Understanding Universal Algebra Using Kleisli-Eilenberg-Moore-Lawvere Diagrams v0.2

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Abstract

This note presents my current understanding of (multisorted/essential/generalized) algebraic theories via what I call KEML-diagrams. Algebraic theories can be represented as monads to which we can associate a Kleisli category, an Eilenberg-Moore category and a Lawvere-style category. KEML-diagrams show clearly how these correspond.

No particular novelty is claimed, except perhaps the presentational novelty of KEML-diagrams. Additionally, our precise definition of Lawvere theories for multisorted and essentially algebraic theories seems to diverge slightly from the literature. Finally, we think there is value in contrasting Arkor and McDermott's higher-order algebraic theories (HOATs) with a much simpler and already powerful notion of HOATs given by extensions of the simply-typed lambda calculus.

Contents

1 Overview

After some introductory remarks and prerequisites in sections 2 and 3, we continue to study simple (i.e. un(i)sorted) algebraic theories (SATs, section 4), multisorted algebraic theories (MATs, section 5), essential and (briefly) generalized algebraic theories (EATs and GATs, section 6), extensions of multisorted algebraic theories (section 7) and higher-order algebraic theories (HOATs, section 8).

Each time, we consider three perspectives on the given class of algebraic theories:

- the *presentation* perspective, where a theory is described very syntactically by listing its operations and the equation axioms they satisfy,
- the *monadic* perspective, where a theory is described as a monad M on a given category $\mathcal{C}$, such that MX is the free model over X (collection of terms containing metavariables from X) of the theory at hand,
- the *Lawverian* perspective, where a theory is described as a category $\mathcal{L}$ whose objects are – roughly – the relevant notion of *arities* and whose morphisms are tuples of terms of the theory.

Each time, we show that monadic and Lawverian theories are actually equivalent notions and that presentations of theories give rise to monadic/Lawverian theories. Moreover, we learn

- that the Kleisli category $\mathsf{Kl}(M)$ of a monadic theory M is equivalent to the opposite of the corresponding Lawvere theory $\mathcal{L}$,[1]
- that the Eilenberg-Moore category $\mathsf{EM}(M)$ of a monadic theory M is equivalent to the category $\mathsf{Model}(\mathcal{L})$ of models of $\mathcal{L}$, which are certain functors $\mathcal{L} \to \mathrm{Set}$,
- that the monad $M : C \to C$ factors over the canonical functor $\mathsf{Kl}(M) \to \mathsf{EM}(M)$ and equivalently over the Yoneda-embedding $\mathbf{y} = \mathrm{Hom}_{\mathcal{L}} : \mathcal{L}^{\mathrm{op}} \to \mathsf{Model}(\mathcal{L})$.

This information is then summarized in what I call a Kleisli-Eilenberg-Moore-Lawvere diagram (KEML-diagram, sections 4.5, 5.5, 6.5, 7.5 and 8.4).

HOATs (section 8) are treated a bit less in-depth. We give our own formulation of what we call λHOATs (defining them as MAT extensions of the simply-typed lambda calculus) and then use a thorough understanding of λHOATs to develop a conceptual understanding of what is going on in Arkor and McDermott's paper on HOATs [AM20].

2 Introductory remarks

- We ignore size issues.
- We don't require our monads to be container functors of finite arity, and correspondingly we don't work with finite products/limits in Lawvere theories.

3 Prerequisites

Notation 3.1. Given a set X and an object $a \in C$, we write $X \cdot a$ for the coproduct $\sum_{x \in X} a$.

Definition 3.2. Given a monad M on C, its Kleisli category $\mathsf{Kl}(M)$ has

- objects $\overline{x}$ where $x \in \mathrm{Obj}(C)$,
- morphisms $\overline{\varphi} : \overline{x} \to \overline{y}$ where $\varphi : x \to My$,
- $\mathrm{id}_{\overline{x}} := \overline{\eta_x}$,
- $\overline{\chi} \circ \overline{\varphi} := \overline{\chi \lll \varphi}$.

There is a functor $J_M : C \to \mathsf{Kl}(M)$ such that $Jx = \overline{x}$ and $J\varphi = \overline{\eta \circ \varphi}$, and a functor $P_M : \mathsf{Kl}(M) \to C$ such that $P_M\overline{x} = Mx$ and $P_M\overline{\varphi} = \mathsf{bind}\,\varphi$. Clearly $P_M J_M = M$.

Theorem 3.3 (Unused)**.** The functor $\mathsf{Monad}(C) \to C/\mathsf{Cat} : M \mapsto (\mathsf{Kl}(M), J_M)$ is fully faithful.

Proof. Given a monad morphism $\zeta : M \to N$, a functor $Z : \mathsf{Kl}(M) \to \mathsf{Kl}(N)$ is given by $Z\overline{x} = \overline{x}$ and $Z\overline{\varphi} = \overline{\zeta \circ \varphi}$.

Faithfulness is obvious: ζ can be retrieved from Z because $Z\overline{\mathrm{id}_{Mx}} = \overline{\zeta_x}$.

To see fullness, pick a morphism $Z : \mathsf{Kl}(M) \to \mathsf{Kl}(N)$ of coslices under C. We define an (a priori unnatural) transformation $\zeta : M \to N$ by $\overline{\zeta_x} = Z\overline{\mathrm{id}_{Mx}}$. Then it is immediate that

$$Z\overline{\varphi} = Z(\overline{\mathrm{id}_{My}} \circ J_M\varphi) = \overline{\zeta_y} \circ J_N\varphi = \overline{\zeta_y \circ \varphi}.$$

We show that ζ is a natural transformation and a monad morphism:

[1] In the case of MATs and EATs we seem to diverge a little from the literature when defining the Lawverian formulation in order for this to be true.

- To see naturality, pick $\varphi : x \to y$. Then

$$\overline{N\varphi \circ \zeta_x} = J_N\varphi \circ \overline{\zeta_x} = J_N\varphi \circ Z\overline{\mathrm{id}_{Mx}} = Z(J_M\varphi \circ \overline{\mathrm{id}_{Mx}}) = Z(\overline{M\varphi}) = \overline{\zeta_y \circ M\varphi}.$$

- To see preservation of unit:

$$\overline{\zeta_x \circ \eta_x^M} = Z(\overline{\eta_x^M}) = Z\mathrm{id}_{\overline{x}} = \mathrm{id}_{\overline{x}} = \overline{\eta_x^N}.$$

- To see preservation of Kleisli composition:

$$\begin{aligned}\overline{\zeta_x \circ (\chi \lll^M \varphi)} &= Z\overline{\chi \lll^M \varphi} \\ &= Z(\overline{\chi} \circ \overline{\varphi}) \\ &= Z\overline{\chi} \circ Z\overline{\varphi} \\ &= \overline{\zeta_z \circ \chi} \circ \overline{\zeta_y \circ \varphi} \\ &= \overline{(\zeta_z \circ \chi) \lll^N (\zeta_z \circ \varphi)}. \qquad \square\end{aligned}$$

4 Simple Algebraic Theories (SATs)

We define SAT presentations (definition 4.1), monadic SATs (definition 4.2) and Lawverian SATs (definition 4.5). We prove:

- Every SAT presentation gives rise to a monadic SAT (proposition 4.3) with the same models (proposition 4.4).
- Every SAT presentation gives rise to a Lawverian SAT (proposition 4.6) with the same models (proposition 4.8).
- Every monadic SAT gives rise to a Lawverian SAT (proposition 4.7) with the same models (proposition 4.9).
- Monadic and Lawverian SATs are essentially the same thing (proposition 4.10).

We summarize in a diagram (section 4.5).

4.1 SAT Presentations

Definition 4.1. A **simple** or **single-sorted algebraic theory presentation** [nLa22a] $\mathfrak{A}$ consists of

- for each arity $n \in \mathbb{N}$, a set of operators $O_{\mathfrak{A}}(n)$,

$$O_{\mathfrak{A}} : \mathbb{N} \to \mathrm{Set},$$

- for each arity $n \in \mathbb{N}$, a set of n-ary axioms, which are pairs (t_1, t_2) of n-ary terms, denoted as $t_1 = t_2$:

$$E_{\mathfrak{A}} : (n \in \mathbb{N}) \to \mathcal{P}(T_{\mathfrak{A}}(n)^2).$$

An n-ary term of $\mathfrak{A}$ ($n \in \mathbb{N}$) is inductively defined to be either a metavariable $\mathbf{m}_i$ ($i \in \{1, \ldots, n\}$) or an expression of the form $o(t_1, \ldots, t_k)$ where o is an operator of arity k and all t_i are also n-ary terms. As such, the set of n-ary terms $T_{\mathfrak{A}}(n)$ is the least fixpoint

$$T_{\mathfrak{A}}(n) = \{\mathbf{m}_1, \ldots, \mathbf{m}_n\} \uplus \sum_{k \in \mathbb{N}, o \in O_{\mathfrak{A}}(k)} T_{\mathfrak{A}}(n)^k.$$

An **algebra** or **model** A of $\mathfrak{A}$ is a set A equipped with a function $[\![o]\!] : A^k \to A$ for every n-ary operator o, satisfying all the axioms. Given $\vec{a} = (a_1, \ldots, a_n) \in A^n$, the interpretation of an n-ary term t is defined recursively by $[\![o(t_1,\dots,t_k)]\!]_{\vec{a}} = [\![o]\!]([\![t_1]\!]_{\vec{a}},\dots,[\![t_k]\!]_{\vec{a}})$ and $[\![\mathbf{m}_i]\!]_{\vec{a}} = a_i$. We say that A satisfies the n-ary axiom $t_1 = t_2$ if $[\![t_1]\!]_{\vec{a}} = [\![t_2]\!]_{\vec{a}}$ for all $\vec{a} \in A^n$.

