2}{2a}\rho d\rho\log\frac{a+\rho}{a-\rho},\quad\text{and is } L = \frac{n^2\rho}{2}\left(1-\frac{a^2-\rho^2}{2a\rho}\log\frac{a+\rho}{a-\rho}\right).$$

When $\rho = a$, we have the whole quantity of line contained within the given sphere $=\frac{1}{2}n^2 a = 2\pi a^3\,.D_0$. Hence, as the number of chords is $\frac{1}{2}n^2$, the average length of all the chords is equal to the radius of the given sphere. Also, dividing by the total volume $\frac{4}{3}\pi a^3$, the average density for the whole sphere is $\frac{3}{2}D_0$. And if all the lines were uniformly distributed with the same density as that at the centre, they would occupy a volume $(2\pi a^3)$ which is equal to that of the cylinder which circumscribes the given sphere.

It will be observed that the density of the lines becomes infinite at the surface of the given sphere. If the chords be all supposed to be produced indefinitely outside the sphere, the external law of density is readily found from the foregoing expressions. In this case $\rho > a$, and in determining the value of dL, we have only to give to z its full limit $\frac{1}{2}\pi$. Thus we find

$$dL = \frac{n^2 \rho d\rho}{2a} \log \frac{\rho + a}{\rho - a}, \text{ and hence } D = \frac{\lambda}{\rho} \log \frac{\rho + a}{\rho - a};$$

and, putting as before $\frac{\rho}{a} = k$, we get $\frac{D}{D_0} = \frac{1}{2k} \log \frac{k+1}{k-1}$.

These forms are remarkably analogous to the expressions we have found for the interior of the sphere, but they materially differ in their character. Thus, by expanding the interior and exterior densities in terms of k, we find

when $\quad k < 1, \ \frac{D}{D_0} = 1 + \frac{k^2}{3} + \frac{k^4}{5} + \&c.$

when $\quad k > 1, \ \frac{D}{D_0} = \frac{1}{k^2} \left(1 + \frac{1}{3k^2} + \frac{1}{5k^4} + \&c.\right)$

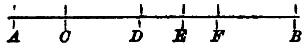
The latter law for remote distances virtually becomes inversely as the square of the distance. Also for reciprocal pairs of points we have the simple relation $D_{\frac{1}{k}} = k^2 D_k$, whatever be the value of k.

NOTE.—An analogous problem for the circle instead of a sphere was proposed by me as the Prize Question in the *Lady's and Gentleman's Diary* for 1866. The law of density for the circle is an elliptic function, and we have found that for the sphere to be a logarithmic function; but the resulting numerical law for the two cases presents a remarkable similarity. The increase of density from the centre outwards is somewhat greater for the sphere, and both of them become infinite on attaining the boundary.

On taking a finite number of points, and making a diagram for the circle, the beautiful clusters of intersections were so striking that I was induced to prepare diagrams on stone, impressions of which are published in the *Diary* for the present year. I have also had photographs taken of a very large one, constructed from 144 points, with a diameter of 17 inches, containing 10,296 connecting lines, and involving 17,178,876 intersections, for which extraordinary production I am indebted to the assiduity and perseverance of my esteemed friend, Mr. John Scott, of Lee.

1910. (Proposed by Professor SYLVESTER.)—AB is a given straight line upon which four points are taken at random. Find the chance that their anharmonic ratio (estimated by the quotient of the whole into the middle by the product of the extreme segments) shall exceed a given quantity.

Solution by STEPHEN WATSON.



Let C, D, E, F be the points, and put AC $= w$, CD $= x$, DE $= y$, CF $= z$, AB $= a$, and m = the given quantity.

Then $\dfrac{zy}{x(z-x-y)} > m$, therefore $y > \dfrac{mx(z-x)}{mx+z}$ $(=r$ suppose$)$,

and obviously y must be $< z-x$; also the limits of x, z, w are respectively $(0, z)$, $(0, a-w)$ and $(0, a)$; moreover the four points can be interchanged in 24 ways, and the measure of the total number of their positions is a^4; hence the required chance is

$$\frac{24}{a^4}\int_0^a dw \int_0^{a-w} dz \int_0^z dx \int_r^{z-x} dy = \frac{24}{a^4}\int_0^a dw \int_0^{a-w} dz \int_0^z \frac{z(z-x)\,dx}{mx+z}$$

$$= \frac{24}{a^4}\int_0^a dw \int_0^{a-w} \left\{\frac{m+1}{m^2}\log(m+1) - \frac{1}{m}\right\} z^2 dz$$

$$= 2\left\{\frac{m+1}{m^2}\log(m+1) - \frac{1}{m}\right\}.$$

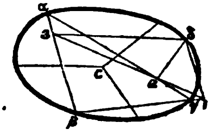
[Professor SYLVESTER remarks that two of the points may be fixed at A and B, by virtue of a general principle: hence the required chance becomes

$$2\int_0^1 \frac{1-x}{1+mx}\,dx = 2\left\{\frac{m+1}{m^2}\log(m+1) - \frac{1}{m}\right\}.$$

Thus the chance of the harmonic ratio being exceeded, that is of the ratio being ultra-harmonic, is $4\log 2 - 2$, that is $\cdot 77258872$, or $\frac{17}{22}$ nearly.]

2395. (Proposed by J. J. WALKER, M.A.)—A proof is required of the following generalization of a well-known theorem:—If from any point on a conic lines be drawn to meet the three sides of any inscribed triangle, parallel to the diameters bisecting those sides respectively, the three points of intersection will be in the same straight line.

I. *Solution by* W. S. McCAY; S. BILLS; *and others.*



Let the eccentric angles of the vertices of the inscribed triangle be (α, β, γ), and δ that of any other point on the ellipse.

Then the equation of the side $\beta\gamma$ is

$$\frac{x}{a}\cos\frac{\beta+\gamma}{2} + \frac{y}{b}\sin\frac{\beta+\gamma}{2} = \cos\frac{\beta-\gamma}{2} \quad \text{..............(1).}$$

Now the relation between the angles that two conjugate diameters make with the axis being $\tan\theta \tan\theta' = -\frac{b^2}{a^2}$, the equation of the line through δ parallel to the diameter bisecting (1) is easily seen to be

$$\frac{x}{a}\sin\frac{\beta+\gamma}{2} - \frac{y}{b}\cos\frac{\beta+\gamma}{2} = \sin\left(\frac{\beta+\gamma}{2}-\delta\right) \quad \ldots\ldots\ldots\ldots(2).$$

The intersection of (1) and (2) is one of the three points that are to be proved collinear; and the theorem would be proved if we could find a line through the intersection of (1) and (2) symmetrical with regard to (α, β, γ). Multiplying (1) by $\sin\frac{1}{2}(\alpha-\delta)$, (2) by $\cos\frac{1}{2}(\alpha-\delta)$ and subtracting, we get, after reducing the right hand side, the equation of a symmetrical line, viz.,

$$\frac{x}{a}\sin\frac{\alpha+\beta+\gamma-\delta}{2} - \frac{y}{b}\cos\frac{\alpha+\beta+\gamma-\delta}{2}$$

$$= 2\sin\frac{\alpha-\delta}{2}\sin\frac{\beta-\delta}{2}\sin\frac{\gamma-\delta}{2} + \sin\frac{\alpha+\beta+\gamma-3\delta}{2}.$$

II. *Solution by* Morgan Jenkins, B.A.

The theorem may be generalized still further thus:—

Let ABC be a triangle inscribed in a conic; A′B′C′ the triangle circumscribed to the conic at the points A, B, C. Then if any three concurrent straight lines RA′, RB′, RC′ meet the polar of R in A″, B″ and C″ respectively, and straight lines through A″, B″, C″ respectively to any point T on the conic, meet BC, CA, AB respectively in α, β, γ; α, β, γ shall be collinear.

By making R the centre of the conic we obtain the theorem in the question.

Let BC, CA, AB meet the polar of R in α', β', γ' respectively,

TA, TB, TC meet the polar of R in $\alpha'', \beta'', \gamma''$ respectively,

and the conic meet the polar of R in P, Q.

Then since α', &c., are the poles of A′A″, &c. respectively,

α', A″; β', B″; γ', C″ are corresponding points of a system in involution of which P, Q are foci ....(1);

and, since the polar of R cuts the inscribed quadrilateral TABC and the conic, therefore α', α''; β', β''; γ', γ''; P, Q are corresponding points of a system in involution........................ (2).

Hence A″, α''; B″, β''; C″, γ'' are corresponding points of another system in involution ....(3).

[We readily obtain (3) from (1) and (2); for if $\alpha', \beta', \ldots$ divide PQ in the ratio $\mu', \nu' \ldots$ respectively; A″, B″... in the ratios D, E... respectively; and α'', β''.. in the ratios μ'', ν''......; then

from (1) we have $D = -\mu'$, $E = -\nu'$, $F = -\rho'$;

from (2) we have $\mu'\mu'' = \nu'\nu'' = \rho'\rho''$;

hence $D\mu'' = E\nu'' = F\rho''$; whence (3) follows].

Therefore $A''\beta'' . B''\gamma . C''\alpha'' = A''\gamma'' . B''\alpha'' . C''\beta''$.

But if F be a perpendicular from T on $\alpha\beta\gamma$

$$(A''\beta'')\, t = 2\frac{T\beta''}{TB} . (TBA''), \&c.$$

Hence $$\frac{(\text{TBA}'')}{(\text{TCA}'')}\cdot\frac{(\text{TCB}'')}{(\text{TAB})}\cdot\frac{(\text{TAC}'')}{(\text{TBC}'')}=1.$$

But $\frac{(\text{TBA}'')}{(\text{TCA}'')}$ is the ratio in which α divides BC, &c. Hence α, β, γ are collinear.

It is to be noticed that if ABC, A″B″C″ be two given triangles, and T a point such that TA″, TB″, TC″ cut BC, ... &c., in three collinear points, the locus of T is given by the equation

$$\frac{(\text{TBA}'')}{(\text{TCA}'')}\frac{(\text{TCB}'')}{(\text{TAB}'')}\frac{(\text{TAC}'')}{(\text{TBC}'')}=1.$$

This locus is a cubic, which degenerates into a conic through ABC and a straight line when A″, B″, C″ are in the same straint line; since A″B″C″ is then a part of the locus.

III. *Solution by the* Proposer.

Let the point be taken as origin, and the three sides be

$$ax+by+c=0,\quad a'x+b'y+c'=0,\quad a''x+b''y+c''=0;$$

then the equation of the conic will be of the form,

$$(a'x+\ldots)(a''x+\ldots)+(ax+\ldots)(a''x+\ldots)+(ax\ldots)(a'x\ldots)=0;$$

or $$\text{A}x^2+\text{B}xy+\text{C}y^2+\ldots.+\ldots.=0,$$

where $\text{A}=a'a''+aa''+aa',\quad \text{C}=b'b''+bb''+bb',$

$\text{B}=a'b''+a''b'+ab''+a''b+ab'+a'b,$

and the absolute term $c'c''+cc''+cc'\equiv 0$, because the origin is on the curve. Write $d=a'b''-a''b',\quad d'=a''b-ab'',\quad d''=ab'-a'b,$

then the equation of the line drawn through the origin parallel to the diameter bisecting the line $ax+by+c=0$ will easily be found to be

$$\{(a+a')d'-(a+a'')d''\}x+\{(b+b')d'-(b+b'')d''\}y=0;$$

and the coordinates of the point of intersection of this line with $ax+by+c=0$ will be found to be

$$x=\{(b+b')d'-(b+b'')d''\}\frac{c}{2d'd''},\quad y=\{(a+a'')d''-(a+a')d'\}\frac{c}{2d'd''}.$$

Similarly, if (x',y'), (x'',y'') are the coordinates of the points in which the sides $a'x+\ldots=0$, $a''x+\ldots=0$ are met by lines drawn through the origin parallel to the diameters bisecting those sides respectively, we have

$$x'=\{(b'+b'')d''-(b'+b)d\}\frac{c'}{2dd''},\quad y'=\{(a'+a)d-(a'+a'')d''\}\frac{c'}{2dd''}$$

$$x''=\{(b''+b')d-(b''+b)d'\}\frac{c''}{2dd'},\quad y''=\{(a''+a)d'-(a''+a')d\}\frac{c''}{2dd'}.$$

From the above values for (x,y), (x',y'), (x'',y'') it will follow that

$$xy'-x'y=-(d^2+d'^2+d''^2-2d'd''-2dd''-2dd')\frac{cc'}{4dd'd''}$$

and, by symmetry,

$$x''y - xy'' = -(d^2 + d'^2 + d''^2 - 2d'd'' - 2dd'' - 2dd')\frac{cc}{4dd'd''},$$

$$x'y'' - x''y' = -(d^2 + d'^2 + d''^2 - 2d'd'' - 2dd'' - 2dd')\frac{c'c''}{4dd'd'},$$

whence

$$xy' - x'y + x''y - xy'' + x'y'' - x''y' = -\frac{d^2 + d'^2 + d''^2 - 2d'd'' - 2dd'' - 2dd'}{4dd'd''}(cc' + cc'' + c'c'');$$

but since $cc' + cc'' + c'c'' = 0$, as remarked before, on account of the origin being on the curve, we have the condition for the points (x, y), (x', y'), (x'', y'') being collinear, satisfied.

In the case of the ellipse the result is easily satisfied by projecting the figure into a circle orthogonally.

IV. *Solution by* Stephen Watson.

Let ABC be any inscribed triangle; take BC $= a$, BA $= c$, as axes, then the equation of the conic is

$$y^2 + Ax^2 + Bxy - cy - aAx = 0,$$

and the coordinates of its centre,

$$x_1 = \frac{cB - 2aA}{B^2 - 4A},\quad y_1 = \frac{A(aB - 2c)}{B^2 - 4A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and if $y = mx$ be any line joining B to any point (x_2, y_2) in the conic

$$x_2 = \frac{cm + aA}{m(m + B) + A},\quad y_2 = mx_2 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

From the above, the equations of the lines through (2), parallel to those through (1), and bisecting BC, CA, AB respectively, are easily found to be

$$y - y_2 = -\frac{2A}{B}(x - x_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$y - y_2 = \frac{cB - 2aA}{aB - 2c}(x - x_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$y - y_2 = -\frac{B}{2}(x - x_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The line through the intersection of (3) and BC, and (5) and AB, is

$$\frac{Ax}{Bm + 2A} + \frac{y}{2m + B} = \tfrac{1}{2}x_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the coordinates (x', y') of the intersection of (6) with the line BC, whose equation is $\left(\frac{x}{a} + \frac{y}{c} = 1\right)$, are

$$x' = \frac{a(Bm + 2A)}{2\{m(m + B) + A\}},\quad y' = \frac{cm(2m + B)}{2\{m(m + B) + A\}},$$

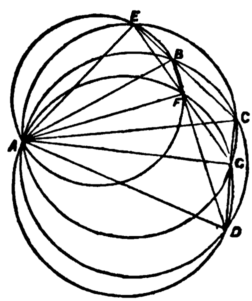
therefore $\frac{x' - x_2}{y' - y_2} = \frac{aB - 2c}{cB - 2aA}$; hence (x', y') fulfil (4), and the lines (3), (4), (5) meet in a point.

2444. (Proposed by A. Renshaw.)—Upon any three chords (AB, AC, AD) of a circle as diameters circles are drawn, cutting one another again in E, F, G; prove that EFG is a straight line.

I. *Solution by* Professor Hirst.

This theorem is the inverse of the well known one that the circle circumscribed to the triangle formed by any three tangents to a parabola passes through the focus. To render this evident, it is merely necessary to take the point A as the centre of inversion. The given circle ABCD then becomes a straight line B′C′D′, and the three circles on AB, AC, AD as diameters become three right lines through B′, C′, D′, respectively, perpendicular to AB′, AC′, AD′, three tangents, therefore, to a parabola of which A is the focus and B′C′D′ the tangent at the vertex.

II. *Solution* by H. Murphy; H. Tomlinson; *the* Proposer; *and others.*



The angles AGD and AGC are both right angles, being those in a semicircle; therefore CGD is a straight line. Similarly CBE and BFD are straight lines. Hence CBD is a triangle inscribed in the original circle, and from a point A on its circumference perpendiculars AE, AF, AG are drawn to its sides; therefore, by a well-known property,* their three feet E, F, G are in the same straight line.

Otherwise: $\angle$ ADC = ABE = AFE, therefore AFE + AFG = ADG + AFG = 2 right angles; therefore E, F, G are in the same straight line.

[An analytical solution is given in Salmon's *Conics*, p. 99, ex. 7; and another in Salmon's *Higher Plane Curves*, p. 240, ex. 2.]

1959. (Proposed by W. S. Burnside, M.A.)—If the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$ are $\alpha, \beta, \gamma, \delta$, find the equation whose roots are $(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)$, &c.

Solution by Matthew Collins, B.A.; S. Bills; *and others.*

We have
$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) = 2\alpha^3-(\alpha+\beta+\gamma+\delta)\alpha^2 + (\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma)-2\beta\gamma\delta;$$
but as $\alpha, \beta, \gamma, \delta$ are the roots of the equation
$$ax^4+4bx^3+6cx^2+4dx+e=0 \quad\text{.....................}\quad (1),$$
we have
$$\Sigma(\alpha) = -\frac{4b}{a}, \quad \Sigma(\beta\gamma\delta) = -\frac{4d}{a}, \quad \beta\gamma\delta = \frac{e}{a\alpha};$$

* See Question 2395, on p. 44 of this volume of the *Reprint*.

hence $y = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) = 2\alpha^3 + \frac{4b}{a}\alpha^2 - \frac{4d}{a} - \frac{2e}{a\alpha}$,

therefore, restoring x for α, and putting $4z$ for y, we have

$$ax^4 + 2bx^3 - 2(d+az)x - e = 0 \quad \ldots\ldots (2).$$

(1) + (2) gives $ax^3 + 3bx^2 + 3cx + (d - az) = 0$ ...... (3);

(1) − (2) gives $bx^3 + 3cx^2 + (3d + az)x + e = 0$ ...... (4).

Now in Meyer Hirsch's *Algebra*, (pp. 119—121) it is shown that the result of eliminating x between the two equations

$$p + qx + rx^2 + sx^3 \text{ and } p' + q'x + r'x^2 + s'x^3 \text{ is}$$

$$(pq'-p'q)(sr'-s'r)(rq'-r'q) + 2(pq'-p'q)(sp'-s'p)(sr'-s'r)$$
$$+ (sp'-s'p)^3 + (pr'-p'r)^2(sr'-s'r)$$
$$+ (pr'-p'r)(sp'-s'p)(sq'-s'q) - (pq'-p'q)(sq'-s'q)^2 = 0 \quad \ldots\ldots (5).$$

Substituting in (5) for p, p', &c., the coefficients in (3) and (4), it becomes

$$9(ac-b^2)(3d^2-3ce-2adz-a^2z^2)(3bd-3c^2+abz)$$
$$+6(ac-b^2)(3d^2-3ce-2adz-a^2z^2)(ae-bd+abz)$$
$$+(ae-bd+abz)^3 + 27(cd-be+acz)^2(ac-b^2)$$
$$+3(cd-be-acz)(ae-bd+abz)(3ad-3bc+a^2z)$$
$$+(3ce-3d^2+2adz+a^2z^2)(3ad-3bc+a^2z)^2 = 0,$$

or

$$a^6z^4 + 8a^3(a^2d-3abc+2b^3)z^3$$
$$+6a^2(a^2ce-3a^2d^2+14abcd-ab^2e-9ac^3-8b^3d+6b^2c^2)z^2$$
$$+\Big\{81(d^2-ce)(bd-c^2)(ac-b^2) + 18(d^2-ce)(ae-bd)(ac-b^2) + (ae-bd)^3$$
$$+9(cd-be)(ae-bd)(ad-bc) + 27(ce-d^2)(ad-bc)^2\Big\} = 0 \quad \ldots (6).$$

[The coefficients in equation (6) may be readily reduced to the forms given in equation (F) of Mr. Burnside's solution. Moreover, equations (B) and (D) of that solution may be deduced from the foregoing as follows:—

Assume $x = v - \frac{b}{a}$; then equations (3) and (4) become

$$a^3v^3 + 3a(ac-b^2)v + 2b^3 - 3abc + a^2d - a^3z = 0 \quad \ldots\ldots (3'),$$

$$a^3bv^3 + 3a^2(ac-b^2)v^2 + (3ab^3 - ba^2bc + 3a^3d + a^4z)v$$
$$-b^4 + 3ab^2c - 3a^2bd + a^3e - a^3bz = 0 \quad \ldots\ldots (4').$$

(4′) − b (3′) gives $3a^2(ac-b^2)v^2 + a(2b^3 - 3abc + a^2d + a^3z)v$

$$-3b^4 + 6ab^2c - 4a^2bd + a^3e = 0 \quad \ldots\ldots (5').$$

Now in Mr. Burnside's notation (3′) and (5′) may be written

$$a^3v^3 + 3aHv + G - a^3z = 0, \quad 3a^2Hv^2 + a(a^3z + 3G) + a^2I - 3H^2 = 0;$$

and they are thus seen to be *identical* with Mr. Burnside's equations.]

II. *Solution by the* PROPOSER.

Since the required equation has for its coefficients functions of the differences of $\alpha, \beta, \gamma, \delta$; the result will not lose its generality by removing the second term from $(a, b, c, d, e)(x, 1)^4 = 0$.

When this is done it is found to be

$$a^4x^4 + 6a^2Hx^2 + 4aGx + a^2I - 3H^2 = 0 \text{ .... (A)},$$

where $G = a^2d - 3abc + 2b^3$, $H = ac - b^2$, $I = ae - 4bd + 3c^2$, and $J = ace + 2bcd - ad^2 - eb^2 - c^3$.

Now it is easily seen that

$$a^3(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) = 4(a^3\alpha^3 + 3aH\alpha + G) = 4a^3z, \text{ suppose;}$$

and α must then be eliminated between the two equations

$$a^3\alpha^3 + 3aH\alpha + G - a^3z = 0, \quad a^4\alpha^4 + 6a^2H\alpha^2 + 4aG\alpha + a^2I - 3H^2 = 0 \text{ .. (B, C)}.$$

These are easily reduced to the two quadratics

$$3a^2H\alpha^2 + a(a^3z + 3G)\alpha + a^2I - 3H^2 = 0 \text{ .... (D)},$$

$$a^2(3G + a^3z)\alpha^2 + (a^3I - 12aH^2)\alpha + 3Ha^3z - 3HG = 0 \text{...... (E)};$$

and the result of the elimination of α between (D) and (E) is

$$a^6z^4 + 8a^3Gz^3 - (18a^3J - 12a^2IH)z^2 + I^3 - 27J^2 = 0 \text{ ...... (F)}.$$

That the coefficient of z must vanish in (F) is otherwise evident from the fact that if the given equation have equal roots, that is if $I^3 - 27J^2 = 0$, two of the roots of the resulting equation must vanish.

2336. (Proposed by S. ROBERTS, M.A.)—Find the locus of points such that the chord of contact of tangents drawn to a given conic from any one of the points is of constant length.

Solution by the REV. J. L. KITCHIN, M.A.; W. CHADWICK; *and others.*

Let (X, Y) be the coordinates of the point from which tangents are drawn; (x_1, y_1), (x_2, y_2) the respective points of contact; then we have

$$\frac{Xx_1}{a^2} + \frac{Yy_1}{b^2} = 1, \quad \frac{Xx_2}{a^2} + \frac{Yy_2}{b^2} = 1; \quad \therefore \quad \frac{X}{a^2}(x_1 - x_2) + \frac{Y}{b^2}(y_1 - y_2) = 0 \text{ ...(1)}.$$

Let ρ, which is of constant length, be the distance of these points from each other; then, if θ denote the angle which ρ makes with axis of x, we have

$$x_1 - x_2 = \rho\cos\theta, \quad y_1 - y_2 = \rho\sin\theta \text{ .... (2)};$$

therefore $$\frac{(x_2 + \rho\cos\theta)^2}{a^2} + \frac{(y_2 + \rho\sin\theta)^2}{b^2} = 1,$$

therefore $$2\left(\frac{x_2\cos\theta}{a^2} + \frac{y_2\sin\theta}{b^2}\right) + \rho\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 0 \text{ .. (3)}.$$

From (1) and (2) we find $\tan\theta = -\dfrac{b^2X}{a^2Y}$; hence substituting in (3) we get

$$2(x_2Y - y_2X) + \frac{(b^2X^2 + a^2Y^2)\rho}{(b^4X^2 + a^4Y^2)^{\frac{1}{2}}} = 0 \text{ .... (4)}$$

From (4) and $\dfrac{Xx_2}{a^2} + \dfrac{Yy_2}{b^2} = 1$, we get

$$x_2 = \frac{a^2b^2X}{b^2X^2+a^2Y^2} - \frac{\rho a^2Y}{2(b^4X^2+a^4Y^2)^{\frac{1}{2}}},\quad y_2 = \frac{a^2b^2Y}{b^2X^2+a^2Y^2} + \frac{\rho b^2X}{2(b^4X^2+a^4Y^2)^{\frac{1}{2}}}.$$

Hence substituting in $\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1$, we get for the equation of the locus

$$\rho^2(b^2X^2+a^2Y^2)^2 = 4(b^4X^2+a^4Y^2)(b^2X^2+a^2Y^2-a^2b^2) \quad \ldots\ldots (5),$$

a curve of the fourth degree. The corresponding locus for the hyperbola is obtained by putting $-b^2$ for b^2. The parabola is a very simple case, and is here omitted.

[*Otherwise*: if (h, k) be the *middle point* of a chord of constant length in the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the *length* of this chord is given by the expression

$$\rho^2 = 4\left(\frac{b^4h^2+a^4k^2}{b^2h^2+a^2k^2}\right)\left(1 - \frac{b^2h^2+a^2k^2}{a^2b^2}\right) \quad \ldots\ldots\ldots\ldots\ldots (A).$$

Moreover, we may write the *equation* of this line in the two following forms, (B) as a chord whose middle point is at the point (h, k), and (C) as the polar of the point (X, Y); viz.,

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}, \ldots\ldots (B), \quad \frac{Xx}{a^2} + \frac{Yy}{b^2} = 1 \quad \ldots\ldots\ldots (C).$$

Comparing (B) with (C) we obtain

$$\frac{h}{X} = \frac{k}{Y} = \frac{b^2h^2+a^2k^2}{a^2b^2} = \left(\frac{b^4h^2+a^4k^2}{b^4X^2+a^4Y^2}\right)^{\frac{1}{2}} = \left(\frac{b^2h^2+a^2k^2}{b^2X^2+a^2Y^2}\right)^{\frac{1}{2}},$$

whence $\frac{b^4h^2+a^4k^2}{b^2h^2+a^2k^2} = \frac{b^4X^2+a^4Y^2}{b^2X^2+a^2Y^2}$, and $\frac{b^2h^2+a^2k^2}{a^2b^2} = \frac{a^2b^2}{b^2X^2+a^2Y^2}$ .. (D).

