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EUREKA

THE JOURNAL OF THE ARCHIMEDEANS

(The Cambridge University Mathematical Society: Junior Branch of the Mathematical Association.)

Editor: *P. Brodetsky, Queens' College.*

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Contents

Editorial

At long last Eureka has reappeared, having hibernated since March, 1942. The reasons for this regrettable delay are many and varied, but the decisive factor seems to have been the paucity of contributions. However, the present editor has inherited sufficient material for an issue this term, and—well, here it is. We realise that the promise given in the last issue that Eureka would appear twice a year has not been kept with any great exactitude, but this may simply be a temporary variation, insignificant statistically. In any case, we apologise profusely, and hope that the present issue keeps to the high standard set in the past.

It will be noticed that, in complete contrast to the last issue, there is here not a single article by a senior member of this, or any other, university. This phenomenon was not caused deliberately, but it shows, at any rate, that undergraduates can and do express themselves, mathematically and otherwise, and that it is not always necessary to call in the help of more learned beings in filling our pages. There was, moreover, to have been an article by one who has just gone down, on the subject of the application of algebra, especially continued fractions, to the study of plant leaves, but unfortunately the pressure of his vitally important research has prevented the author from writing out his paper, which some of us heard as a lecture last term.

Complaints, contributions, requests for copies of this and other issues etc., should be addressed to P. Brodetsky, Queens' College, Cambridge, as the Editor has been completely unable to find any trace of the whereabouts of his official address.

Archimedeans' Activities

THE MUSIC GROUP

The Music Group met frequently last year, for the most part to listen to gramophone records, but there were four concerts by members of the Group. The records were chosen with a marked bias in favour of orchestral music, and the concerts were hampered by an excess of pianists and a scarcity of competent instrumentalists. Despite this the Music Group was probably the most successful of the three, largely owing to the efforts of Mr. Davis, who is to be congratulated on the way in which he overcame all difficulties (which included dealing with reluctant radiograms and obtaining buns for tea).

A. O. L. A.

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THE PLAY-READING GROUP

The Play-reading Group met weekly since its inception in January, and was run by Mr. Lighthill. A great variety of plays was read, the authors ranging from Shaw to Shakespeare; most of the reading took place in Mr. Glauert's rooms, which were kindly lent for the purpose.

A. O. L. A.

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THE ARCHIMEDEANS' BRIDGE GROUP

Meetings of this Group were held regularly during the Michaelmas and Lent terms on alternate Fridays in Newnham, attendance varying from 4 to 16 members. Some excitement was caused about the middle of the Lent term by the introduction of duplicate bridge at one of the meetings, but apart from a few enthusiasts not many supported this form of the game. Miss Coad Pryor has agreed to run the Group this year, and it is hoped that there will be some new members, as most of the old members have gone down.

V. A.

N.U.S. Congress, 1943

The Annual Congress of the National Union of Students was held in April of last year in London. Sir Archibald Sinclair opened the proceedings at a plenary session in Kingsway Hall, and the Congress then split up into its various Commissions. In addition to the six Faculty Commissions of Science, Medicine, Social Science, Education, Arts and Engineering, there were Works Commissions, which discussed such aspects of student life as military training, faculty reform, war-work, etc. Representatives reported on these Commissions at the final plenary session in Holborn Hall.

Mathematicians were mainly interested in the Science Commission, which was addressed at King's College by a woman research-worker, and then at Birkbeck by Dr. Martin Ruhemann. The first meeting served the practical purpose of answering undergraduates' questions on scientific research, and the second developed into a lively discussion on the social functions of science.

A local "follow-up" meeting was held in Cambridge last term at Newnham, when Dr. Ruhemann again addressed a Science Commission. This meeting was organised by the Undergraduate Council, to which most of the credit for its success must go.

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A Proof that Every Equation has a Root

By F. J. Dyson

THERE are so many proofs of the theorem that every equation has a root, that it seems almost criminal to produce another. I can however say two things in my defence: first, the proof I shall give is probably not a new one; second, if my proof is new it has a certain advantage over other proofs in using only the most elementary arguments.

Let $F(z) \equiv z^n + a_1 z^{n-1} + \ldots + a_n = 0$ be any equation of degree n. We prove that it has at least one root $z = x + iy$ in the complex plane.