A **morphism of algebras/models** is a function $f : A \to B$ such that $f \circ [\![o]\!] = [\![o]\!] \circ f^k : A^k \to B$ for every k-ary operator o.

The **syntax** of $\mathfrak{A}$ is the initial object of the category of $\mathfrak{A}$-models. It is the set of nullary terms of $\mathfrak{A}$, divided by the equivalence relation generated by the axioms.

We do not consider equality, equivalence, or homomorphism of SAT *presentations*. Instead, before relating different SATs, we more towards a more extensional means of defining them. A first one is using monads (section 4.2).

4.1.1 Type Analogy

SATs can be thought of as quotient inductive types (of the SAT's terms) in a simply typed language. Simply typed, so we do not get to assume any equalities dependent on arguments. Quotient, so we can have axioms.

4.1.2 Examples

The following are examples of theories that can be presented as SATs:

- Monoids, groups, rings, fields, …
- Modules over a fixed ring, vector spaces over a fixed field, actions of a fixed group, …

  In the case of e.g. vector spaces over $\mathbb{K}$, we will have a binary addition operator and for every $k \in \mathbb{K}$ a unary scalar multiplication operator.
- Typed arithmetic in the sense of TAPL [Pie02] with only booleans.

4.2 Monadic SATs

Definition 4.2. A **monadic SAT** is a monad on $\mathcal{C} := \mathsf{Set}$.

Proposition 4.3. Every SAT presentation $\mathfrak{A}$ gives rise to a monadic SAT $M_{\mathfrak{A}}$.

Construction. We first define a *syntax endofunctor* $F_{\mathfrak{A}} : \mathcal{C} \to \mathcal{C}$ by

$$F_{\mathfrak{A}}\, X = \sum_{k\in\mathbb{N}, o\in O_{\mathfrak{A}}(k)} X^k.$$

Next, let $F^*_{\mathfrak{A}}$ be the free monad over $F_{\mathfrak{A}}\, X$. Effectively, $F^*_{\mathfrak{A}}\, X$ is now the set of X-ary terms.

This is a model of $\mathfrak{A}^*$, the SAT presentation obtained from $\mathfrak{A}$ by removing all axioms. Now let $\sim$ be the strictest reflexive and transitive X-indexed congruence on $F^*_{\mathfrak{A}}\, X$ that satisfies all the axioms. By congruence we mean that $M_{\mathfrak{A}}\, X := F^*_{\mathfrak{A}}\, X/{\sim_X}$ defines a monad. Then we take this as the monad associated to $\mathfrak{A}$. Effectively, $M_{\mathfrak{A}}\, X$ is now the set of X-ary terms modulo the axioms. □

Proposition 4.4. The category $\mathsf{EM}(M_{\mathfrak{A}})$ of Eilenberg-Moore algebras of $M_{\mathfrak{A}}$ is isomorphic to the category $\mathsf{Model}(\mathfrak{A})$ of models of $\mathfrak{A}$, and the isomorphism commutes with the forgetful functors to $\mathcal{C}$ on the nose.

Proof. We first establish the following:

$$\mathsf{EM}(M_{\mathfrak{A}}) \subseteq_1 \mathsf{EM}(F^*_{\mathfrak{A}}) \cong_2 \mathsf{Alg}(F) \cong_3 \mathsf{Model}(\mathfrak{A}^*) \supseteq_4 \mathsf{Model}(\mathfrak{A}),$$

where the inclusions denote fully faithful embeddings injective on objects.

1. The quotient morphism $[\sqcup] : F^*_{\mathfrak{A}} \to M_{\mathfrak{A}}$ constitutes a monad morphism so that every EM-algebra of $M_{\mathfrak{A}}$ is trivially an EM-algebra of $F^*_{\mathfrak{A}}$, and similar for morphisms. Injectivity on objects follows from surjectivity of $[\sqcup]$. The embedding is faithful as both categories are faithful subcategories of $\mathcal{C}$. Fullness follows from surjectivity of $[\sqcup]$.

2. This is a well-known property of free monads.
3. This is trivial.
4. The inclusion of objects is trivial. The definition of a morphism of models bears no mention of the set of axioms, so the inclusion of morphisms is automatically fully faithful.

So it remains to be shown that an object of the central three isomorphic categories is an EM-algebra of $M_{\mathfrak{A}}$ (i.e. respects $\sim$) if and only if it is a model of $\mathfrak{A}$ (i.e. respects the axioms that generate $\sim$). This is automatic as equality of terms' interpretation in the carrier is automatically a reflexive and transitive congruence. □

4.3 Lawverian SATs from Monadic SATs

Definition 4.5. A **Lawverian SAT** is a category $\mathcal{L}$ satisfying the following properties:

- it has products, algebraically,
- it contains a designated object $\star$,
- every object is an (algebraically obtained) product of $\star$'s up to a designated isomorphism (i.e. there is an algebraic operation for factorizing any object as an algebraically obtained product of $\star$'s up to a designated isomorphism).

A **morphism of Lawverian SATs** is a product-, $\star$- and factorization-preserving functor $\mathcal{L} \to \mathcal{L}'$. This constitutes a 2-category Law.

A **model** of $\mathcal{L}$ is a product-preserving functor $\mathcal{L} \to \mathrm{Set}$. The forgetful functor $U : \mathsf{Model}(\mathcal{L}) \to C$ is defined by $U\,T = T\,\star$.

Proposition 4.6. Every SAT presentation $\mathfrak{A}$ gives rise to a Lawverian SAT $\mathcal{L}_{\mathfrak{A}}$.

Construction. This follows immediately from the following proposition. □

Proposition 4.7. Every monadic SAT M gives rise to a Lawverian SAT $\mathcal{L}_M$, functorially.

Construction. We define $\mathcal{L}_M := \mathsf{Kl}(M)^{\mathrm{op}}$, the opposite Kleisli category of M:

- Products are given by coproducts in $\mathsf{Kl}(M)$, which are in turn given by coproducts in $C = \mathrm{Set}$, which are disjoint unions.
- We define $\star = \overline{\top}$ where $\top = \{()\}$.
- The object $\overline{X}$ can be written as
$$\overline{X} \cong \overline{X \cdot \top} = \star^X.$$
□

Let us consider what this means. By the universal property of the product, the full Hom-structure of $\mathcal{L}_M$ is given by the Hom-sets with codomain $\star$. There we have:

$$\mathrm{Hom}_{\mathcal{L}_M}\left(\star^X, \star\right) \cong \mathrm{Hom}_{\mathcal{L}_M}\left(\overline{X}, \star\right) = \mathrm{Hom}_{\mathsf{Kl}(M)}\left(\overline{\top}, \overline{X}\right) = \mathrm{Hom}_C(\top, MX) \cong MX,$$

which we interpreted before as the set of X-ary terms modulo equality. Hence, more generally, we have

$$\mathrm{Hom}_{\mathcal{L}_M}\left(\star^Y, \star^X\right) \cong (MY)^X,$$

the set of X-tuples of Y-ary terms. The identity (monadic unit) corresponds to the tuple of all variables, and composition (Kleisli composition) amounts to substitution. The functor $J_M^{\mathrm{op}} : C^{\mathrm{op}} \to \mathcal{L}_M$ creates *pure* morphisms (i.e. morphisms without monadic effects), which are the renamings.

Proposition 4.8. The category $\mathsf{Model}(\mathcal{L}_{\mathfrak{A}})$ of models of $\mathcal{L}_{\mathfrak{A}}$ is equivalent to the category $\mathsf{Model}(\mathfrak{A})$ of models of $\mathfrak{A}$, and the equivalence commutes with the forgetful functors to C.

Proof. This follows immediately from the following proposition. □

Proposition 4.9. The category $\mathsf{Model}(\mathcal{L}_M)$ of models of $\mathcal{L}_M$ is equivalent to the category $\mathsf{EM}(M)$ of EM-algebras of M, and the equivalence commutes with the forgetful functors to C.

Proof. $\boxed{\rightarrow}$ Given a model $T \in \mathsf{Model}(\mathcal{L}_M)$, i.e. a product-preserving functor $\mathcal{L}_M \to \mathsf{Set}$, we build an EM-algebra A. For the carrier, we (must) take $A := T\star$. Recall that

$$\iota : (MY)^X \cong \mathrm{Hom}_{\mathcal{L}}(\star^Y, \star^X) : f \mapsto \overline{f}$$

so that the functorial action of T creates an operation

$$\ggg\!\!>^A : (MY)^X \to A^Y \to A^X,$$

i.e. $T\,\overline{f}\,g = f \ggg\!\!>^A g$. By functoriality and product preservation w.r.t. $\star^X$ (functoriality and coproduct preservation w.r.t. X), $\ggg\!\!>^A$ is a componentwise application of some 'A-specific bind' operation

$$\gg\!=^A : MY \to A^Y \to A,$$

i.e. $(f \ggg\!\!>^A g)(x) = f(x) \gg\!=^A g$.