Substituting from (D) in (A) we obtain, for the required locus, the same equation (5) as in the foregoing solution.]

NOTE ON QUESTION 1990. BY PROFESSOR CAYLEY.

The theorem of paragraph 4,* (ascribed by Professor Sylvester to Mr. Crofton,) that "if a circle and a straight line be cut by any transversal in three points, these will be the foci of a system of Cartesian ovals having double contact with one another at two fixed points," may be enunciated under a more complete form, as follows :—

If in a given circle the chords PP_1, BC meet in A, then each of the two Cartesians, foci A, B, C, which pass through P, will also pass through P_1; and moreover, if α, α' be the diametrals of the chord PP_1 (that is, the extre-

* See *Reprint*, Vol. VI., p. 88

mities of the diameter at right angles to PP_1) then the tangents at P, P_1 to one of the Cartesians will be αP, αP_1 respectively, and to the other of them $\alpha' P$, $\alpha' P_1$ respectively, these tangents being thus independent of the position of the chord BC; and thence also thus;

Given the points A, B, C *in lined*, and the point P;

through P, B, C draw a circle (A) and let PA meet this in P_1,
„ P, C, A „ (B) „ PB „ P_2,
„ P, A, B „ (C) „ PC „ P_3,

then each of the Cartesians, foci A, B, C, which pass through P will also pass through P_1, P_2, P_3; and if

α, α' are the diametrals of PP_1 in circle (A),
β, β' „ PP_2 „ (B),
γ, γ' „ PP_3 „ (C),

then (the points of the several pairs being properly selected) the points (α, β, γ) and the points (α', β', γ') will each lie in a line through P, viz., the lines $P\alpha\beta\gamma$ and $P\alpha'\beta'\gamma'$ will be the tangents at P to the two Cartesians respectively.

The two Cartesians meet in the points P, P_1, P_2, P_3, and in the symmetrically situated points in regard to the axis ABC; the theorem contains as part of itself the well-known property that the two Cartesians cut at right angles at each of their points of intersection; it gives moreover the construction for the following problem:—given the foci A, B, C, and one intersection P of a pair of triconfocal Cartesians, to find the remaining intersections, and the tangents at each of the intersections.

AT RANDOM: No. VI. By W. S. B. Woolhouse, F.R.A.S.

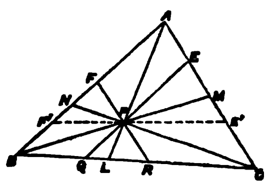
In an article contained in the August Number of the *Educational Times*, (*Reprint*, Vol. VIII., p. 35), Professor Whitworth seems to labour under some misapprehension as to an inaccuracy which has arisen in the statement of my Question (2420) proposed in the June Number. It is, in fact, a pure inadvertency, which I should not have failed to discover in drawing up a solution to the question, and has no reference whatever to the contorted interpretation of language which Professor Whitworth has incautiously chosen to assume. In the rough working of the question I was under the impression that the permutations of each combination were included, but it appears that only the combinations, inclusive of repetitions, are taken into account. The question therefore, instead of "Three numbers," &c., should have been stated "A set of three numbers," &c. It is, however, unnecessary to make any formal correction, as I perceive the substance of the question has been given by Professor Whitworth as the second part of his question (2434) in the July Number. I may, however, be permitted to observe that in his enunciation of this second part an inaccuracy occurs; the phrase "each one less than n" should be "each one not exceeding n." But I cannot bring my mind to attribute this oversight to looseness of language, notwithstanding Professor Whitworth has at the same time so pointedly reminded us that "by due care we can generally so express ourselves as to preclude any doubt about the meaning of our most unqualified statements."

Professor Whitworth further remarks, that "a precisely similar ambiguity occurs in Mr. Woolhouse's Question (2433) in the July Number." He asks, do I "mean that five points *are* selected at random, or that a selection of five points *is* made at random?" It would appear to me that in making this animadversion Professor Whitworth had not given the subject that attention which he might have done on assuming the office of a critic. Where the number of possible selections is infinite, he ought to have satisfied himself that it is virtually the same whether the five points are selected separately or as a group, and that the ambiguity he fancied was a myth of his own imagination.

Before concluding these few lines I may observe that Professor Whitworth and myself are in perfect agreement as to the general interpretation and usually defined signification of the phrase "at random;" and it is only right to add that I am obliged to Professor Whitworth for having suggested the inadvertency in the enunciation of Question (2420).

2359. (Proposed by EXHUMATUS.)—A point is taken at random in each side of a triangle ABC, so forming a second triangle DEF; determine the locus of those points within ABC which have a given probability p of being enclosed within the triangle DEF.

I. *Solution by* STEPHEN WATSON.



Let P be a point in the required locus, through which draw lines as in the diagram, and put respectively BL, LC, CM, &c. = a_1, a_2, b_1, &c., AL, AP, PL = l, l_1, l_2, and AE, AF = x, y. Then we easily find

$$LQ = \frac{a_2 l_2 x}{b l_1 - l x},\quad LR = \frac{a_1 l_2 y}{c l_1 - l y}.$$

Now P will lie within the triangle DEF when D is any point on QR; hence when E and F lie on AM and AN, the number of favourable positions of D, E, F, is

$$\int_0^{c_1}\int_0^{b_2} (LQ + LR)\, dy\, dx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When E and F lie on AM and BN, R lies beyond C, and the favourable positions are

$$\int_0^{b_2} (a_2 + LQ)\, c_2 dx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly, when E and F lie on CM and AN, the favourable positions are

$$\int_0^{c_1} (a_1 + LR)\, b_1 dy \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

When E and F lie on CM and BN. If E be at E, F may lie anywhere on F′N, and D on BC; hence the favourable positions are

$$\int_{b_2}^{b} (\text{AF}' - c_1)\, adx = \int_{b_2}^{b} \left(\frac{a_2 c l_1 x}{alx - a_1 b l_1} - c_1\right) adx \ldots\ldots\ldots\ldots (4).$$

The above include the whole of the favourable cases, and the total number of positions is abc. Hence the chance of P lying within the triangle DEF, for every position of D, E, F is

$$\frac{(1)+(2)+(3)+(4)}{abc} = \frac{1}{abc}\left\{a_2b_2c_2 - a_2b_1c_1 - \frac{l_2}{l}(a_2b_2c + a_1bc_1) + \frac{a_2b_1cl_1}{l}\right\}$$
$$+ \frac{l_1}{al^2}\left\{a_2l_2 \log \frac{a_1bl_1}{a_2b_2l_2} + a_1l_2 \log \frac{a_2cl_1}{a_1c_1l_2} + \frac{a_1a_2l_1}{a}\log\frac{bc}{b_2c_1}\right\} = p \ldots (5).$$

Now (x, y, z) being the triangular coordinates of P, so that $x+y+z=1$, we have

$\frac{l_2}{l} = x, \frac{l_1}{l} = 1-x$; also $\frac{a_1}{a_2} = \frac{\Delta \text{ APB}}{\Delta \text{ APC}} = \frac{z}{y}$, $\therefore \frac{a_1}{a} = \frac{z}{1-x}$, and $\frac{a_2}{a} = \frac{y}{1-x}$;

with similar values for $\frac{b_1}{b}, \frac{b_2}{b}$, &c.; hence (5) reduces to

$$x(1-x)\log\left(\frac{1-x}{x}\right) + y(1-y)\log\left(\frac{1-y}{y}\right) + z(1-z)\log\left(\frac{1-z}{z}\right) = p,$$

or
$$\left(\frac{1-x}{x}\right)^{x(1-x)}\left(\frac{1-y}{y}\right)^{y(1-y)}\left(\frac{1-z}{z}\right)^{z(1-z)} = \epsilon^{p} \ldots\ldots\ldots (6),$$

which is the equation of the locus required.

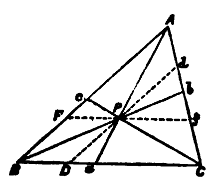
Cor. 1. If AL, BM, CN be the internal bisectors of the angles A, B, C, then $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{1}{2s}$; and (6) gives

$$p = \frac{a(b+c)}{4s^2}\log\left(\frac{b+c}{a}\right) + \frac{b(c+a)}{4s^2}\log\left(\frac{c+a}{b}\right) + \frac{c(a+b)}{4s^2}\log\left(\frac{a+b}{c}\right),$$

which is the chance that the point of the intersection of internal bisectors lies within the triangle DEF.

Cor. 2. If AL, BM, CN bisect the sides, then $x = y = z = \frac{1}{3}$; and (6) gives $p = \frac{2}{9}\log 2$, which is the chance that the point of intersection of the bisectors of the sides lies within the triangle DEF.

II. *Solution by* Morgan Jenkins, B.A.



Let P be a point within the triangle; a, b, c the points in which AP, BP, CP meet BC, CA, AB, respectively; D any point in Ba; F any point in BA; d, f the points in which DP, FP meet AC. Then if F be in Bc, the favourable cases will be when E falls within df; but if F be in cA, the favourable cases will be when E falls within Cd.

Now, using triangular coordinates, the several points may be denoted as follows :—

$$(\text{P})\ldots(x, y, z);\quad (\text{D})\ldots(0, \rho, 1-\rho);\quad (\text{F})\ldots(1-\tau, \tau, 0);$$

$$(d)\ldots\left(\frac{\rho x}{\rho-y}, 0, \frac{\rho(y+z)-y}{\rho-y}\right);\quad (f)\ldots\left(\frac{\tau(x+y)-y}{\tau-y}, 0, \frac{\tau z}{\tau-y}\right).$$

The chance of P being within DEF, if D be any point in Ba, will be

$$\frac{\delta(\text{BD})}{\text{BC}}\cdot\frac{\delta(\text{BF})}{\text{BA}}\cdot\frac{df}{\text{AC}}+\frac{\delta(\text{BD})}{\text{BC}}\cdot\frac{\delta(\text{BF})}{\text{BA}}\cdot\frac{\text{C}d}{\text{AC}}.$$

Now BD $=(1-\rho)$ BC, and BF $=(1-\tau)$ BA; hence the sum of these chances, when D lies anywhere in Ba, is

$$\int_{\frac{y}{y+z}}^{1}\int_{\frac{y}{x+y}}^{1}\left(\frac{\rho x}{\rho-y}-\frac{\tau(x+y)-y}{\tau-y}\right)d\rho d\tau+\int_{\frac{y}{y+z}}^{1}\int_{0}^{\frac{y}{x+y}}\frac{\rho x}{\rho-y}\,d\rho d\tau$$

$$=\int_{\frac{y}{y+z}}^{1}\frac{\rho x}{\rho-y}\,d\rho-\frac{z}{y+z}\int_{\frac{y}{x+y}}^{1}\frac{\tau(x+y)-y}{\tau-y}\,d\tau$$

$$=xy\log\frac{(x+z)(y+z)}{xy}+\frac{yz^2}{y+z}\log\frac{(x+y)(x+z)}{yz}.$$

It will be found that the chance when D lies in aC may be obtained from the foregoing expression by interchanging y and z; hence we have

$$p=xy\log\frac{(x+z)(y+z)}{xy}+xz\log\frac{(x+y)(y+z)}{xz}+\frac{yz^2+y^2z}{y+z}\log\frac{(x+z)(x+y)}{yz}$$

$$=x(1-x)\log\left(\frac{1-x}{x}\right)+y(1-y)\log\left(\frac{1-y}{y}\right)+z(1-z)\log\left(\frac{1-z}{z}\right);$$

therefore $\left(\frac{1-x}{x}\right)^{x(1-x)}\left(\frac{1-y}{y}\right)^{y(1-y)}\left(\frac{1-z}{z}\right)^{z(1-z)}=\epsilon^p$

is the equation, in triangular coordinates, of the required locus.

2397. (Proposed by Professor WHITWORTH.)—If a body of homogeneous fluid be thoroughly shaken and allowed to come to rest again, the chance that no particle of the fluid now occupies its original position is ϵ^{-1}, where ϵ is the base of Napierian logarithms.*

I. *Solution by the* PROPOSER; *and* C. M. INGLEBY, LL.D.

First suppose there were a finite number n of particles in the fluid; and let them be distinguished severally by the letters $\alpha, \beta, \gamma\ldots\kappa$. Let N de-

* For the history of this problem see Todhunter's *History of the Theory of Probability*, Arts. 160–162, 198, 204, 430, 627, &c.

note the number of ways of arranging the n particles; also let (A) express the condition that a is in its original place, and (a) the condition that a is out of its original place. Let (B) and (b) denote the corresponding conditions for β, and so on.

Then we have $N = \Pi(n)$; but $N(A) = \Pi(n-1)$; hence by subtraction $N(a) = \Pi(n) - \Pi(n-1)$. But the condition (B) would leave us $(n-1)$ things instead of n to deal with. Hence writing $(n-1)$ for n in the last equation, we have $N(aB) = \Pi(n-1) - \Pi(n-2)$.

Therefore, subtracting, $N(ab) = \Pi(n) - 2\Pi(n-1) + \Pi(n-2)$.

Similarly, writing $n-1$ for n in this equation, we get

$$N(abC) = \Pi(n-1) - 2\Pi(n-2) + \Pi(n-3),$$

and subtracting, $N(abc) = \Pi(n) - 3\Pi(n-1) + 3\Pi(n-2) - \Pi(n-3)$, and so on. Hence we obtain

$$N(abc\ldots k) = \Pi(n) - \frac{n}{1}\Pi(n-1) + \frac{n(n-1)}{1.2}\Pi(n-2) - \text{\&c. to } (n+1) \text{ terms.}$$

$$= \Pi(n)\left\{1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \ldots\ldots \text{to } (n+1) \text{ terms}\right\}.$$

Therefore the chance that *all* the n particles are misplaced is

$$1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \ldots\ldots \text{to } (n+1) \text{ terms,}$$

which, when n becomes infinite, becomes the expansion of e^{-1}.

II. *Solution by* W. S. B. WOOLHOUSE, F.R.A.S.

1. The number of particles of the fluid is here assumed to be infinite, or indefinitely great. To determine the question generally, suppose the number to be finite. Consider a primary group of n things or symbols to be taken in a specified order, and on the $\Pi(n)$ permutations of the group being compared with it, let $\phi(n)$ denote the number of ways in which the n symbols are wholly displaced. Then it is evident that the number of ways having 1 symbol in its original place and $n-1$ symbols wholly displaced is equal to $n\phi(n-1)$; the number of ways having 2 symbols in their original places and the remaining $n-2$ symbols wholly displaced is equal to $\frac{1}{2}n(n-1)\phi(n-2)$; and so on. The total number is that of the permutations. That is,

$$\phi(n) + n\phi(n-1) + \tfrac{1}{2}n(n-1)\phi(n-2) \ldots . + 1 = \Pi(n).$$

Similarly are obtained the equalities

$$\phi(n-1) + (n-1)\phi(n-2) \ldots . + 1 = \Pi(n-1)$$
$$\phi(n-2) \ldots . + 1 = \Pi(n-2)$$
$$\text{\&c.} \qquad \text{\&c.}$$

Hence, by eliminating the depressed functions $\phi(n-1)$, $\phi(n-2)$, &c., from these relations, we get

$$\phi(n) = \Pi(n) - n\Pi(n-1) + n\frac{n-1}{2}\Pi(n-2) - n\frac{n-1}{2}\cdot\frac{n-2}{3}\Pi(n-3) + \text{\&c.}$$

$$= \Pi(n)\left(1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4}\text{ \&c. to } n+1 \text{ terms}\right)$$

$$= \Pi(n)\left(\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4}\text{ \&c. to } n-1 \text{ terms}\right).$$

Therefore, if one of the $\Pi(n)$ permutations of the group be taken indiscriminately and compared with the primary arrangement, the chance that all the symbols are severally displaced is

$$p = \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \text{\&c. to } n-1 \text{ terms.}$$

According to the established formula

$$\epsilon^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2.3} + \frac{x^4}{2.3.4} - \text{\&c.}$$

we have $\epsilon^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \text{\&c.}$

Thus it appears that the chance of the n symbols being wholly displaced is precisely equal to the expansion of ϵ^{-1} to $n+1$ terms, which is a remarkable result. It is further remarkable that, as the number n is consecutively increased, the value of the probability is alternately increased and diminished. In fact, it oscillates on both sides at the same time that it approximates nearer and nearer to its ultimate value ϵ^{-1}.

2. *Otherwise:* as there are $n-1$ positions of displacement for any one symbol, the probability of displacement for each symbol is $\frac{n-1}{n}$, and the probability that all the n symbols shall be displaced is

$$p = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$$

$$= 1 - n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} - \frac{n(n-1)(n-2)}{2.3} \cdot \frac{1}{n^3} + \text{\&c.}$$

And when n is increased indefinitely, this becomes

$$p = 1 - 1 + \frac{1}{2} - \frac{1}{2.3} - \text{\&c.} = \epsilon^{-1},$$

the same as before.

Or if $\frac{1}{n} = m$; then $p = (1-m)^{\frac{1}{m}}$, and

$$\log p = \frac{1}{m} \log(1-m) = \frac{1}{m} \int \frac{-dm}{1-m}$$

$$= \frac{1}{m} \int -dm\,(1 + m + m^2 + \text{\&c.})$$

$$= -\left(1 + \frac{m}{2} + \frac{m^2}{3} + \text{\&c.}\right)$$

$$= -\left(1 + \frac{1}{2n} + \frac{1}{3n^2} + \text{\&c.}\right)$$

$$\therefore p = \epsilon^{-\left(1 + \frac{1}{2n} + \frac{1}{3n^2} + \text{\&c.}\right)}$$

And when $n = \infty$, $p = \epsilon^{-1}$.

NOTE.—It should be stated, that the methods of solution last given are not quite strict. The probability of displacement of any specified symbol is undoubtedly $\frac{n-1}{n}$, but the probability of the displacement of all the symbols, when n is a finite number, is not accurately equal to $\left(\frac{n-1}{n}\right)^n$, since these several simultaneous displacements are not independent. For it is obvious that the displacement of any one symbol must necessitate the displacement of the symbol which appertains to the place it occupies. The inaccuracy thus involved diminishes as the number n is increased, and becomes zero when n is infinite. But unless this is proved to be so, the solution is not conclusive even for this particular case. It will however be unnecessary here to adduce any such proof as the solution first given is perfectly general and accurate throughout.

2426. (Proposed by C. TAYLOR, M.A.) — If F_n denote the fraction $\frac{1+a^2+a^4....+a^{2n}}{a+a^3+...+a^{2n-1}}$, prove that, in general, if n is not $=1$,

(1)...$F_n > \frac{n+1}{n} + \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2$, (2)...$F_n > \frac{1}{2}\,\frac{n+1}{n}\left(a+\frac{1}{a}\right)$, (3)...$\frac{n}{n+1}F_n$ increases with n.

Solution by C. M. INGLEBY, LL.D.

1. Summing the series, for numerator and denominator, in F_n, and converting the resulting fraction into a continued fraction, as in my solution of Quest. 1521, (*Reprint*, Vol. V., p. 99,) we obtain the expression,

$$F_n = a+\frac{1}{a}-\frac{1}{F_{n-1}} \quad ....(1)$$

$$= \left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2+2-\frac{1}{F_{n-1}} = \frac{n+1}{n}+\frac{n-1}{n}+\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2-\frac{1}{F_{n-}}$$

$$= \frac{n+1}{n}+\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2+\left(\frac{n-1}{n}-\frac{1}{F_{n-1}}\right) ....(2).$$

But, by Quest. 1521, $F_{n-1} > \frac{n}{n-1}$;

$\therefore \frac{n-1}{n}-\frac{1}{F_{n-1}}$ is positive; $\therefore$ from (2), $F_n > \frac{n+1}{n}+\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2$.

2. Since $F_{n-1} > \frac{n}{n-1}$ and $a+\frac{1}{a} > 2$;

$$\therefore F_{n-1} > \frac{n}{n-1}\cdot\frac{2}{a+\frac{1}{a}}, \quad \therefore \frac{1}{F_{n-1}} < \frac{n-1}{2n}\left(a+\frac{1}{a}\right);$$

substituting, then, for $\frac{1}{F_{n-1}}$ in (1) the quantity $\frac{n-1}{2n}\left(a+\frac{1}{a}\right)$, the equation (1) becomes an inequality, viz.,

$$F_n > a+\frac{1}{a}-\frac{n-1}{2n}\left(a+\frac{1}{a}\right), \therefore F_n > \frac{1}{2}\frac{n+1}{n}\left(a+\frac{1}{a}\right).$$

3. This is a corollary from (2). Of course, if $\frac{n}{n+1}F_n$ increases with n, it is $> \frac{n-1}{n}F_{n-1}$ and this $< \frac{n-2}{n-1}F_{n-2}$, and so forth; till at last we have $\frac{2}{3}\frac{1+a^2+a^4}{a+a^3} > \frac{1}{2}\left(a+\frac{1}{a}\right)$; so that $\frac{n}{n+1}F_n > \frac{1}{2}\left(a+\frac{1}{a}\right)$. Conversely, if $\frac{n}{n+1}F_n$ do not increase with n, and we exclude equations, (as we must, since we should have at last a general equation between a function of n and a quantity that does not involve it), we have

$$\frac{n}{n+1}F_n < \frac{1}{2}\left(a+\frac{1}{a}\right), \text{ or } F_n < \frac{1}{2}\frac{n+1}{n}\left(a+\frac{1}{a}\right),$$

which is absurd, since the converse is established by Art. 2.

2447. (Proposed by Professor SYLVESTER.)—If in the development of $\sin^{-1}.x$, viz.,

$$\frac{x^1}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\ldots\ldots$$

every number which occurs is increased by unity, prove that the result, *i.e.*,

$$\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\cdot\frac{x^6}{6}+\ldots\ldots$$

is equal to $\frac{1}{2}(\sin^{-1}x)^2$.

I. *Solution by* THOMAS SAVAGE, M.A.

Since $(\sin^{-1}x)^2$ is unaffected by a change in the sign of x, and vanishes with x, we may assume that

$$u = \tfrac{1}{2}(\sin^{-1}x)^2 = A_2\frac{x^2}{2}+A_4\frac{x^4}{4}+A_6\frac{x^6}{6}+\ldots.$$

Now $\int u\,dx = \frac{1}{2}x(\sin^{-1}x)^2+(1-x^2)^{\frac{1}{2}}\sin^{-1}x-x = xu+(1-x^2)\frac{du}{dx}-x$,

$$\therefore\ A_2\frac{x^3}{2.3}+A_4\frac{x^5}{4.5}+\ldots = A_2\frac{x^3}{2}+A_4\frac{x^5}{4}+\ldots+(1-x^2)(A_2x+A_4x^3+\ldots)-x,$$

whence, by equating the coefficients of corresponding powers of x, we have

$A_2 = 1$ and $\dfrac{A_{2n}}{2n(2n+1)} = A_{2n+2} - A_{2n} + \dfrac{A_{2n}}{2n}$ or $A_{2n+2} = \dfrac{2n}{2n+1} . A_{2n}$.

Thus we obtain $\quad A_4 = \dfrac{2}{3},\ A_6 = \dfrac{2.4}{3.5},\ A_8 = \dfrac{2.4.6}{3.5.7}$, &c.

Therefore $\quad \frac{1}{2}(\sin^{-1}x)^2 = \dfrac{x^2}{2} + \dfrac{2}{3}\dfrac{x^4}{4} + \dfrac{2.4}{3.5}\cdot\dfrac{x^6}{6} + \ldots.$

II. *Solution by* M. Jenkins, B.A.; H. Tomlinson; *and others.*

We have $\dfrac{d}{dx}\left\{\frac{1}{2}(\sin^{-1}x)^2\right\} = (\sin^{-1}x)(1-x^2)^{-\frac{1}{2}}$

and $\quad \dfrac{d^2}{dx^2}\left\{(\frac{1}{2}\sin^{-1}x)^2\right\} = (1-x^2)^{-1} + x(\sin^{-1}x)(1-x^2)^{-\frac{3}{2}}$

$$= (1-x^2)^{-1}\left\{1 + x\frac{d}{dx}.\tfrac{1}{2}(\sin^{-1}x)^2\right\} \ldots\ldots\ldots (1).$$

Assume that $\dfrac{d}{dx}\left\{(\frac{1}{2}\sin^{-1}x)^2\right\} = \sum\limits_{r=0}^{r=\infty} A_{2r+1}\, x^{2r+1}$ where $A_1 = 1$ .. (2),

since it is evident that the series cannot contain *even* powers of x; then

$$\frac{d^2}{dx^2}\left\{(\tfrac{1}{2}\sin^{-1}x)^2\right\} = \sum_{r=0}^{r=\infty}(2r+1)\,A_{2r+1}\,x^{2r};$$

therefore from (1) and (2) we obtain

$$1 + \sum_{r=0}^{r=\infty}\left\{(2r+3)\,A_{2r+3} - (2r+1)\,A_{2r+1}\right\} x^{2r+2}$$

$$= 1 + \sum_{r=0}^{r=\infty} A_{2r+1}\, x^{2r+2}.$$

Hence $A_{2r+3} = \dfrac{2r+2}{2r+3} A_{2r+1}$; and therefore, since $A_1 = 1$, we have

$$A_3 = \frac{2}{3}, \text{ and } A_{2r+1} = \frac{2.4\ldots 2r}{3.5\ldots(2r+1)}.$$

Hence by integration of (2) we obtain

$$\tfrac{1}{2}(\sin^{-1}x)^2 = \tfrac{1}{2}x^2 + \sum_{1}^{\infty}\frac{2.4\ldots 2r}{3.5\ldots(2r+1)}\cdot\frac{x^{2r+2}}{2r+2},$$

which proves the theorem.

[Another investigation is given in Todhunter's *Differential Calculus*, Art. 121 (4th ed.) For a more general form of the property, see Question 2481.]

