

# Flowsnake Earth

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## Abstract

By folding and gluing the edges of a collection of one or more regular hexagons, there are several ways to build closed polyhedra. The surface of the sphere can be mapped conformally to the faces of these polyhedra, and therefore onto the hexagons. The Gosper island is a two-dimensional fractal shape generated from a hexagon which seamlessly tiles the plane in a triangular lattice, and can be recursively subdivided into 7 congruent Gosper island shapes similar to the original. The flowsnake is a space-filling fractal curve which fills the Gosper island. When we combine these ideas we can construct several map projections from the sphere onto a collection of Gosper island shapes, each yielding as an inverse a “sphere-filling” fractal curve, and a recursive subdivision of the sphere into smaller Gosper island shapes in a succession of triangular lattices.

## Introduction

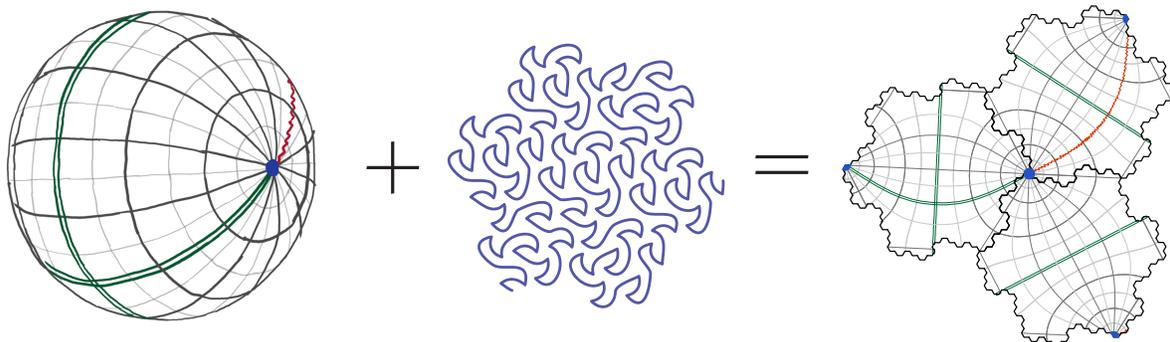
As a bona fide dabbler, in this paper I will attempt to find out just how far one project can go in rejecting tradition for the sake of whimsy. Not content to cast aside the standard map projections, in this paper we will do away with the expected shape of a map, the standard geographical coordinate system, rectangular coordinates themselves, the definitions of “hexagon” and “polyhedron”, the base ten number system, and the concept of a “curve.” Or in short, we will see how many of my favorite mathematical objects and ideas we can mash together by tiling a sphere with hexagons, tweaking each hexagon’s boundary into the fractal Gosper island shape, and covering each with a flowsnake fractal curve. This will allow us to make new maps of the globe onto puzzle-like interlocking pieces, and design 3-dimensional sculptures of squiggly curves which fill a spherical surface.

## Coordinates, Tilings & Grids

At the heart of the connection between these seemingly disparate ideas is the notion of a coordinate system, a link between arithmetic and geometry, representing geometric points using numbers so that we can compare, measure, and transform geometric spaces using discrete calculations.

The standard coordinate system on the sphere is the latitude/longitude grid, which uses angle measures from the sphere’s center relative to some distinguished polar axis and distinguished arc from pole to pole (the “prime meridian”). On the Earth or any other spinning sphere, the axis of rotation is a natural choice to define the poles, and we conventionally pick the prime meridian to pass through London, England.

Considered as a map from a sphere to a rectangle (see Figure 10), this coordinate system does not preserve shapes, has scale which varies dramatically, and includes singularities at the poles which do not map



**Figure 1:** *Earth + Flowsnake*

uniquely to points on the rectangle. To mitigate these defects, a wide variety of map projections and coordinate systems have been designed, usually based on mapping the sphere to a planar disk or polygon, a cylinder, or a polyhedron. For an overview of related topics, see Snyder [1] and Popko [2]. A flippant summary is that spheres, being curved, are tricky to map. To understand grids on the sphere the easiest place to start is the plane.