- Since T preserves the identity, we know that $\eta \ggg\!\!>^A g = g$ and hence, applying to x, that $\eta(x) \gg\!=^A g = g(x)$,
- Since T preserves composition, we know that

  $$(f \ggg\!\!> g) \ggg\!\!>^A h = f \ggg\!\!>^A (g \ggg\!\!>^A h)$$

  and hence, considered componentwise, that

  $$(my \gg\!= g) \gg\!=^A h = my \gg\!=^A (g \ggg\!\!>^A h).$$

- By naturality w.r.t. Y, we know that

  $$(Mg \circ f) \ggg\!\!>^A h = f \ggg\!\!>^A (h \circ g)$$

  and hence, considered componentwise, that

  $$(Mg)(my) \gg\!=^A h = my \gg\!=^A (h \circ g).$$

We now define the algebra morphism

$$\alpha : MA \to A : ma \mapsto ma \gg\!=^A \mathrm{id},$$

i.e. $\alpha = \mathrm{id}_{MA} \ggg\!\!>^A \mathrm{id}_A$. Then we have

$$\begin{aligned}
\alpha(\eta(a)) &= \eta(a) \gg\!=^A \mathrm{id} = a,\\
\alpha(\mu(mma)) &= (mma \gg\!= \mathrm{id}) \gg\!=^A \mathrm{id}\\
&= mma \gg\!=^A (\mathrm{id} \ggg\!\!>^A \mathrm{id})\\
&= mma \gg\!=^A \alpha = M\alpha(mma) \gg\!=^A \mathrm{id} = (\alpha \circ M\alpha)(mma),
\end{aligned}$$

so indeed we have an EM-algebra for M. To see that this constitutes a functor $\mathsf{Model}(\mathcal{L}_M) \to \mathsf{EM}(M)$, assume a natural transformation $\theta : T \to T'$ and write $f = \theta_\star : A \to A'$. We have

$$f(\alpha(ma)) = f(ma \gg\!=^A \mathrm{id}) = ma \gg\!=^{A'} (f \circ \mathrm{id}) = Mf(ma) \gg\!=^{A'} \mathrm{id} = \alpha'(Mf(ma)).$$

$\boxed{\leftarrow}$ Given an EM-algebra $A \in \mathsf{EM}(M)$, we build a product-preserving functor $T : \mathcal{L}_M \to \mathrm{Set}$. We define $T\star^X := A^X$. To make this functorial, we require an operation $\ggg^A$ as above, satisfying the identity and composition laws proven above. (The naturality law follows automatically as naturality w.r.t. Y is just composition with a pure morphism.)

We define

$$\begin{aligned} & f \ggg^A g := \alpha \circ Mg \circ f \\ \Rightarrow \qquad & mx \gg=^A g = (\alpha \circ Mg)(mx). \end{aligned}$$

Then we have

$$\begin{aligned} \eta(x) \gg=^A f &= (\alpha \circ Mf \circ \eta)(x) = (\alpha \circ \eta \circ f)(mx) = f(mx), \\ (f \ggg g) \ggg^A h &= \alpha \circ Mh \circ \mu \circ Mg \circ f \\ &= \alpha \circ \mu \circ MMh \circ Mg \circ f \\ &= \alpha \circ M(\alpha \circ Mh \circ g) \circ f \\ &= f \ggg^A (\alpha \circ Mh \circ g) = f \ggg^A (g \ggg^A h), \end{aligned}$$

so indeed we have a model of $\mathcal{L}_M$. To see that this constitutes a functor $\mathsf{EM}(M) \to \mathsf{Model}(\mathcal{L}_M)$, assume an algebra morphism $f : A \to A'$. We shall construct a natural transformation $\theta : T \to T'$. On objects, we set $\theta_{\star^X} = f^X : A^X \to (A')^X$. On morphisms, we have

$$f \circ (g \ggg^A h) = (f \circ \alpha \circ Mh \circ g) = (\alpha' \circ M(f \circ h) \circ g)(mx) = g \ggg^{A'} f \circ h.$$

$\boxed{\simeq}$ It is straightforward to check that the two functors constructed above constitute an equivalence of categories $\mathsf{Model}(\mathcal{L}_M) \simeq \mathsf{EM}(M)$. □

4.4 Monadic SATs from Lawverian SATs

Proposition 4.10. The functor $\mathsf{Monad}(\mathrm{Set}) \to \mathsf{Law} : M \mapsto \mathcal{L}_M$ is a 2-equivalence.

Proof. $\boxed{\leftarrow}$ To a Lawverian SAT $\mathcal{L}$, we associate a monad $M_\mathcal{L}$ on $C = \mathrm{Set}$ given by

- $M_\mathcal{L}X := \mathrm{Hom}_\mathcal{L}(\star^X, \star)$,
- $(M_\mathcal{L}f)(mx) : mx \circ \star^f = mx \circ (\pi_{f(x)})_x$
- $\eta(x) := \pi_x$,
- $\mu(mmx) := mmx \circ (mx)_{mx}$.

The unit and multiplication are indeed natural:

$$\begin{aligned} (M_\mathcal{L}f \circ \eta)(x) &= (M_\mathcal{L}f)(\pi_x) = \pi_x \circ (\pi_{f(x)})_x = \pi_{f(x)} = (\eta \circ f)(x), \\ (M_\mathcal{L}f \circ \mu)(mmx) &= (M_\mathcal{L}f)(mmx \circ (mx)_{mx}) \\ &= mmx \circ (mx)_{mx} \circ (\pi_{f(x)})_x, \\ (\mu \circ M_\mathcal{L}M_\mathcal{L}f)(mmx) &= \mu(mmx \circ (\pi_{(M_\mathcal{L}f)(mx)})_{mx}) \\ &= mmx \circ (\pi_{(M_\mathcal{L}f)(mx)})_{mx} \circ (my)_{my} \\ &= mmx \circ (\pi_{(M_\mathcal{L}f)(mx)} \circ (my)_{my})_{mx} \\ &= mmx \circ ((M_\mathcal{L}f)(mx))_{mx} \\ &= mmx \circ (mx \circ (\pi_{f(x)})_x)_{mx} \\ &= mmx \circ (mx)_{mx} \circ (\pi_{f(x)})_x, \end{aligned}$$

and satisfy the monad laws:

$$
\begin{aligned}
(\mu \circ \eta)(mx) &= \mu(\pi_{mx}) = \pi_{mx} \circ (mx')_{mx'} = mx, \\
(\mu \circ M_{\mathcal{L}}\eta)(mx) &= \mu(mx \circ (\pi_{\eta(x)})_x) \\
&= mx \circ (\pi_{\eta(x)})_x \circ (mx')_{mx'} \\
&= mx \circ (\pi_{\eta(x)} \circ (mx')_{mx'})_x \\
&= mx \circ (\eta(x))_x = mx \circ (\pi_x)_x = mx \circ \mathrm{id} = mx, \\
(\mu \circ \mu)(mmmx) &= \mu(mmmx \circ (mmx)_{mmx}) \\
&= mmmx \circ (mmx)_{mmx} \circ (mx)_{mx} \\
(\mu \circ M_{\mathcal{L}}\mu)(mmmx) &= \mu(mmmx \circ (\pi_{\mu(mmx)})_{mmx}) \\
&= mmmx \circ (\pi_{\mu(mmx)})_{mmx} \circ (mx)_{mx} \\
&= mmmx \circ (\pi_{\mu(mmx)} \circ (mx)_{mx})_{mmx} \\
&= mmmx \circ (\mu(mmx))_{mmx} \\
&= mmmx \circ (mmx \circ (mx)_{mx})_{mmx} \\
&= mmmx \circ (mmx)_{mmx} \circ (mx)_{mx}.
\end{aligned}
$$

$\boxed{\text{Law}}$ We can now observe that $\mathcal{L} \simeq \mathsf{Kl}(M_{\mathcal{L}})^{\mathrm{op}} = \mathcal{L}_{M_{\mathcal{L}}}$. Indeed, we have $\mathrm{Hom}_{\mathcal{L}}(\star^Y, \star^X) \cong \mathrm{Hom}_{\mathcal{L}}(\star^Y, \star)^X = (M_{\mathcal{L}}Y)^X$. Under this isomorphism,

- the identity morphism at $\star^X$ is sent to $(\lambda x.\pi_x) = \eta$,
- for $f \in \mathrm{Hom}_{\mathcal{L}}(\star^Y, \star^X)$ and $g \in \mathrm{Hom}_{\mathcal{L}}(\star^Z, \star^Y)$, the composite morphism $f \circ g$ is sent to $(\lambda x.\pi_x \circ f \circ g)$ which should be equal to the Kleisli composition of $(\lambda x.\pi_x \circ f)$ and $(\lambda y.\pi_y \circ g)$.

$$
\begin{aligned}
&(\lambda x.\pi_x \circ f) \ggg (\lambda y.\pi_y \circ g) \\
&= \mu \circ M_{\mathcal{L}}(\lambda y.\pi_y \circ g) \circ (\lambda x.\pi_x \circ f) && \text{(Def. } \ggg\text{)} \\
&= \lambda x.(\mu \circ M_{\mathcal{L}}(\lambda y.\pi_y \circ g) \circ (\lambda x'.\pi_{x'} \circ f))(x) && \text{(}\eta\text{-expand function)} \\
&= \lambda x.(\mu \circ M_{\mathcal{L}}(\lambda y.\pi_y \circ g))(\pi_x \circ f) && \text{(Apply last function)} \\
&= \lambda x.\mu\Big(\pi_x \circ f \circ (\pi_{\pi_y \circ g})_y\Big) && \text{(Map over } M_{\mathcal{L}}\text{)} \\
&= \lambda x.\pi_x \circ f \circ (\pi_{\pi_y \circ g})_y \circ (mz)_{mz} && \text{(Def. } \mu\text{)} \\
&= \lambda x.\pi_x \circ f \circ (\pi_{\pi_y \circ g} \circ (mz)_{mz})_y && \text{(Precompose componentwise)} \\
&= \lambda x.\pi_x \circ f \circ (\pi_y \circ g)_y && \text{(Projection from tuple)} \\
&= \lambda x.\pi_x \circ f \circ g. && \text{(}\eta\text{-contract tuple)}
\end{aligned}
$$

So indeed $\mathcal{L} \simeq \mathsf{Kl}(M_{\mathcal{L}})^{\mathrm{op}} = \mathcal{L}_{M_{\mathcal{L}}}$.