Writing $z = x + iy$, we have

$$F(z) \equiv P(x, y) + iQ(x, y)$$

where P and Q are polynomials in x and y with real coefficients. The terms of highest degree in $P(x, y)$ are

$$x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \ldots,$$

and vanish on n straight lines through the origin making angles

$\frac{\pi}{2n}, \frac{3\pi}{2n}, \ldots, \frac{(2n-1)\pi}{2n}$ with the x-axis. Thus the graph of the equation $P(x, y) = 0$ has n real asymptotes making the same angles $\frac{\pi}{2n}, \ldots, \frac{(2n-1)\pi}{2n}$ with the x-axis. Similarly the graph of $Q(x, y) = 0$ has n real asymptotes making angles $0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(n-1)\pi}{n}$ with the x-axis. Thus if we take a sufficiently large circle C in the x-y-plane, centre the origin, C will cut the graph of $P = 0$ at just $2n$ points $P_1, P_2, \ldots, P_{2n}$, and the graph of $Q = 0$ at $2n$ points $Q_1, \ldots, Q_{2n}$; and these $4n$ points will lie alternately round the circle. Further, if the circle C is large, the value of P on the circle will actually change sign at the points P_i, and the value of Q at the points Q_i.

We say that two points R and S are "connected" if it is possible to join them by a path lying entirely inside C and not cutting the graph of $P = 0$. Let every pair of points Q_aQ_b which are connected be joined by a path of this kind. Among all these point-pairs let Q_a, Q_b be chosen as having the smallest possible angular separation on the circle. The sign of P is the same at Q_a and Q_b, and so Q_a and Q_b are separated on the circle by an even number of points P_i; thus there exists a point Q_c lying between Q_a and Q_b on C, such that P has opposite signs at Q_a and Q_c. If Q_c were connected to any other point Q_d, the path joining Q_c to Q_d could not intersect the path joining Q_a to Q_b, because P takes opposite signs on the two paths. Thus Q_d would also lie between Q_a and Q_b on C, and the angular separation between Q_c and Q_d would be less than that between Q_a and Q_b, contrary to hypothesis. Therefore the point Q_c is not connected to any other Q_d.

Let R be the region formed by all points which are connected to Q_c. The boundary of R will consist of (i) arcs of the type P_iP_{i+1} on C, and (ii) arcs of the graph of $P = 0$. Since no point Q_d can lie in R, there can be only one arc P_iP_{i+1} of C in the boundary of R, namely the arc containing Q_c. Thus the boundary of R will consist of

(i) the arc P_iP_{i+1} on C;

(ii) a connected arc A of the graph of P, joining P_i to P_{i+1} inside C, and perhaps

(iii) isolated pieces of the graph of P.

Q is a continuous function, and has opposite signs at the ends P_i and P_{i+1} of the arc A. Hence Q vanishes at some point (x, y) on A, and P also vanishes at this point. Thus $z = x + iy$ is a root of $F(z) \equiv P + iQ = 0$.

Q. E. D.

Pyramid Patience

By B. D. Price

Form of the Puzzle.—There are three upright pegs; on one is a pile of square blocks forming a pyramid, the smallest block being at the top. There are eight blocks, grading in size.

The problem is to move the pyramid completely to another peg, moving only the top block off one peg on to the top of the pile on another. A block may not be immediately on top of a smaller block.

Notation.—The blocks will be numbered (1) (smallest) to (8) (largest), (p) being the general block. Pegs are numbered 1, 2, 3, starting on 1 and ending on 3.

Theorem 1.

A puzzle of n blocks requires $2^n - 1$ moves.

For suppose this holds for a particular value m.

Then if one be added (say, a larger one than m) the puzzle may be solved:—

	Moves
Move the top m blocks from 1 to 2	$2^m - 1$
Move $(m + 1)$ from 1 to 3	1
Move the m blocks from 2 to 3	$2^m - 1$
Total moves	$2^{m+1} - 1$

Thus if the theorem holds for m it holds for $m + 1$.

But it holds for $m = 1$.

Thus it holds for any positive integral number of blocks.

For an eight-block puzzle, the number of moves is $2^8 - 1 = 255$.

Theorem 2.

(1) moves every 2 moves.
(2) ,, ,, 2^2 ,,
↓
In general, (p) ,, ,, 2^p ,,

Choose a particular move in which (1) is moved from a to b. Then the next move does not involve b, as the smallest piece (1) is on top of the peg b, and moving (1) again would be redundant.