**Planar Tilings.** Three types of regular polygon can single-handedly tile the plane: the regular triangle, square, and hexagon. Of these, the square grid is by far the dominant choice throughout human art, mathematics, and engineering, but a tiling of hexagons has much to recommend it: it gives each cell more directly adjacent neighbors (6 vs. 4), reduces the total lengths of cell walls (hence its use in honeycombs and chicken wire), is closer to uniform with respect to angle measure, and in a discrete sampling problem such as image representation reduces the severity of grid-aligned aliasing artifacts. A vibrant niche has sprung up around image-processing with hexagonal pixels, see e.g. Middleton and Sivaswamy's monograph [3].

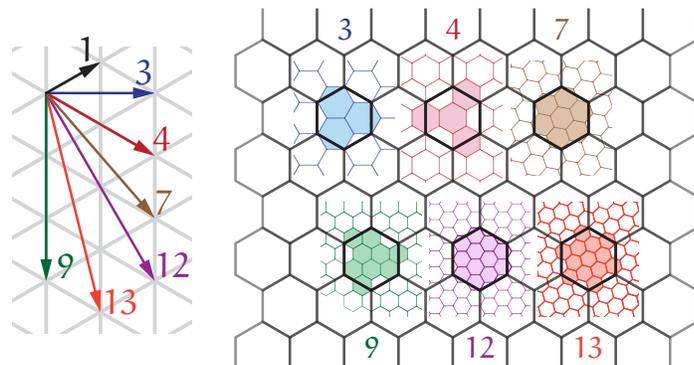
Square grids do however have three striking advantages: they are entrenched in the culture, ubiquitous and familiar; they easily separate into two perpendicular components, a property which generalizes nicely to any dimension; and every square can easily be recursively subdivided.

**Triangle Grids & Lösschian Numbers.** As every high school student learns, the Pythagorean Theorem relates the squared magnitude of any arbitrary displacement vector on a square grid with unit vectors  $x$  and  $y$  separated by a quarter turn: given any coefficients  $a$  and  $b$ ,  $(ax + by)^2 = a^2 + b^2$ .

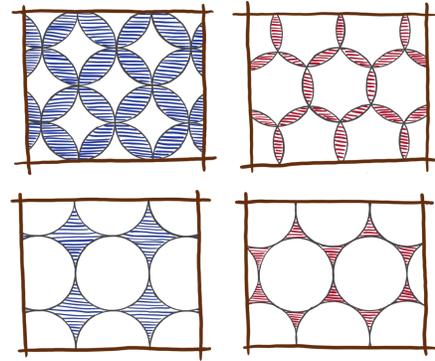
In the case of a triangular grid a similar relation holds. If we use two unit vectors  $x$  and  $y$  which are separated by a third of a full turn, then the analogous expression is  $(ax + by)^2 = a^2 + b^2 - ab$ .

Natural numbers of this form are called Lösschian numbers, after Lössch [4], a geographer and economist who used various triangular grids as an abstract model for geographic networks and distribution of economically important places. The first few are shown at the left of Figure 3. The OEIS [5] lists other connections.

In either a square grid or a triangle grid, any displacement between two grid points defines a new grid of a coarser scale. Given any hexagonal tiling, we can create a finer tiling with  $n$  times as many hexagons for any Lösschian number  $n$ , as seen in Figure 3. This is the idea behind Goldberg polyhedra [6], which typically start with a dodecahedron and then make a Lösschian refinement of the grid. For instance the classic soccer ball shape is the result of a factor-of-3 refinement to the dodecahedron. As we will see, the recursive factor-of-7 refinement of the hexagon is the basis of our fractal shapes in this paper.



**Figure 3:** The simplest few refinements of a hexagon tiling of the plane. Each collection of small shaded hexagons has the same area as one of the larger hexagons.



**Figure 2:** Circles arranged in a hexagonal pattern (right) tile more efficiently than circles in a square pattern (left).

**The Gosper Island & The Flowsnake.** As mentioned, a square can be recursively subdivided into smaller squares, for example in half in each dimension. Each division increases the total number of smallest-scale squares by a factor of 4. Such subdivisions of squares are used frequently in science and engineering, with cells usually ordered either by interleaving bits of the two coordinates (Z order), or by using a fractal Hilbert curve. See Figure 4.