This equivalence is natural w.r.t. $\mathcal{L}$: if we have a morphism of Lawverian SATs $L : \mathcal{L} \to \mathcal{L}'$, then we get a monad morphism $\omega_L : M_{\mathcal{L}} \to M_{\mathcal{L}'} : \mathcal{C} \to \mathcal{C}$ such that $\omega_L(mx) = L\, mx$. This leads to a functor $L_{\omega_L} : \mathcal{L}_{M_{\mathcal{L}}} \to \mathcal{L}_{M_{\mathcal{L}'}}$ which is easily seen to fit in a commutative diagram

$$
\begin{tikzcd}
\mathcal{L} \arrow[r, "\simeq"] \arrow[d, "L"'] & \mathcal{L}_{M_{\mathcal{L}}} \arrow[d, "L_{\omega_L}"'] \\
\mathcal{L}' \arrow[r, "\simeq"] & \mathcal{L}_{M_{\mathcal{L}'}}.
\end{tikzcd}
$$

$\boxed{\text{Monad(Set)}}$ For any monadic SAT M, we have an isomorphism of monads $M \cong M_{\mathcal{L}_M}$. Indeed, we have

$$\iota : M_{\mathcal{L}_M}X = \mathrm{Hom}_{\mathcal{L}_M}(\star^X, \star) = \mathrm{Hom}_{\mathsf{Kl}(M)}(\top, X \cdot \top) \cong MX.$$

Writing a prime ($'$) for the operations of $M_{\mathcal{L}_M}$, we have

$$\begin{aligned}
\iota(\eta'(x)) &= \iota(\pi_x^{\mathcal{L}_M}) = \iota(\iota_x^{\mathsf{Kl}(M)}) = \eta(x) \in MX. \\
\iota(mx \gg=' f) &= (\iota \circ \mu' \circ M' f)(mx) \\
&= (\iota \circ \mu')(mx \circ (\pi_{f(x)})_x) \\
&= \iota(mx \circ (\pi_{f(x)})_x \circ (my)_m y) \\
&= \iota(mx \circ (\pi_{f(x)} \circ (my)_m y)_x) \\
&= \iota(mx \circ (f(x))_x) \\
&= \iota([f(x)]_x \circ mx) && \text{dualize} \\
&= \iota(mx) \gg= (\iota \circ f).
\end{aligned}$$

This isomorphism of monads is easily seen to be natural w.r.t. M. ◻

4.5 The KEML-diagram for SATs

We can summarize our findings about SATs in a diagram. Assume that a monadic SAT M and a Lawverian SAT $\mathcal{L}$ correspond. Then we obtain a commuting diagram as follows:

$$\begin{tikzcd}
\mathsf{Set} \arrow[rrr, "M"] \arrow[dd, "X \mapsto \star^X"'] \arrow[dr, "J_M"] & & & \mathsf{Set} \\
& \mathsf{Kl}(M) \arrow[r, "\overline{X} \mapsto (MX,\mu)"] \arrow[dl, dash, "\simeq"] & \mathsf{EM}(M) \arrow[ur, "U"'] \arrow[dr, dash, "\simeq"] & \\
\mathcal{L}^{\mathrm{op}} \arrow[rrr, "\mathrm{Hom}_{\mathcal{L}}"'] & & & \mathsf{Model}(\mathcal{L}) \arrow[uu, "U : T \mapsto T\star"']
\end{tikzcd}$$

The existence of each of the arrows is clear; in the case of the equivalence $\mathcal{L}^{\mathrm{op}} \simeq \mathsf{Kl}(M)$ it follows from the correspondence of $\mathcal{L}$ and M (propositions 4.7 and 4.10). The commutation of the triangle on the left is clear by construction of $\mathcal{L}_M$ (proposition 4.7). Commutation of the upper trapezium is a general theorem about monads. Commutation of the right triangle follows from proposition 4.9, and commutation of the lower trapezium follows from its proof.

4.6 Discussion

4.6.1 The Empty SAT presentation

A SAT presentation with zero operators gives rise to the identity monad.

- Then $J_{\mathrm{Id}} : \mathrm{Set} \cong \mathsf{Kl}(\mathrm{Id})$ is an isomorphism,
- In the Lawverian SAT, *all* morphisms are tuples of projections, so that $X \mapsto \star^X : \mathrm{Set} \simeq \mathcal{L}^{\mathrm{op}}$ is an equivalence,
- Being an EM-algebra is void, so $U : \mathsf{EM}(\mathrm{Id}) \cong \mathrm{Set}$ is an isomorphism,
- Product preservation determines $T \in \mathsf{Model}(\mathcal{L})$ fully from $T\star$ so $U : \mathsf{Model}(\mathcal{L}) \simeq \mathrm{Set} : T \mapsto T\star$ is an equivalence,
- Then $\mathrm{Hom}_{\mathcal{L}} : \mathcal{L}^{\mathrm{op}} \simeq \mathsf{Model}(\mathcal{L})$ is also an equivalence.

5 Multisorted Algebraic Theories (MATs)

We define MAT presentations (definition 5.1), monadic MATs (definition 5.2) and Lawverian MATs (definition 5.5). We prove:

- Every MAT presentation gives rise to a monadic MAT (proposition 5.3) with the same models (proposition 5.4).
- Every MAT presentation gives rise to a Lawverian MAT (proposition 5.6) with the same models (proposition 5.8).
- Every monadic MAT gives rise to a Lawverian MAT (proposition 5.7) with the same models (proposition 5.9).
- Monadic and Lawverian MATs are essentially the same thing (proposition 5.10).

We summarize in a diagram (section 5.5).

5.1 MAT Presentations

Definition 5.1. A **multisorted algebraic theory presentation** [nLa22a] $\mathfrak{A}$ consists of:

- a set $S_\mathfrak{A}$ of sorts,
- for each arity $\vec{r} \in \mathsf{List}\, S$ and output sort $s \in S$, a set of operators $O_\mathfrak{A}(\vec{r}, s)$,

$$O_\mathfrak{A} : \mathsf{List}\, S \times S \to \mathsf{Set},$$

- for each arity $\vec{r} \in \mathsf{List}\, S$ and output sort $s \in S$, a set of $\vec{r}$-ary axioms of sort s, which are pairs (t_1, t_2) of $\vec{r}$-ary terms of sort s, denoted as $t_1 = t_2$:

$$E_\mathfrak{A} : ((\vec{r}, s) \in \mathsf{List}\, S \times S) \to \mathcal{P}(T_\mathfrak{A}(\vec{r}, s)^2).$$

An $\vec{r}$-ary term of output sort s is inductively defined to be either a metavariable $\mathbf{m}_i$ where $r_i = s$, or an expression of the form $o(t_1, \ldots, t_k)$, where o has arity $\vec{q}$ of length k and output sort s and every t_i is an $\vec{r}$-ary term of output sort q_i:

$$T_\mathfrak{A}(\vec{r}, s) = \{\mathbf{m}_i \mid r_i = s\} \uplus \sum_{\vec{q} \in \mathsf{List}\, A, o \in O_\mathfrak{A}(\vec{q}, s)} \prod_i T_\mathfrak{A}(\vec{r}, q_i).$$

An **algebra** or **model** A of $\mathfrak{A}$ consists of a set A_s for every sort s, and functions $[\![o]\!] : (\prod_i A_{r_i}) \to A_s$ for every $\vec{r}$-ary operator o of output sort s, satisfying the axioms. Given $\vec{a} = (a_1, \ldots, a_n) \in \prod_i A_{r_i}$, the interpretation of an $\vec{r}$-ary term t is defined recursively by $[\![o(t_1, \ldots, t_k)]\!]_{\vec{a}} = [\![o]\!]([\![t_1]\!]_{\vec{a}}, \ldots, [\![t_k]\!]_{\vec{a}})$ and $[\![\mathbf{m}_i]\!]_{\vec{a}} = a_i$. We say that A satisfies the axiom $t_1 = t_2$ of arity $\vec{r}$ if $[\![t_1]\!]_{\vec{a}} = [\![t_2]\!]_{\vec{a}}$ for all $\vec{a} \in \prod_i A_{r_i}$.