2452. (Proposed by Professor SYLVESTER.)—Prove that if P, Q be two points chosen at random inside a tetrahedron ABCD, the chance is an even one of ABCPQ forming the angles of a *convex* solid.

I. *Solution by the* PROPOSER.

If we divide the whole of the tetrahedron ABCD into partial tetrahedra of which P is the vertex and each face in turn the base, it is clear that Q will lie in one of these. In like manner P will lie in one of the tetrahedra having Q for its vertex and one of the faces of ABCD for its base. Hence the sum of the chances of PQABC, PQABD, PQACD, PQBCD forming reentrant solids is *twice certainty*, and consequently the chance of PQABC being reentrant when ABCD is a regular tetrahedron is $\frac{1}{2}$, and the principle of hyperperspective invariable association leads to the belief that the conclusion remains good whatever the form of the limiting tetrahedron.

So in like manner we see instantaneously that the chance is $\frac{2}{3}$ of P, Q taken at random inside a triangle ABC forming with A, B the angles of a reentrant quadrilateral, at all events when ABC is equilateral, and then by perspective whatever the form of ABC.

NOTE.—The method of invariable association above suggested is open to very serious and possibly irremovable objections in point of legitimacy, whether for the triangle or the tetrahedron. The result itself is of course correct; in fact the method used for the triangle in the solution of Question 2371 (*Reprint*, Vol. VIII., p. 36) gives for the chance of non-convexity in the case of the tetrahedron

$$2\int_0^h dx\,.\,x^2(h-x) \div \int_0^h dx\,.\,x^2h, \text{ which is } \tfrac{1}{2}.$$

II. *Solution by* PROFESSOR EVERETT.

If the solid ABCPQ is not convex, either P is within ABCQ or Q within ABCP. These two events are mutually exclusive and equally probable, therefore the chance of ABCPQ being not convex is double the chance of Q being within ABCP.

Let h and $h-x$ denote the distances of D and P respectively from the base ABC. For each value of x the chance of Q being within ABCP is $\frac{h-x}{h}$; but the probabilities of different values of x are as x^2; hence the chance of Q being within ABCP is

$$\int_0^h dx\,\frac{h-x}{h}\,x^2 \div \int_0^h dx\,.\,x^2 = \frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{3}} = \tfrac{1}{4},$$

and the chance of ABCPQ being not convex is double of this, or $\frac{1}{2}$.

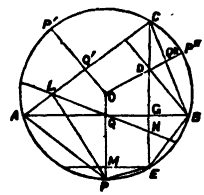
If instead of a tetrahedron ABCD the limiting solid be a prism on ABC as base, the chance of ABCPQ being not convex is

$$2\int_0^h dx\,\frac{h-x}{3h} \div \int_0^h dx = \frac{2}{3}\left(1-\frac{1}{2}\right) = \frac{1}{3}.$$

The former solution holds good for any pyramid or cone, and the latter for any prism or cylinder.

2405. (Proposed by H. MURPHY.)—If from the middles of the arcs which subtend the three sides of an inscribed triangle, three sets of perpendiculars be drawn to the three sides of the triangle, and their feet joined, a new triangle will be formed; prove that the sum of the segments of the perpendiculars of the original triangle, between their point of intersection and the sides of the new triangle, is equal to the excess of the diameter of the circumscribing circle above the radius of the inscribed circle.

Solution by the PROPOSER; J. DALE; *and others.*



From P, the middle of the arc AB, draw the perpendiculars PL and PQ; then the line LQ gives the direction of one of the sides of the triangle referred to in the Question. Now, (Euc. I., 26,) DG = GE = QM; but in the triangles NQG and PME, the angle NQG = LQA = LPA, which is the complement of the angle PAL, and MEP is the complement of the same angle; therefore, (Euc. I., 26,) PM = NG; hence PQ = ND. In the same way P′Q′ and P″Q″ may be proved equal to the other two segments of the perpendiculars; but

$$\begin{aligned} PQ + P'Q' + P''Q'' &= R\left\{3 - (\cos A + \cos B + \cos C)\right\} \\ &= 2R - R(\cos A + \cos B + \cos C - 1) \\ &= 2R - r. \end{aligned}$$

COR.—If P were taken anywhere in the arc AB, we should have, similarly, PQ = DN, which is the solution of an Honour Question, proposed in 1866, in Trinity College, Dublin.

2460. (Proposed by T. COTTERILL, M.A.)—Find the conic which is the locus of a point such that the lines joining it to a pair of conjugate foci are harmonicals to the conic. The circular points at infinity being considered as conjugate foci, the three conics thus obtained and the conic in question have four points in common.

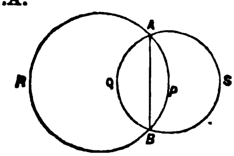
Solution by PROFESSOR HIRST.

Let A, A′; B, B′; C, C′ be the three pairs of opposite intersections of any quadrilateral circumscribed to the given conic (Σ). Through A draw any right line x and connect its pole with A′ by a line x'. The pole of x', of course, lies in x; and x and x' are harmonicals relative to (Σ). But they are also corresponding rays of two homographic pencils (for to each ray x corresponds but one ray x' and *vice versâ*), hence the locus of their intersection x is a conic (X) which touches in A and A′ the connectors of these points with the pole of AA′. Moreover since, when either of the lines x or x'

touches the conic (Σ) the other passes through its point of contact, it is obvious that (X) passes through the four points of contact of the circumscribed quadrilateral. The same is true of the conic (Y) [or (Z)] which is the locus of the intersection of harmonicals passing through B and B′ [or C and C′]. Hence the three conics (X), (Y), (Z) pass through the same four points of (Σ).

When A and A′ coincide with the circular points at infinity we have obviously the theorem in the question.

NOTE WITH REFERENCE TO QUESTION 1829 (*Reprint*, Vol. IV., p. 77.)
BY SAMUEL ROBERTS, M.A.



It may be worth while to notice that if in the first case of a circle, PARB be the given circle, and with centre P thereon and radius $2k$ we draw a circle SAQB meeting the given circle in A and B; then, joining AB, the probability sought in the question is expressed by

$$\frac{2\,\text{Area AQB} + \text{Area APB}}{\text{Area PARB}}.$$

The corresponding expression for a sphere also gives the probability in that case: that is to say, we have for the sought probability

$$\frac{2\,\text{Vol. AQB} + \text{Vol. APB}}{\text{Vol. PARB.}},$$

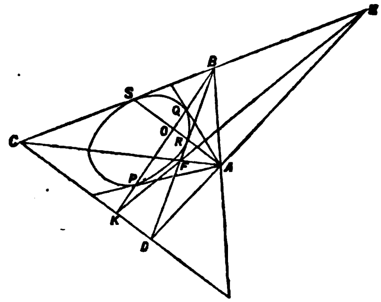
supposing the figure to be turned about RP.

I have not been able to obtain a satisfactory explanation of this result which ought to enable us to solve the problems without integration by taking a fixed point on the surface of the sphere or the circumference of the circle.

We seem however to get a clue to the matter by considering that if AQB be filled up with concentric arcs and APB with arcs touching at P and having their centres on RP, an element of an arc to radius ρ in AQB will subtend at P twice the angle which a corresponding element of an arc to radius ρ in APB will subtend at the same point. This would seem to imply that the elements of AQB must count twice when we seek the probability that a point taken at random in the circle PARB shall be not further than $2k$ from P.

2343. (Proposed by W. K. CLIFFORD, B.A.)—A is any point within or without a conic, B any point on its polar, CD a fixed straight line. Tangents BC, BD are drawn cutting CD in C, D. AD, AC meet BC, BD in E, F; show that EF is a fixed straight line and meets CD on the polar of A.

I. *Solution by the* Rev. J. L. Kitchin, M.A.



Let the conic, for convenience, be an ellipse, and A *without* the curve. Then it is plain that AOB is a triangle self-conjugate with respect to the conic.

Taking this as the triangle of reference, we may write the equation to the conic

$l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0$, or

$m^2\beta^2 = (n\gamma + l\alpha)(n\gamma - l\alpha)$...(1);

therefore $n\gamma + l\alpha = 0$ ......(2)

and $n\gamma - l\alpha = 0$, ...(3)

are the equations of tangents from B, viz., BSC, BRD.

Assumed the equation to CD

$$p\alpha + q\beta + r\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

then since AD passes through the intersection of (3) and (4), and also through the point ($\beta = 0$, $\gamma = 0$), we get for its equation

$$lq\beta + (pn + rl)\gamma = 0 \ldots\ldots\ldots\ldots\ldots (5).$$

Similarly the equation of AC is

$$lq\beta - (pn - rl)\gamma = 0 \quad \ldots\ldots\ldots\ldots (6).$$

Now EF passes through (2) and (5), and also through (3) and (6); hence its equation is $\quad -p\alpha + q\beta + r\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots (7)$, which shows that EF is a *fixed* line: also it evidently passes through the intersection of $\alpha = 0$ and (4), or EFK is a straight line.

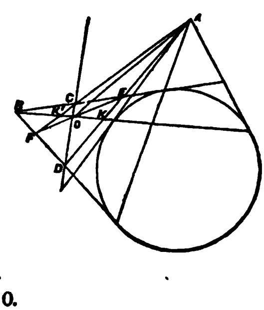
Obviously we might draw a figure, and the lines corresponding to those in the above diagram, either with B or O to start from; and the same conclusion would follow in each case. Hence the point may be either within or without the conic. Again, the only use made of the ellipse is to make the figure simpler; hence what is proved is true of any conic section.

It is easy to write the remaining two equations for the lines from B and O; viz., these are

$$p\alpha - q\beta + r\gamma = 0 \ldots. (8), \quad \text{and } p\alpha + q\beta - r\gamma = 0 \ldots. (9).$$

Of these, (8) passes through the intersection of CD and $\beta = 0$, or the polar of B; and (9) through the intersection of CD and $\gamma = 0$, or the polar of O.

II. *Solution by* J. Dale; H. Tomlinson; *and others.*



Let the polar of A meet the given line CD, and the lines AD, AC in O, K, K′ respectively, and join OA. Then the range (AKED) = (AK′FC), since both are harmonic; also OA is common to both ranges; moreover, (OK, OK′) and (OD, OC) are coincident; therefore OE and OF′ are in the same straight line; and since KK′, CD, and OA are constant in position, EF is also constant in position.

The same may be shown analytically, by referring the conic to any self-conjugate triangle, A and its polar being represented by ($y = 0$, $z = 0$) and $x = 0$, the equation being

$$L^2x^2 + M^2y^2 + N^2z^2 = 0.$$

The tangents to the conic from any point B on the polar of A will be

$$L(M^2y'^2+N^2z'^2)^{\frac{1}{2}}x+MNz'y+MNy'z=0,$$
$$-L(M^2y'^2+N^2z'^2)^{\frac{1}{2}}x+MNz'y+MNy'z=0.$$

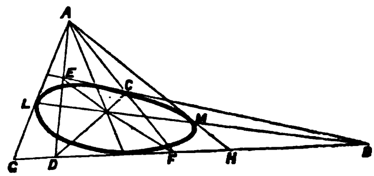
Let the equation to CD be $lx+my+nz=0$; then the equations to AED AFC are respectively

$$L(My'^2+Nz'^2)^{\frac{1}{2}}(my+nz)+lMN(z'y+y'z)=0,$$
$$L(My'^2+Nz'^2)^{\frac{1}{2}}(my+nz)+lMN(z'y+y'z)=0;$$

and thence we find the equation to EF to be $-lx+my+nz=0$, which represents a fixed straight line passing through the intersection of the line $lx+my+nz=0$ and $x=0$, that is, through the intersection of CD and the polar of A.

III. *Solution by the* Rev. R. H. Wright, M.A.

Taking AGH as the triangle of reference, where GH, HA, AG are respectively represented by $\alpha=0$, $\beta=0$, $\gamma=0$, the equations of the several lines may be expressed as follows:—



Conic.......................$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0$;

Polar LM...................$-l\alpha+m\beta+n\gamma=0$;

Tangent BC................ $l\alpha-2m\beta+2n\gamma=0$;

AD.... $\beta-k\gamma=0$; AF.... $\beta-h\gamma=0$;

EF.... $l(h-k)\alpha+2(n+mk)\beta-2h(n+mk)\gamma=0$;

DC.... $l(k-h)\alpha+2(n+mh)\beta-2k(n+mh)\gamma=0$;

By subtracting the two last equations, we have for the intersection of CD and EF,

$$-l\alpha+m\beta+n\gamma=0,$$

which is the polar of A; therefore the property is established. In the same manner we may obtain the reciprocal property.

Cor. 1.—From this it is manifest that if a conic be inscribed in a triangle; a tangent at the point where the line which joins the vertex with the point of contact of the opposite side cuts the conic, the line which joins the points of contact with the other two sides, and the base of the triangle, are concurrent.

Cor. 2.—The locus of intersection of EF and CD will be a straight line, the polar of A.

IV. *Solution by* W. H. Laverty.

Project the conic into a circle, and A into its centre, then the properties being entirely about position, will hold good in the general case, if they do in the particular one. But B will be somewhere on the line at infinity, and BEC, BFD will be parallel tangents. Also EF is evidently parallel to CD, and at a fixed distance from it, that is, EF is a fixed right line. And it meets CD on the line at infinity, which is the polar of A.

2280. (Proposed by M. W. CROFTON, B.A.)—A Cartesian oval or ellipse is cut by an *axial* circle [*i.e.* a circle bisected by the axis] in two points whose vectorial coordinates referred to two of the foci are (r, r') and (s, s'). Show that if a circle concentric to the former be drawn *touching* the curve, the vectorial coordinates of either point of contact are $\left\{\frac{1}{2}(r+s),\ \frac{1}{2}(r'+s')\right\}$.

I. *Solution by* JAMES DALE.

Taking one of the foci as origin, and putting c for the distance between the foci, the polar equation of a Cartesian oval is

$$lr+m\,(r^2-2cr\cos\theta+c^2)^{\frac{1}{2}}=c\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation of an axial circle whose radius is k, and whose centre is at a distance d from the origin, is

$$r^2-2dr\cos\theta+d^2-k^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating $\cos\theta$ from (1) and (2) gives

$$\left\{\,l^2d+m^2(c-d)\,\right\}r^2-2lcdr+\text{const.}=0\ \ldots\ldots\ldots\ldots(3);$$

therefore if r and s be the radii common to the circle and curve, we have

$$\tfrac{1}{2}(r+s)=\frac{lcd}{l^2d+m^2(c-d)},\ \text{and similarly}\ \tfrac{1}{2}(r'+s')=\frac{mc\,(c-d)}{l^2d+m^2(c-d)}.$$

If (2) be taken to represent a tangential circle, then the roots of (3) must be equal; so that, if R be the coordinate of the point of contact, we have

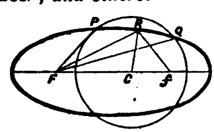
$$R=\frac{lcd}{l^2d+m^2(c-d)},\ \text{and}\ S=\frac{mc\,(c-d)}{l^2d+m^2(c-d)};$$

therefore $\qquad R=\frac{1}{2}(r+s),\ \ S=\frac{1}{2}(r'+s').$

In the ellipse $l=m=e$; therefore $R=\dfrac{d}{e}$, and $S=\dfrac{c-d}{e}$.

II. *Solution by* S. WATSON; H. TOMLINSON; *and others.*

Let F, f be the foci, C the common centre of the axial and touching circles, the former cutting the curve in P, Q, the latter touching it in R. Put $\angle\,\text{QF}f=\alpha$, $\angle\,\text{PF}f=\beta$, $\text{F}f=2c=2ae$, and $\text{FC}=d$; then $r(a-c\cos\alpha)=b^2$,



therefore $r\cos\alpha=\dfrac{ar-b^2}{c}$. Similarly, $s\cos\beta=\dfrac{as-b^2}{c}$. But CQ = CP;

therefore $\qquad r^2+d^2-2rd\cos\alpha=s^2+d^2-2sd\cos\beta,$

therefore $\qquad r^2-s^2=2d\,(r\cos\alpha-s\cos\beta)=2d\,(r-s)\,\dfrac{a}{c}$

therefore $\qquad r+s=2\,\dfrac{ad}{c}=\dfrac{2d}{e}\ldots\ldots\ldots\ldots\ldots\ldots(1).$

Now RC, being a normal at R, bisects the angle FRF; hence

$$\text{FR}:f\text{R}=\text{FC}:f\text{C},\ \text{therefore FR}:d=2a:2ae,$$

therefore $\qquad \text{FR}=\dfrac{d}{e}=\frac{1}{2}(r+s)$, by (1).

Similarly, $f\text{R}=\frac{1}{2}(r'+s')$.

2237. (Proposed by M. W. Crofton, B.A.)—Prove that the equation of the osculating circle at the extremity of the major axis of the ellipse $r+r'=2a$ is $\frac{r^2}{a+c}+\frac{r'^2}{a-c}=2a$.

I. *Solution by* W. Chadwick; J. Dale; H. Tomlinson; *and others.*

Since the equation of the ellipse is put in the form $r+r'=2a$, we have

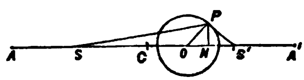
$$r=\left\{y^2+(x+c)^2\right\}^{\frac{1}{2}},\quad r'=\left\{y^2+(x-c)^2\right\}^{\frac{1}{2}}.$$

Now the radius of the osculating circle at the extremity of the major axis is $\frac{b^2}{a}$, and therefore the equation to this circle is

$$y^2+\left(x-\frac{c^2}{a}\right)^2=\frac{b^4}{a^2},\text{ or }\frac{y^2+(x+c)^2}{a+c}+\frac{y^2+(x-c)^2}{a-c}=2a,$$

that is
$$\frac{r^2}{a+c}+\frac{r'^2}{a-c}=2a.$$

II. *Solution by* W. H. Laverty.



The centre of the osculating circle will evidently lie on the major axis. Then, first, to find the equation to a circle having its centre at the point $(r=k,\ r'=2c-k)$, we have, if $SO=k$, $SC=CS'=c$, $OP=\rho$, $SP=r$, $S'P=r'$,

$$\cos SOP+\cos S'OP=0,\quad\text{or}\quad\frac{k^2+\rho^2-r^2}{k}+\frac{(2c-k)^2+\rho^2-r'^2}{2c-k}=0;$$

whence we see that the equation to the circle may be written under the general form

$$kr^2+k'r'^2=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the osculating circle passes through the points

$$(r=a+c,\ r'=a-c),\quad (r=a+c-p,\ r'=a-c+p),$$

where p is a very small quantity; whence we have

$$k(a+c)^2+k'(a-c)^2=1;\ \text{and}\ k(a+c-p)^2+k'(a-c+p)^2=1.$$

Eliminating k, k', between these two equations and equation (1), we find

$$\frac{r^2}{a+c}+\frac{r'^2}{a-c}=2a.$$

On forming the Magic Square of Numbers from 1 to $(12n+6)^2$.

By W. H. Laverty.

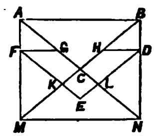
The magic square of an odd number $(6n+3)$ is by well-known rules easily attainable. Let P be the sum of any one of its rows, columns, or diagonals, so that

$$P=\frac{6n+3}{2}\left\{(6n+3)^2+1\right\}.$$

Now change (m) any one of the numbers from 1 to $(6n+3)^2$ into a small square of the form

$$\left.\begin{matrix}(4m-1)\,.\,(4m-2)\\(4m-3)\,.\,(4m)\end{matrix}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\alpha).$$

Now separate the constituents of the squares (α) by lines, so that the large square (A) becomes one having $\{(12n+6)\}$ numbers in each row; and let us examine what alterations we must make in order that (A) may be a magic square. Firstly, it is evident that the two *rows* of (α) have equal sums and consequently that the *rows* also of (A) will have equal sums. Secondly; in (α) the right hand column is greater by two than the left hand column; therefore in (A) the columns will be alternately greater and less than the proper sum $8P-3(6n+3)$ ($=Q$ suppose); by $\{(6n+3)\}$. Thirdly; one diagonal in (α) is less than the other by 4; therefore in (A) one will be less, and the other greater than Q by $2(6n+3)$. Now to remedy this; look at the figure; where $CH=CL=CK=CG=\frac{1}{3}CB=$ etc.; $BD=\frac{1}{3}BN=2n+1$. If in all the (α) squares, contained in the hexagon CAFEDBC; the places of $(4m)$ and $(4m-3)$ be interchanged; let us examine the result. Firstly; the *rows* do not change. Secondly; each of the too-small and too-great columns respectively in (A) has been increased or diminished by $\frac{6n+3}{3}\,.\,3$; the very result required. Also, the diagonals, which have been increased or diminished through $\frac{2}{3}$ of their length, have been increased or diminished by $\frac{2}{3}(6n+3)\,.\,3$; likewise the very result wanted. Hence we have a perfect magic square.



In the particular case of the following Question 2243, we have $n=0$; and the square stands as in the *first* of the arrangements there given. In the above figure (MB) is not a diagonal *line*, but a diagonal formed of (α) squares, similarly for the points B, D, &c.

2243. (Proposed by S. Bills.)—Place the numbers 1 to 36 in six rows, six in a row, so that each row, vertical, lateral, and diagonal, shall make 111.

Solution by W. H. Laverty; *and the* Proposer.

Of the two following arrangements, the *first* may be obtained by making $n=0$ in the foregoing paper;—

31	30	3	2	23	22
32	29	1	4	24	21
11	10	19	18	27	26
9	12	20	17	25	28
15	14	35	34	7	6
18	16	33	36	5	8

5	32	3	34	1	36
12	29	10	27	8	25
15	20	23	13	19	21
18	17	14	24	22	16
28	7	30	11	26	9
3	6	31	2	35	4

Note on Question 1990. By M. W. Crofton, B.A.

The theorem in paragraph 4 will be found stated as a note in a paper on the Cartesian ovals which I have given in the *Proceedings of the London Mathematical Society,* No. VI. The theorems enunciated by Professor Cayley in connexion with it (*Reprint,* Vol. VIII., p. 51) will be found (in substance), with their proofs, in the text of the above Paper. That αP, αP_1 are tangents to one Cartesian, and $\alpha' P$, $\alpha' P_1$ to the other, is an evident consequence of the theorem in Quest. 1924 (*Educational Times* for March, 1866):—that a Cartesian is always equally inclined to the circle through any two foci and the straight line through the third.

These theorems are cases of the following more general ones regarding bicircular quartics:—

If two given circles meet in P, P_1, and they be cut by any arbitrary circle in four points 1234; then the two (complete) bicircular quartics, foci 1234, which pass through P, also pass through P_1; and if α, α' be the two centres of similitude of the given circles, then αP, αP_1 will be tangents to one bicircular quartic; $\alpha' P$, $\alpha' P_1$ those to the other; these tangents being thus quite independent of the positions of the foci 1234 on the two given circles.

Hence, given two circles, the four points where any arbitrary circle meets them are the foci of one of a system of bicircular quartics which have double contact with each other at the two intersections of the given circles.

Several results connected with these theorems are given in a paper which I presented to the London Mathematical Society at one of their recent meetings.

On a Criterion of Divisibility by 7 and 13. By M. Jenkins, B.A.

In the *Proceedings of the Royal Society* (Vol. VII., No. 3, p. 42, Ap. 27, 1854,) it is noticed by the Rev. James Booth, LL.D., F.R.S., that a number of six places of figures, consisting of a repetition of a period of any three figures, is divisible by 7 and 13. Example 507507.

A useful criterion of divisibility by these two numbers is thus suggested. It is given in the following rule by making $p=7$ or 13, $r=10$, and therefore $t=6$, $\frac{1}{2}t=3$.

Let r be the scale of numbers, p any uneven prime number, t the index to which r appertains, and suppose that t is even.

Then to find out if any number of more than $\frac{1}{2}t$ digits is divisible by p.

"Add sufficient cyphers to the left hand of the number to make its number of digits a multiple of $\frac{1}{2}t$; subtract the extreme left-hand digit from the next $\frac{1}{2}t$th digit, and the residue from the next $\frac{1}{2}t$th digit, and so on until we obtain a final residue; form similar final residues beginning with the second, third, &c. digits from the left hand; then the divisibility by p of the number formed by the $\frac{1}{2}t$ final residues taken in order is a criterion of the divisibility by p of the given number."

The subtraction must be algebraical, and therefore the criterion number may contain negative as well as positive digits. The process gives a number which is a criterion of divisibility by every prime divisor of $r^{\frac{1}{2}t}+1$; if, however, we are considering only a single prime divisor p, the final residues may be made positive by the addition of any multiple of p during the process, and may also be made less than r if p be less than r.

Examples : 57,398,362 is divisible by 7 because 021 is divisible by 7 and leaves a remainder 8 (mod. 13).

The criterion number of 645,268 is (−4, 2, 3) or 377 for both 7 and 13; or 923 for 13 or 323 for 7.

A similar criterion exists when t is odd; t taking the place of $\frac{1}{2}t$ and addition being used instead of subtraction.

The proof of the correctness of the criteria is simple, depending on the facts that $r^{\frac{1}{2}t}+1$ in the first case, and $r^{t}-1$ in the latter is divisible by p.

A *sufficient* condition for t being even when $r=10$ and p is prime, is $p \equiv \pm 7, \ \pm 11, \ \pm 17$, or ± 19 (mod. 40), since in that case, by a theorem of Euler's, $10^{\frac{1}{2}(p-1)} \equiv -1$ (mod. p), and therefore t is a divisor of $p-1$ contaning the same power of 2.

The antitheses with respect to the nature of the recurring r-mal of p^{-1} are shown below.

t even.

(I.) A number is divisible by p if it contains twice ($\frac{1}{2}t$) digits *repetitive*, i.e. $\frac{1}{2}t$ and $\frac{1}{2}t$ respectively in direct order.

(II.) Period of p^{-1} in scale r consists of twice ($\frac{1}{2}t$) digits *complementary* to $r-1$.

t odd.

(III.) twice (t) digits *complementary* to p.

(IV.) t figures non-complementary, or twice (t) figures *repetitive*.

Note on the Four-Point Problem. By W. S. B. Woolhouse, F.R.A.S.

An approximate solution of this problem may be obtained as follows :—

It is just the same whether four points A, B, C, D are taken at random, or two lines AB, CD are taken at random. In the latter case, the relative position of the lines as regards intersection evidently resolves into three cases :—

1. The lines may already intersect;
2. The lines may intersect on the production of one of them;
3. The lines may intersect on the production of both.