Grids of hexagonal cells have one apparent problem as a coordinate system for the plane: it is impossible to subdivide a single hexagon

into congruent smaller hexagons. If we try, either some of the smaller hexagons must overlap, or gaps are left in the larger hexagon.

The Gosper island shape is a way to work around this problem. Instead of using a hexagon as our base shape, we instead start with a grid of hexagons, but as we recursively subdivide each hexagon into 7, we modify the original shape to match the boundaries of the smaller tiles. If we carry this process out to its infinite limit, the shape we are left with is a fractal, the so-called Gosper island. Like hexagons, Gosper island shapes can be used to tile the plane, and when the centers of each tile are connected, the resulting network is a triangular lattice. Unlike hexagons, however, the Gosper island shape can be recursively subdivided into 7 congruent pieces similar to the original. See Figure 5, and also Figure 11.

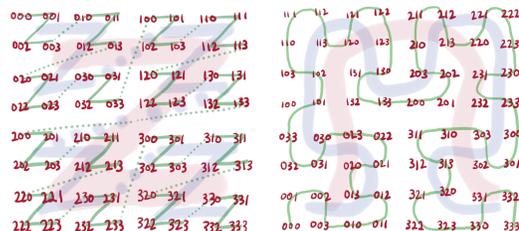
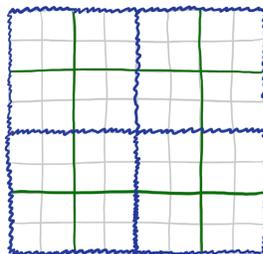


Figure 4: Z and Hilbert curve pixel ordering.

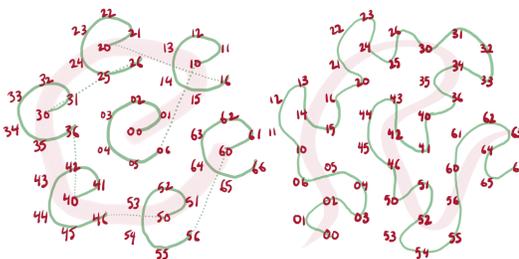
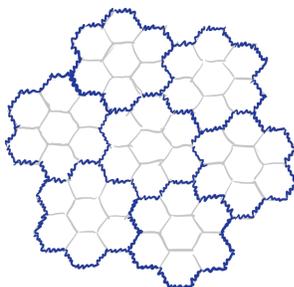


Figure 5: Spiral and flowsnake pixel ordering.

Analogous to the Z order in square subdivision, the most obvious ordering for points in the Gosper island is the so-called spiral order. Like with the square case, any point inside the Gosper island can be described with arbitrary precision using a numeral, this time in base 7. A book [3] and dozens of academic papers have been written using this spiral representation of points, primarily for applications in image processing.

Just as with the Hilbert curve in the square case, it is possible to order the points in the subdivided Gosper island shape such that they form a continuous curve. Bill Gosper, the inventor of this curve, called it the *flowsnake* as a play on the word (and shape of a) *snowflake*, as described by Gardner [7]. My favorite book about the Hilbert curve, the flowsnake, and other similar fractal curves is Ventrella’s [8].

### Folding Up Hexagons

We can cover a sphere by drawing several hexagons on a flat, sufficiently elastic sheet, cutting them out, gluing their edges together until they form a closed shape, and then stretching them over a rigid sphere.

To make our resulting polyhedron topologically equivalent to the sphere, we must match one defining topological feature: the sum of angular defects at each vertex must be equal to 2 full turns.

We can see this property in action by examining the cube (8 corners, each with  $\frac{1}{4}$  turn defect) or the icosahedron (12 corners, each with  $\frac{1}{6}$  turn of defect). Or easiest, we can start with any planar polygon and its mirror image: Glue the two halves together along their matching edges, then blow up the interior to obtain a topological sphere. The total angular defect will be twice the sum of the exterior angles of each polygon half, or two full turns.