Thus the next move is a to c or c to a.

If the next move is a to c, the next move again cannot be a to b or c to b as (1) is on top of b. Also c to a would be redundant. Thus the next move must involve (1) moving from b to a or c. Thus (1) is moved every other move, or every 2 moves.

Similarly for c to a.

Now remove (1) from the problem. (2) now moves every 2 moves. Replacing (1) we get that (1) moves every 2 moves.

(2) ,, ,, 2^2 ,,

And in general by a similar argument

(p) moves every 2^p moves.

THEOREM 3.

If the block (p) be followed during the solution of the problem:—
If p *is odd*, the moves follow the rotation

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow;$$

if p *is even*, $1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow$.

[Which is quite arbitrary— *e.g.*, this is so for a puzzle of $2p$ pieces, in moving from 1 to 3. *Proof anyone?*]

From theorems 2 and 3 it follows that the general solution may be written out thus:—

Block moved	Number of move in complete solution 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	2		3		1		2		3		1		2		3		1		2		3		1	
2		3				2				1				3				2				1		
3				2								3								1				
4								3																2
5																2								

And so on, the figures in the diagram referring to the peg moved to in the particular move.

THE m^{th} MOVE. [All these results are inferred from Th. 1–3 with the help of the above diagram.]

Let $m = 2^{p-1}a$ (a odd),

$$b = \frac{a+1}{2}.$$

The m^{th} move is the b^{th} move of the block (p).

$b + 1 = 3c + d$ ($d < 3$).

The move m is given by:—

d	p odd	p even
0	3 to 1	3 to 2
1	1 to 2	2 to 1
2	2 to 3	1 to 3

Let $m - 2^{q-1} = (e-1)2^q + f$ ($f < 2^q$).

Then the last move of the (q) piece, after the m^{th} move has been completed, was its e^{th} move.

Analysing e as we did b,

$$e + 1 = 3g + h \qquad (h < 3),$$

and the following table gives where (q) is after the m^{th} move:—

h	q odd	q even
0	1	2
1	2	1
2	3	3

(Numbers in columns 2, 3 refer to the pegs.)

Thus the m^{th} move is completely resolved.

Points of Interest.

After $2^n - 1$ moves, all the n pieces moved are on the same peg—see Theorem 1 and the solution diagram.

Each additional piece more than doubles the necessary number of moves.

Solution of the problem of n pieces with more than $2^n - 1$ moves involves retracing moves.

Can anyone evolve a rule for the next move, merely by looking at the puzzle?

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A Remark on the Schröder-Bernstein Theorem

By G. Kreisel

In Littlewood's *Theory of Real Functions* (pp. 6–8) a proof of this theorem is given which at first sight makes any application to particular cases seem very remote. The proof below deals with the question from a slightly different angle. For simplicity I use Littlewood's notation.

Also I define "ϕ" to be the correlating function (A $\rightarrow$ A′) and if C is a sub-set of A $\phi(C) =_{Def.} \in(n = \phi(y) \equiv y \epsilon C)$. Thus $\phi(A) = A'$.

I consider the set $\sum_{0}^{\infty}\phi^n(D) = P$. This is evidently K_o. (Every K contains D. Since $K \supset K^1$ $K \supset \phi(D)$ and $\therefore$ generally if $K \supset \phi^n(D)$ $K \supset \phi^{n+1}(D)$. Further, P is a K. $\therefore$ It is the smallest K.)

I define the following similarity relation Ω.

$\phi(\xi) \epsilon \phi(K_o) \rightarrow \Omega(\phi(\xi)) = \xi$; $\phi(\xi) \epsilon [\phi(A) - \phi(K_o)] \rightarrow \Omega(\phi(\xi)) = \phi(\xi)$.
$\therefore \Omega(A') = A_1$, q.e.d.

The proof incidentally shows exactly what need be known of "ϕ" and "D" for the method of proof to be applicable. In a context like Littlewood's book the following observation is appropriate: the concepts of the theory of integers which I have used in the proof can be developed quite easily without Schröder-Bernstein (by induction *à la* Hilbert-Bernays, Vol. I).