The apparent probabilities of these several cases, on consideration, appear to be nearly equal, and the second alone is that in which the quadrilateral is reentrant. This probability is therefore $\frac{1}{3}$, at least approximately. It is here unnecessary to assume any boundary.

1911. (Proposed by Professor Cayley.)—Given four points, and also the "conic of centres"—viz. the conic which is the locus of the centres of the several conics which pass through the four given points; then if a conic through the four given points has for its centre a given point on the conic of centres, it is required to find a construction for the asymptotes of this conic.

I. *Solution by the* Proposer.

1. Consider four given points, and in connection therewith a given line IJ; the locus of the poles of IJ, in regard to the several conics which pass through the four points, is a conic, the "conic of poles." Consider a particular conic Θ, through the four points; the pole of IJ in regard to the conic Θ is a point C on the conic of poles, and the tangents from C to the conic Θ meet the conic of poles in two points H, K; the chord of intersection HK passes through the point Π which is the pole of IJ in regard to the conic of poles. Moreover, the polars of a point C′, in regard to the several conics through the four points, meet in a point Ω′, the "common pole" of C′, and in particular if C′ be the point C on the conic of poles, then the common pole is a point Ω on the line IJ; this being so, the line HK passes (as already mentioned) through Π, and the lines HK and ΠΩ are harmonics in regard to the conic of poles.

2. Assuming the foregoing properties, then, given the four points, the line IJ, the conic of poles, and the point C on this conic; we may construct Π the pole of IJ in regard to the conic of poles; and also Ω the common pole of C; the line HK is then given as a line passing through Π, and harmonic to ΠΩ in regard to the conic of poles; this line meets the conic of poles in the points H, K; and then CH, CK are the tangents from C to a conic Θ which passes through the four points.

3. In particular if IJ be the line infinity, then the conic of poles is the conic of centres; Π is the centre of this conic; Ω is as before the common pole of C; HK is given as the diameter of the conic of centres, conjugate to ΠΩ; H, K are the extremities of this diameter; and then CH, CK are the asymptotes of the conic through the four points, which has the point C for its centre; and the asymptotes are therefore constructed as required. If the points H, K are imaginary, the asymptotes will be also imaginary; the conic Θ is in this case an ellipse.

4. It is hardly necessary to remark, in regard to the construction of the point Ω, that we have among the conics through the four points, three pairs of lines meeting in points P, Q, R respectively (it is clear that the conic of poles passes through these three points); the harmonics of CP, CQ, CR in regard to the three pairs of lines respectively meet in a point, which is the required point Ω. In the particular case where the point C is on the conic of centres, the three harmonics are parallel; it is therefore sufficient to construct *one* of them; and the line HK is then the diameter of the conic of poles, conjugate to the harmonic so constructed.

5. It remains to prove the properties assumed in (1). We may take $z=0$ for the equation of the line IJ, $x=0$, $y=0$ for the equations of the tangents to the conic Θ at its intersections with the line IJ, so that we have $(x=0, y=0)$ for the coordinates of the point C; the equation of the conic Θ will be of the form $z^2-xy=0$, and the four points may then be taken to be the intersections of the conic $z^2-xy=0$, and the arbitrary conic

$$(a, b, c, f, g, h)(x, y, z)^2=0.$$

The equation of the conic of centres is found to be

$$x(ax+hy+gz)-y(hx+by+fz)=0, \text{ or } ax^2-by^2+gzx-hxy=0;$$

or, as it may also be written,

$$(2a, -2b, 0, -f, g, 0)(x, y, z)^2=0;$$

and it is convenient to remark that the equation in line coordinates (or condition that this conic may be touched by the line $\xi x+\eta y+\zeta z=0$) is

$$(-f^2, -g^2, -4ab, 2af, 2bg, -fg)(\xi, \eta, \zeta)^2=0.$$

The line $x=0$ meets the conic of poles in the point $x=0$, $by+fz=0$, and the line $y=0$ meets the same conic in the point $y=0$, $ax+gz=0$; hence the line HK, which is the line joining these two points, has for its equation

$$afx+bgy+fgz=0;$$

and it only remains to be shown that this line passes through the point Π, and is the harmonic of the line $\Pi\Omega$ in regard to the conic of centres. The point Π is the pole of the line $z=0$ in regard to the conic of centres, its coordinates are at once found to be

$$x:y:z=bg:af:-2ab;$$

and we thence see that Π is a point on the line HK. The point Ω is given as the intersection of the polars of C in regard to the conics $z^2-xy=0$, and $(a, b, c, f, g, h)(x, y, z)^2=0$ respectively; that is, as the intersection of the lines $z=0$, and $gx+fy+cz=0$; its coordinates therefore are

$$x:y:z=-f:g:0.$$

Hence the equation of the line $\Pi\Omega$ is

$$2abg\,x+2abf\,y+(af^2+bg^2)\,z=0.$$

Now, in general, if we have a conic the line-equation whereof is $(A, B, C, F, G, H)(\xi, \eta, \zeta)^2=0$, then the condition in order that, in regard thereto, the lines $\lambda x+\mu y+\nu z=0$ and $\lambda' x+\mu' y+\nu' z=0$ may be harmonics, is

$$(A, B, C, F, G, H)(\lambda, \mu, \nu)(\lambda', \mu', \nu')=0;$$

that is,

$$A\lambda\lambda'+B\mu\mu'+C\nu\nu'+F(\mu\nu'+\mu'\nu)+G(\nu\lambda'+\nu'\lambda)+H(\lambda\mu'+\lambda'\mu)=0.$$

Hence, in order that the two lines HK and $\Pi\Omega$ may be harmonics in regard to the conic of centres, we should have

$$(-f^2, -g^2, -4ab, 2af, 2bg, -fg)(af, bg, fg)(2abg, 2abf, af^2+bg^2)=0.$$

But developing, and omitting the common factor $abfg$, which enters into all the terms, this equation is

$$-(2af^2)-(2bg^2)-4(af^2+bg^2)+\left\{4af^2+2(af^2+bg^2)\right\}$$
$$+\left\{4bg^2+2(af^2+bg^2)\right\}-2(af^2+bg^2)=0,$$

which is identically true; and the lines HK and $\Pi\Omega$ are therefore harmonics in regard to the conic of centres.

II. *Solution by the* Rev. R. TOWNSEND, F.R.S.

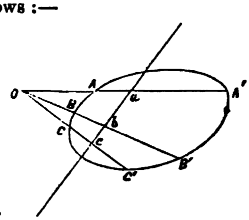
Professor CAYLEY's question is evidently equivalent to the following:—"Given the centre O of a conic, and three points A, B, C on the curve, to determine the directions, real or imaginary, of the asymptotes"—which may be solved immediately as follows:—If X, Y, Z be the three points at infinity on, and U, V, W the three middle points of, the three sides BC, CA, AB of the triangle ABC; then, the three pairs of lines OX and OU, OY and OV, OZ and OW being evidently pairs of conjugate diameters of the conic, the two double rays, real or imaginary, OM and ON, of the involution they determine, as dividing harmonically the three angles they form, are consequently the two asymptotes required.

The more general problem—"Given a point and line P and L pole and polar with respect to a conic, and three points A, B, C on the curve, to determine the directions, real or imaginary, of the two tangents from P to the curve"—may be solved in precisely the same manner. For, if X, Y, Z be the three intersections of L with, and U, V, W the three points harmonically conjugate to, them with respect to the three sides BC, CA, AB of the triangle ABC; then, the three pairs of lines PX and PU, PY and PV, PZ and PW being evidently pairs of conjugate lines with respect to the conic, the two double rays, real or imaginary, PM and PN, of the involution they determine, as dividing harmonically the three angles they form, are consequently the two tangents required.

The reciprocal problem—"Given a point and line P and L pole and polar with respect to a conic, and three tangents A, B, C to the curve, to determine the positions, real or imaginary, of the two intersections of L with the curve," with the particular case of it—"Given the centre O of a conic and three tangents A, B, C to the curve, to determine the directions, real or imaginary, of the two points at infinity,"—may of course be solved by the reciprocal of the preceding construction.

III. *Solution by* F. D. Thomson, M.A.

In the proposed problem, we know the centre of the conic and four points through which the conic passes, and we have to draw the tangents to the conic from the centre. The problem is, therefore, a particular case of the following:—"Given a point and its polar with respect to a conic passing through three given points, to draw the tangent through the given point." The more general problem may be solved as follows:—



Let O be the given point, *abc* its polar, A, B, C the given points; then since the polar *abc* divides harmonically the chord through O, the points A', B', C', where OA, OB, OC meet the curve again, are known.

Also (C . ABA'B') = (C' . ABA'B').

Take the segment on *abc*; then we get two homographic sections $\alpha\beta\alpha'\beta'$, $\alpha_1\beta_1\alpha_1'\beta_1'$, suppose, the *double points* of which will be the points where *abc* meets the curve, that is, the points of contact of tangents through O.

In the particular case proposed, *abc* is the line at infinity, and $OA' = -OA$, $OB' = -OB$, $OC' = -OC$; and if we draw lines $O\alpha$, $O\beta$, $O\alpha'$, $O\beta'$ .... parallel to CA, CB, CA', CB' respectively, and $O\alpha_1$, $O\beta_1$, $O\alpha_1'$, $O\beta_1'$ parallel to C'A, C'B, C'A', C'B'..... ; then $(O\,.\,\alpha\beta\alpha'\beta'\ldots)$, $(O\,.\,\alpha_1\beta_1\alpha'_1\beta'_1\ldots)$ form two homographic pencils, the double rays of which will be the asymptotes required.

2378. (Proposed by the Editor.)—Find the average (1) of the areas of all the circles which can be drawn within the circumference of a given circle, and (2) of the volumes of all the spheres which can be drawn within the surface of a given sphere; also find the probability (3) that any one of the circles taken at random from (1) will exceed a given circle, and (4) that any one of the spheres taken at random from (2) will exceed a given sphere.

Solution by S. BILLS; S. WATSON; *and others.*

1. Let a be the radius of the given circle, r the radius of a variable circle, x the distance of its centre from the centre of the given circle, N the number of circles that can be drawn within the given circle, S the sum of all their areas, and A the average required. Then we have

$$N = \int_0^a (a-x)\,.\,2\pi x dx = \tfrac{1}{3}\pi a^3,\quad S = \int_0^a 2\pi x dx \int_0^{a-x} \pi r^2 dr = \tfrac{1}{30}\pi^2 a^4;$$

therefore $$A = \frac{S}{N} = \frac{\pi a^2}{10} = \tfrac{1}{10}\text{ (area of given circle)}.$$

2. Adopting a similar notation for the sphere, we have

$$N = \int_0^a (a-x)\,.\,4\pi x^2 dx = \tfrac{1}{3}\pi a^4,\quad S = \int_0^a 4\pi x^2 dx \int_0^{a-x} \tfrac{4}{3}\pi r^3 dr = \tfrac{4}{315}\pi^2 a^7;$$

therefore $$A = \frac{S}{N} = \tfrac{4}{105}\pi a^3 = \tfrac{1}{35}\text{ (volume of given sphere)}.$$

3. Let N_1 be the number of circles, greater than a circle of radius c, that can be drawn within the given circle; then we have

$$N_1 = \int_0^{a-c} (a-x-c)\,.\,2\pi x dx = \tfrac{1}{3}\pi (a-c)^3;$$

hence the probability (p) that a circle drawn at random in (1) will have a radius greater than c is

$$p = \frac{N_1}{N} = \left(\frac{a-c}{a}\right)^3.$$

4. Using the same notation for the sphere, we have

$$N_1 = \int_0^{a-c} (a-x-c)\,.\,4\pi x^2 dx = \tfrac{1}{3}\pi (a-c)^4;$$

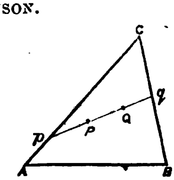
therefore, in this case, the required probability is

$$p = \frac{N_1}{N} = \left(\frac{a-c}{a}\right)^4.$$

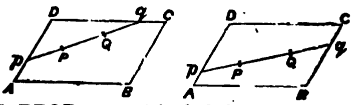
2371. (Proposed by Professor CAYLEY.)—If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

Solution by the REV. M. M. U. WILKINSON.

1. Produce PQ both ways till it meets the periphery of the triangle in p, q. If AB be the side on which lies neither p nor q, APQB will be convex, but APQC, BPQC will be reentrant. Therefore the chance of the quadrilateral being reentrant is twice the chance of its being convex, and is therefore $\frac{2}{3}$.



2. Taking the case of a parallelogram on AB, the same method shows that, if the sides, in which p and q lie, meet in D, then DAPQ, DCPQ are reentrant,



and BAPQ, CBPQ convex. (APQC, BPQD are not included in the conditions of the problem.) If the sides, in which p and q lie, are parallel, as AD, BC, then DPQC, APQB are convex, and DAPQ, BPQC reentrant. The chance of reentrance is therefore $\frac{1}{2}$.

2416. (Proposed by the Rev. R. Townsend, F.R.S.)—Find the point in a plane the sum of the squares of whose distances from any number of given lines in the plane shall be the minimum.

Solution by James Dale.

Let (l_1, m_1, n_1), (l_2, m_2, n_2), ... (l_n, m_n, n_n) be the given lines, $d_1, d_2, d_3 \ldots\ldots d_n$ the distances of the required point (x, y, z) from these n lines; then

$$d_1^2 = \frac{(l_1x + m_1y + n_1z)^2}{\left\{l_1, m_1, n_1\right\}^2},\; d_2^2 = \frac{(l_2x + m_2y + n_3z)^2}{\left\{l_2, m_2, n_3\right\}^2}, \ldots$$

$$d_n^2 = \frac{(l_nx + m_ny + n_nz)^2}{\left\{l_n, m_n, n_n\right\}^2}, \text{ where}$$

$$\left\{l_r, m_r, n_r\right\}^2 = l_r^2 + m_r^2 + n_r^2 - 2m_rn_r \cos A - 2n_rl_r \cos B - 2l_rm_r \cos C.$$

Adding these, and equating the differential of the sum to zero, we get

$$\left\{\Sigma\left(\frac{l^2}{\left\{l, m, n\right\}^2}\right)x + \Sigma\left(\frac{lm}{\left\{l, m, n\right\}^2}\right)y + \Sigma\left(\frac{ln}{\left\{l, m, n\right\}^2}\right)z\right\}dx$$

$$+ \ldots\, dy + \ldots\, dz = 0,$$

and as $ax + by + cz = 2\Delta$, we have $adx + bdy + cdz = 0$;

$$\therefore\; \frac{\Sigma\left(\frac{l^2}{\left\{l, m, n\right\}^2}\right)x + \Sigma\left(\frac{lm}{\left\{l, m, n\right\}^2}\right)y + \Sigma\left(\frac{ln}{\left\{l, m, n\right\}^2}\right)z}{a}$$

$$= \frac{\Sigma\left(\frac{ml}{\left\{l, m, n\right\}^2}\right)x + \Sigma\left(\frac{m^2}{\left\{l, m, n\right\}^2}\right)y + \Sigma\left(\frac{mn}{\left\{l, m, n\right\}^2}\right)z}{b}$$

$$= \frac{\Sigma\left(\frac{nl}{\left\{l, m, n\right\}^2}\right)x + \Sigma\left(\frac{nm}{\left\{l, m, n\right\}^2}\right)y + \Sigma\left(\frac{n^2}{\left\{l, m, n\right\}^2}\right)z}{c}.$$

These equations, together with $ax + by + cz = 2\Delta$, determine x, y, z.

When the number of lines is three, taking them for the sides of the triangle of reference, the equations reduce to $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

2261. (Proposed by S. BILLS.)—Sixty-three school-boys walk out three abreast for thirty-one days in succession; it is required to arrange them daily so that no two of them shall walk twice abreast.

Solution by W. S. B. WOOLHOUSE, F.R.A.S.

Let the sixty-three school-boys be denoted by h; 1, (1); 2, (2); 3, (3); ... 31, (31); and let us endeavour so to arrange these symbols in threes that, after performing a (31) cyclical operation, by successively increasing the numerals by unity, no pair of symbols shall be repeated, and the conditions of the question shall thus be realized. As an auxiliary let r be a primitive root to modulus 31; that is, a number such that the successive powers $r^0, r, r^2, r^3, \ldots r^{29}$ when severally divided by 31 shall all leave different remainders. Then these thirty remainders must necessarily exhibit all the consecutive numbers 1 to 30. In what follows let it be distinctly understood that all values employed are to be estimated according to this modulus, and that the number 31 and all multiples of it are to be rejected from all values that ordinarily exceed such amount. According to the even and odd powers of r, it will be convenient to partition the numbers 1 to 30 in two systems, viz.,

$$\left.\begin{array}{ll}\text{Even system} & r^0, r^2, r^4, \ldots . r^{28} \\ \text{Odd system} & r, \; r^3, r^5, \ldots . r^{29}\end{array}\right\} \ldots(\theta).$$

Suppose now that the sixty-three symbols have for their primitive arrangement either of the two following forms A and B:—

A

h	31	(31)
r	r^{11}	r^{21}
r^3	r^{13}	r^{23}
r^5	r^{15}	r^{25}
r^7	r^{17}	r^{27}
r^9	r^{19}	r^{29}
r^0	(r^{α})	(r^{β})
r^2	$(r^{\alpha+2})$	$(r^{\beta+2})$
r^4	$(r^{\alpha+4})$	$(r^{\beta+4})$
⋮	⋮	⋮
r^{28}	$(r^{\alpha+28})$	$(r^{\beta+28})$

B

h	31	(31)
r^0	r^{10}	r^{20}
r^2	r^{12}	r^{22}
r^4	r^{14}	r^{24}
r^6	r^{16}	r^{26}
r^8	r^{18}	r^{28}
r	$(r^{\alpha+1})$	$(r^{\beta+1})$
r^3	$(r^{\alpha+3})$	$(r^{\beta+3})$
r^5	$(r^{\alpha+5})$	$(r^{\beta+5})$
⋮	⋮	⋮
r^{29}	$(r^{\alpha+29})$	$(r^{\beta+29})$.

The only possibility of repetition of symbols or duads would be amongst the last 15 triads of symbols which are placed below the second horizontal rule. Of these each vertical column will evidently constitute one of the two systems (θ) in a direct order of succession. Those included within parentheses, that is the second and third vertical columns, must be made to form different systems, and therefore α and β must be unlike as to even and odd.

Again, if the numbers in the first column be subtracted from those in the second and third, we get two sets of differences, viz.,

A		B	
$r^{\alpha}-r^{0}$	$r^{\beta}-r^{0}$	$r^{\alpha+1}-r$	$r^{\beta+1}-r$
$r^{\alpha+2}-r^{2}$	$r^{\beta+2}-r^{2}$	$r^{\alpha+3}-r^{3}$	$r^{\beta+3}-r^{3}$
$r^{\alpha+4}-r^{4}$	$r^{\beta+4}-r^{4}$	$r^{\alpha+5}-r^{5}$	$r^{\beta+5}-r^{5}$
$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
$r^{\alpha+28}-r^{28}$	$r^{\beta+28}-r^{28}$	$r^{\alpha+29}-r^{29}$	$r^{\beta+29}-r^{29}$

and it is evident that in the cyclical process no repetition of a duad can occur unless values in one of these two sets of differences should be identical with values contained in the other set. But each of these sets of differences progressing by the common ratio r^2 must necessarily also constitute one of the two systems (θ), and hence the values will be all different if each set appertains to a different system. Hence if we make $r^{\alpha}-1 = r^{\alpha'}$ and $r^{\beta}-1 = r^{\beta'}$ the indices α', β' must be unlike as to even and odd.

We have therefore only to comply with the two following requirements:

α, β to be unlike as to even and odd, and neither of them to be zero;

α', β' also to be unlike as to even and odd.

The number 31 has four reciprocal pairs of primitive roots, viz., (3, 21), (11, 17), (12, 13), (22, 24). To work out the calculation numerically, take $r = 3$, and we readily form the following table showing the values of r^{α}, r^{β}, α', β' classed in two divisions according to the even and odd systems.

α	r^{α}	α'	α	r^{α}	α'
0	1	..	1	3	24
2	9	12	3	27	5
4	19	26	5	26	10
6	16	21	7	17	6
8	20	4	9	29	16
10	25	13	11	13	19
12	8	28	13	24	27
14	10	2	15	30	9
16	28	3	17	22	29
18	4	1	19	12	23
20	5	18	21	15	22
22	14	11	23	11	14
24	2	30	25	6	20
26	18	7	27	23	17
28	7	25	29	21	8
β	r^{β}	β'	β	r^{β}	β'

By inspecting this table the suitable cooperating values of α, β, which satisfy the foregoing requirements, are readily extracted as follows:—

$$\left\{\begin{array}{lll} \alpha \text{ even}, & \alpha' \text{ even}, & \alpha = 2,\ 4,\ 8,\ 12,\ 14,\ 20,\ 24 \\ \beta \text{ odd}, & \beta' \text{ odd}, & \beta = 3,\ 11,\ 13,\ 15,\ 17,\ 19,\ 27 \end{array}\right\}$$

$$\left\{\begin{array}{lll} \alpha \text{ odd}, & \alpha' \text{ even}, & \alpha = 1,\ 5,\ 7,\ 9,\ 21,\ 23,\ 25,\ 29 \\ \beta \text{ even}, & \beta' \text{ odd}, & \beta = 6,\ 10,\ 16,\ 18,\ 22,\ 26,\ 28 \end{array}\right\}.$$

The corresponding values of r^α, r^β are

$$\left\{\begin{array}{l} r^\alpha = 9,\ 19,\ 20,\ 8,\ 10,\ 5,\ 2 \\ r^\beta = 27,\ 13,\ 24,\ 30,\ 22,\ 12,\ 23 \end{array}\right\}$$

$$\left\{\begin{array}{l} r^\alpha = 3,\ 26,\ 17,\ 29,\ 15,\ 11,\ 6,\ 21 \\ r^\beta = 16,\ 25,\ 28,\ 4,\ 14,\ 18,\ 7 \end{array}\right\},$$

and in either of these cases (r^α) (r^β) may be any number contained in the first line associated with any number in the second line. The number of sets of these values is therefore $7.7 + 8.7 = 105$; and as the forms A and B may be both resorted to with each of them, the number of distinct constructions is 210.

As an example, take $\alpha = 2$, $\beta = 11$, or $r^\alpha = 9$, $r^\beta = 13$; then the primitive arrangements A and B, in which the several numbers are filled in vertically from the preceding table, are the ollowing:—

A

h	31	(31)
3	13	15
27	24	11
26	30	6
17	22	23
29	12	21
1	(9)	(13)
9	(19)	(24)
19	(16)	(30)
16	(20)	(22)
20	(25)	(12)
25	(8)	(15)
8	(10)	(11)
10	(28)	(6)
28	(4)	(23)
4	(5)	(21)
5	(14)	(3)
14	(2)	(27)
2	(18)	(26)
18	(7)	(17)
7	(1)	(29)

B

h	31	(31)
1	25	5
9	8	14
19	10	2
16	28	18
20	4	7
3	(27)	(8)
27	(26)	(10)
26	(17)	(28)
17	(29)	(4)
29	(13)	(5)
13	(24)	(14)
24	(30)	(2)
30	(22)	(18)
22	(12)	(7)
12	(15)	(1)
15	(11)	(9)
11	(6)	(19)
6	(23)	(16)
23	(21)	(20)
21	(3)	(25)