A **morphism of algebras/models** is a pointwise function $f_s : A_s \to B_s$ such that $f_s \circ [\![o]\!] = [\![o]\!] \circ \prod_i f_{r_i} : \prod_i A_{r_i} \to B$ for every $\vec{r}$-ary operator o of sort s.

The **syntax** of $\mathfrak{A}$ is the initial object of the category of $\mathfrak{A}$-models. It is the S-indexed set of nullary terms of $\mathfrak{A}$, divided by the equivalence relation generated by the axioms.

We do not consider equality, equivalence, or homomorphism of MAT *presentations*. Instead, before relating different MATs, we more towards a more extensional means of defining them. A first one is using monads (section 5.2).

5.1.1 Type Analogy

MATs can be thought of as mutual quotient inductive types (of the MAT's terms for every sort) in a simply typed language. Simply typed, so we do not get to assume any equalities dependent on arguments. Quotient, so we can have axioms.

5.1.2 Examples

The following are examples of theories that can be presented as MATs:

- Examples of two-sorted algebraic theories are dependent pairs of:
  - A field and a vector space over it,
  - A ring and a module over it,
  - A group and an action of it on a set.
- Presheaves over a fixed base category $\mathcal{W}$ (one sort for every object of $\mathcal{W}$ and one unary operator for every morphism),
- Typed arithmetic in the sense of TAPL [Pie02], where the sorts are the types,
- The simply type λ-calculus (STLC) with variables formalized as de Bruijn indices and with an explicit substitution operator. There is a set of types Ty and a set of contexts Ctx and the sorts are

$$\{\mathrm{Sub}(\Gamma, \Delta) \mid \Gamma, \Delta \in \mathrm{Ctx}\} \uplus \{\mathrm{Tm}(\Gamma, T) \mid \Gamma \in \mathrm{Ctx}, T \in \mathrm{Ty}\}.$$

5.2 Monadic MATs

Definition 5.2. A **monadic MAT** with set of sorts S is a monad on $C := \mathrm{Set}^S$.

Proposition 5.3. Every MAT presentation $\mathfrak{A}$ gives rise to a monadic MAT $M_{\mathfrak{A}}$.

Construction. We first define a *syntax endofunctor* $F_{\mathfrak{A}} : C \to C$ by

$$F_{\mathfrak{A}}\, X\, s = \sum_{o \in \omega^{-1}(s)} \prod_i X\, \alpha(o)_i.$$

Next, let $F^*_{\mathfrak{A}}$ be the free monad over $F_{\mathfrak{A}}\, X$. Effectively, $F^*_{\mathfrak{A}}\, X\, s$ is now the set of s-sorted terms with r-sorted variables from $X\, r$.

This is a model of $\mathfrak{A}^*$, the MAT presentation obtained from $\mathfrak{A}$ by removing all axioms. Now let $\sim$ be the strictest reflexive and transitive (X, s)-indexed congruence on $F_{\mathfrak{A}}\, X\, s$ that satisfies all the axioms. By congruence we mean that $M_{\mathfrak{A}}\, X\, s := F_{\mathfrak{A}}\, X\, s/{\sim_s}$ defines a monad. Then we take this as the monad associated to $\mathfrak{A}$. Effectively, $M_{\mathfrak{A}}\, X\, s$ is now the set of s-sorted terms with r-sorted variables from $X\, r$, modulo the axioms. □

Proposition 5.4. The category $\mathsf{EM}(M_{\mathfrak{A}})$ of Eilenberg-Moore algebras of $M_{\mathfrak{A}}$ is isomorphic to the category $\mathsf{Model}(\mathfrak{A})$ of models of $\mathfrak{A}$, and the isomorphism commutes with the forgetful functors to C on the nose.

Proof. The proof could literally be copy-pasted from proposition 4.4. □

5.3 Lawverian MATs from Monadic MATs

Definition 5.5. A **Lawverian MAT** with set of sorts S is a category $\mathcal{L}$ satisfying the following properties:

- it has products, algebraically,
- it is equipped with a function $\langle \sqcup \rangle : S \to \mathrm{Obj}(\mathcal{L})$,
- every object is an (algebraically obtained) product of objects in the image of $\langle \sqcup \rangle$ up to a designated isomorphism (i.e. there is an algebraic operation for factorizing any object as an algebraically obtained product of $\langle \sqcup \rangle$'s up to a designated isomorphism).

A **morphism of Lawverian MATs** with set of sorts S is a product-, $\langle \sqcup \rangle$- and factorization-preserving functor $\mathcal{L} \to \mathcal{L}'$. This constitutes a 2-category $\mathsf{Law}(S)$.

A **model** of $\mathcal{L}$ is a product-preserving functor $\mathcal{L} \to \mathrm{Set}$. The forgetful functor $U : \mathsf{Model}(\mathcal{L}) \to C$ is defined by $U\, T\, s = T\, \langle s \rangle$.

Other authors [nLa22a] instead simply consider categories that have (finite) products. We force things a bit to make $\mathcal{L}^{\mathrm{op}}$ equivalent to the Kleisli category of the corresponding monadic MAT.

Proposition 5.6. Every MAT presentation $\mathfrak{A}$ gives rise to a Lawverian MAT $\mathcal{L}_{\mathfrak{A}}$.

Construction. This follows immediately from the following proposition. □

Proposition 5.7. Every monadic MAT M gives rise to a Lawverian MAT $\mathcal{L}_M$.

Construction. We define $\mathcal{L}_M := \mathsf{Kl}(M)^{\mathrm{op}}$, the opposite Kleisli category of M:

- We define $\langle s \rangle = \overline{\delta_s}$ where $\delta_s\, r = \{\mathsf{refl} \mid r = s\}$.
- Products are given by coproducts in $\mathsf{Kl}(M)$, which are in turn given by coproducts in C, which are pointwise disjoint unions.
- The object $\overline{X}$ can be written as

$$\overline{X} \cong \overline{\sum_{s \in S} X\, s \cdot \delta_s} = \prod_{s \in S} \langle s \rangle^{X\, s}. \qquad \square$$

Let us consider what this means. By the universal property of the product, the full Hom-structure of $\mathcal{L}_M$ is given by the Hom-sets with codomain $\langle s \rangle$ for some $s \in S$. There we have:

$$\mathrm{Hom}_{\mathcal{L}_M}\left(\prod_{r \in S} \langle r \rangle^{X\, r}, \langle s \rangle\right) \cong \mathrm{Hom}_{\mathcal{L}_M}\left(\overline{X}, \langle s \rangle\right) = \mathrm{Hom}_{\mathsf{Kl}(M)}\left(\overline{\delta_s}, \overline{X}\right) = \mathrm{Hom}_C(\delta_s, MX) \cong MX\, s,$$

which we interpreted before as the set of terms of sort s with r-sorted variables in $X\, r$, modulo equality. More generally, we have

$$\mathrm{Hom}_{\mathcal{L}_M}\left(\prod_{r \in S} \langle r \rangle^{Y\, r}, \prod_{s \in S} \langle s \rangle^{X\, s}\right) \cong \prod_{s \in S} (MY\, s)^{X\, s} = \mathrm{Hom}_C(X, MY).$$

The identity corresponds to the the tuple of variables, and composition amounts to substitution. The functor $J_M^{\mathrm{op}} : C^{\mathrm{op}} \to \mathcal{L}_M$ creates *pure* morphisms (i.e. morphisms without monadic effects), which are the renamings.

Proposition 5.8. The category $\mathsf{Model}(\mathcal{L}_{\mathfrak{A}})$ of models of $\mathcal{L}_{\mathfrak{A}}$ is equivalent to the category of models $\mathsf{Model}(\mathfrak{A})$ of $\mathfrak{A}$, and the equivalence commutes with the forgetful functors to C.

Proof. This follows immediately from the following proposition. □

Proposition 5.9. The category $\mathsf{Model}(\mathcal{L}_M)$ of models of $\mathcal{L}_M$ is equivalent to the category $\mathsf{EM}(M)$ of EM-algebras of M, and the equivalence commutes with the forgetful functors to C.

Proof. $\boxed{\to}$ Given a model $T \in \mathsf{Model}(\mathcal{L}_M)$, i.e. a product-preserving functor $\mathcal{L}_M \to \mathsf{Set}$, we build an EM-algebra A. For the carrier, we (must) take $A\, s := T\, \langle s \rangle$ so that

$$T\left(\prod_{s \in S} \langle s \rangle^{X\, s}\right) \cong \prod_{s \in S} T\langle s \rangle^{X\, s} = \prod_{s \in S} (A\, s)^{X\, s} = \mathrm{Hom}_C(X, A).$$

Recall that

$$\iota : \mathrm{Hom}_C(X, MY) \cong \mathrm{Hom}_{\mathcal{L}}\left(\prod_{r \in S} \langle r \rangle^{Y\, r}, \prod_{s \in S} \langle s \rangle^{X\, s}\right) : f \mapsto \overline{f}$$

so that the functorial action of T creates an operation

$$\gg\!\!=^A : \mathrm{Hom}_C(X, MY) \to \mathrm{Hom}_C(Y, A) \to \mathrm{Hom}_C(X, A),$$

i.e. $T\,\overline{f}\,g = f \mathbin{>\!\!=\!\!\!>}^A g$. By functoriality and coproduct preservation w.r.t. X, $\mathbin{>\!\!=\!\!\!>}^A$ is a componentwise application of some 'A-specific bind' operation

$$\gg\!=^A : \prod_{s \in S} MY\,s \to \mathrm{Hom}_C(Y, A) \to A\,s,$$

i.e. $(f \mathbin{>\!\!=\!\!\!>}^A g)(s, x) = f_s(x) \gg\!=^A_s g$.