If we want to arrive at two full turns of angular defect in a shape constructed from regular hexagons, which have  $\frac{1}{3}$  turn interior angles, then we must attach exactly 1, 2, or 3 hexagons at each vertex, and the total defect at the corners must add up to 6 missing hexagons. See Figure 6.

The simplest case is to glue two hexagons back to back so that each hexagon borders the other along all six of its edges. We can map this shape to a sphere by mapping each hexagon to one hemisphere. This gives us the symmetry of a hexagonal dihedron.

The next simplest case is to make one hexagon into a two-faced triangle by folding three of the corners to meet at the center. We can map the central triangle to one hemisphere, and the triangle formed by the three corners to the other hemisphere. This symmetry is the triangular dihedron.

We can place three hexagons edge to edge in a loop and then fold down the top and bottom corners to make a squat triangular prism. On the sphere, each hexagon corresponds to a  $\frac{1}{3}$  turn slice of longitudes.

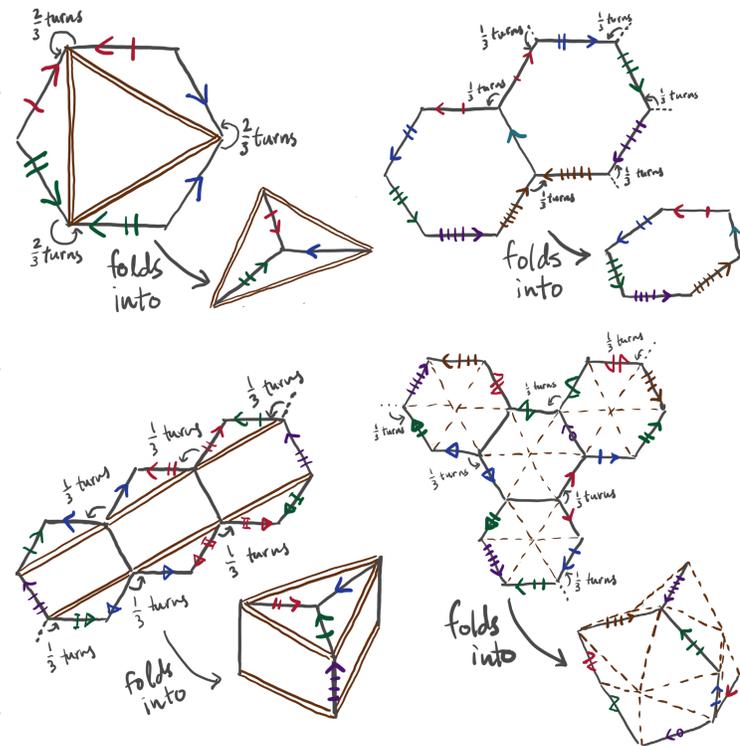
With four hexagons, we can connect a hexagon to each of the other three along two adjacent edges. On the sphere, each of these hexagons corresponds to one face of a spherical tetrahedron, with the centers of its edges bent into additional corners. Alternately we can consider the central triangle of each hexagon to correspond to one octant of the sphere. If we crease and fold these flat hexagons into a polyhedron, we can make either a (non-convex) tetrakis hexahedron (a shape formed by gluing a square pyramid to each face of a cube, see Figure 6), or an octahedron, depending on where we place the creases. Unfortunately neither of these shapes is easy to visualize from line drawings; I urge readers to make paper models.

This four-folded-hexagon octahedron case is especially noteworthy beyond the scope of this paper because it admits a nice recursive subdivision by powers of 4, allowing us to make a  $2^n \times 2^n$  grid of hexagonal pixels over the sphere for any natural number  $n$  (computer memory is usually designed to efficiently handle blocks of  $2^n$  numbers rather than powers of 7), making it a potentially effective interchange format for spherical imagery or video, as I intend to describe in detail in a future paper.

The only place I've ever seen all of these symmetries suggested for grids on the sphere is Purser and Rančić's NOAA office note [9], though the grids they construct are somewhat different than those here.

It is of course possible to go beyond four hexagons, but as far as I can tell these four symmetry systems are the only ones yielding nicely symmetrical maps. The obvious way to glue five hexagons together is to place three edge-to-edge in a loop as in the three-hexagon case, and then glue the fourth and fifth hexagons to the exposed top and bottom edges, centering one on each pole. The result is a taller triangular prism.