On a Geometric Trifle

By G. Kreisel

A FAMILIAR algebraic theorem states that in general two quadratic forms of order n have n singular members (two quadratic forms $x'Ax = 0$, $x'Bx = 0$, being "general" if the roots of $| A - \lambda B | = 0$ are distinct). The translation into geometric language is immediate (and interesting).

It is natural to ask for a proof of the geometric theorem in the language proper to it; here it is: the theorem I propose to prove states that

A system of quadrics in $[n]$ defined by two nondegenerate quadrics which do not touch contains $n + 1$ cones whose vertices are linearly independent, and define a self-conjugate simplex; further, no vertex lies on either of the two quadrics.

(The second half of the theorem is of infinitesimal interest, but helps in the induction argument.)

I observe first that an algebraic (1,1) correspondence in any number of dimensions has at least one self-corresponding point; if $v + 1$ points of a subspace $[v]$ of $[n]$ are self-corresponding every point of $[v]$ remains fixed in the transformation. (At this stage it is needless to stress that I deal with an algebraically closed field.)

Next, I consider the collineation defined by the following rule: Given P, I take the polar prime π of P w.r.t. one quadric Σ and call the pole of π w.r.t. the other quadric S P'. Then $P \rightarrow P'$ is (1, 1) and algebraic.

Lemma I. A self-corresponding point cannot lie on either of the quadrics.

Suppose it does: then the tangent primes of the point on the two quadrics are identical so that the quadrics touch. But I excluded this case from the start.

Lemma II. The vertex V of a cone through the intersection of the two quadrics is a self-corresponding point of the correspondence.

If a generating line of the cone meets the quadrics in R_1T, the harmonic conjugate of V w.r.t. R_1T lies on the polar of V w.r.t. Σ and S. There are n linearly independent generators, which fix the polar space.

Lemma III. A self-corresponding point V of the collineation is the vertex of a cone of the family depending on Σ and S.

Take P on the intersection of Σ and S. Let PV intersect Σ in Q_1, S in Q_2. Since the harmonic conjugate of V w.r.t. P,Q_1 separates

P,Q_2 harmonically w.r.t. V, $Q_1 = Q_2$ or else P,V coincide, which is impossible by lemma I.

Lemma IV. I now make use of the induction hypothesis. The polar prime of V intersects the two quadrics in $2V^2_{n-2}$'s which do not touch.

Otherwise there would be a point on the two quadrics in $[n - 1]$ whose polar prime would go through V (by the pole-polar property) and also through the common tangent space of the V^2_{n-2}'s at P, so that the quadrics Σ,S would touch.

The theorem follows if one observes that the n self-corresponding points of the analogous collineation for the two V^2_{n-2}'s are also self-corresponding for the one defined above, and that the theorem is true for [2].

REMARKS:

(1) Another statement of the geometric theorem would be: if the base locus of a pencil of quadrics has no double point, etc.

(2) The cones of lemma III are actually quadratic cones. For this it is sufficient to prove that a line not passing through the vertex meets the cone in two points; the plane defined by a line and V intersects the quadrics in 2 conics whose points of intersection lie on two lines through V.

(3) It is seen from lemma III and the final step that the argument works for any self-corresponding point not on the quadrics. This settles the hackneyed case of the cones through two intersecting conics in space, *i.e.* the intersections of two planes and a quadric.

Let the conics intersect on A,B. Consider the polar line p of AB w.r.t. the quadric. p meets the quadric in P_1,P_2, the planes in Q_1,Q_2. Then if O_1,O_2 are the harmonic pair common to P_1P_2, Q_1Q_2 they are self-corresponding points of my collineation. Since the other vertices of cones would lie on AB, O_1,O_2 determine the only proper cones.

If $\overline{AB}$ touches the quadric the only vertex lies on the polar line of $\overline{AB}$ w.r.t. the quadric. It is the pole of the plane, defined by AB and its polar line, w.r.t. the original plane pair.

(4) The theorem incidentally furnishes the following corollary:

The necessary and sufficient condition for the base locus of two quadrics to be simple is that the appropriate characteristic roots are distinct.

If, and only if, the base locus is simple do there exist exactly $n + 1$ cones. $n + 1$ cones exist if, and only if, the a.c. roots are distinct.

The deduction is left to the reader.

Some Guesses in the Theory of Partitions

By F. J. Dyson

Professor Littlewood, when he makes use of an algebraic identity, always saves himself the trouble of proving it; he maintains that an identity, if true, can be verified in a few lines by anybody obtuse enough to feel the need of verification. My object in the following pages is to confute this assertion.