- Since T preserves the identity, we know that $\eta \mathbin{>\!\!=\!\!\!>}^A g = g$ and hence, applying to x, that $\eta_s(x) \gg\!=^A_s g = g_s(x)$,
- Since T preserves composition, we know that

  $$(f \mathbin{>\!\!=\!\!\!>} g) \mathbin{>\!\!=\!\!\!>}^A h = f \mathbin{>\!\!=\!\!\!>}^A (g \mathbin{>\!\!=\!\!\!>}^A h)$$

  and hence, considered componentwise, that

  $$(my \gg\!=_s g) \gg\!=^A_s h = my \gg\!=^A_s (g \mathbin{>\!\!=\!\!\!>}^A h).$$

- By naturality w.r.t. Y, we know that

  $$(Mg \circ f) \mathbin{>\!\!=\!\!\!>}^A h = f \mathbin{>\!\!=\!\!\!>}^A (h \circ g)$$

  and hence, considered componentwise, that

  $$(Mg)_s(my) \gg\!=^A_s h = my \gg\!=^A_s (h \circ g).$$

We now define the algebra morphism

$$\alpha : \mathrm{Hom}_C(MA, A) : (s, ma) \mapsto ma \gg\!=^A_s \mathrm{id},$$

i.e. $\alpha = \mathrm{id}_{MA} \mathbin{>\!\!=\!\!\!>}^A \mathrm{id}_A$. Then we have

$$\begin{aligned}
\alpha_s(\eta_s(a)) &= \eta_s(a) \gg\!=^A_s \mathrm{id} = a, \\
\alpha_s(\mu_s(mma)) &= (mma \gg\!=_s \mathrm{id}) \gg\!=^A_s \mathrm{id} \\
&= mma \gg\!=^A_s (\mathrm{id} \mathbin{>\!\!=\!\!\!>}^A_s \mathrm{id}) \\
&= mma \gg\!=^A_s \alpha = (M\alpha)_s(mma) \gg\!=^A_s \mathrm{id} = (\alpha \circ M\alpha)_s(mma),
\end{aligned}$$

so indeed we have an EM-algebra for M. To see that this constitutes a functor $\mathrm{Model}(\mathcal{L}_M) \to \mathsf{EM}(M)$, assume a natural transformation $\theta : T \to T'$ and write $f = \theta_{\langle \sqcup \rangle} : \mathrm{Hom}_C(A, A')$. We have

$$f_s(\alpha_s(ma)) = f_s(ma \gg\!=^A_s \mathrm{id}) = ma \gg\!=^{A'}_s (f \circ \mathrm{id}) = (Mf)_s(ma) \gg\!=^{A'}_s \mathrm{id} = \alpha'_s((Mf)_s(ma)).$$

$\boxed{\leftarrow}$ Given an EM-algebra $A \in \mathsf{EM}(M)$, we build a product-preserving functor $T : \mathcal{L}_M \to \mathrm{Set}$. We define $T\big(\prod_{s \in S} \langle s \rangle^{X\,s}\big) := \mathrm{Hom}_C(X, A)$. To make this functorial, we require an operation $\mathbin{>\!\!=\!\!\!>}^A$ as above, satisfying the identity and composition laws proven above. (The naturality law follows automatically as naturality w.r.t. Y is just composition with a pure morphism.)

We define

$$\begin{aligned}
& f \mathbin{>\!\!=\!\!\!>}^A g := \alpha \circ Mg \circ f \\
\Rightarrow \qquad & mx \gg\!=^A_s g = (\alpha \circ Mg)_s(mx).
\end{aligned}$$

Then we have

$$\begin{aligned}
\eta_s(x) \gg\!=^A_s f &= (\alpha \circ Mf \circ \eta)_s(x) = (\alpha \circ \eta \circ f)_s(mx) = f_s(mx), \\
(f \mathbin{>\!\!=\!\!\!>} g) \mathbin{>\!\!=\!\!\!>}^A h &= \alpha \circ Mh \circ \mu \circ Mg \circ f \\
&= \alpha \circ \mu \circ MMh \circ Mg \circ f \\
&= \alpha \circ M(\alpha \circ Mh \circ g) \circ f \\
&= f \mathbin{>\!\!=\!\!\!>}^A (\alpha \circ Mh \circ g) = f \mathbin{>\!\!=\!\!\!>}^A (g \mathbin{>\!\!=\!\!\!>}^A h),
\end{aligned}$$

so indeed we have a model of $\mathcal{L}_M$. To see that this constitutes a functor $\mathsf{EM}(M) \to \mathsf{Model}(\mathcal{L}_M)$, assume an algebra morphism $f : \mathrm{Hom}_C(A, A')$. We shall construct a natural transformation $\theta : T \to T'$. On objects, we set $\theta_{\prod_{s\in S}\langle s\rangle^{X\,s}} = (f \circ \sqcup) : \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, A')$. On morphisms, we have

$$f \circ (g \ggg^A h) = (f \circ \alpha \circ Mh \circ g) = (\alpha' \circ M(f \circ h) \circ g)(mx) = g \ggg^{A'} f \circ h.$$

$\boxed{\simeq}$ It is straightforward to check that the two functors constructed above constitute an equivalence of categories $\mathsf{Model}(\mathcal{L}_M) \simeq \mathsf{EM}(M)$. □

5.4 Monadic MATs from Lawverian MATs

Proposition 5.10. The functor $\mathsf{Monad}(\mathrm{Set}^S) \to \mathsf{Law}(S) : M \mapsto \mathcal{L}_M$ is a 2-equivalence.

Proof. $\boxed{\leftarrow}$ To a Lawverian MAT $\mathcal{L}$, we associate a monad $M_\mathcal{L}$ on $C = \mathrm{Set}^S$ given by

- $M_\mathcal{L} X\, s := \mathrm{Hom}_\mathcal{L}\big(\prod_{r\in S}\langle r\rangle^{X\,r}, \langle s\rangle\big)$,
- $(M_\mathcal{L} f)_s(mx) := mx \circ \prod_{r\in S}\langle r\rangle^{f\,r} = mx \circ (\pi_{(r,f_r(x))})_{(r,x)}$
- $\eta_s(x) := \pi_{(s,x)}$,
- $\mu_s(mmx) := mmx \circ (mx)_{(r,mx)}$.

The unit and multiplication are indeed natural:

$$\begin{aligned}
(M_\mathcal{L} f \circ \eta)_s(x) &= (M_\mathcal{L} f)_s(\pi_{s,x}) = \pi_{s,x} \circ (\pi_{r,f_r(x')})_{r,x'} = \pi_{s,f_s(x)} = (\eta \circ f)_s(x),\\
(M_\mathcal{L} f \circ \mu)_s(mmx) &= (M_\mathcal{L} f)_s(mmx \circ (mx)_{r,mx})\\
&= mmx \circ (mx)_{r,mx} \circ (\pi_{f_q(x)})_{q,x},\\
(\mu \circ M_\mathcal{L} M_\mathcal{L} f)_s(mmx) &= \mu_s(mmx \circ (\pi_{r,(M_\mathcal{L} f)_r(mx)})_{r,mx})\\
&= mmx \circ (\pi_{r,(M_\mathcal{L} f)_r(mx)})_{r,mx} \circ (my)_{q,my}\\
&= mmx \circ (\pi_{r,(M_\mathcal{L} f)_r(mx)} \circ (my)_{q,my})_{r,mx}\\
&= mmx \circ ((M_\mathcal{L} f)_r(mx))_{r,mx}\\
&= mmx \circ (mx \circ (\pi_{q,f(x)})_{q,x})_{r,mx}\\
&= mmx \circ (mx)_{r,mx} \circ (\pi_{q,f(x)})_{q,x},
\end{aligned}$$

and satisfy the monad laws:

$$\begin{aligned}
(\mu \circ \eta)_s(mx) &= \mu_s(\pi_{s,mx}) = \pi_{s,mx} \circ (mx')_{r,mx'} = mx,\\
(\mu \circ M_\mathcal{L}\eta)_s(mx) &= \mu_s(mx \circ (\pi_{r,\eta_r(x)})_{r,x})\\
&= mx \circ (\pi_{r,\eta_r(x)})_{r,x} \circ (mx')_{q,mx'}\\
&= mx \circ (\pi_{r,\eta_r(x)} \circ (mx')_{q,mx'})_{r,x}\\
&= mx \circ (\eta_r(x))_{r,x} = mx \circ (\pi_{r,x})_{r,x} = mx \circ \mathrm{id} = mx,\\
(\mu \circ \mu)_s(mmmx) &= \mu_s(mmmx \circ (mmx)_{r,mmx})\\
&= mmmx \circ (mmx)_{r,mmx} \circ (mx)_{q,mx}\\
(\mu \circ M_\mathcal{L}\mu)_s(mmmx) &= \mu_s(mmmx \circ (\pi_{r,\mu_r(mmx)})_{r,mmx})\\
&= mmmx \circ (\pi_{r,\mu_r(mmx)})_{r,mmx} \circ (mx)_{q,mx}\\
&= mmmx \circ (\pi_{r,\mu_r(mmx)} \circ (mx)_{q,mx})_{r,mmx}\\
&= mmmx \circ (\mu_r(mmx))_{r,mmx}\\
&= mmmx \circ (mmx \circ (mx)_{q,mx})_{r,mmx}\\
&= mmmx \circ (mmx)_{r,mmx} \circ (mx)_{q,mx}.
\end{aligned}$$