Given any of our tilings we can also use the Lőschian subdivisions shown in Figure 3 to make a finer mesh. For example, we might start with our dihedral two-hexagon shape, and divide each face into 13 parts, for a total of 26 hexagons or 26 Gosper island shapes tiling the sphere. The way we subdivide grids—centered on vertices of our initial hexagons rather than face centers—sets the present paper apart from Goldberg's construction [6] and other commonly used refinements of grids on a sphere.



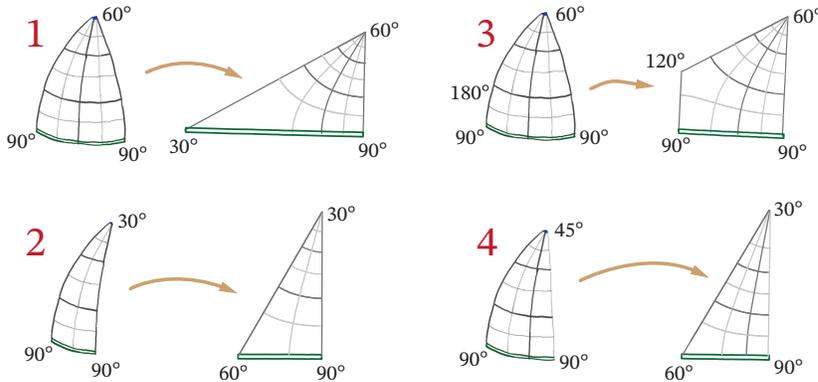
**Figure 6:** One, two, three, or four hexagons folded and glued into a closed shape. In each case, the angular defect sums to 2 turns.

## Conformal Maps

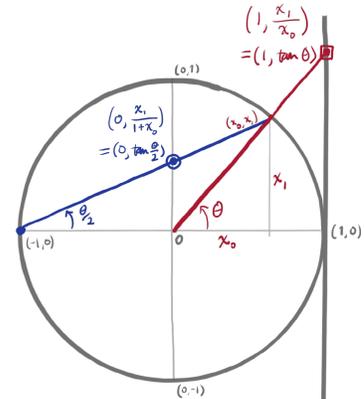
After we have settled on a basic symmetry pattern for our map we must fill in other details of the projection. For instance we can choose to preserve the area scale of the map across the projection, or we can try to optimize for preserving a distance scale between arbitrary points on the map.

As an alternative, conformal maps—except possibly along a boundary or isolated singular points—preserve angles and local shapes. That is, an infinitesimal circle in the domain of the map will be projected onto an infinitesimal circle in the codomain.

The earliest conformal map is the stereographic projection, known to the ancient Greeks two thousand years ago. Given an arbitrary  $n$ -sphere, the stereographic projection is a perspective projection through one



**Figure 7:** The “basic triangles” for each map in Figure 11, showing how spherical triangles map to planar polygons. The angular defect at a vertex is the difference between the angle on the sphere and the angle in the plane.



**Figure 8:** The tangent (square) and stereographic projection or half-angle tangent (circle).

pole on the sphere onto an  $n$ -dimensional hyperplane through the equator. It maps the surface of the sphere minus a single point onto the entirety of the hyperplane. (To see how the formulae relate to this geometrical idea look at the one-dimensional case. See Figure 8.)

The Mercator projection, developed in the 16th century, is an even more famous conformal map. Instead of projecting the sphere onto a plane the Mercator projection maps the sphere minus the poles onto an infinite two-ended cylinder. The projection is defined so that north always points along the cylinder and angles are preserved, turning rhumb lines (paths with constant compass bearing) into straight helix shapes on the cylinder. This made it an essential tool for maritime navigation before GPS. See Pérez-Duarte and Swart [10].

But conformal projections can also map between arbitrary two-dimensional shapes. The celebrated Riemann mapping theorem, a gem of 19th century mathematics, states roughly that there is a conformal map between any two simply connected shapes in the complex number plane, unique up to trivial transformations.