In order to save space, I must refer my readers to the first three pages of chapter XIX of Hardy and Wright's *Introduction to the Theory of Numbers* for a detailed account of the idea of a partition, and for a description of the way in which the properties of partitions are represented in the form of algebraic identities. I will always refer to this chapter by the symbol (A). The plan of my argument is as follows. After a few preliminaries I state certain properties of partitions which I am unable to prove: these guesses are then transformed into algebraic identities which are also unproved, although there is conclusive numerical evidence in their support; finally, I indulge in some even vaguer guesses concerning the existence of identities which I am not only unable to prove but also unable to state. I think this should be enough to disillusion anyone who takes Professor Littlewood's innocent view of the difficulties of algebra. Needless to say, I strongly recommend my readers to supply the missing proofs, or, even better, the missing identities.

* * * *

The total number of partitions of an integer n into a sum of positive integral parts is denoted by $p(n)$. The "generating function" of $p(n)$ is the infinite series

(1) $$P = \sum_{n=0}^{\infty} p(n)x^n,$$

which is a function of the variable x regular in $|x| < 1$. The form of P is given by two identities of Euler

(2) $$P^{-1} = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \ldots,$$

(3) $$P^{-1} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3+1)} = 1 - x - x^2 + x^5 + x^7 - \ldots,$$

which are proved in (A).

There are three beautiful arithmetical properties of $p(n)$, which were discovered, and later proved, by Ramanujan, namely:—

(4) $p(5n + 4) \equiv 0 \pmod 5$,

(5) $p(7n + 5) \equiv 0 \pmod 7$,

(6) $p(11n + 6) \equiv 0 \pmod{11}$.

They appear as theorems 359–361 in (A), and can be proved analytically without much difficulty, using identities like (3); in fact, there are at least four different proofs of (4) and (5).

It would be satisfying to have a direct proof of (4). By this I mean, that although we can prove (in four ways) that the partitions of $5n + 4$ can be divided into five equally numerous subclasses, it is unsatisfactory to receive from the proofs no concrete idea of how the division is to be made. We require a proof which will not appeal to generating functions, but will demonstrate by cross-examination of the partitions themselves the existence of five exclusive, exhaustive and equally numerous subclasses. In what follows I shall not give such a proof, but I shall take the first step towards it, as will appear.

The result of subtracting the number of parts in a partition from the largest part we call the "*rank*" of the partition. It is easy to see that the ranks of partitions of n will take the values

$$n-1,\ n-3,\ n-4,\ \ldots,\ 2,\ 1,\ 0,\ -1,\ -2,\ \ldots,\ 4-n,\ 3-n,\ 1-n,$$

and no others. The number of partitions of n with rank m we denote by $N(m, n)$. The number of partitions of n whose rank is congruent to m modulo q we denote by $N(m, q, n)$. Thus

$$(7)\quad N(m, q, n) = \sum_{r=-\infty}^{\infty} N(m + rq, n).$$

The conjecture which I am making is

$$(8)\quad N(0, 5, 5n+4) = N(1, 5, 5n + 4) = N(2, 5, 5n + 4) = N(3, 5, 5n + 4) = N(4, 5, 5n + 4);$$

or, in words, the partitions of $5n + 4$ are divided into five equally numerous classes according to the five possible values of the least positive residue of their ranks modulo 5. In the same way we have

$$(9)\quad N(0, 7, 7n + 5) = N(1, 7, 7n + 5) = \ldots = N(6, 7, 7n + 5).$$

The truth of (4) and (5) would follow at once, if (8) and (9) could be proved. But the corresponding conjecture with modulus 11 is definitely false.

There is in the theory of partitions a "principle of conjugacy," explained in (A), p. 272. This principle includes a duality relation between the number of parts and the largest part in a partition, and thus partitions of rank m are in a relation of duality with partitions of rank $-m$. It can thus easily be proved that

$$(10)\quad N(m, n) = N(-m, n),$$

$$(11)\quad N(m, q, n) = N(q - m, q, n).$$

Hence (8) reduces to only two independent identities, and (9) to three.