The cyclical development of the first of these, showing all the required thirty-one arrangements, is produced by successively increasing all the numbers by an unit, and rejecting 31 when necessary, as the following will show:—

h	31	(31)	*h*	1	(1)	*h*	2	(2)	*h*	3	(3)	*h*	4	(4)
3	13	15	4	14	16	5	15	17	6	16	18	7	17	19
27	24	11	28	25	12	29	26	13	30	27	14	31	28	15
26	30	6	27	31	7	28	1	8	29	2	9	30	3	10
17	22	23	18	23	24	19	24	25	20	25	26	21	26	27
29	12	21	30	13	22	31	14	23	1	15	24	2	16	25
1	(9)	(13)	2	(10)	(14)	3	(11)	(15)	4	(12)	(16)	5	(13)	(17)
9	(19)	(24)	10	(20)	(25)	11	(21)	(26)	12	(22)	(27)	13	(23)	(28)
19	(16)	(30)	20	(17)	(31)	21	(18)	(1)	22	(19)	(2)	23	(20)	(3)
16	(20)	(22)	17	(21)	(23)	18	(22)	(24)	19	(23)	(25)	20	(24)	(26)
20	(25)	(12)	21	(26)	(13)	22	(27)	(14)	23	(28)	(15)	24	(29)	(16)
25	(8)	(15)	26	(9)	(16)	27	(10)	(17)	28	(11)	(18)	29	(12)	(19)
8	(10)	(11)	9	(11)	(12)	10	(12)	(13)	11	(13)	(14)	12	(14)	(15)
10	(28)	(6)	11	(29)	(7)	12	(30)	(8)	13	(31)	(9)	14	(1)	(10)
28	(4)	(23)	29	(5)	(24)	30	(6)	(25)	31	(7)	(26)	1	(8)	(27)
4	(5)	(21)	5	(6)	(22)	6	(7)	(23)	7	(8)	(24)	8	(9)	(25)
5	(14)	(3)	6	(15)	(4)	7	(16)	(5)	8	(17)	(6)	9	(18)	(7)
14	(2)	(27)	15	(3)	(28)	16	(4)	(29)	17	(5)	(30)	18	(6)	(31)
2	(18)	(26)	3	(19)	(27)	4	(20)	(28)	5	(21)	(29)	6	(22)	(30)
18	(7)	(17)	19	(8)	(18)	20	(9)	(19)	21	(10)	(20)	22	(11)	(21)
7	(1)	(29)	8	(2)	(30)	9	(3)	(31)	10	(4)	(1)	11	(5)	(2)

h	5	(5)	*h*	6	(6)	*h*	7	(7)	*h*	8	(8)	*h*	9	(9)
8	18	20	9	19	21	10	20	22	11	21	23	12	22	24
1	29	16	2	30	17	3	31	18	4	1	19	5	2	20
31	4	11	1	5	12	2	6	13	3	7	14	4	8	15
22	27	28	23	28	29	24	29	30	25	30	31	26	31	1
3	17	26	4	18	27	5	19	28	6	20	29	7	21	30
6	(14)	(18)	7	(15)	(19)	8	(16)	(20)	9	(17)	(21)	10	(18)	(22)
14	(24)	(29)	15	(25)	(30)	16	(26)	(31)	17	(27)	(1)	18	(28)	(2)
24	(21)	(4)	25	(22)	(5)	26	(23)	(6)	27	(24)	(7)	28	(25)	(8)
21	(25)	(27)	22	(26)	(28)	23	(27)	(29)	24	(28)	(30)	25	(29)	(31)
25	(30)	(17)	26	(31)	(18)	27	(1)	(19)	28	(2)	(20)	29	(3)	(21)
30	(13)	(20)	31	(14)	(21)	1	(15)	(22)	2	(16)	(23)	3	(17)	(24)
13	(15)	(16)	14	(16)	(17)	15	(17)	(18)	16	(18)	(19)	17	(19)	(20)
15	(2)	(11)	16	(3)	(12)	17	(4)	(13)	18	(5)	(14)	19	(6)	(15)
2	(9)	(28)	3	(10)	(29)	4	(11)	(30)	5	(12)	(31)	6	(13)	(1)
9	(10)	(26)	10	(11)	(27)	11	(12)	(28)	12	(13)	(29)	13	(14)	(30)
10	(19)	(8)	11	(20)	(9)	12	(21)	(10)	13	(22)	(11)	14	(23)	(12)
19	(7)	(1)	20	(8)	(2)	21	(9)	(3)	22	(10)	(4)	23	(11)	(5)
7	(23)	(31)	8	(24)	(1)	9	(25)	(2)	10	(26)	(3)	11	(27)	(4)
23	(12)	(22)	24	(13)	(23)	25	(14)	(24)	22	(15)	(25)	27	(16)	(26)
12	(6)	(3)	13	(7)	(4)	14	(8)	(5)	15	(9)	(6)	16	(10)	(7)

h	10	(10)	*h*	11	(11)	*h*	12	(12)	*h*	13	(13)	*h*	14	(14)
13	23	25	14	24	26	15	25	27	16	26	28	17	27	29
6	3	21	7	4	22	8	5	23	9	6	24	10	7	25
5	9	16	6	10	17	7	11	18	8	12	19	9	13	20
27	1	2	28	2	3	29	3	4	30	4	5	31	5	6
8	22	31	9	23	1	10	24	2	11	25	3	12	26	4
11	(19)	(23)	12	(20)	(24)	13	(21)	(25)	14	(22)	(26)	15	(23)	(27)
19	(29)	(3)	20	(30)	(4)	21	(31)	(5)	22	(1)	(6)	23	(2)	(7)
29	(26)	(9)	30	(27)	(10)	31	(28)	(11)	1	(29)	(12)	2	(30)	(13)
26	(30)	(1)	27	(31)	(2)	28	(1)	(3)	29	(2)	(4)	30	(3)	(5)
30	(4)	(22)	31	(5)	(23)	1	(6)	(24)	2	(7)	(25)	3	(8)	(26)
4	(18)	(25)	5	(19)	(26)	6	(20)	(27)	7	(21)	(28)	8	(22)	(29)
18	(20)	(21)	19	(21)	(22)	20	(22)	(23)	21	(23)	(24)	22	(24)	(25)
20	(7)	(16)	21	(8)	(17)	22	(9)	(18)	23	(10)	(19)	24	(11)	(20)
7	(14)	(2)	8	(15)	(3)	9	(16)	(4)	10	(17)	(5)	11	(18)	(6)
14	(15)	(31)	15	(16)	(1)	16	(17)	(2)	17	(18)	(3)	18	(19)	(4)
15	(24)	(13)	16	(25)	(14)	17	(26)	(15)	18	(27)	(16)	19	(28)	(17)
24	(12)	(6)	25	(13)	(7)	26	(14)	(8)	27	(15)	(9)	28	(16)	(10)
12	(28)	(5)	13	(29)	(6)	14	(30)	(7)	15	(31)	(8)	16	(1)	(9)
28	(17)	(27)	29	(18)	(28)	30	(19)	(29)	31	(20)	(30)	1	(21)	(31)
17	(11)	(8)	18	(12)	(9)	19	(13)	(10)	20	(14)	(11)	21	(15)	(12)

h	15	(15)	*h*	16	(16)	*h*	17	(17)	*h*	18	(18)	*h*	19	(19)
18	28	30	19	29	31	20	30	1	21	31	2	22	1	3
11	8	26	12	9	27	13	10	28	14	11	29	15	12	30
10	14	21	11	15	22	12	16	23	13	17	24	14	18	25
1	6	7	2	7	8	3	8	9	4	9	10	5	10	11
13	27	5	14	28	6	15	29	7	16	30	8	17	31	9
16	(24)	(28)	17	(25)	(29)	18	(26)	(30)	19	(27)	(31)	20	(28)	(1)
24	(3)	(8)	25	(4)	(9)	26	(5)	(10)	27	(6)	(11)	28	(7)	(12)
3	(31)	(14)	4	(1)	(15)	5	(2)	(16)	6	(3)	(17)	7	(4)	(18)
31	(4)	(6)	1	(5)	(7)	2	(6)	(8)	3	(7)	(9)	4	(8)	(10)
4	(9)	(27)	5	(10)	(28)	6	(11)	(29)	7	(12)	(30)	8	(13)	(31)
9	(23)	(30)	10	(24)	(31)	11	(25)	(1)	12	(26)	(2)	13	(27)	(3)
23	(25)	(26)	24	(26)	(27)	25	(27)	(28)	26	(28)	(29)	27	(29)	(30)
25	(12)	(21)	26	(13)	(22)	27	(14)	(23)	28	(15)	(24)	29	(16)	(25)
12	(19)	(7)	13	(20)	(8)	14	(21)	(9)	15	(22)	(10)	16	(23)	(11)
19	(20)	(5)	20	(21)	(6)	21	(22)	(7)	22	(23)	(8)	23	(24)	(9)
20	(29)	(18)	21	(30)	(19)	22	(31)	(20)	23	(1)	(21)	24	(2)	(22)
29	(17)	(11)	30	(18)	(12)	31	(19)	(13)	1	(20)	(14)	2	(21)	(15)
17	(2)	(10)	18	(3)	(11)	19	(4)	(12)	20	(5)	(13)	21	(6)	(14)
2	(22)	(1)	3	(23)	(2)	4	(24)	(3)	5	(25)	(4)	6	(26)	(5)
22	(16)	(13)	23	(17)	(14)	24	(18)	(15)	25	(19)	(16)	26	(20)	(17)

h	20	(20)	*h*	21	(21)	*h*	22	(22)	*h*	23	(23)	*h*	24	(24)
23	2	4	24	3	5	25	4	6	26	5	7	27	6	8
16	13	31	17	14	1	18	15	2	19	16	3	20	17	4
15	19	26	16	20	27	17	21	28	18	22	29	19	23	30
6	11	12	7	12	13	8	13	14	9	14	15	10	15	16
18	1	10	19	2	11	20	3	12	21	4	13	22	5	14
21	(29)	(2)	22	(30)	(3)	23	(31)	(4)	24	(1)	(5)	25	(2)	(6)
29	(8)	(13)	30	(9)	(14)	31	(10)	(15)	1	(11)	(16)	2	(12)	(17)
8	(5)	(19)	9	(6)	(20)	10	(7)	(21)	11	(8)	(22)	12	(9)	(23)
5	(9)	(11)	6	(10)	(12)	7	(11)	(13)	8	(12)	(14)	9	(13)	(15)
9	(14)	(1)	10	(15)	(2)	11	(16)	(3)	12	(17)	(4)	13	(18)	(5)
14	(28)	(4)	15	(29)	(5)	16	(30)	(6)	17	(31)	(7)	18	(1)	(8)
28	(30)	(31)	29	(31)	(1)	30	(1)	(2)	31	(2)	(3)	1	(3)	(4)
30	(17)	(26)	31	(18)	(27)	1	(19)	(28)	2	(20)	(29)	3	(21)	(30)
17	(24)	(12)	18	(25)	(13)	19	(26)	(14)	20	(27)	(15)	21	(28)	(16)
24	(25)	(10)	25	(26)	(11)	26	(27)	(12)	27	(28)	(13)	28	(29)	(14)
25	(3)	(23)	26	(4)	(24)	27	(5)	(25)	28	(6)	(26)	29	(7)	(27)
3	(22)	(16)	4	(23)	(17)	5	(24)	(18)	6	(25)	(19)	7	(26)	(20)
22	(7)	(15)	23	(8)	(16)	24	(9)	(17)	25	(10)	(18)	26	(11)	(19)
7	(27)	(6)	8	(28)	(7)	9	(29)	(8)	10	(30)	(9)	11	(31)	(10)
27	(21)	(18)	28	(22)	(19)	29	(23)	(20)	30	(24)	(21)	31	(25)	(22)

h	25	(25)	*h*	26	(26)	*h*	27	(27)	*h*	28	(28)	*h*	29	(29)	*h*	30	(30)
28	7	9	29	8	10	30	9	11	31	10	12	1	11	13	2	12	14
21	18	5	22	19	6	23	20	7	24	21	8	25	22	9	26	23	10
20	24	31	21	25	1	22	26	2	23	27	3	24	28	4	25	29	5
11	16	17	12	17	18	13	18	19	14	19	20	15	20	21	16	21	22
23	6	15	24	7	16	25	8	17	26	9	18	27	10	19	28	11	20
26	(3)	(7)	27	(4)	(8)	28	(5)	(9)	29	(6)	(10)	30	(7)	(11)	31	(8)	(12)
3	(13)	(18)	4	(14)	(19)	5	(15)	(20)	6	(16)	(21)	7	(17)	(22)	8	(18)	(23)
13	(10)	(24)	14	(11)	(25)	15	(12)	(26)	16	(13)	(27)	17	(14)	(28)	18	(15)	(29)
10	(14)	(16)	11	(15)	(17)	12	(16)	(18)	13	(17)	(19)	14	(18)	(20)	15	(19)	(21)
14	(19)	(6)	15	(20)	(7)	16	(21)	(8)	17	(22)	(9)	18	(23)	(10)	19	(24)	(11)
19	(2)	(9)	20	(3)	(10)	21	(4)	(11)	22	(5)	(12)	23	(6)	(13)	24	(7)	(14)
2	(4)	(5)	3	(5)	(6)	4	(6)	(7)	5	(7)	(8)	6	(8)	(9)	7	(9)	(10)
4	(22)	(31)	5	(23)	(1)	6	(24)	(2)	7	(25)	(3)	8	(26)	(4)	9	(27)	(5)
22	(29)	(17)	23	(30)	(18)	24	(31)	(19)	25	(1)	(20)	26	(2)	(21)	27	(3)	(22)
29	(30)	(15)	30	(31)	(16)	31	(1)	(17)	1	(2)	(18)	2	(3)	(19)	3	(4)	(20)
30	(8)	(28)	31	(9)	(29)	1	(10)	(30)	2	(11)	(31)	3	(12)	(1)	4	(13)	(2)
8	(27)	(21)	9	(28)	(22)	10	(29)	(23)	11	(30)	(24)	12	(31)	(25)	13	(1)	(26)
27	(12)	(20)	28	(13)	(21)	29	(14)	(22)	30	(15)	(23)	31	(16)	(24)	1	(17)	(25)
12	(1)	(11)	13	(2)	(12)	14	(3)	(13)	15	(4)	(14)	16	(5)	(15)	17	(6)	(16)
1	(26)	(23)	2	(27)	(24)	3	(28)	(25)	4	(29)	(26)	5	(30)	(27)	6	(31)	(28)

It may be observed that if a different notation be required, a consecutive series of numbers, or any arbitrary set of symbols, or other designations, may be substituted for the first arrangement, and the two collated together will supply a key for putting down all the others.

We may also remark that the same process will resolve the problem generally for any other number, provided only that the modulus $m = 6k + 1$ be a prime number, the number of school-boys being then $n = 2m + 1$.

Extension of the Solution of Question 2261. By S. Bills.

Suppose 255 school boys to walk out for 127 days, 3 abreast, and let it be required to arrange them so that no two shall walk twice abreast.

Referring to the foregoing Solution of Question 2261, and employing Mr. Woolhouse's notation, I find 3 and 85 to be a pair of reciprocal roots of r^{α} to modulus 127.

Taking $r = 3$, the succession of remainders arising from dividing r^{α} by 127, while α takes every value from 1 to 126 will be in order as follows.

(Read this list downwards.)

3	22	119	26	106	100	56	72	20	62
9	66	103	78	64	46	41	89	60	59
27	71	55	107	65	11	123	13	53	50
81	86	38	67	68	33	115	39	32	23
116	4	114	74	77	99	91	117	96	69
94	12	88	95	104	43	19	97	34	80
28	36	10	31	58	2	57	37	102	113
84	108	30	93	47	6	44	111	52	85
125	70	90	25	14	18	5	79	29	1
121	83	16	75	42	54	15	110	87	
109	122	48	98	126	35	45	76	7	
73	112	17	40	124	105	8	101	21	
92	82	51	120	118	61	24	49	63	

Now if we take $\alpha = 2$ and $\beta = 7$; then $\alpha' = 90$ and $\beta' = 3$; and $r^{\alpha} = 9$, $r^{\beta} = 28$; and the necessary conditions, α and β unlike as to sign; and also α' and β' unlike as to sign will be satisfied, (see Solution of Quest. 2261).

Mr. Woolhouse's forms A and B, extended for 255 symbols, will be as follows:—

A

h	127	(127)
r	r^{43}	r^{85}
r^3	r^{45}	r^{87}
r^5	r^{47}	r^{89}
⋮	⋮	⋮
r^{41}	r^{83}	r^{125}
r^0	(r^{α})	(r^{β})
r^2	$(r^{\alpha+2})$	$(r^{\beta+2})$
r^4	$(r^{\alpha+4})$	$(r^{\beta+4})$
⋮	⋮	⋮
r^{124}	$(r^{\alpha+124})$	$(r^{\beta+124})$

B

h	127	(127)
r^0	r^{42}	r^{84}
r^2	r^{44}	r^{86}
r^4	r^{46}	r^{88}
r^6	r^{48}	r^{90}
⋮	⋮	⋮
r^{40}	r^{82}	r^{124}
r	$(r^{\alpha+1})$	$(r^{\beta+1})$
r^3	$(r^{\alpha+3})$	$(r^{\beta+3})$
r^5	$(r^{\alpha+5})$	$(r^{\beta+5})$
⋮	⋮	⋮
r^{125}	$(r^{\alpha+125})$	$(r^{\beta+125})$

Now taking $r=3$, $\alpha=2$, and $\beta=7$; substituting in A, and remembering to take out the proper remainders from the above table, we find the primitive arrangement to be as follows. And by using the form B we should obtain another primitive arrangement. To save room however I shall not develop the subject further than to obtain one primitive arrangement.

∞	127	(127)	1	(9)	(28)	31	(25)	(106)	72	(13)	(111)
			9	(81)	(125)	25	(98)	(65)	13	(117)	(110)
3	67	57	81	(94)	(109)	98	(120)	(77)	117	(37)	(101)
27	95	5	94	(84)	(92)	120	(64)	(58)	37	(79)	(20)
116	93	45	84	(121)	(66)	64	(68)	(14)	79	(76)	(53)
28	75	24	121	(73)	(86)	68	(104)	(126)	76	(49)	(96)
125	40	89	73	(22)	(12)	104	(47)	(118)	49	(60)	(102)
109	106	39	22	(71)	(108)	47	(42)	(46)	60	(32)	(29)
92	65	97	71	(4)	(83)	42	(124)	(33)	32	(34)	(7)
66	77	111	4	(36)	(112)	124	(100)	(43)	34	(52)	(63)
86	58	110	36	(70)	(119)	100	(11)	(6)	52	(87)	(59)
12	14	101	70	(122)	(55)	11	(99)	(54)	87	(21)	(23)
108	126	20	122	(82)	(114)	99	(2)	(105)	21	(62)	(80)
83	118	53	82	(103)	(10)	2	(18)	(56)	62	(50)	(85)
112	46	96	103	(38)	(90)	18	(35)	(123)	50	(69)	(3)
119	33	102	38	(88)	(48)	35	(61)	(91)	69	(113)	(27)
55	43	29	88	(30)	(51)	61	(41)	(57)	113	(1)	(116)
114	6	7	30	(16)	(78)	41	(115)	(5)			
10	54	63	16	(17)	(67)	115	(19)	(45)			
90	105	59	17	(26)	(95)	19	(44)	(24)			
48	56	23	26	(107)	(93)	44	(15)	(89)			
51	123	80	107	(74)	(75)	15	(8)	(39)			
78	91	85	74	(31)	(40)	8	(72)	(97)			

By passing this primitive arrangement through a cycle of 127 we should obtain the required arrangements for every day during the 127 days.

This question admits of numerous constructions, but it is unnecessary to develop it further as it necessarily leads to such lengthened results, and the method is the same for each.

The above arrangement may be relied upon as accurate.

Note on the Four-Point Problem. By T. Savage, M.A.

1. Let P, Q, R, S be four points taken at random. Then the chance that the terminated line PQ will intersect the terminated line RS is one-third of the chance that the four points are the angles of a convex quadrilateral. For if the quadrilateral be reentrant, they cannot intersect. If the quadrilateral be convex, they will intersect only when Q is the angle opposite to P, and any one of the three Q, R, S may with equal probability be the opposite angle.

2. Now the chance that the terminated lines PQ, RS will intersect is the compound chance that P and Q lie on opposite sides of RS, and that R, S

lie on opposite sides of PQ. If then we may assume that infinite space is divided into two equal portions by any straight line, this chance is $\frac{1}{2} \times \frac{1}{2}$, and the chance of a convex quadrilateral is $\frac{3}{4}$.

3. The above assumption is undoubtedly true when the straight line passes through any point at a finite distance, but probably in that case only. The following appears to be free from this objection. If we examine four points which form a reentrant quadrilateral, we see that three of the six pairs of points lie on the same side of the line joining the other pair, but that in a convex quadrilateral four of the six pairs have this property. Hence if x be the chance that four points form a convex quadrilateral, the chance that two random points P, Q lie on the same side of the line joining two other random points R, S is $\frac{2}{3}x + \frac{1}{2}(1-x)$ or $\frac{1}{2} + \frac{1}{6}x$; and the chance that P, Q lie on opposite sides of RS is $\frac{1}{2} - \frac{1}{6}x$. Hence we must have $3(\frac{1}{2} - \frac{1}{6}x)^2 = x$, which gives us $x = 9 - 6\sqrt{2}$.

4. The same result is obtained if we consider that the only ways in which we can obtain a convex quadrilateral are by supposing P, Q to be on the same or opposite sides of RS, according as R, S are on the same or opposite sides of PQ. This would give us the equation

$$(\tfrac{1}{2} + \tfrac{1}{6}x)^2 + (\tfrac{1}{2} - \tfrac{1}{6}x)^2 = x,$$

which is the same as the former.

5. The chance that P, Q fall on opposite sides of a random line RS is then, it appears, $\sqrt{2} - 1$. That this chance must be less than $\frac{1}{2}$ seems to be evident from the following. If any line divide space into two portions in the proportion of $x : y$, the chance that two random points lie on the same side of the line is $\frac{x^2 + y^2}{(x+y)^2}$; that they lie on opposite sides it is $\frac{2xy}{(x+y)^2}$, and always less than the former, unless $x = y$. The above would appear to show that the average proportion of $x : y$ is $\sqrt{2} + 1 : 1$.

6. In (2), (3), and (4) it is assumed that the two events,

P and Q on opposite sides of RS, and
R and S on opposite sides of PQ,

are independent. It is difficult completely to justify this, but it is evident that the probability of the second depends simply on the ratio of the two portions into which we consider space to be divided by the line PQ; and I do not see how the probable value of this can be affected by our knowledge that PQ joins two points taken at random, one on one side of the line RS and the other on the opposite side. As there is, however, doubtless a weak point here, I will proceed to show that the chance of a convex quadrilateral is less than $\frac{3}{4}$. Let λ be the chance that R, S are on opposite sides of PQ when P, Q are known to be on opposite sides of RS; and μ the same chance when P, Q are known to be on the same side of RS. Then, from the reasoning in (5), it is clear that neither λ nor μ can exceed $\frac{1}{2}$, and the modified forms of the equations in (3) and (4) are

$$3(\tfrac{1}{2} - \tfrac{1}{6}x)\lambda = x, \quad \text{and} \quad (\tfrac{1}{2} - \tfrac{1}{6}x)\lambda + (\tfrac{1}{2} + \tfrac{1}{6}x)(1 - u) = x;$$

from which, by eliminating x, we find that

$$(\lambda + 1)(\mu + 1) = 2,$$

which gives us $\frac{1}{3}$ as an inferior limit to the value of either λ or μ. If we express x in terms of (1) λ or (2) μ, we find

$$x = \frac{3\lambda}{2 + \lambda}, \text{ or } = \frac{3(1-\mu)}{3 + \mu}.$$

By substituting either for λ its greatest or for μ its least value, we find that the true value of x must be less than $\frac{3}{5}$.

7. It is curious to observe that just as we obtain the maximum value of λ or μ by supposing space to be divided into two equal portions by any straight line, so we appear to obtain the minimum value by assuming that the intersection of two random lines is a random point. For let PQ, RS meet in O. Then if O be a random point, of the three O, P, Q any one may with equal probability be the middle point, and therefore the chance that O is the middle point is $\frac{1}{3}$. The discordance between the results obtained by the methods of § 3 and § 4, when we make either of these assumptions, appears to be a sufficient proof of their falsity.

1971. (Proposed by C. W. MERRIFIELD, F.R.S.)—In a given tetrahedron to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges.

Quaternion Solution by T.

Let α, β, γ be three (vector) edges of the tetrahedron, one corner being the origin. Let ρ be the vector of the origin of the sought rectangular system, which may be called i, j, k (unknown vectors.) The condition that i, drawn from ρ, intersects α is

$$S.i\alpha\rho = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

That it intersects the opposite edge, whose equation is $\varpi = \beta + x(\beta-\gamma)$, the condition is

$$S.i(\beta-\gamma)(\rho-\beta) = 0, \text{ or } Si\left\{(\beta-\gamma)\rho-\beta\gamma\right\} = 0 \quad \ldots. (2).$$

There are two other equations like (1), and two like (2), which can be at once written down.

Put $\beta-\gamma=\alpha_1$, $\gamma-\alpha=\beta_1$, $\alpha-\beta=\gamma_1$, $V\beta\gamma=\alpha_2$, $V\gamma\alpha=\beta_2$, $V\alpha\beta=\gamma_2$, $V\alpha_1\alpha = \alpha_3$, $V\beta_1\beta = \beta_3$, $V\gamma_1\gamma = \gamma_3$;

and the six become

$$S.i\alpha\rho = 0, \quad S.i\alpha_1\rho - Si\alpha_2 = 0,$$
$$S.j\beta\rho = 0, \quad S.j\beta_1\rho - Sj\beta_2 = 0,$$
$$S.k\gamma\rho = 0, \quad S.k\gamma_1\rho - Sk\gamma_2 = 0.$$

The two in i give $\quad i \parallel \alpha S\alpha_2\rho - \rho(S\alpha\alpha_2 + S\alpha_3\rho)$.

Similarly, $\quad j \parallel \beta S\beta_2\rho - \rho(S\beta\beta_2 + S\beta_3\rho)$, and $k \parallel \gamma S\gamma_2\rho - \rho(S\gamma\gamma_2 + S\gamma_3\rho)$.

The conditions of rectangularity, viz.,

$$Sij = 0, \quad Sjk = 0, \quad Ski = 0,$$

at once give three equations of the fourth order, the first of which is

$$0 = S\alpha\beta\, S\alpha_2\rho\, S\beta_2\rho - S\alpha\rho\, S\alpha_2\rho\,(S\beta\beta_2 + S\beta_3\rho) - S\beta\rho\, S\beta_2\rho\,(S\alpha\alpha_2 + S\alpha_3\rho)$$
$$+ \rho^2 (S\alpha\alpha_2 + S\alpha_3\rho)(S\beta\beta_2 + S\beta_3\rho).$$

The required origin of the rectangular system is thus given as the intersection of three surfaces of the fourth order. How many such points of intersection there may be, I have not endeavoured to find.

2466. (Proposed by H. MURPHY.)—If four points A, B, C, D be either in the same plane or not, and if the three rectangles AB . CD, AC . DB, AD . BC be taken; the sum of any two of them is greater than the third, except when the points lie on the circumference of a circle.

Solution by PROFESSOR CAYLEY.

Write for shortness $BC = f$, $CA = g$, $AB = h$; $AD = a$, $BD = b$, $CD = c$; then, Lemma, if r be the radius of the sphere circumscribed about the tetrahedron ABCD, we have

$$4r^2\left\{\begin{array}{l} -a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2 \\ +(a^2f^2+b^2c^2)(g^2+h^2-f^2) \\ +(b^2g^2+c^2a^2)(h^2+f^2-g^2) \\ +(c^2h^2+a^2b^2)(f^2+g^2-h^2) \end{array}\right\} = \begin{array}{l} 2b^2c^2g^2h^2+2c^2a^2h^2f^2+2a^2b^2f^2g^2 \\ -a^4f^4-b^4g^4-c^4h^4, \end{array}$$

where the left-hand side is $=576V^2r^2$, if V be the volume of the tetrahedron.

Suppose first that the points are not in the same plane, then the left-hand side $(=576\,V^2r^2)$ is positive; therefore the right-hand side is also positive, or putting for shortness, $af = \alpha$, $bg = \beta$, $ch = \gamma$, we have

$2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4 = +$, that is, $4\beta^2\gamma^2-(\alpha^2-\beta^2-\gamma^2)^2 = +$, and thence $\alpha<\beta+\gamma$; for if α were equal to or greater than $\beta+\gamma$, say $\alpha = \beta+\gamma+x$, the left-hand side would be $4\beta^2\gamma^2-\left\{2\beta\gamma+2(\beta+\gamma)x+x^2\right\}^2$, which vanishes if $x = 0$, and is negative for x positive. Similarly $\beta<\gamma+\alpha$, $\gamma<\alpha+\beta$; and the theorem is thus proved for the case where the four points are not in a plane.