We can thus develop such projections straight-forwardly using known complex-valued functions. In each case we need to make a conformal map from the complex plane onto the appropriate domain treated as a portion of the complex plane, and we are guaranteed to have the correct map from the sphere by composing it with the stereographic projection. See Needham [11], and Figure 10.

Two decades after Riemann, Schwarz and Christoffel independently showed how to explicitly construct conformal projections from an infinite half-plane to any planar polygon. Such projections are now called Schwarz–Christoffel maps. Driscoll and Trefethen [12] have built software for constructing even difficult examples of these maps.

To develop our desired map projections to Gosper island shapes, we first project the sphere onto a collection of hexagons. One straight-forward way to make these maps is to find a “slice” of the sphere from pole to pole between two longitudes which we can map onto a polygon representing part of one of our hexagons. In particular, we can form maps from the hemisphere to a regular hexagon, and from the hemisphere to two orientations of rhombuses with a  $\frac{1}{3}$  turn corner. See Figure 10.

### Sphere-Filling Curves

Once we have a map projection from the sphere to some collection of planar shapes which we can cover with some grid in the plane, we can easily invert that map and project our grid back onto the sphere. Indeed, this is how every spherical coordinate system works. Since our grids are recursively refinable, this provides a way to number points on the sphere to arbitrary precision. This is the principle behind so-called discrete global grid systems [13], which typically start with the symmetry of the icosahedron.

If we know how to make an area-filling fractal curve in our planar shapes, we can project the curve back onto the sphere. We might call such a curve sphere-filling. Six Hilbert curves projected onto the faces

Conformal mappings from the sphere to the plane (stereographic projection) and the cylinder (Mercator projection). Schwarz–Christoffel mappings from the half plane to the hexagon and to two different orientations of rhombus.