Fortunately, this reduction of our capital is more than offset by other considerations. In fact, (8) and (9) are only the leading and most interesting members in a whole series of similar identities, as listed below:—

(12) $N(1, 5, 5n + 1) = N(2, 5, 5n + 1)$,

(13) $N(0, 5, 5n + 2) = N(2, 5, 5n + 2)$,

(8) $N(0, 5, 5n + 4) = N(1, 5, 5n + 4) = N(2, 5, 5n + 4)$,

(14) $N(2, 7, 7n) = N(3, 7, 7n)$,

(15) $N(1, 7, 7n + 1) = N(2, 7, 7n + 1) = N(3, 7, 7n + 1)$,

(16) $N(0, 7\ 7n + 2) = N(3\ 7\ 7n + 2)$,

(17) $N(0, 7, 7n+3) = N(2, 7, 7n+3)$, $N(1, 7, 7n+3) = N(3, 7, 7n+3)$,

(18) $N(0, 7, 7n + 4) = N(1, 7, 7n + 4) = N(3, 7, 7n + 4)$,

(9) $N(0, 7, 7n+5) = N(1, 7, 7n+5) = N(2, 7, 7n+5) = N(3, 7, 7n+5)$,

(19) $N(0, 7, 7n+6) + N(1, 7, 7n+6) = N(2, 7, 7n+6) + N(3, 7, 7n+6)$.

Of these relations, only (8) and (9) give any arithmetical properties of $p(n)$. The rest of the series is interesting only because it may throw some light on (8) and (9); as yet, however, I have been unable to find any plan behind the apparently haphazard distribution of these identities.

*   *   *   *

I now proceed to put the equations into algebraic form by means of generating functions. The algebraic form is useful for numerical computations, and also seems to offer the best prospect of arriving at proofs. I shall omit the calculations, but on the basis of formulae to be found in (A) the generating function $G(m) = \sum_{n=0}^{\infty} N(m, n)x^n$ takes the form

(20) $$G(m) = P \sum_{r=1}^{\infty} (-1)^{r-1} \left(x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)}\right) x^{mr},$$

where P is given by (1). This form is valid when $m \geq 0$ and, with certain reservations, when $m < 0$ also; but when $m < 0$ it is simpler to use the relation

(21) $$G(m) = G(-m),$$

deducible from (10). (20) and (21) can thus be combined in the formula

(22) $$G(m) = P \sum_{r=1}^{\infty} (-1)^{r-1} \left(x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)}\right) x^{r|m|},$$

valid for all values of m. The series on the right of (22) is simple in form, and is of the type called "false theta-functions" by Professor Rogers, if that is any consolation.

The generating function of $N(m, q, n)$ is

(23) $$G(m, q) = \sum_{n=0}^{\infty} N(m, q, n)x^n = \sum_{s=-\infty}^{\infty} G(m + sq),$$

by (7). We suppose that q is a positive integer, and that $0 < m < q$. Then we substitute from (22) into (23), and the summation with respect to s can be performed in finite terms, giving the final result

$$(24) \quad G(m, q) = P\sum_{r=1}^{\infty}(-1)^{r-1}\frac{(x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)})(x^{mr} + x^{(q-m)r})}{(1 - x^{qr})}.$$

The coefficients in P have been tabulated as far as x^{600}, and the coefficients in the series on the right of (24) are all very small; (24) therefore affords much the quickest way of calculating the values of $N(m, q, n)$ numerically. The equations (12) — (19) can be expressed in analytical form by means of (24); as an example we take the equation $N(1, 7, n) = N(3, 7, n)$, which leads to the following statement.

(25) *In the power-series*

$$P\sum_{r=1}^{\infty}(-1)^{r-1}\frac{(x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)})(x^{r} - x^{3r} - x^{4r} + x^{6r})}{(1 - x^{7r})}$$

the coefficients of x^{7n+1}, x^{7n+3}, x^{7n+4}, x^{7n+5} *vanish identically.*

* * * *

It is interesting for several reasons to examine the numerical evidence in some detail. First comes a table of the values of the two differences

$$a = N(0, 5, n) - N(2, 5, n), \; b = N(1, 5, n) - N(2, 5, n)$$

for values of n up to 50.