$\boxed{\mathrm{Law}(S)}$ We can now observe that $\mathcal{L} \simeq \mathsf{Kl}(M_{\mathcal{L}})^{\mathrm{op}} = \mathcal{L}_{M_{\mathcal{L}}}$. Indeed, we have

$$\mathrm{Hom}_{\mathcal{L}}\left(\prod_{r\in S}\langle r\rangle^{Y\,r}, \prod_{s\in S}\langle s\rangle^{X\,s}\right) \cong \prod_{s\in S}\mathrm{Hom}_{\mathcal{L}}\left(\prod_{r\in S}\langle r\rangle^{Y\,r}, \langle s\rangle\right)^{X\,s} = \prod_{s\in S}(M_{\mathcal{L}}Y\,s)^{X\,s} = \mathrm{Hom}_{\mathcal{C}}(X, M_{\mathcal{L}}Y).$$

Under this isomorphism,

- the identity morphism at $\prod_{s\in S}\langle s\rangle^{X\,s}$ is sent to $(\lambda s.\lambda x.\pi_{(s,x)}) = \eta$,
- for
$$f \in \mathrm{Hom}_{\mathcal{L}}\left(\prod_{r\in S}\langle r\rangle^{Y\,r}, \prod_{s\in S}\langle s\rangle^{X\,s}\right), \qquad g \in \mathrm{Hom}_{\mathcal{L}}\left(\prod_{r\in S}\langle r\rangle^{Z\,r}, \prod_{s\in S}\langle s\rangle^{Y\,s}\right),$$
the composite morphism $f \circ g$ is sent to $(\lambda s.\lambda x.\pi_{(s,x)} \circ f \circ g)$ which should be equal to the Kleisli composition of $(\lambda s.\lambda x.\pi_{(s,x)} \circ f)$ and $(\lambda t.\lambda y.\pi_{(t,y)} \circ g)$.

$$\begin{aligned}
&(\lambda s.\lambda x.\pi_{(s,x)} \circ f) \ggg (\lambda t.\lambda y.\pi_{(t,y)} \circ g) \\
&= \mu \circ M_{\mathcal{L}}(\lambda t.\lambda y.\pi_{(t,y)} \circ g) \circ (\lambda s.\lambda x.\pi_{(s,x)} \circ f) && \text{(Def. } \ggg\text{)} \\
&= \lambda s.\lambda x.(\mu \circ M_{\mathcal{L}}(\lambda t.\lambda y.\pi_{(t,y)} \circ g) \circ (\lambda s'.\lambda x'.\pi_{(s',x')} \circ f))_s(x) && (\eta\text{-expand function}) \\
&= \lambda s.\lambda x.(\mu \circ M_{\mathcal{L}}(\lambda t.\lambda y.\pi_{(t,y)} \circ g))_s(\pi_{(s,x)} \circ f) && \text{(Apply last function)} \\
&= \lambda s.\lambda x.\mu_s\left(\pi_{(s,x)} \circ f \circ (\pi_{t,\pi_{(t,y)}\circ g})_{(t,y)}\right) && \text{(Map over } M_{\mathcal{L}}\text{)} \\
&= \lambda s.\lambda x.\pi_{(s,x)} \circ f \circ (\pi_{t,\pi_{(t,y)}\circ g})_{(t,y)} \circ (mz)_{(u,mz)} && \text{(Def. } \mu\text{)} \\
&= \lambda s.\lambda x.\pi_{(s,x)} \circ f \circ (\pi_{t,\pi_{(t,y)}\circ g} \circ (mz)_{(u,mz)})_{(t,y)} && \text{(Precompose componentwise)} \\
&= \lambda s.\lambda x.\pi_{(s,x)} \circ f \circ (\pi_{(t,y)} \circ g)_{t,y} && \text{(Projection from tuple)} \\
&= \lambda s.\lambda x.\pi_{(s,x)} \circ f \circ g. && (\eta\text{-contract tuple})
\end{aligned}$$

So indeed $\mathcal{L} \simeq \mathsf{Kl}(M_{\mathcal{L}})^{\mathrm{op}} = \mathcal{L}_{M_{\mathcal{L}}}$.

This equivalence is natural w.r.t. $\mathcal{L}$: if we have a morphism of Lawverian MATs $L : \mathcal{L} \to \mathcal{L}'$, then we get a monad morphism $\omega_L : M_{\mathcal{L}} \to M_{\mathcal{L}'} : \mathcal{C} \to \mathcal{C}$ such that $(\omega_L)_s(mx) = L\,mx$. This leads to a functor $L_{\omega_L} : \mathcal{L}_{M_{\mathcal{L}}} \to \mathcal{L}_{M_{\mathcal{L}'}}$ which is easily seen to fit in a commutative diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\simeq"] \arrow[d, "L"'] & \mathcal{L}_{M_{\mathcal{L}}} \arrow[d, "L_{\omega_L}"'] \\
\mathcal{L}' \arrow[r, "\simeq"] & \mathcal{L}_{M_{\mathcal{L}'}}.
\end{tikzcd}$$

$\boxed{\mathrm{Monad}(\mathrm{Set}^S)}$ For any monadic MAT M, we have an isomorphism of monads $M \cong M_{\mathcal{L}_M}$. Indeed, we have

$$\iota_s : M_{\mathcal{L}_M}X\,s = \mathrm{Hom}_{\mathcal{L}_M}\left(\prod_{r\in S}\langle r\rangle^{X\,r}, \langle s\rangle\right) = \mathrm{Hom}_{\mathsf{Kl}(M)}\left(\delta_s, \sum_{r\in S}X\,r\cdot\delta_r\right) \cong MX\,s.$$

Writing a prime ($'$) for the operations of $M_{\mathcal{L}_M}$, we have

$$
\begin{aligned}
\iota_s(\eta'_s(x)) &= \iota_s(\pi^{\mathcal{L}_M}_{s,x}) = \iota(\iota^{\mathsf{Kl}(M)}_{s,x}) = \eta_s(x) \in MX. \\
&= mmx \ggg [mx]_{r,mx} \\
\iota_s(mx \gg\!=_s' f) &= (\iota \circ \mu' \circ M'f)_s(mx) \\
&= (\iota \circ \mu')_s(mx \circ (\pi_{r,f(x)})_{r,x}) \\
&= \iota_s(mx \circ (\pi_{r,f(x)})_{r,x} \circ (my)_{t,my}) \\
&= \iota_s(mx \circ (\pi_{r,f(x)} \circ (my)_{t,my})_{r,x}) \\
&= \iota_s(mx \circ (f(x))_{r,x}) \\
&= \iota_s([f(x)]_{r,x} \circ mx) && \text{dualize} \\
&= \iota_s(mx) \gg\!=_s (\iota \circ f).
\end{aligned}
$$

This isomorphism of monads is easily seen to be natural w.r.t. M. □

5.5 The KEML-diagram for MATs

We can summarize our findings about MATs in a diagram. Assume that a monadic MAT M and a Lawverian MAT $\mathcal{L}$ correspond. Then we obtain a commuting diagram as follows:

$$
\begin{tikzcd}
\mathsf{Set}^S \arrow[rrr, "M"] \arrow[dr, "J_M"] \arrow[dd, "X \mapsto \prod_{s \in S} \langle s \rangle^{X\,s}"'] & & & \mathsf{Set}^S \\
& \mathsf{Kl}(M) \arrow[r, "\overline{X} \mapsto (MX,\mu)"] \arrow[dl, dash, "\simeq"] & \mathsf{EM}(M) \arrow[ur, "U"] \arrow[dr, dash, "\simeq"'] & \\
\mathcal{L}^{\mathrm{op}} \arrow[rrr, "\mathrm{Hom}_{\mathcal{L}}"'] & & & \mathsf{Model}(\mathcal{L}) \arrow[uu, "U : T \mapsto \lambda s.T\langle s \rangle"']
\end{tikzcd}
$$

The existence of each of the arrows is clear; in the case of the equivalence $\mathcal{L}^{\mathrm{op}} \simeq \mathsf{Kl}(M)$ it follows from the correspondence of $\mathcal{L}$ and M (propositions 5.7 and 5.10). The commutation of the triangle on the left is clear by construction of $\mathcal{L}_M$ (proposition 5.7). Commutation of the upper trapezium is a general theorem about monads. Commutation of the right triangle follows from proposition 5.9, and commutation of the lower trapezium follows from its proof.

5.6 Discussion

5.6.1 The Empty MAT presentation

A MAT presentation with zero operators gives rise to the identity monad.