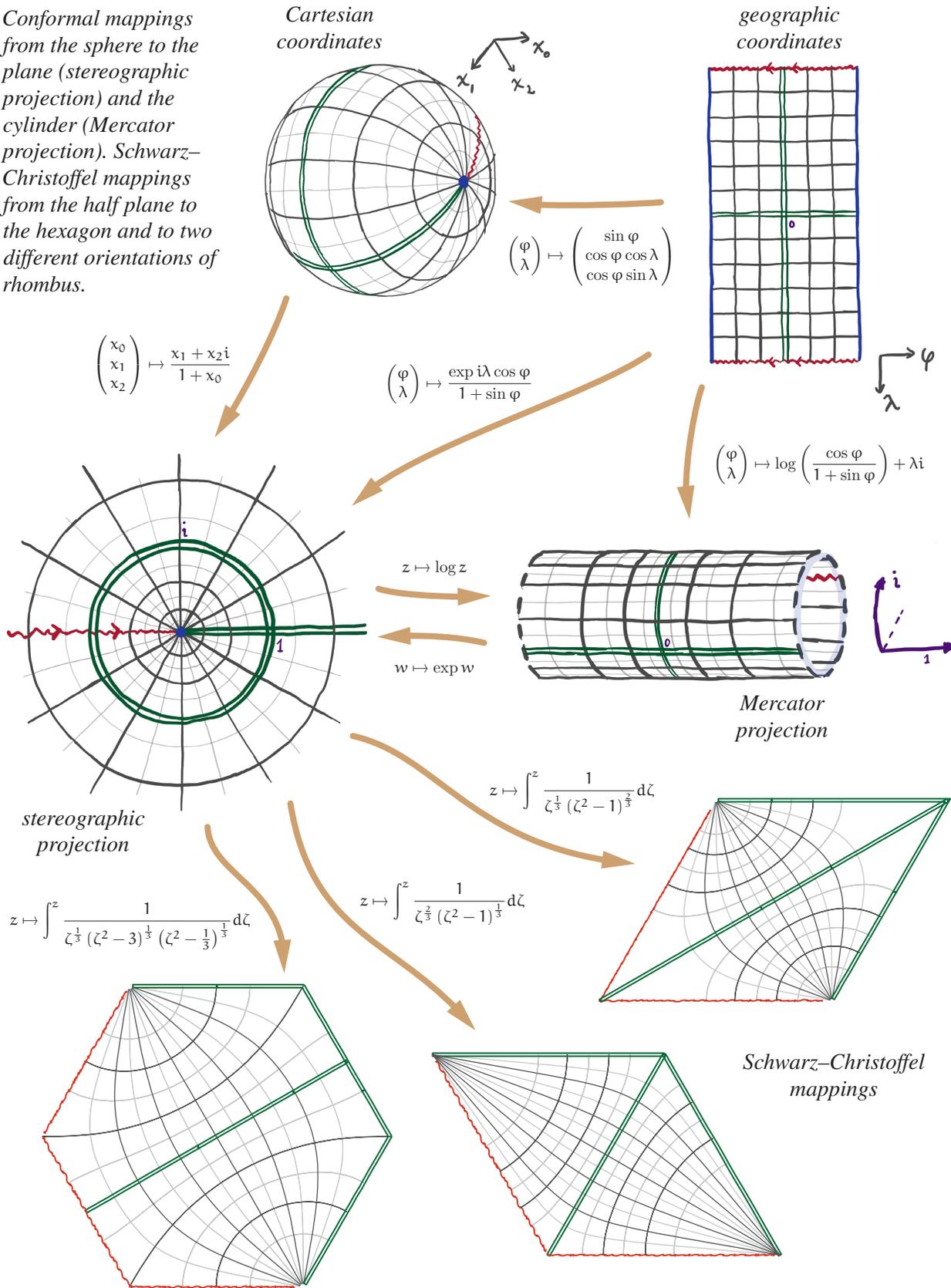
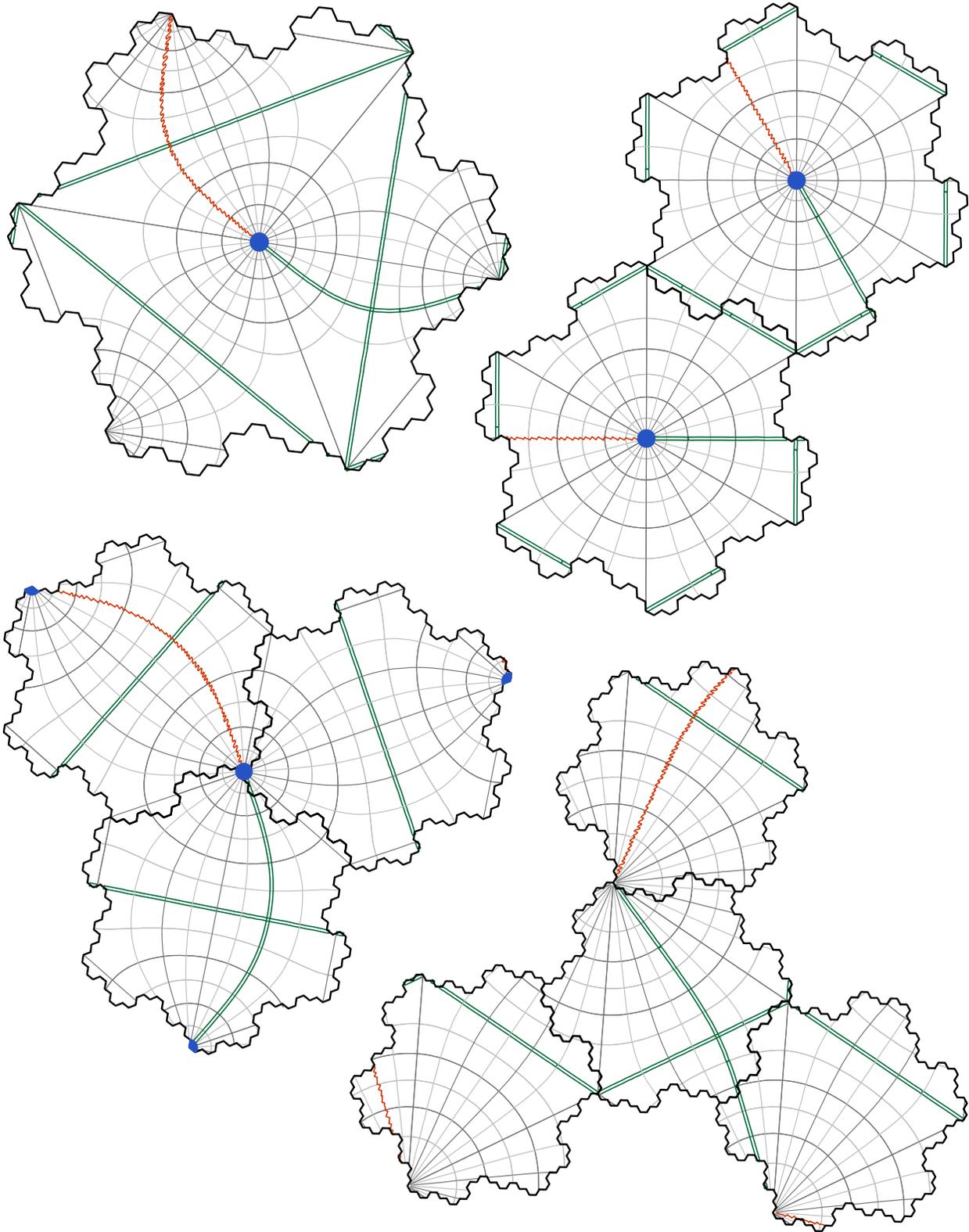