Starting from this general case, if we imagine the point D continually to approach and ultimately to coincide with the plane ABC, but so as not to be in the circle ABC, then the expression $2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4$, which does not vanish in the limit, is throughout equal to the positive quantity $576\,V^2r^2$ (in the limit V is $=0$ and $r = \infty$, but Vr is finite, and of course V^2r^2 is positive), that is, the expression in question is $= +$, and the theorem follows as before. Of course when the four points are in a circle, then the expression is $= 0$, and consequently one of the quantities α, β, γ is equal to the sum of the other two.

The lemma is at once proved by means of my theorem for the relation between the distances of five points in space, [*Cambridge Mathematical Journal*, Vol. II. (1841) p. 269,] viz., if the point 1 is the centre of the circumscribed sphere, and the points 2, 3, 4, 5, are the points A, B, C, D respectively, then the relation in question, viz.,

$$\left|\begin{array}{cccccc} 0, & (12)^2, & (13)^2, & (14)^2, & (15)^2, & 1 \\ (21)^2, & 0, & (23)^2, & (24)^2, & (25)^2, & 1 \\ (31)^2, & (32)^2, & 0, & (34)^2, & (35)^2, & 1 \\ (41)^2, & (42)^2, & (43)^2, & 0, & (45)^2, & 1 \\ (51)^2, & (52)^2, & (53)^2, & (54)^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{array}\right| = 0$$

becomes

$$\begin{vmatrix} 0, & r^2, & r^2, & r^2, & r^2, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Multiplying the last line by $-r^2$ and adding it to the first line, this is

$$\begin{vmatrix} -r^2, & 0, & 0, & 0, & 0, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

and then proceeding in the same way with the first and last columns, the equation is

$$\begin{vmatrix} -2r^2, & 0, & 0, & 0, & 0, & 1 \\ 0\ , & 0, & h^2, & g^2, & a^2, & 1 \\ 0\ , & h^2, & 0, & f^2, & b^2, & 1 \\ 0\ , & g^2, & f^2, & 0, & c^2, & 1 \\ 0\ , & a^2, & b^2, & c^2, & 0, & 1 \\ 1\ , & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is in fact the equation of the Lemma. See my papers in the *Quarterly Journal of Mathematics*, Vol. III. (1859), pp. 275—277, and Vol. V. (1861), pp. 381—384.

COR.—It appears by the demonstration that for any four points not in the same plane, the expression

$$-a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2+(a^2f^2+b^2c^2)(g^2+h^2-f^2)$$
$$+(b^2g^2+c^2a^2)(h^2+f^2-g^2)+(c^2h^2+a^2b^2)(f^2+g^2-h^2)$$

is always positive.

1496. (Proposed by MATTHEW COLLINS, B.A.)—Prove that a triangular pyramid whose vertices are A, B, C, D, and a parallelepiped formed from it as follows, have the same centre of gravity; viz., through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and BD; and the third pair of faces are obtained from the remaining two opposite edges AD and BC.

Solution by W. S. B. WOOLHOUSE, F.R.A.S.

Each pair of parallel planes determined as described in the Question are those which pass through two opposite edges of the tetrahedron, and the line of shortest distance between the edges is normal to the planes. The parallelism of the planes proves the figure to be a parallelepiped. Now, if with respect to the parallelepiped, planes be drawn through the centre of gravity parallel to each pair of opposite faces, they will manifestly bisect every line which connects those faces, and therefore they will respectively bisect the three lines which join the middle points of the opposite edges of the tetrahedron which lie in those faces. But the three last mentioned lines are known to mutually bisect one another in the centre of gravity of the tetrahedron. Therefore the tetrahedron and parallelepiped have the same centre of gravity.

2459. (Proposed by R. BALL, M.A.)—If ω be a cube root of unity, express the roots of the equation

$$ax^3+3bx^2+\frac{3}{a}\left\{b^2+\omega\,(ac-b^2)\right\}x+\frac{1}{a^2}\left\{a^2d+3\,(\omega-1)\,(ab\;-b^3)\right\}=0,$$

in terms of α, β, γ the roots of $ax^3+3bx^2+3cx+d=0$.

Solution by the REV. ROBERT HARLEY, F.R.S.

Writing in the first cubic X, and in the second X′, in place of $ax+b$, we have

$$X^3+3\omega\,(ac-b^2)\,X+a^2d-3abc+2b^3=0 \;\ldots\ldots\ldots\ldots (1),$$

$$X'^3+3\,(ac-b^2)\,X'+a^2d-3abc+2b^3=0 \;\ldots\ldots\ldots\ldots (2).$$

Now (2)−(1) gives $X'^3-X^3+3\,(ac-b^2)\,(X'-\omega X)=0$;

or $(X'-\omega X)\left\{X'^2-\omega XX'+\omega^2X^2+3\,(ac-b^2)\right\}=0$;

whence, either $X=\omega^2X'$,

or $X=\frac{1}{2}\omega^2\left\{X'+\sqrt{(-3)}\sqrt{[X'^2+4\,(ac-b^2)]}\right\}$.

The latter result is incongruous with the condition $\Sigma X=0$, and must therefore be rejected. The former gives

$$x=\omega^2x'-b\,(1-\omega\;),$$

where $x'=\alpha$, β or γ.

1805. (Proposed by Professor SYLVESTER.)—Prove that in general

$$\int_0^l dx\int_0^x dy\,F(x,y)=\int_0^l dx\int_0^x dy\,F(l-y,\,l-x)\ldots\ldots\ldots(1),$$

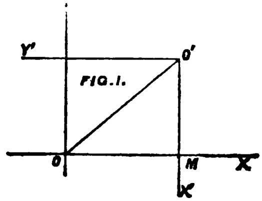
and hence show instantaneously that

$$\int_0^\pi d\theta\int_0^\theta d\theta'\,(\sin\theta\sin\theta')^{2i-1}\sin(\theta-\theta')=\frac{1\,.\,3\,.\,5\ldots\ldots(4i-1)}{2\,.\,4\,6\ldots\ldots 4i}\,\frac{\pi}{i}\ldots(3);$$

and (3) apply the latter integral to prove that the chance of four points taken at random in a circle forming the apices of a reentrant quadrilateral is $\frac{35}{12\pi^2}$.

Solution by THOMAS SAVAGE, M.A.

1. The first of the above integrals is the volume included between the surface $z = F(x, y)$, the plane of xy, and the three planes perpendicular to the plane of xy, and having as their traces on this plane the three lines $y = 0$, $y = x$, $x = l$. If we transfer the origin (Fig. 1) to a point (O′) whose coordinates are $(l, l, 0)$, and take for our new axes of x and y two lines (O′X′, O′Y′) parallel respectively to (OY, OX) but drawn in the opposite direction, it is clear that the equation to the surface becomes $z = F(l-y, l-x)$, and that to find the same volume as before, the limits of the integration with respect to y and x will be $(0, x)$, $(0, l)$ respectively. Hence the truth of the first equation is manifest.



2. A particular case of the above general formula is evidently

$$\int_0^{\pi} d\theta \int_0^{\theta} d\theta' (\sin\theta \sin\theta')^{2i-1} \cos\theta \sin\theta'$$

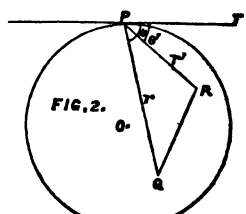
$$= -\int_0^{\pi} d\theta \int_0^{\theta} d\theta' (\sin\theta \sin\theta')^{2i-1} \sin\theta \cos\theta';$$

$$\therefore \int_0^{\pi} d\theta \int_0^{\theta} d\theta' (\sin\theta \sin\theta')^{2i-1} \sin(\theta-\theta')$$

$$= 2\int_0^{\pi} d\theta \int_0^{\theta} d\theta' (\sin\theta \sin\theta')^{2i-1} \sin\theta \cos\theta'$$

$$= \frac{1}{i}\int_0^{\pi} (\sin\theta)^{4i} d\theta = \frac{1.3.5....(4i-1)}{2.4.6....4i} \cdot \frac{\pi}{i}.$$

3. Let P, Q, R, S be four points taken at random within a circle whose centre is O. One of these four points (P suppose) must be more distant from the centre than the others. Then Q, R, S are three points taken at random within the circle whose radius is OP, and the chance that PQRS should be the apices of a reentrant quadrilateral is the chance that one of the three points Q, R, S should fall within the triangle formed by the other two and P, or is three times the average area of the triangle PQR divided by the area of the circle whose radius is OP. Referring the circle to polar coordinates, P being the pole, and the tangent at P the initial line, the average area of PQR is easily seen to be



$$\frac{\iiiint \frac{1}{2} rr' \sin(\theta-\theta') . rdrd\theta . r'dr'd\theta'}{\iiiint rdrd\theta . r'dr'd\theta'}$$

the limits of r, r', θ', θ being respectively

$(0, 2\text{OP} \sin\theta)$, $(0, 2\text{OP} \sin\theta')$, $(0, \theta)$, $(0, \pi)$.

Hence the average area of the triangle PQR is

$$\frac{\frac{1}{2}(\mathrm{OP})^6}{\frac{1}{2}\pi^2(\mathrm{OP})^4}\int_0^\pi d\theta\int_0^\theta d\theta'\,.\,\frac{64}{9}(\sin\theta\sin\theta')^3\sin(\theta-\theta')$$

$$=\frac{64(\mathrm{OP})^2}{9\pi^2}\cdot\frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}=\frac{35(\mathrm{OP})^2}{36\pi};$$

and the chance of a reentrant quadrilateral is

$$\frac{3}{\pi(\mathrm{OP})^2}\cdot\frac{35(\mathrm{OP})^2}{36\pi}=\frac{35}{12\pi^2}.$$

2356. (Proposed by CAPTAIN CLARKE, R.E., F.R.S.)—A tetrahedron is thrown into the air and a small shot fired through it; show that the chance of one of the faces (which is marked) being perforated, is $2F \div S$, where F is the area of the marked face and S the whole surface of the tetrahedron.

Solution by W. S. B. WOOLHOUSE, F.R.A.S.

The relation stated in the Question may be proved to be true generally for any polyhedron. If we conceive a large number of shots to be fixed through the polyhedron, and suppose the polyhedron to occupy every possible position with respect to the direction in which the shots are thrown, a little consideration will show that the number of shots which perforate two faces whose areas are F, F′ will be proportional to those areas. For, in integrating for the various positions of the plane of F it is evident that the plane of F′ will equally require to be integrated through the same positions; and as the orthographic projections are proportional to the areas, it is hence manifest that the resulting number of perforations must be also proportional to those areas. Hence if only one perforation were made, the chance of its being in the face F would be $F \div S$. But as two distinct faces are perforated by the shot, one in entering and another in quitting the polyhedron, the chance of the assigned face F being one of them is $2F \div S$.

1843. (Proposed by the EDITOR.)—1. Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle.

2. Four points being taken at random within a sphere, find the chance that the sphere drawn through them will lie wholly within the given sphere.

Solution by SAMUEL ROBERTS, M.A.

1. The centre of a circle through three favourable points must of course lie within the given circle, and any point within this circle may be taken as

the centre of a system of interior circles. Suppose we take any such point P, c being its distance from the centre, the radius of a circle about P must not exceed $r-c$, r being the radius of the given circle.

Now if a circle be drawn about P as centre, with the radius t, the points on its circumference can be combined together three and three in $\frac{4}{3}\pi^3 t^3$ ways, and $\int_0^{r-c} \frac{4}{3}\pi^3 t^3 dt$ gives the total number of such combinations relative to P. The probability required in this case is therefore

$$p = \left(\tfrac{8}{3}\pi^4 \int_0^r \int_0^{r-c} cdc\,.\,t^3dt\right) \div \left(\frac{\pi^3 r^6}{6}\right) = \frac{4\pi}{r^6}\int_0^r (r-c)^4\,cdc = \frac{2\pi}{15}.$$

2. Adopting a similar notation for the sphere, the number of favourable combinations relative to the point P is $\int_0^{r-c} \frac{32}{3}\pi^4 t^8 dt$, and consequently the probability required in this case is

$$p = \left(\tfrac{128}{3}\pi^5 \int_0^r \int_0^{r-c} c^2dc\,.\,t^8dt\right) \div \left(\frac{32\pi^4 r^{12}}{243}\right) = \frac{36\pi}{r^{12}}\int_0^r (r-c)^9\,c^2dc = \frac{3\pi}{55}.$$

[Mr. ROBERTS remarks that the problem may be readily extended to hyper-space of n dimensions; then, the content of the hyper-sphere being

$$\frac{\pi^{\frac{1}{2}n} r^n}{\Gamma(\frac{1}{2}n+1)}, \text{ the result is } \frac{n^n \pi^{\frac{1}{2}n}}{r^{n(n+1)}\Gamma(\frac{1}{2}n+1)}\int_0^r (r-c)^{n^2} c^{n-1} dc,$$

which includes all the cases for integral values of n.]

II. *Solution by* W. S. B. WOOLHOUSE, F.R.A.S.

1. The required chance being obviously the same for all circles, will remain unaltered if the given circle be augmented by a concentric annulus; and it will therefore follow that the same probability must result if the new cases thence arising be treated separately. It will be observed that the investigation is affected by this peculiarity, that the new cases which satisfy the proposed condition are principally derived from a conversion of a portion of the old combinations which were previously unfavourable. Thus the newly acquired favourable cases are those in which the circle passing through any three points passes into the annulus without going beyond it; whereas the total new cases are those in which one, or more, of the three points is situated in the annulus. If we now conceive the annulus to be diminished without limit, the total new cases will ultimately become those which have one of the points in the periphery of the given circle, and the additional favourable cases will be those in which the circle drawn through the three points is tangential to the given circle. To estimate these favourable cases, let R denote the radius of the given circle, and ρ that of the circle described through the three points; then, as the points may be anywhere in the circumference, the positions are $(2\pi\rho)^3$. Also, as the circles are in contact, the positions of

the centre, from ρ to $\rho+d\rho$, are $2\pi(R-\rho)\,d\rho$. Therefore the estimated number of favourable positions is

$$(2\pi)^4\int\rho^3d\rho\,(R-\rho) = (2\pi)^4\,R^5\left(\tfrac{1}{4}-\tfrac{1}{5}\right) = \tfrac{4}{5}\pi^4R^5.$$

The total number of new cases, bearing in mind that each of the three points may be in the periphery and the remaining two on the surface, is $3\,(2\pi R)\,.\,(\pi R^2)^2 = 6\pi^3R^5$. Hence by division the required probability is $\frac{2\pi}{15}$.

2. If four points be taken at random within a sphere, the chance that the sphere drawn through them shall be wholly within the given sphere may be obtained in a similar manner.

Here the estimated number of acquired favourable positions is

$$\int(4\pi\rho^2)^4\,.\,4\pi\,(R-\rho)^2d\rho = (4\pi)^5\int\rho^8d\rho\,(R^2-2R\rho+\rho^2)$$
$$= (4\pi)^5\,R^{11}\left(\frac{1}{9}-\frac{2}{10}+\frac{1}{11}\right) = \frac{(4\pi)^5}{495}\,R^{11}.$$

And the total number of new cases is

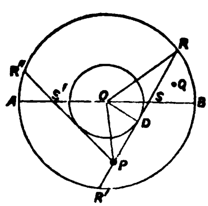
$$4\,(4\pi R^2)\,.\left(\frac{4}{3}\pi R^3\right)^3 = \frac{4^5\pi^4}{3^3}\,R^{11}.$$

Therefore the probability of the sphere being included within the given one is $\frac{3\pi}{55}$.

N.B.—This solution is only approximative, the estimated points not being in strictness equally distributed. An accurate solution of the question would be very complicated.

1849. (Proposed by Professor SYLVESTER.)—Two points are taken at random, one on each side of a given diameter of a circle; what is the probability that the chord drawn through them shall not exceed a given length?

Solution by STEPHEN WATSON.



Let AOB be the diameter, RR′ a chord of the given length, touching a concentric circle in D, cutting AB in S, and passing through P, one of the random points. Put $OB=a$, $OP=x$, $\angle\,ROD=\alpha$, $\angle\,POB=\phi$, $\angle\,DOP=\theta$; then the chord through PQ will be less than RR′ so long as Q lies within either of the spaces BSR, AS′R″, the area of the first of which is

$$BSR = \tfrac{1}{2}a^2\left\{\alpha+\theta-\phi+\cos^2\alpha\tan(\phi-\theta)-\sin\alpha\cos\alpha\right\}\ \ldots\ldots\ldots\ (1).$$

Now the result obtained by taking BSR alone must be doubled, because AS′R″ obviously passes through the same variations of area; also the limits of ϕ are from 0 to $\alpha+\theta$, of x from $a\cos\alpha$ to a, and (since $x\cos\theta=a\cos\alpha$) of θ from 0 to α; moreover an element of the circle at P is $xdxd\phi$, and the total number of positions of P and Q is expressed by $\frac{1}{4}a^4\pi^2$; hence the required chance is

$$p = \frac{8}{a^4\pi^2}\int_{a\cos\alpha}^{a}\int_{0}^{\alpha+\theta}(1)\; x\,dx\,d\phi$$

$$= \frac{2}{a^2\pi^2}\int_{a\cos\alpha}^{a}\left\{(\alpha+\theta)^2-(\alpha+\theta)\sin 2\alpha+2\cos^2\alpha\log\left(\frac{\cos\theta}{\cos\alpha}\right)\right\}x\,dx \quad \ldots\ldots (2)$$

Now remembering that $x\cos\theta = a\cos\alpha$, we have

$\int x\,dx = \frac{1}{2}x^2$, $\int \theta\, x\,dx = \frac{1}{2}(\theta x^2 - a^2\cos^2\alpha\tan\theta)$ ....(3, 4),

$\int \theta^2 x\,dx = \frac{1}{2}x^2\theta^2 - a^2\cos^2\alpha\,(\theta\tan\theta + \log\cos\theta)$ ....(5),

$\int \log\left(\frac{\cos\theta}{\cos\alpha}\right) x\,dx = \int\log\left(\frac{a}{x}\right)x\,dx = \frac{1}{2}x^2\log\left(\frac{a}{x}\right)+\frac{1}{4}x^2$ ....(6).

Taking (3), (4), (5), (6) between the above stated limits of x and θ, and substituting the results in (2), it becomes

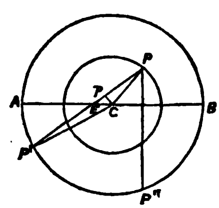
$$p = \frac{1}{\pi^2}\left\{\alpha(\alpha-\sin 2\alpha)(3+\sin^2\alpha)+\sin^2\alpha\cos^2\alpha\,(3-2\log\cos\alpha)\right\}.$$

[Putting, in Mr. Watson's general result, $\alpha = \frac{1}{6}\pi$, we find that the chance of the chord being not greater than the *radius* is

$$p = \frac{13}{144} - \frac{13\sqrt{3}}{48\pi} + \frac{3}{16\pi^2}\left(3+\log\frac{4}{3}\right), \text{ or } p = \cdot 00342, \text{ nearly.}$$

The value of the analogous chance when the two points are taken at random *anywhere within the circle* is ·0117, as shown in Professor SYLVESTER's solution of Question 1818, on p. 20 of Vol. V. of the *Reprint*.]

II. *Solution by* SAMUEL ROBERTS, M.A.



Let AB be the given diameter of a circle whose radius is unity. Describe an inner concentric circle, and let P be a point on its upper semi-circumference. Through P there can be drawn in the original circle two chords of a given length. Let PP′ be one of these chords, meeting the diameter at E, which must by the conditions of the question lie between the centre and A or B. For the particular position of P, AEP′ is the favourable area outside PP′. Now it may be inferred from the symmetry of the figure that the favourable areas outside PP″, the other chord, will be the same in inverted order as those outside PP′, so that we need only double the result relative to PP′ alone.

Draw Cp perpendicular to PP′, and let $\angle\,\text{PCB} = \phi$, $\angle\,\text{PC}p = \theta$, $\angle\,\text{P'C}p = \alpha$; then we have $\angle\,\text{ACP'} = \phi+\alpha+\theta-\pi$, $\angle\,\text{CEP} = \phi+\theta-\frac{1}{2}\pi$;

therefore area $\text{ACP'} = \dfrac{\phi+\alpha+\theta-\pi}{2} - \dfrac{\cos\alpha\sin(\phi+\alpha+\theta)}{2\cos(\phi+\theta)}$.

This expression is to be multiplied by $\rho d\rho d\phi$ and integrated as to ϕ between the limits $(\pi-\alpha-\theta, \pi)$ and as to ρ between the limits $(\cos\alpha, 1)$.

The first integration gives

$$\left\{\frac{(\alpha+\theta)^2}{4}-\frac{(\alpha+\theta)\cos\alpha\sin\alpha}{2}+\frac{\cos^2\alpha}{2}(\log\cos\theta-\log\cos\alpha)\right\}\rho d\rho.$$

Substituting for $\rho d\rho$ its equivalent $\frac{\cos^2\alpha\sin\theta\, d\theta}{\cos^3\theta}$, we have to integrate

$$\cos^2\alpha\int_0^\alpha\left\{\frac{\theta^2}{4}+\frac{\alpha-\cos\alpha\sin\alpha}{2}\theta+\frac{\alpha^2-2\alpha\cos\alpha\sin\alpha-2\cos^2\alpha\log\cos\alpha}{4}+\frac{\cos^2\alpha\log\cos\theta}{2}\right\}\frac{\sin\theta\, d\theta}{\cos^3\theta}.$$

But
$$\int\frac{\theta^2\sin\theta\, d\theta}{\cos^3\theta}=\frac{1}{2}\left(\frac{\theta^2}{\cos^2\theta}-2\theta\tan\theta-2\log\cos\theta\right),$$

$$\int\frac{\theta\sin\theta\, d\theta}{\cos^3\theta}=\frac{1}{2}\left(\frac{\theta}{\cos^2\theta}-\tan\theta\right),$$

$$\int\frac{\sin\theta\, d\theta}{\cos^3\theta}=\frac{1}{2\cos^2\theta},\quad\int\frac{\sin\theta\log\cos\theta\, d\theta}{\cos^3\theta}=\frac{1}{2}\left(\frac{\log\cos\theta}{\cos^2\theta}+\frac{1}{2\cos^2\theta}\right).$$

Taking these integrals between the specified limits, we get

$$\tfrac{1}{8}\left\{\alpha(\alpha-\sin 2\alpha)(3+\sin^2\alpha)+\sin^2\alpha\cos^2\alpha(3-2\log\cos\alpha)\right\}.$$

This must be multiplied by 2 and divided by $\frac{1}{4}\pi^2$, giving for the required probability the same value as in the foregoing solution.

2340. (Proposed by R. TUCKER, M.A.) – The circles of curvature at the extremities of a pair of semi-conjugate diameters of an ellipse are drawn; find (1) the envelop of their radical axis and (2) the curve upon which their points of intersection lie.

Solution by the PROPOSER; *the* REV. J. L. KITCHIN, M.A.; *and others.*

1. Let θ be the eccentric angle of the extremity of one diameter, then $x=a\cos\theta$ and $y=b\sin\theta$.

Now the equation to the circle of curvature is

$$(X-\alpha)^2+(Y-\beta)^2=\rho^2 \quad\text{.....................(1)},$$

but $\alpha=ae^2\cos^3\theta$, $\beta=-\frac{a^2e^2}{b}\sin^3\theta$, and $\rho=-\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{ab}$,

hence the equation (1) becomes

$$X^2+Y^2-2ae^2\cos^3\theta\,.\,X+\frac{2a^2e^2}{b}\sin^3\theta\,.\,Y=3a^2e^2\sin^2\theta+2b^2-a^2 \quad\text{......(2)}.$$

The equation to the other circle will be obtained by changing θ into $\frac{1}{2}\pi+\theta$, whence we obtain

$$X^2+Y^2+2ae^2\sin^3\theta \,.\, X+\frac{2a^2e^2}{b}\cos^3\theta \,.\, Y=3a^2e^2\cos^2\theta+2b^2-a^2 \quad \ldots(3).$$

The equation to radical axis of (2) and (3) is

$$bX(\cos^3\theta+\sin^3\theta)+aY(\cos^3\theta-\sin^3\theta)=\tfrac{3}{2}ab\cos 2\theta \ldots\ldots\ldots\ldots(4).$$

Differentiating, we have

$$bX(\cos\theta-\sin\theta)+aY(\cos\theta+\sin\theta)=2ab \ldots\ldots\ldots\ldots(5).$$

Put $\cos\theta-\sin\theta=\lambda$ and $\cos\theta+\sin\theta=\mu$; then (4) and (5) become

$$\mu bX(1+\lambda^2)+\lambda aY(1+\mu^2)=3\lambda\mu ab, \quad \lambda bX+\mu aY=2ab \ldots(6,7);$$

that is, making use of (7) in (6), the equation of the envelop is obtained by eliminating λ and μ between the three equations

$$\frac{bX}{\lambda}+\frac{aY}{\mu}=ab, \quad \lambda bX+\mu aY=2ab, \quad \lambda^2+\mu^2=2 \ldots\ldots(8,9,10).$$

Multiplying (8) by (10) and equating the result to (9), we get

$$-\frac{\lambda^3}{\mu^3}=\frac{bX}{aY},$$

hence we find that the curve required has for its equation

$$\left\{\pm(bX)^{\frac{2}{3}}\mp(aY)^{\frac{2}{3}}\right\}\left\{(bX)^{\frac{2}{3}}+(aY)^{\frac{2}{3}}\right\}^{\frac{1}{2}}=2^{\frac{1}{2}}ab.$$

2. To find the curve on which the points of intersection lie, we have to eliminate λ and μ between the two equations

$$ae^2X\lambda(1+\mu^2)-\frac{a^2e^2}{b}Y\mu(1+\lambda^2)=2(X^2+Y^2)-(a^2+b^2),$$

$$\mu b(1+\lambda^2)X+a\lambda(1+\mu^2)Y=3ab\lambda\mu, \quad \lambda^2+\mu^2=2,$$

the first equation being obtained by adding (2) and (3) together; that is, we have to eliminate λ and μ between three equations of the form

$$A\lambda(1+\mu^2)+B\mu(1+\lambda^2)=c, \quad A'\mu(1+\lambda^2)+B'\lambda(1+\mu^2)=c'\lambda\mu, \quad \lambda^2+\mu^2=2;$$

that is to say, between three equations of the form

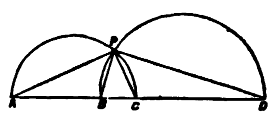
$$\kappa\mu(1+\lambda^2)=cB'-Ac'\lambda\mu, \quad -\kappa\lambda(1+\mu^2)=cA'-Bc'\lambda\mu, \quad \lambda^2+\mu^2=2 \ldots(11,12,13).$$

Squaring and adding (11) and (12) with the help of (13) we get a quadratic for the determination of $\lambda\mu$; hence λ, μ are known, and the curve can be determined.