n	a	b	n	a	b	n	a	b	n	a	b	n	a	b
1	1	0	2	0	1	3	0	−1	4	0	0	5	0	1
6	1	0	7	0	0	8	−1	−1	9	0	0	10	0	1
11	1	0	12	0	1	13	0	−2	14	0	0	15	−1	1
16	1	0	17	0	1	18	−1	−2	19	0	0	20	0	1
21	2	0	22	0	1	23	−1	−2	24	0	0	25	−1	2
26	1	0	27	0	0	28	−1	−3	29	0	0	30	0	2
31	2	0	32	0	2	33	−1	−3	34	0	0	35	−1	2
36	2	0	37	0	1	38	−2	−4	39	0	0	40	−1	2
41	3	0	42	0	2	43	−1	−4	44	0	0	45	−2	3
46	3	0	47	0	2	48	−2	−5	49	0	0	50	0	3

What is remarkable about this table, apart from the columns of zeros, is the regularity of behaviour of a and b within each arithmetic progression of common difference 5, and also the smallness of the values. If the partitions of 48 were distributed "at random" into five classes, we should expect statistically that the numbers of partitions in each pair of classes would differ by anything from 100 to 250. Clearly, then, the values of a and b, namely -2 and -5, require some explanation. It seems certain that there remain to be discovered alternative forms for the generating functions of a and b, which will make it intuitive when these coefficients vanish, when they are positive, when negative, and why in general they are so small. And exactly the same remarks apply to the coefficients relating to the modulus 7.

In the case of modulus 7, we obtain from equations (12)–(19) some striking congruence properties of $p(n)$. We write

$$c = \mathrm{N}(0, 7, n) - \mathrm{N}(3, 7, n), \; d = \mathrm{N}(1, 7, n) - \mathrm{N}(3, 7, n),$$
$$e = \mathrm{N}(2, 7, n) - \mathrm{N}(3, 7, n).$$

Then, by (11), $p(n) \equiv c + 2d + 2e \pmod 7$.

Now using (12)–(19), we find

$$(26) \begin{cases} \text{when } n \equiv 1, \; p(n) \equiv c \pmod 7, \\ \text{when } n \equiv 2, \; p(n) \equiv 2d + 2e \pmod 7, \\ \text{when } n \equiv 3, \; p(n) \equiv 3c \pmod 7, \\ \text{when } n \equiv 4, \; p(n) \equiv -5e \pmod 7. \end{cases}$$

Below is a table of the actual least positive residues of $p(n)$ (mod 7) for various values of n.

n	1	8	15	22	29	36	43	50	57	64
lpr	1	1	1	1	1	1	2	1	2	2
n	2	9	16	23	30	37	44	51	58	65
lpr	2	2	0	2	4	0	2	4	2	2
n	3	10	17	24	31	38	45	52	59	66
lpr	3	0	3	0	3	3	3	0	3	3
n	4	11	18	25	32	39	46	53	60	67
lpr	5	0	0	5	5	0	5	0	5	5

It will be seen that these residues exhibit a strong regularity, which is sufficiently explained by the congruence relations (26) together with the fact that the values of c, d and e are initially very small.

For comparison I append a similar table of the least positive residues of $p(n)$ (mod 11) for various values of n.

n	1	12	23	34	45	56	67	78	89	100
lpr	1	0	1	1	1	0	1	0	1	1
n	2	13	24	35	46	57	68	79	90	101
lpr	2	2	2	0	2	2	2	2	2	2
n	3	14	25	36	47	58	69	80	91	102
lpr	3	3	0	3	3	0	3	3	3	3
n	4	15	26	37	48	59	70	81	92	103
lpr	5	0	5	0	5	0	5	0	5	5
n	5	16	27	38	49	60	71	82	93	104
lpr	7	0	7	0	0	7	7	0	7	0

The regularity of this table is of precisely the same character as the regularity of the previous one. One is thus led irresistibly to the conclusion that there must be some analogue modulo 11 to the relations (26).

I hold in fact:

That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and denote by $M(m, q, n)$ the number of partitions of n whose crank is congruent to m modulo q;

that $M(m, q, n) = M(q - m, q, n)$;

that

$$M(0, 11, 11n + 6) = M(1, 11, 11n + 6) = M(2, 11, 11n + 6) \\ = M(3, 11, 11n + 6) = M(4, 11, 11n + 6);$$

that numerous other relations exist analogous to (12)–(19), and in particular

$$M(1, 11, 11n+1) = M(2, 11, 11n+1) = M(3, 11, 11n+1) \\ = M(4, 11, 11n + 1);$$

that $M(m, 11, n)$ has a generating function not completely different in form from (24);

that the values of the differences such as $M(0, 11, n) - M(4, 11, n)$ are always extremely small compared with $p(n)$.

Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

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Short Vision

By A. C. Falconer

Thought is the only way which leads to life
All else is hollow spheres
Reflecting back
In heavy imitation
And blurred degeneration
A senseless image of our world of thought.

Man *thinks* he is the thought which gives him life!
He binds a sheaf and claims it as himself!
He *is* a ring through which pass swinging ropes
Which merely move a little as he slips.

The Ropes are Thought
The Space is Time
Could he but see, then he might climb.

Book Review

Generalised Foreign Politics. By LEWIS F. RICHARDSON, F.R.S. (C.U.P., 8s. 6d.)

This is a book to which neither Professor Hardy nor Professor Hogben could take exception. The title is sufficient indication of its wide practical interest, but the mathematics is complete in itself; some of the theorems relate to unobservable worlds, *e.g.*, "If each pair of three nations be separately stable, the triplet may nevertheless be unstable." The author suggests substitution of the word "dog" for "nation," in which case the theorem is verifiable.

The whole work is intended to represent what would happen if "instinct and tradition were allowed to act uncontrolled," *i.e.*, if nobody read *Generalised Foreign Politics.* Each nation is represented by a single variable, which indicates in some way its warlike preparations and aggressiveness. The precise meaning is interpreted at leisure by the author in the statistical sections of the book, which show a very close agreement with the theory (particularly for the 1910–1914 arms race).

He considers first the case of two nations only. Nations increase their armaments more rapidly if their neighbours are heavily armed, but the cost of upkeep has a restraining effect; he assumes then that:—

$$\frac{dx}{dt} = ky - \alpha x + g, \qquad \frac{dy}{dt} = lx - \beta y + h,$$

where k and l are positive "defence coefficients," α and β are positive "fatigue coefficients," g and h being constants to represent "fixed grievances." Negative x and y are interpreted as co-operation and trade. War occurs when x and y tend to positive infinity with t. The equations can be solved by the usual methods and it is easily seen that if $\alpha\beta > kl$, $x \to x_0$ and $y \to y_0$ as $t \to \infty$, where:—

$$ky_0 - \alpha x_0 + g = 0$$
$$\text{and} \quad lx_0 - \beta y_0 + h = 0.$$

If $\alpha\beta < kl$, the system is in equilibrium at (x_0, y_0), called "the point of balance of power," but from any other starting point the system will tend either to war or to extreme co-operation. The author neglects the case $\alpha\beta = kl$; here there is no point of balance of power, and the system is unstable for all starting-points unless also $\frac{\alpha}{l} = -\frac{g}{h}$. It is interesting to note that neither unilateral disarmament nor mutual disarmament without satisfaction are permanent.

There is an additional difficulty when we come to n nations, as the war-like preparations of any one is really a vector, with components directed against the other nations severally. However, if we take the total war-like preparations x_i of the i^{th} nation and its total grievance g_i, the equations:—

$$\frac{dx_i}{dt} = g_i + \sum_{j=1}^{j=n} k_{ij} x_j \, (i = 1 \text{ to } n)$$

follow from the more accurate equations which allow for "directed intentions." k_{ii} is negative, being a fatigue coefficient, while k_i is positive for $i \neq j$. It can be shown that a sufficient condition for stability of peace is that the matrix $[\frac{1}{2}(k_{ij} + k_{ji})]$ has no positive latent root. The world in 1935 was unstable. The only problem for which directed intentions are needed is that of pacifism. It is found that if there are two pugnacious nations and one pacifist nation (that is, a nation whose defence coefficients are zero and whose grievances are ≤ 0), the preparations of the pugnacious nations against the pacifist nation tend to a constant, which depends only upon old grievances against the pacifist nation.

The good agreement between this theory and fact shows that foreign politics in peace-time run in grooves which can only be changed by an effort; it also shows that if the defence coefficients were negative, as many statesmen appear to think, arms races could not occur. One hopes that in time foreign politics will become a branch of Pure Mathematics and that Part III of the Tripos will contain questions such as:—"Prove that given any just-stable world another can in general be formed from it by making one pair of nations more equal and another pair less equal, and indicate the cases of failure."

A. O. L. A.

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