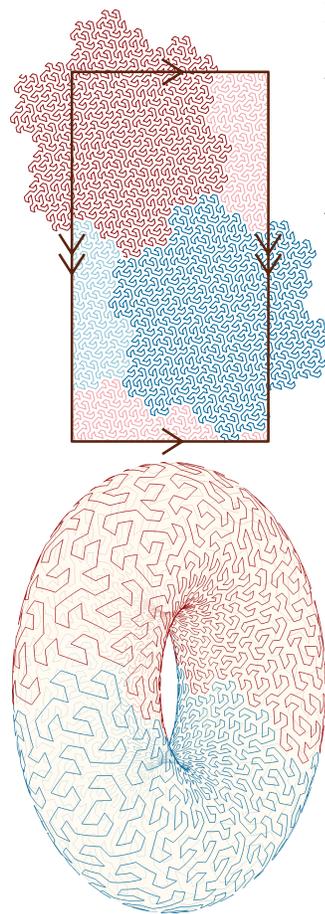
Figure 10: Projecting from the sphere to parts of hexagons using conformal mappings.



**Figure 11:** Projections of the globe onto one, two, three, or four Gosper Island shapes. These maps have the symmetries of the triangular dihedron, hexagonal dihedron, triangular prism, and octahedron, respectively.

of a cube and thence projected to the sphere have been used in both engineering and art, as in Google’s S2 geography library and Segerman’s 3D-printed sculptures [14]. Other fractal curves have been projected onto parts of a sphere, as in Robert Fathauer’s lovely sculpture “Radial Development” [15]. This paper may be the first to suggest projecting a collection of flowsnake curves onto a sphere or making world maps from Gosper islands.

**Beyond Spheres.** A sphere isn’t the only surface a collection of Gosper island shapes might tile. By gluing opposite ends, a single hexagon can be turned into a topological torus. Embedding it in three-dimensional space gets a bit tricky however (see Sullivan [16]); if two or more hexagons are used—or two Gosper island shapes—to tile the torus, it can be made into a familiar donut shape, now with a fractal glaze. See Figure 12.



**Figure 12:** Two flowsnakes covering the surface of a torus can be connected into a single continuous loop. Top: the planar torus shown in a rectangle. Bottom: the torus conformally embedded in 3-dimensional space, with orientation chosen so that the flowsnakes are congruent.

### Other Possible Art Applications

Beyond puzzle-like world maps and sculptures of spherical fractal curves, these tools could be used for other art projects. Spherical illustrations, photographs, or videos could be projected onto hexagons or Gosper islands, the way German et al. [17] and others have suggested with various other map projections. A flowsnake might be painted onto a spherical or toroidal lampshade to project fractal shadows on the wall. Every crater on the moon could be assigned a numerical code in base 7 using the flowsnake grid, and turned into a diatonic-scale melody.

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