2386. (Proposed by G. O. HANLON.)—If the side AD of a square be divided harmonically in B and C, and squares be placed on AB, BC, CD, falling without the square on AD, prove that the circles round the four squares pass through a common point.

Solutions by J. DALE; *the* PROPOSER; R. TUCKER, M.A.; *and others.*

1. Describe semicircles on AC, BD cutting in P, and join PB, PC; then as (P . ABCD) is a harmonic pencil, of which the conjugate rays PA, PC, and PB, PD are at right angles, therefore each of the



angles APB, BPC, CPD is half a right angle; and circles described on the segments AB, BC, CD on the side towards P, and on AD on the side opposite P, each containing half a right angle, will pass through P; and these circles will circumscribe the squares placed on AB, BC, CD, AD.

2. *Otherwise:* let ADEF be the square on AD; then FB and EC meet on the circle round ADEF, say in the point P. Then FPA, FPE, EPD are evidently each half a right angle, from which the proof immediately follows (that is, the circles all pass through P).

3. *Or again:* let AB = h, AC = k, AD = a; then, taking A as the origin of rectangular coordinates, the given line being the axis of x, the equations to the circles circumscribing the squares placed on AB, BC, CD, and AD towards the opposite side, are

$$x^2+y^2 = hx+hy \quad \text{.........} (1),$$
$$x^2+y^2 = (k+h)\,x+(k-h)\,y-hk \quad \text{.........} (2),$$
$$x^2+y^2 = (a+k)\,x+(a-k)\,y-ak \quad \text{.........} (3),$$
$$x^2+y^2 = ax-ay \quad \text{.........} (4).$$

The radical axes of (1), (2), (3) with respect to (4) are

$$(a-h)\,x-(a+h)y \qquad = 0,$$
$$(-a+h+k)\,x+(a-h+k)\,y-hk = 0,$$
$$kx+(2a-k)y \quad -ak = 0,$$

and the condition that these lines meet in a point reduces to $\frac{1}{a}+\frac{1}{h}=\frac{2}{k}$; therefore when h, k, a are in harmonical progression, the circles meet in a point.

2117. (Proposed by R. TUCKER, M.A.)—A point P is given (1) on the base of a triangle, (2) on a circle; draw a transversal through another given point Q to cut (1) or (2) so that the intercept may subtend a given or a maximum angle at P.

Solution by JAMES DALE.

1. Let QRS be the required transversal cutting the sides in R and S so that RPS is equal to the given angle. On PQ as diameter describe a circle, cutting the transversal in T; draw PD, PE perpendicular to AC, AB; and join PT, DE.

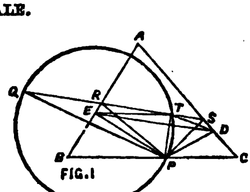


FIG. I

Then the angles SPT, RPT are respectively equal to ADT, AET,

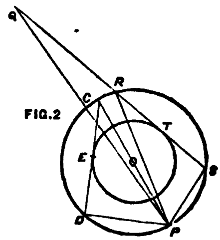
therefore $\angle$ BAC + ADT + AET = BAC + RPS = DTE = a constant.

Hence the following construction. From P draw perpendiculars on the sides of the triangle; on the line joining the feet of these perpendiculars describe a segment of a circle containing an angle equal to the sum of the given angle and BAC; and join Q to either of the points in which the circle is cut by the circle described on PQ as diameter. The maximum angle is the supplement of BAC, the arc DTE in this case coinciding with DE.

2. In the case of the circle, draw any chord CD so that CPD may be equal to the given angle; describe a concentric circle touching CD, and from Q draw a tangent to this latter circle, and cutting the given circle in RS. Then since chords equidistant from the centre of a circle are equal, and equal chords subtend equal angles, it follows that the angle RPS is equal to the given angle.

The maximum angle is equal to the angle in the segment formed by joining QP.

FIG. 2



2429. (Proposed by R. Tucker, M.A.)—An ellipse (A) and hyperbola (B) have the same transverse and conjugate diameters; prove that the polar of any point on (B) with reference to (A) touches (B).

I. *Solution by* James Dale.

Let a, b be the transverse and conjugate axes; then taking for axes of reference the asymptotes of the hyperbola, which are a pair of equal conjugate diameters of the ellipse, the equations of the two curves are

$$x^2+y^2=\tfrac{1}{2}(a^2+b^2),\ \ xy=\tfrac{1}{4}(a^2+b^2)\ \ldots\ldots\ldots\ldots\ldots\ (A, B).$$

The polar of any point (h, k) with respect to (A) is $hx+ky=\frac{1}{2}(a^2+b^2)$, and this polar touches B if $hk=\frac{1}{4}(a^2+b^2)$, that is, if (h, k) is a point on B; therefore (B) is its own reciprocal with respect to (A).

II. *Solution by the* Proposer.

Let the ellipse and hyperbola be given by the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ \ldots\ (A, B).$$

Now the polar of any point (x', y') on (B) with reference to (A) will have for its equation

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$

the envelope of which is readily found to be the hyperbola (B) or its conjugate.

If the conics be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \;\ldots\ldots (B),$$
$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0 \;\ldots\ldots (A),$$

the curve envelope will be given as the result of the elimination of (x', y') (a point on B) between (B) and

$$x'(a'x+h'y+g')+y'(h'x+b'y+f')+g'x+f'y+c'=0.$$

The resulting equation is

$$\begin{aligned} & x'\left[(ab'-hh')y+(ah'-a'h)x+af'-bg'\right] \\ +\, & y'\left[(hb'-h'b)y+(hh'-a'b)x+hf'-bg'\right] \\ +\, & (b'g-h'f)y+(h'g-a'f)x+f'g-fg'=0. \end{aligned}$$

2438. (Proposed by R. TUCKER, M.A.)—Two parabolas turned in opposite directions have the same axis and coincident vertices; if from the vertex as centre, and diameter a mean proportional between the parameters of the curves, a circle be described, prove that the polar of any point on one with reference to the circle touches the other.

I. *Solution by* W. H. LAVERTY; H. TOMLINSON; *and others.*

Let the equations to the three curves be

$$y^2=4ax,\;\; y^2=-4bx,\;\; x^2+y^2=4ab \;\ldots\ldots\ldots\ldots (1, 2, 3).$$

Then the polar of (x', y') with reference to (3) is

$$yy'=-xx'+4ab \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and if (4) is coincident with $y=mx-\dfrac{b}{m}$ which is a tangent to (2), we have $m=-\dfrac{x'}{y'}$, and $-\dfrac{b}{m}=\dfrac{4ab}{y'}$; or eliminating m, $y'^2=4ax'$, and (x', y') is a point on (1).

II. *Solution by the* PROPOSER.

Let $y^2=4ax$, and $x^2+y^2=r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (1, 2)$, be the equations to the parabola and circle; then the polar of any point (x', y') on (1) with reference to (2) will be given by

$$xx'+yy'=r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the envelope of which is readily found to be $y^2=-\dfrac{r^2}{a}x$, and the product of the parameters is $=4a\left(\dfrac{r^2}{a}\right)=4r^2$, which proves the property stated.

1985. (Proposed by MATTHEW COLLINS, B.A.)—Find what integral value of x renders $\frac{3x^7-29}{547}$ = a whole number, or else prove it impossible.

Solution by SAMUEL BILLS.

If $\frac{3x^7-29}{547}$ = a whole number be possible for any very *large* numbers, it is well known that it will also be possible for some either positive or negative values of x *less* than $\frac{547}{2}$, that is for some value of x not *greater* than 273.

Now by using various artifices and contractions, I have tried *all* the numbers for x, both positive and negative, as far as 273, and none of them satisfy the required condition; I therefore conclude that the relation is impossible.

2339. (Proposed by T. COTTERILL, M.A.)—1. If two triangles in a plane are similar, but right and left in position; then the lines drawn from the angular points of one triangle parallel, perpendicular, or isoclinal to the corresponding sides of the other triangle, meet in a point. Find its locus, if the first triangle remain fixed, while the second moves in any manner in the plane.

2. Analytically, we may assume the sums of the corresponding angles to vanish.

Solution by SAMUEL ROBERTS, M.A.

This proposition depends on the movement in rotation of the second triangle, and not on its translation in space. Moreover, if the lines corresponding to a given position of the second triangle are drawn (say) perpendicular to its sides, those lines will be parallel to sides of the second triangle in another position, and isoclinal in any position of that triangle. Let us take lines parallel to the sides of the second triangle. Denote the drawn lines by 1′, 2′, 3′, and the sides of the first triangle (fixed) by 1, 2, 3; then we have

$$\angle(1'2') = \angle(12), \quad \angle(2'3') = \angle(23), \quad \angle(3'1') = \pi - \angle(31),$$

supposing the figures are appropriated as the relative positions of the triangles require. It is very easy to see that the locus of the points is the circumscribing circle.

If we refer the triangles to two independent axes (x, y), we find that (A, B, C), (A′, B′, C′) take the forms $[\gamma-\beta, \pi-(\gamma-\alpha), (\beta-\alpha)]$, $[\beta-\gamma, \pi-(\alpha-\gamma), \alpha-\beta]$, and 2π produces the same results as 0.

2471. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Let a given surface, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation situated in its plane; and, corresponding to an abscissa x, let y, y' be the respective distances of the boundary above and below the axis; then, if h, k denote the radii of gyration round the axes, and M the total area, we shall have

$$h^2 = \int x^2 dx \frac{y+y'}{M}, \quad k^2 = \tfrac{1}{3}\int dx \frac{y^3+y'^3}{M}.$$

Let also $$A = \int x^2 dx \frac{y^3+y'^3}{M}, \quad B = \int x dx \frac{y\int y^2 dx + y'\int y'^2 dx}{M},$$

and $$C = \tfrac{1}{3}A + 3B; \text{ then}$$

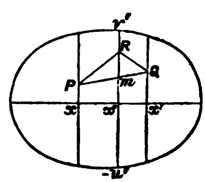
1. If *three* points be taken at random, on the given surface, the average area of the triangle connecting them, in parts of the total area, is $(\Delta) = \frac{h^2k^2 + C}{M^2}$; and the average square of the area, expressed in parts of the square of the total area, is $(\Delta^2) = \frac{3}{2}\frac{h^2k}{M^2}$.

2. If *four* points be taken at random, the probability that the quadrilateral shall be reentrant $= 4(\Delta)$.

3. If *five* points be taken at random on the surface, the probability of

a convex pentagon	=	$1 - 10(\Delta) + 10(\Delta^2)$,
one reentrant point	=	$10(\Delta) - 20(\Delta^2)$,
two reentrant points	=	$10(\Delta^2)$.

Solution by the Proposer.



In the direction of the axis of x the three points may be taken in the order P, R, Q, as in the diagram, since all other arrangements will eventually be included, if we afterwards suppose the position of the points to be permuted, and in consequence multiply the result by 6. Let the coordinates of P, Q, R be (x, y), (x', y'), (x'', y''), and let β denote the ordinate of the point m in which the side PQ is intersected by y'', produced if necessary; then we shall have

$$\beta = \frac{y(x'-x'') + y'(x''-x)}{x'-x}, \quad \Delta = \frac{x'-x}{2}(y''-\beta), \quad \int \Delta\, dy'' = \frac{x'-x}{4}(y''-\beta)^2.$$

When R is above the point m, or $y'' > \beta$, the integral gives $\frac{1}{4}(x'-x)(v''-\beta)^2$; and when R is below m, the integral, made also positive, is $\frac{1}{4}(x'-x)(u''+\beta)^2$, where v'', u'' refer to the upper and lower boundary. Therefore when R, or y'', varies from boundary to boundary, the value of the integral is

$$\int \Delta\, dy'' = \tfrac{1}{4}(x'-x)\left\{(v''-\beta)^2 + (u''+\beta)^2\right\}$$
$$= \tfrac{1}{4}(x'-x)\left\{v''^2 + u''^2 - 2(v''-u'')\beta + 2\beta^2\right\}.$$

To abbreviate, put $v''^2 + u''^2 = 2\gamma''^2$, $v'' - u'' = 2\epsilon''$, $v'' + u'' = 2\mu''$....(a);
then $\int \Delta\, dy'' = \frac{1}{2}(x'-x)\gamma''^2 - (x'-x)\beta\epsilon'' + \frac{1}{2}(x'-x)\beta^2 =$

$$\tfrac{1}{2}(x'-x)\gamma''^2 - \{y(x'-x'') + y'(x''-x)\}\epsilon'' + \frac{\{y(x'-x'') + y'(x''-x)\}^2}{2(x'-x)};$$

$$\therefore \iint \Delta\, dx''dy'' = \tfrac{1}{2}(x'-x)\int\gamma''^2 dx'' - (y'-y)\int\epsilon'' x'' dx'' - (yx'-y'x)\int\epsilon'' dx'' + \frac{\{y(x'-x'') + y'(x''-x)\}^3}{6(x'-x)(y'-y)}.$$

Now by hypothesis, as regards the abscissas, the point R lies between P and Q, so that the limits of x'' are x and x'. On taking the preceding integral between the limits, $x'' = x...x'$, the last term gives

$$\frac{y'^3(x'-x)^3 - y^3(x'-x)^3}{6(x'-x)(y'-y)} = \frac{(x'-x)^2}{6}(y'^2 + yy' + y^2).$$

Therefore, between the stated limits, we have

$$\iint \Delta\, dx''\, dy'' = \frac{x'-x}{2}\left(\int\gamma'^2 dx' - \int\gamma^2 dx\right) - (y'-y)\left(\int\epsilon' x' dx' - \int\epsilon x\, dx\right)$$
$$- (yx'-y'x)\left(\int\epsilon' dx' - \int\epsilon\, dx\right) + \frac{(x'-x)^2}{6}(y'^2 + yy' + y^2).$$

Again, the limits of y are $-u'....v'$;

$$\therefore \iiint \Delta\, dy'\, dx''\, dy'' = \mu'(x'-x)\left(\int\gamma'^2 dx' - \int\gamma^2 dx\right)$$
$$-2\mu'(\epsilon'-y)\left(\int\epsilon' x' dx' - \int\epsilon x\, dx\right)$$
$$-2\mu'(yx'-\epsilon' x)\left(\int\epsilon' dx' - \int\epsilon\, dx\right)$$
$$+2\mu'\frac{(x'-x)^2}{6}\left(\frac{v'^2 - v'u' + u'^2}{3} + \epsilon' y + y^2\right).$$

Also the limits of y are $-u....v$;

$$\therefore \iiiint \Delta\, dy\, dy'\, dx''\, dy'' = 4\mu\mu'\frac{x'-x}{2}\left(\int\gamma'^2 dx' - \int\gamma^2 dx\right)$$
$$-4\mu\mu'(\epsilon'-\epsilon)\left(\int\epsilon' x' dx' - \int\epsilon x\, dx\right)$$
$$-4\mu\mu'(\epsilon x' - \epsilon' x)\left(\int\epsilon' dx' - \int\epsilon\, dx\right)$$
$$+4\mu\mu'\frac{(x'-x)^2}{6}\left(\frac{v^2 - vu + u^2}{3} + \epsilon\epsilon' + \frac{v'^2 - v'u' + u'^2}{3}\right)$$

(or, putting $\frac{v^2 - vu + u^2}{3} = \lambda^2$, $\frac{v'^2 - v'u' + u'^2}{3} = \lambda'^2$, it becomes)

$$= 2\mu\mu'\left(x'\int\gamma'^2 dx' + x\int\gamma^2 dx\right) - 2\mu\mu'\left(x\int\gamma'^2 dx' + x'\int\gamma^2 dx\right)$$
$$- 4\mu\mu'\left(\epsilon'\int\epsilon' x' dx' + \epsilon\int\epsilon x\, dx\right) + 4\mu\mu'\left(\epsilon\int\epsilon' x' dx' + \epsilon'\int\epsilon x\, dx\right)$$
$$- 4\mu\mu'\left(\epsilon x'\int\epsilon' dx' + \epsilon' x\int\epsilon\, dx\right) + 4\mu\mu'\left(\epsilon' x\int\epsilon' dx' + \epsilon x'\int\epsilon\, dx\right)$$
$$+ \frac{2\mu\mu'}{3}\left\{(x^2\lambda'^2 + x'^2\lambda^2) + (x'^2\lambda'^2 + x^2\lambda^2) + \epsilon\epsilon'(x'^2 + x^2) - 2xx'(\lambda'^2 + \lambda^2) - 2xx'\epsilon\epsilon'\right\}.$$

This expression is a symmetrical function of x, x', and it is here purposely arranged in pairs of terms of the form

$$X = F'xf'x' + f'xF'x',$$

each of which pairs will, on further integration, give

$$\int X\,dx' = F'xfx' + f'x\,Fx' \quad (x' = x) \quad = F'xfx + Fxf'x,$$

and therefore $$\iint X\,dx\,dx' = Fx\,.fx.$$

Hence, by observing this formula, the remaining double integration of the foregoing expression is directly effected, and we get

$$\iiiint\!\!\iint \Delta\,dx\,dx'\,dy\,dy'\,dx'\,dy'' =$$

$$2\int\mu\,dx\,.\int(\mu x\,dx\int\gamma^2\,dx) - 2\int\mu x\,dx\,.\int(\mu\,dx\int\gamma^2\,dx)$$

$$-4\int\mu\,dx\,.\int(\mu\epsilon\,dx\int\epsilon x\,dx) + 4\int\mu\epsilon\,dx\,.\int(\mu\,dx\int\epsilon x\,dx)$$

$$-4\int\mu\epsilon\,dx\,.\int(\mu x\,dx\int\epsilon\,dx) + 4\int\mu x\,dx\,.\int(\mu\epsilon\,dx\int\epsilon\,dx)$$

$$+\tfrac{2}{3}\Big\{\int\mu x^2\,dx\,.\int\mu\lambda^2\,dx + \int\mu\,dx\,.\int\mu\lambda^2x^2\,dx + \int\mu\epsilon\,dx\,.\int\mu\epsilon\,x^2\,dx$$

$$-2\int\mu x\,dx\,.\int\mu\lambda^2x\,dx - \int\mu\epsilon x\,dx\,.\int\mu\epsilon x\,dx\Big\}\ldots\ldots(b).$$

According to the notation adopted in the question, M is the total area, and regarding the surface dynamically as a thin lamina, h, k are the radii of gyration round the axes;

$$\therefore\quad M = \iint dx\,dy = \int dx\,(v+u) = 2\int\mu\,dx,$$

$$M\,.\,h^2 = \iint x^2\,dx\,dy = \int x^2\,dx\,(v+u) = 2\int\mu x^2\,dx,$$

$$M\,.\,k^2 = \iint y^2\,dx\,dy = \tfrac{1}{3}\int dx\,(v^3+u^3) = 2\int\mu\lambda^2\,dx.$$

In all that precedes, the position of the coordinate axes is wholly unrestricted. If the origin be at the centre of gravity, we shall also have

$$0 = \iint x\,dx\,dy = \int x\,dx\,(v+u) = 2\int\mu x\,dx,$$

$$0 = \iint y\,dx\,dy = \tfrac{1}{2}\int dx\,(v^2-u^2) = 2\int\mu\epsilon\,dx.$$

Also, if the axes be principal axes of rotation, we shall have furthermore

$$0 = \iint xy\,dx\,dy = \tfrac{1}{2}\int x\,dx\,(v^2-u^2) = 2\int\mu\epsilon x\,dx.$$

The integral we have found, therefore, reduces down to the following:

$$\iiiint\!\!\iint(\quad) = M\int(\mu x\,dx\int\gamma^2dx) - 2M\int(\mu\epsilon\,dx\int\epsilon x\,dx) + \tfrac{1}{3}M^2h^2k^2 + \tfrac{1}{3}M\int\mu\lambda^2x^2dx.$$

But $\int(\mu\epsilon\,dx\int\epsilon x\,dx) + \int(\epsilon x\,dx\int\mu\epsilon\,dx) = \int\mu\epsilon\,dx\,.\int\epsilon x\,dx = 0$

gives $$\int(\mu\epsilon\,dx\int\epsilon x\,dx) = -\int(\epsilon x\,dx\int\mu\epsilon\,dx);$$

$$\therefore\ \iiiint\!\!\iint(\quad) = M\int(\mu x\,dx\int\gamma^2\,dx) + 2M\int(\epsilon x\,dx\int\mu\epsilon\,dx)$$

$$+\tfrac{1}{3}M^2h^2k^2 + \tfrac{1}{3}M\int\mu\lambda^2x^2\,dx$$

$$= \tfrac{1}{4}M\int\Big\{(v+u)\,x\,dx\int(v^2+u^2)\,dx\Big\} + \tfrac{1}{4}M\int\Big\{(v-u)\,x\,dx\int(v^2-u^2)\,dx\Big\}$$

$$+\tfrac{1}{6}M^2h^2k^2 + \tfrac{1}{18}M\int x^2\,dx\,(v^3+u^3)$$

$$= \tfrac{1}{2}M\int(vx\,dx\int v^2\,dx) + \tfrac{1}{2}M\int(ux\,dx\int u^2\,dx)$$

$$+\tfrac{1}{6}M^2h^2k^2 + \tfrac{1}{18}M\int x^2dx\,(v^3+u^3)\ldots(c).$$

By now permuting the three points, so as to obtain every position and combination, the sum of the areas is $6\iiiiiint(\ )$; and as each point traverses the area M, the number of positions is M^3. Therefore the average area is $\frac{6\iiiiiint(\ )}{M^3}$; and when expressed in parts of the total area, it is $(\Delta) = \frac{6\iiiiiint(\ )}{M^4}$. Hence, substituting y, y', respectively, for v, u, conformably with the notation in the question, we finally get

$$(\Delta) = \frac{3\int(ydx\int y^2dx) + 3\int(y'x\,dx\int y'^2dx)}{M^3} + \frac{h^2k^2}{M^2} + \frac{1}{3M^3}\int x^2\,dx\,(y^3+y'^3)$$

$$= \frac{3B}{M^2} + \frac{h^2k^2}{M^2} + \frac{1}{3}\cdot\frac{A}{M^2} = \frac{h^2k^2+C}{M^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d),$$

which establishes the formulæ first stated.

To determine (Δ^2), the average value of Δ^2 divided by M^2, we have

$$\Delta = \tfrac{1}{2}(x'-x)(y''-\beta) = \tfrac{1}{2}y''(x'-x) - \tfrac{1}{2}y(x'-x'') - \tfrac{1}{2}y'(x''-x)$$

$$= \tfrac{1}{2}y''(x'-x) - \tfrac{1}{2}x''(y'-y) + \tfrac{1}{2}(xy'-x'y)$$

$$\therefore\quad \Delta^2 = \tfrac{1}{4}x''^2(y'-y)^2 + \tfrac{1}{4}y''^2(x'-x)^2 + \tfrac{1}{4}(xy'-x'y)^2$$

$$+ Hx'' + Ky'' + Lx''y'' \ldots. (e),$$

where H, K, L are factors independent of the point (x'', y''), which factors will go out on integration with respect to that point. The average value of Δ^2 when one point $(x''y'')$ alone varies is

$$\Delta^2_{(1)} = \frac{\iint \Delta^2\,dx''\,dy''}{M} = \frac{\iint (e)\,dx''\,dy''}{M}.$$

Now, as h, k are the radii of gyration round the axes of x, y, we have

$$\iint x''^2\,dx''\,dy'' = M\,.\,h^2,\quad \iint y''^2\,dx''\,dy'' = M\,.\,k^2;$$

and, as the figure is referred to its centre of gravity and the principal axes of rotation through that centre as axes of coordinates, we shall also have

$$\iint x''\,dx''\,dy'' = 0,\quad \iint y''\,dx''\,dy'' = 0,\quad \iint x''y''\,dx''\,dy'' = 0.$$

Therefore from (e) we immediately find

$$\Delta^2_{(1)} = \tfrac{1}{4}h^2(y'^2-2yy'+y^2) + \tfrac{1}{4}k^2(x'^2-2xx'+x^2) + \tfrac{1}{4}(x^2y'^2-2xy\,x'y'+x'^2y^2)$$

$$\ldots\ldots (f).$$

Similarly the average value of Δ^2, when two points vary on the surface, is

$$\Delta^2_{(2)} = \frac{\iint (f)\,dx'\,dy'}{M} = \tfrac{1}{4}h^2(k^2_{*}+y^2) + \tfrac{1}{4}k^2(h^2_{*}+x^2) + \tfrac{1}{4}(k^2x^2_{*}+h^2y^2)$$

$$= \frac{h^2k^2}{2}\left(\frac{x^2}{h^2} + \frac{y^2}{k^2} + 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (g).$$

And, when all three points vary on the surface,

$$\Delta^2_{(3)} = \frac{\iint (g)\,dx\,dy}{M} = \frac{h^2k^2}{2}(1+1+1) = \frac{3}{2}h^2k^2 \ldots\ldots\ldots\ldots (h).$$

Hence, dividing by M^2, we get $(\Delta^2) = \frac{3}{2} \cdot \frac{h^2 k^2}{M^2}$ ....(k).

Now suppose *four* points P, Q, R, S to be taken on the surface. If the fourth point S be taken at random, since it may indiscriminately fall upon any point of the surface (M), the probability that it shall fall within the triangle PQR is evidently accurately expressed by the fraction $\frac{\text{triangle PQR}}{M}$. And when P, Q, R take all positions on the given surface, this probability obviously becomes $= \frac{\text{average triangle PQR}}{M} = (\Delta)$. Hence, as each point may be separately considered as the fourth, we have, generally,

$$
\begin{aligned}
(\Delta) &= \text{prob. of S falling in PQR,} \\
&= \quad ,, \quad \text{P} \quad ,, \quad \text{QRS,} \\
&= \quad ,, \quad \text{Q} \quad ,, \quad \text{RSP,} \\
&= \quad ,, \quad \text{R} \quad ,, \quad \text{SPQ.}
\end{aligned}
$$

These separate probabilities are identical in value, since the points admit of being permuted. The four conditions implied are also individually exclusive, and the sum of the values obviously makes up the complete probability that the quadrilateral shall be reentrant. Therefore if four points be taken at random on the given surface, the probability that the quadrilateral shall be reentrant $= 4(\Delta)$; that is, four times the average triangle expressed in parts of the total area.

Lastly, suppose that *five* points P, Q, R, S, T are taken at random on the given surface. Four assigned points P, Q, R, S will be the corners of a quadrilateral that is either convex or reentrant. If PQRS be convex, its area will be half the sum of the component triangles, viz.

$$\text{area} = \tfrac{1}{2}(\text{PQR} + \text{QRS} + \text{RSP} + \text{SPQ}) \text{ ....} (l),$$

and the fifth point T will be reentrant if it fall within this area. If PQRS be a reentrant quadrilateral, the area expressed by (l) will be that of the triangle which includes the reentrant point; and again the fifth point T will also be reentrant if it fall within this particular area. Hence the probability of the fifth point T being reentrant is in all cases equal to the area (l) divided by the total area; and when the points P, Q, R, S severally take all positions on the surface, this last-mentioned quotient becomes $= 2(\Delta)$. The other four points having each of them the same probability of being reentrant, we find by collecting these several probabilities that the total probability that one of the points shall be reentrant plus twice the probability that a pair of points shall be reentrant is equal to $10(\Delta)$.

Again, the probability that S, T shall be a pair of reentrant points is evidently $\frac{\text{PQR}}{M} \times \frac{\text{PQR}}{M} = \frac{(\text{PQR})^2}{M^2}$. When P, Q, R take all positions, this becomes $= (\Delta^2)$; and a like probability will obtain for every pair of the five points. Therefore the probability of a pair of reentrant points is $= 10(\Delta^2)$.

Hence also the probability of only one reentrant point $= 10(\Delta) - 20(\Delta^2)$. And the probability of a convex pentagon $= 1 - 10(\Delta) + 10(\Delta^2)$.

NOTE.—When the axis of x is diametral, then

$$h^2 = \frac{2}{M}\int yx^2 dx, \qquad k^2 = \frac{2}{3M}\int y^3 dx;$$

$$A = \frac{2}{M}\int y^3 x^2 dx, \qquad B = \frac{2}{M}\int yx dx \int y^2 dx.$$

And in the formula for B, since $\int yxdx \, . \, c = 0$, the inner integral may be estimated from any epoch.

If the axis of y be also diametral, the calculation of the integrals may be further simplified by taking them for only one quadrant and doubling the results just stated.

Since the values depend upon the distribution of areas alone, it is evident that they will remain unchanged when the figure is modified in any way by orthogonal projection.

As examples of the practical application of the formulæ, a table is annexed showing the numerical results for four principal figures.

EXAMPLES.

Form of Boundary.	Triangle.	Square or Rectangle.	Regular Hexagon.	Circle or Ellipse.
Value of (Δ)	$\frac{1}{12}$	$\frac{11}{144}$	$\frac{289}{3888}$	$\frac{35}{48\pi^2}$
Value of (Δ^2)	$\frac{1}{72}$	$\frac{1}{96}$	$\frac{25}{2592}$	$\frac{3}{32\pi^2}$
Prob. of a reent. quadrilateral ..	$\frac{1}{3}$	$\frac{11}{36}$	$\frac{289}{972}$	$\frac{35}{12\pi^2}$
Prob. of a convex quadrilateral ..	$\frac{2}{3}$	$\frac{25}{36}$	$\frac{683}{972}$	$1-\frac{35}{12\pi^2}$
Prob. of a convex pentagon	$\frac{11}{36}$	$\frac{49}{144}$	$\frac{1373}{3888}$	$1-\frac{305}{48\pi^2}$
Prob. of one reentrant point....	$\frac{20}{36}$	$\frac{80}{144}$	$\frac{2140}{3888}$	$\frac{260}{48\pi^2}$
Prob. of two reentrant points ..	$\frac{5}{36}$	$\frac{15}{144}$	$\frac{375}{3888}$	$\frac{45}{48\pi^2}$
Value of (Δ)	·08333	·07639	·07433	·07388
Value of (Δ^2)	·01389	·01042	·00964	·00950
Prob. of a reent. quadrilateral ..	·3333	·3056	·2973	·2955
Prob. of a convex quadrilateral..	·6667	·6944	·7027	·7045
Prob. of a convex pentagon	·3056	·3403	·3531	·3562
Prob. of one reentrant point. ..	·5555	·5555	·5504	·5488
Prob. of two reentrant points ..	·1389	·1042	·0965	·0950

2472. (Proposed by Professor CAYLEY.)—Through four points on a circle to draw a conic, such that an axis may pass through the centre of the circle.

Solution by the PROPOSER.

Let the equation of the conic be $(a, b, c, f, g, h)(x, y, 1)^2 = 0$, then if as usual the inverse coefficients are represented by (A, B, C, F, G, H), the equation of the two axes is

$$(a-b)(Cx-G)(Cy-F)+h[(Cx-G)^2-(Cy-F)^2] = 0,$$

whence if an axis pass through the origin

$$(a-b)\,FG+h(G^2-F^2) = 0.$$

Consider now the circle $x^2+y^2-1 = 0$ and on it the four points in which it is intersected by the conic $(a, b, c, f, g, h)(x, y, 1)^2 = 0$; then for any conics through the four points we have

$$(a, b, c, f, g, h)(x, y, 1)^2+\lambda(x^2+y^2-1) = 0;$$

so that, taking this for the equation of the required conic, and representing it by

$$(a', b', c', f', g', h')(x, y, 1)^2 = 0,$$

the values of the coefficients are

$$a' = a+\lambda,\ b' = b+\lambda,\ c' = c-\lambda,\ f' = f,\ g' = g,\ h' = h,$$

and we thence have

$$F' = F-\lambda f,\ \ G' = G-\lambda g,\ \ a'-b' = a-b,\ \ h' = h.$$

The required relation is $(a'-b')\,F'G'+h'(G'^2-F'^2) = 0$,

that is $(a-b)(F-\lambda f)(G-\lambda g)+h\left\{(G-\lambda g)^2-(F-\lambda f)^2\right\} = 0$,

a quadric equation in λ; and substituting for λ each of its two values, we have the two required conics

$$(a, b, c, f, g, h)(x, y, 1)^2+\lambda(x^2+y^2-1) = 0,$$

for each of which an axis passes through the centre of the circle.

2473. (Proposed by Professor SYLVESTER.)—Show (1) how to plant 81 trees so as to form 800 rows of 3 trees in a row; (2) how to plant 10 trees so as to form 10 rows of three trees in a row.

Solution by the PROPOSER.

1. Take any cubic curve and P any point on it. Draw PQ a tangent at P cutting the curve in Q, QR a tangent at Q cutting the curve in R, join PR cutting the curve in S, QS cutting it in T, PT cutting it in U, QU cutting it in V, and so on, until m points are obtained at which trees are supposed to be planted. Then according as m is of the form $4k+3$, $4k+2$, $4k+1$, $4k$, the number (μ) of collineations between the points taken 3 and 3 together may, by the Theory of Unicursal Derivation (laid before the London Mathematical Society by the Proposer), be shown to be $2k^2+2k$, $2k^2+k$, $2k^2$, $2k^2-k$ respectively. In the first case proposed, we have $k = 20$, and $m = 4k+1$; therefore $\mu = 2k^2 = 800$.

2. Another interesting case pointed out by Professor CAYLEY is where $k = 2$, $m = 4k+2 = 10$, and $\mu = 2k^2+k = 10$. This is the case of 10 trees

planted so as to form 10 rows, a variation upon the old problem of planting 9 trees so as to form 9 rows.

NOTE.—If we give to P, Q, R, S, T, &c., the names 1, 2, 4, 5, 7, &c., leaving out multiples of 3, then any 3 points whose numerical names λ, μ, ν satisfy the equation $\lambda^4+\mu^4+\nu^4-2\lambda^2\mu^2-2\mu^2\nu^2-2\nu^2\lambda^2=0$ will lie in the same straight line.

[In the *Mathematician*, Vol. II., p. 278, Mr. WOOLHOUSE has shown that 15 points may be placed so as to form 26 rows of 3 in a row.]

1927. (Proposed by W. S. BURNSIDE, M.A.)—To find the conic of least eccentricity that can be drawn through four given points.

I. *Solution by* H. R. GREER, B.A.

Let the points be defined by the intersections of two conics U and V which are given in the form $\dfrac{l}{x}+\dfrac{m}{y}+\dfrac{n}{z}=0$, and $\dfrac{l'}{x}+\dfrac{m'}{y}+\dfrac{n'}{z}=0$; and let the coordinates of their fourth point of intersection be proportional to $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$; so that we shall have $l\alpha+m\beta+n\gamma=0$, $l'\alpha+m'\beta+n'\gamma=0$. Now it is demonstrable that there exists a method of geometrical derivation according to which a line being represented by a conic through three fixed points, the line at infinity shall be represented by the circle through the same; and furthermore, that if the perpendiculars from any point on the sides of the fixed triangle be (α, β, γ), those from the corresponding point will be proportional to $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$; and also, that the angle between the asymptotes of any conic through the fundamental points will be equal to that in the segment cut off from the circumscribing circle by the line corresponding to the conic. Assuming these principles, which are built upon a particular method of *Quadric Inversion*, it is evident that, in the case before us, we must proceed as follows. Form the equation of the shortest chord of the circumscribing circle through (α, β, γ), and in the result write for (x, y, z) their reciprocals. Now the equation of the line joining (α, β, γ) to the centre of the circumscribing circle is $\begin{vmatrix} x & \alpha & \cos A \\ y & \beta & \cos B \\ z & \gamma & \cos C \end{vmatrix}=0$, say $Lx+My+Nz=0$; and that of the line through (α, β, γ) perpendicular to this is, say, $\begin{vmatrix} x & \alpha & \lambda \\ y & \beta & \mu \\ z & \gamma & \nu \end{vmatrix}=0$, where $\lambda=L-M\cos C-N\cos B$, and cyclically for μ and ν. But $L=\beta\cos C-\gamma\cos B$, and cyclically for M and N; therefore, after a little reduction,

$$\lambda=\alpha(\sin^2 B-\sin^2 C)+\beta\,.\sin A\sin B-\gamma\sin A\sin C;$$

therefore, writing $\sin A : \sin B : \sin C = a : b : c$, the equation of the required shortest chord is

$$\begin{vmatrix} x & \alpha & \alpha(b^2-c^2)+ab\beta-ca\gamma \\ y & \beta & \beta(c^2-a^2)+bc\gamma-ab\alpha \\ z & \gamma & \gamma(a^2-b^2)+ca\alpha-bc\beta \end{vmatrix} = 0.$$

The equation of the sought conic is got, consequently, by writing in the above for x, y, z, yz, zx, xy respectively; and for α, β, γ, $mn'-m'n$, &c. &c.

II. *Solution by the* Proposer.

If the four points do not form a reentrant quadrilateral, an ellipse can be described through the points. This being so, let

$U \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0$, and $V = (a_1, b_1, c_1, f_1, g_1, h_1)(x, y, z)^2 = 0$

be the equations of any two conics passing through these points, then the equation of any other conic of the system is of the form $U + kV = 0$.

Now, put $A = bc - f^2$, &c.,

$$\theta \equiv A \sin^2 A + B \sin^2 B + C \sin^2 C + 2F \sin B \sin C + 2G \sin C \sin A + 2H \sin A \sin B,$$

$$\theta' \equiv a + b + c - 2f \cos A - 2g \cos B - 2h \cos C;$$

also let θ_1 and θ'_1 be the similar quantities relative to V, and Θ, Θ' relative to $U + kV$. Then we have

$$\Theta \equiv \theta + k\phi + k^2\theta_1, \quad \Theta' \equiv \theta' + k\theta'_1,$$

where $\phi \equiv (bc_1 + b_1c - 2ff_1)\sin^2 A + (ca_1 + c_1a - 2gg_1)\sin^2 B$
$+ (ab_1 + a_1b - 2hh_1)\sin^2 C + 2(gh_1 + g_1h - af_1 - a_1f)\sin B \sin C$
$+ 2(hf_1 + h_1f - bg_1 - b_1g)\sin C \sin A + 2(fg_1 + f_1g - ch_1 - c_1h)\sin A \sin B.$*

Now, from the identity $(a, b, c, f, g, h)(x, y, z)^2 \equiv \lambda(X^2 + Y^2 - e^2Z^2)$, where $Z = 0$ is the equation of the directrix, and e the eccentricity, we have

$\theta' = \lambda(2 - e^2)$, $\theta = \lambda^2(1 - e^2)$, giving $\dfrac{\theta'^2 - 4\theta}{\theta} = \dfrac{e^4}{1 - e^2}$; whence we determine k, so that the eccentricity of the conic $U + kV = 0$ shall be a minimum by equating to zero the differential of $\dfrac{\Theta'^2}{\Theta}$ with respect to k. This being done, we find

$$(\theta' + k\theta'_1)(\phi + 2k\theta_1) = 2\theta'_1(\theta + k\phi + k^2\theta_1),$$

or $\theta'\phi - 2\theta\theta'_1 = k(\theta'_1\phi - 2\theta'\theta_1)$, which determines k when the eccentricity e of the conic $U + kV = 0$ is a minimum.

Note.—The condition that the four points ($U = 0$, $V = 0$) lie on a circle may be readily expressed in terms of $\theta, \theta', \theta_1, \theta'_1, \phi$, as follows:—

In general if $U = 0$ represents a circle $\theta'^2 = 4\theta$; therefore when $U + kV = 0$ is the equation of a circle, $k^2(\theta_1'^2 - 4\theta_1) + 2k(\theta'\theta'_1 - 2\phi) + \theta'^2 - 4\theta = 0$ determines k, and it is plain that k can have but one value, whence the condition required is

$$(\theta'^2 - 4\theta)(\theta_1'^2 - 4\theta_1) = (\theta'\theta'_1 - 2\phi)^2.$$

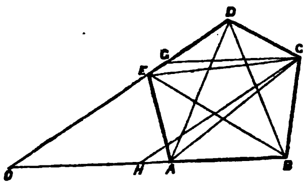
[* The geometrical meaning of ϕ is that the conics $U = 0$, $V = 0$ cut the line at infinity harmonically.]

2418. (Proposed by Dr. SHAW.)—If the vertices of a pentagon be A, B, C, D, E; and if a, b, c, d, e be the areas of the triangles EAB, ABC, BCD, CDE, DEA; show that the area P of the pentagon is given by the quadratic equation

$$P^2-(a+b+c+d)\,P+(ab+bc+cd+de+ea)=0.$$

I. *Solution by* STEPHEN WATSON.

Produce any two of the sides, as BA, DE, to meet in O, and draw CG parallel to AB and CH to DE. Put $OA=a_1$, $OB=a_2$, $OE=b_1$, $OD=b_2$, $CG=\alpha$, $CH=\beta$, and $\sin\angle O=m$. Then



$$P=\tfrac{1}{2}m\,(a_2\beta+b_2\alpha-a_1b_1)$$
$$=\tfrac{1}{2}m\,(a_2b_2-a_1b_1)+e\ldots\ldots(1).$$

Also
$$\left.\begin{array}{l} a=\tfrac{1}{2}mb_1\,(a_2-a_1)\\ b=\tfrac{1}{2}m\beta\,\,(a_2-a_1)\\ d=\tfrac{1}{2}m\alpha\,\,(b_2-b_1)\\ e=\tfrac{1}{2}ma_1\,(b_2-b_1)\end{array}\right\}\ldots\ldots(2).$$

From (2) we easily get

$$a_1=\frac{ea}{d},\ b_1=\frac{a\beta}{b},\ a_2=\frac{ea}{d}+\frac{2b}{m\beta},\ b_2=\frac{a\beta}{b}+\frac{2d}{m\alpha};$$

hence by substitution (1) becomes

$$P=b+d+\tfrac{1}{2}\left(\frac{ad+be-ae}{bd}\right)\alpha\beta m=a+c+e+\frac{2bd}{\alpha\beta m};$$

whence eliminating $\alpha\beta m$, we have, as required,

$$P^2-(a+b+c+d+e)\,P+(ab+bc+cd+de+ea)=0.$$

II. *Solution by* JAMES DALE.

The six angles made at A by the lines AB, AC, AD, AE have their sines connected by the following well-known relation,

$$\sin BAC\,.\,\sin DAE+\sin BAE\,.\,\sin CAD=\sin BAD\,.\,\sin CAE;$$

whence, multiplying each term by ($\tfrac{1}{4}$AB . AC . AD . AE), we get

$$\triangle ABC\,.\,\triangle ADE+\triangle ABE\,.\,\triangle ACD=\triangle ABD\,.\,\triangle ACE;$$

or putting a_1, b_1, c_1, d_1, e_1 for the areas of the triangles CAD, DBE, ECA, ADB, BEC, we have the following conditions:—

$$aa_1+be=c_1d_1\ldots\ldots(1)\qquad a_1+b+e=P\ldots\ldots(6)$$
$$bb_1+ca=d_1e_1\ldots\ldots(2)\qquad b_1+c+a=P\ldots\ldots(7)$$
$$cc_1+db=e_1a_1\ldots\ldots(3)\qquad c_1+d+b=P\ldots\ldots(8)$$
$$dd_1+ec=a_1b_1\ldots\ldots(4)\qquad d_1+e+c=P\ldots\ldots(9)$$
$$ee_1+ad=b_1c_1\ldots\ldots(5)\qquad e_1+a+d=P\ldots\ldots(10).$$

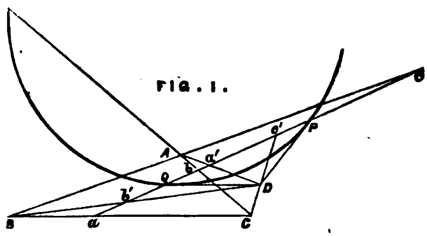
From (8) and (9),
$$c_1d_1=P^2-(b+c+d+e)\,P+(bc+cd+de+be)$$
$$=aa_1+be,\ \text{by (1)}$$
$$=aP-ab-ea+be,\ \text{by (6)};$$

therefore
$$P^2-(a+b+c+d+e)\,P+(ab+bc+cd+de+ea)=0.$$

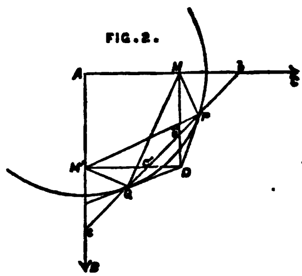
[For another solution see Hymer's *Trigonometry*, 3rd ed., p. 146.]

2376. (Proposed by T. Cotterill, M.A.)—1. If ABC is a self-conjugate triangle of a conic, to which DP, DQ are tangents at P and Q, then any conic through ABCD cuts PQ harmonically. 2. Hence, if DM, DM′ are the perpendiculars on the axes, DM, DM bisect the angles PMQ, PM′Q.

Solution by James Dale.



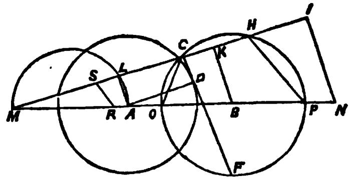
1. Let the chord PQ cut BC, CA, AB in a, b, c; join DA, DB, DC cutting PQ in a', b', c'; then the points aa', bb', cc' form a system in involution; and as each of the triangles Daa', Dbb', Dcc' is self-conjugate with respect to the conic, it follows that P, Q are the double points of the system $aa'\,bb'\,cc'$. If now any conic through ABCD cut PQ in P′, Q′, then these points also belong to the system $aa'\,bb'\,cc'$, and therefore form a harmonic range with the double points P, Q.



2. Suppose now that AB, AC are the axes, then BC passes to infinity. Let PQ cut the axes in b, c, and DM, DM′ perpendicular respectively to AC, AB cut PQ in b', c'; join PM, QM; PM′, QM′; then PQ is divided harmonically in b, b'; and as the conjugate rays Mb, M′b' are at right angles, it follows that MP, MQ make equal angles with MD; so also M′C, M′C′ make equal angles with M′D.

1973. (Proposed by the Rev. R. Townsend, M.A.)—Draw a chord of one of two given circles so as to touch and be cut in a given ratio at its point of contact with the other.

Solution by William Hopps.



Let A, B be the centres of the given circles, and suppose the tangent to the circle (A) at D, to meet the circle (B) at C and F, so that CD : DF $=m:n$, the given ratio. Draw AB, AD, and through C draw a line parallel to AD to meet BA produced at M, and the circle (B) at H; also draw AL, BK each perpendicular to CF. Then CD : DF $=m:n$, by hypothesis,

therefore CD : (CD + DF) $=$ AL : CF $=m:(m+n)$,

therefore AL : $\frac{1}{2}$ CF (or BK) $=$ MA : MB $=m:\frac{1}{2}(m+n)$;

hence MA : MB $-$ MA (or AB) $=m:\frac{1}{2}(n-m)=$ a given ratio.

But AB is given, therefore MA is also given, and likewise a semicircle thereon passes through L, because MLA is a right angle; moreover LC=AD, which is given. We have, therefore, to draw a line from M to cut the semicircle on MA in L, and the circle (B) in C, so that the part LC intercepted by their circumferences shall be equal to AD, the radius of the given circle (A).

Produce MH until HI=LC or AD; at I draw a line perpendicular to MI to meet AB produced at N, and let O, P be the intersections of the circle (B) with MN. Then HI=LC and KH=CK; hence KI=LK, and therefore BN=AB; consequently N is a given point. Draw CO, HP, and divide MA at R, so that MN : MP=MA : MR; also draw RS parallel to HP, as in the diagram. Then we have

$$MI : MN = ML : MA,\ MP : MH = MR : MS,\ MN : MP = MA : MR\ (\text{Con.});$$

hence compounding these equal ratios we get

$$MI : MH = ML : MS, \text{ therefore } HI \text{ (or } LC) : MH = SL : MS;$$

therefore $$LC : SL = MH : MS = MP : MR.$$

Now LC, MR, MP are given, therefore SL is also given, consequently SC or MC−MS is given. But ∠ MRS=MPH=MCO, hence S, R, O, C are in the circumference of a circle, therefore MC . MS=MO . MR=a given space. That is, the difference and rectangle of MC, MS are given to determine them, which is a well-known elementary problem.

It may be observed that to draw from M a line to cut the semicircle on MA in L, and meet the circle B at C, so that LC shall be of given length, is a problem in Apollonius's lost treatise *De Inclinationibus*, restored by Ghetaldus and others. (See Bonnycastle's *Geometry*, 6th edition, p. 367; also Hutton's or Barlow's *Mathematical Dictionary*, under the name Apollonius.)

Note on a Criterion of Divisibility. By M. Jenkins, B.A.

The rule for finding a criterion of divisibility by continued subtraction of digits, beginning at the left hand (given on p. 69 of this volume of the *Reprint*), may give rise to inconveniently high numbers to be considered as digits of the scale of notation: *e.g.* if the digits situated at the requisite interval be 9, 1, 9, 1, 9, 1, &c. The continued subtraction then gives successively −8, +17, −16, +25, &c.

The rule, however, may be put in a different form, and generalized thus:—

Let r be the base of the scale of notation; p the proposed divisor, supposed prime to r; N the dividend, a criterion of whose divisibility by p is required; and let $r^T - a$ be divisible by p, or $r^T \equiv a \pmod{p}$, where a may be supposed positive or negative, but of absolute magnitude not greater than $\frac{1}{2}p$; and T may be taken to be the least value of the index consistent with the congruence.

Mark off from the right hand of N divisions of T digits, and let $s_n, s_{n-1}, \ldots s_1, s_0$ be the magnitudes of the divisions standing by themselves (s_n, the extreme left-hand division, consisting possibly of less than T digits): then $s_n a^n + s_{n-1} a^{n-1} + \ldots + s_1 a + s_0$ is a criterion of the divisibility of N by p.

This criterion might be expressed to be the magnitude of the divisions considered as digits in the scale of a.

The correctness of the rule is evident from the fact that if s'_m be the local magnitude of the $(m+1)$th division,

$$s'_m = s_m r^{mT}; \text{ and therefore } s'_m \equiv s_m a^m \text{ [mod. } (r^T - a)\text{]};$$

therefore $N \equiv s_n a^n + \dots + s_1 a_1 + s_0$ [mod. $(r^T - a)$]; *a fortiori* (mod. p).

Let t be the number of digits in the period of p^{-1}, expressed as an r-mal: then if $T = t$, $a = +1$; and we have the rule, "Add all the divisions together for a criterion."

If t be even, and $T = \frac{1}{2}t$, $a = -1$; then the rule becomes:—Subtract the sum of the divisions of even rank from the sum of those of uneven rank; the difference with the proper sign is congruential to the given number N.

The analogy of these two rules to the ordinary ones for 9 and 11 is obvious: by the continued application of them a number of not greater than t or $\frac{1}{2}t$ digits, as the case may be, can always, if t be known, be speedily found congruential to N. But the utility of the further application of the general rule to a number of t' digits, where t' is not $> t$ (t odd) or $\frac{1}{2}t$ (t even) is the probability of finding a congruence $r^T \equiv a$ (mod. p) such that a and the greatest integer in $t' \div T$ shall both be small.

Examples.—Find a criterion of divisibility by 7 and 13, and one for divisibility by 37 of

57, 002 | 637, 095 | 430, 019

	430,	019
	637,	095
	57,	002
$10^3 \equiv -1$ (mod. 7 or 13)	1124	116
	116	
	1,008	
	1	
	7	

The given number is therefore divisible by 7, and leaves a least residue -7, or a remainder 6, after division by 13. For the divisor 37, we have, since $10^3 \equiv +1$ (mod. 37),

$$N \equiv 1124 + 116 \equiv 124 \times 10 \equiv 13 \times 10 \equiv 19 \text{ (mod. 37).}$$

Find a criterion of divisibility by 17 of

53 | 7826 | 5918 | 4379 | 6210 | 4389

6210	4389
+53	7826
6264	12215
−5918	4379
~~3~~,45	~~78~~,36
−6	−156
39	−120
	156
	36

The original number is first reduced to 8 digits, since $10^8 \equiv -1$ (mod. 17); then, since $10^2 \equiv -2$, and $10^4 \equiv +4$ (mod. 17), we multiply 3 by -2 and add to 45; 78 by -2 and add to 36; 39 by 4 and add to -120; this process gives the criterion 36.

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