- Then $J_{\mathrm{Id}} : \mathsf{Set}^S \cong \mathsf{Kl}(\mathrm{Id})$ is an isomorphism,
- In the Lawverian MAT, *all* morphisms are tuples of projections, so that $X \mapsto \prod_{s \in S} \langle s \rangle^{X\,s} : \mathsf{Set}^S \simeq \mathcal{L}^{\mathrm{op}}$ is an equivalence,
- Being an EM-algebra is void, so $U : \mathsf{EM}(\mathrm{Id}) \cong \mathsf{Set}^S$ is an isomorphism,
- Product preservation determines $T \in \mathsf{Model}(\mathcal{L})$ fully from all $T\langle s \rangle$ so $U : \mathsf{Model}(\mathcal{L}) \simeq \mathsf{Set}^S : T \mapsto \lambda s.T\langle s \rangle$ is an equivalence,
- Then $\mathrm{Hom}_{\mathcal{L}} : \mathcal{L}^{\mathrm{op}} \simeq \mathsf{Model}(\mathcal{L})$ is also an equivalence.

6 Essentially and Generalized Algebraic Theories (EATs and GATs)

EATs [AHR99, nLa21] differ from MATs (only) in that operations may be partial: they may require that the arguments satisfy certain equalities depending on them.

GATs [Car86, Car78] differ from MATs in that sorts can now be dependent. With a trick[2], we can moreover allow GAT operations to be partial like EAT operations.

In both cases, the consequence is that the *arity* (by now better called premise) of an operator may mention operators. The question is then: *do we allow circular dependencies?* In Cartmell's original definition of GATs [Car86, Car78], this is allowed. In Adámek et al.'s definition of EATs, it is not. We observe that circular dependencies are not allowed in contemporary approaches such as Kaposi and Kovács' formalization of HIITs [KK18] and will not allow it here either. By consequence and in line with [KKA19], we will discuss the algebraic aspects of EATs/GATs by induction on the number of operators.

In fact, we will not really discuss GATs at all: as already observed by Cartmell [Car86], GATs are essentially EATs. Indeed, by replacing every dependent sort with a non-dependent sort that we think of as its total space, turning the dependencies into operations (to be thought of as projections from the total space), we obtain an EAT. For example, when we formalize dependent type theory, instead of having a sort $\mathrm{Ty}(\Gamma)$ depending on a context Γ, we would have an operation $\mathrm{getCtx} : \mathrm{Ty} \to \mathrm{Ctx}$.

Overview We define EAT presentations (definition 6.1), monadic EATs (definition 6.2) and Lawverian EATs (definition 6.5). We prove:

- An EAT presentation may give rise to a monadic EAT (definition 6.3) with the same models (proposition 6.4).
- Every monadic EAT gives rise to a Lawverian EAT (proposition 6.6) with the same models (proposition 6.7).
- Monadic and Lawverian EATs are essentially the same thing (proposition 6.8).

We summarize in a diagram (section 6.5).

6.1 EAT Presentations

We give a more general definition of EAT presentations than Adámek et al. [AHR99]; our definition reduces to theirs if the prior theory is a MAT.

Definition 6.1. An **essentially algebraic theory presentation** $\mathfrak{A}$ consists of:

- a set of sorts $S_{\mathfrak{A}}$
- a prior EAT or MAT $\mathfrak{P}$ such that $S_{\mathfrak{P}} = S_{\mathfrak{A}}$,
- for each arity $\vec{r} \in \mathsf{List}\, S$, output sort $s \in S$ and list of conditions $K \in C_{\mathfrak{P}}(\vec{r})$, a set of operators $O_{\mathfrak{A}}(\vec{r}, s, K)$,
$$O_{\mathfrak{A}} : (\vec{r} \in \mathsf{List}\, S) \times (s \in S) \times C_{\mathfrak{P}}(\vec{r}) \to \mathsf{Set}.$$
The set $C_{\mathfrak{P}}(\vec{r}) = \{[\,]\}$ isd defined below;
- for each arity $\vec{r} \in \mathsf{List}\, S$, output sort $s \in S$ and list of conditions $K \in C_{\mathfrak{P}}(\vec{r})$, a set of axioms, which are pairs (t_1, t_2), denoted as $t_1 = t_2$:
$$E_{\mathfrak{A}} : (\vec{r} \in \mathsf{List}\, S) \times (s \in S) \times (K \in C_{\mathfrak{P}}(\vec{r})) \to \mathcal{P}\big(T_{\mathfrak{A}}(\vec{r}, s, K)^2\big).$$

Note that the conditions are phrased in the prior theory, whereas the axioms are phrased in the current theory.

[2]by including sorts of equality proofs

An $\vec{r}$-ary term of output sort s and conditions K is inductively defined to be either a metavariable $\mathbf{m}_i$ where $r_i = s$, or an expression of the form $o(t_1, \dots, t_k)$, where o has arity $\vec{q}$ of length k, conditions L and output sort s and every t_i is an $\vec{r}$-ary term of output sort q_i with conditions K, and the conditions $L[t_1, \dots, t_k]$ need to follow from K:

$$T_{\mathfrak{A}}(\vec{r}, s, K) = \{\mathbf{m}_i \mid r_i = s\} \uplus \sum_{\vec{q} \in \mathsf{List}\, A, L \in C_{\mathfrak{P}}(\vec{q}), o \in O_{\mathfrak{A}}(\vec{q}, s, L)} \{(t_1, \dots, t_k) \mid t_i \in T_{\mathfrak{A}}(\vec{r}, q_i, K) \text{ and } K \Rightarrow L[t_1, \dots, t_k]\}.$$

Let us not dwell on the meaning of substitution and implication of conditions.

The set of lists of conditions $C_{\mathfrak{P}}(\vec{r})$ is defined as follows:

- If $\mathfrak{P}$ is a MAT, then it is the following least fixpoint:

$$C_{\mathfrak{P}}(\vec{r}) = \{[]\} \uplus \left\{(K, (t_1, t_2)) \,\middle|\, K \in C_{\mathfrak{P}}(\vec{r}) \text{ and } s \in S \text{ and } t_1, t_2 \in T_{\mathfrak{P}}(\vec{r}, s)\right\} \cong \mathsf{List} \sum_{s \in S} (T_{\mathfrak{P}}(\vec{r}, s))^2.$$

- If $\mathfrak{P}$ is an EAT with prior theory $\mathfrak{Q}$, then it is the following least fixpoint:

$$C_{\mathfrak{P}}(\vec{r}) = C_{\mathfrak{Q}}(\vec{r}) \uplus \left\{(K, (t_1, t_2)) \,\middle|\, K \in C_{\mathfrak{P}}(\vec{r}) \text{ and } s \in S \text{ and } t_1, t_2 \in T_{\mathfrak{P}}(\vec{r}, s, K)\right\}.$$

In case $\mathfrak{A}$ occurs as a prior theory to another EAT, we need to define the set of lists of conditions $C_{\mathfrak{A}}(\vec{r})$. We define it as the following least fixpoint:

$$C_{\mathfrak{A}}(\vec{r}) = C_{\mathfrak{P}}(\vec{r}) \uplus \{(K, (t_1, t_2)) \mid K \in C_{\mathfrak{A}}(\vec{r}) \text{ and } s \in S \text{ and } t_1, t_2 \in T_{\mathfrak{A}}(\vec{r}, s, K)\}.$$

An **algebra** or **model** A of $\mathfrak{A}$ is a model of $\mathfrak{P}$ equipped with functions

$$[\![o]\!] : \left\{\vec{a} \in \prod_i A_{r_i} \,\middle|\, \forall (t_1, t_2) \in K. [\![t_1]\!]_{\vec{a}} = [\![t_2]\!]_{\vec{a}}\right\} \to A_s$$

for every $o \in O_{\mathfrak{A}}(\vec{r}, s, K)$, satisfying the axioms. Given $\vec{a} = (a_1, \dots, a_n) \in \prod_i A_{r_i}$, the interpretation of an $\vec{r}$-ary term t is defined recursively by $[\![o(t_1, \dots, t_k)]\!]_{\vec{a}} = [\![o]\!]([\![t_1]\!]_{\vec{a}}, \dots, [\![t_k]\!]_{\vec{a}})$ and $[\![\mathbf{m}_i]\!]_{\vec{a}} = a_i$. We say that A satisfies the axiom $t_1 = t_2$ of arity $\vec{r}$ and conditions K if $[\![t_1]\!]_{\vec{a}} = [\![t_2]\!]_{\vec{a}}$ for all $\vec{a} \in \prod_i A_{r_i}$ satisfying K.

A **morphism of algebras/models** is a morphism in $\mathsf{Model}(\mathcal{P})$ such that $f_s \circ [\![o]\!] = [\![o]\!] \circ \prod_i f_{r_i} : \prod_i A_{r_i} \to B$ for every $\vec{r}$-ary operator o of sort s.

The **syntax** of $\mathfrak{A}$ is the initial object of the category of $\mathfrak{A}$-models. It is the S-indexed set of nullary terms of $\mathfrak{A}$, divided by the equivalence relation generated by the axioms.

Note that if the prior theory is empty, then there can be no conditions so the EAT is really a MAT. In this sense, instead of allowing the prior theory to be a MAT, we could allow it to be the empty EAT.

6.1.1 Type Analogy

EATs can be thought of as mutual quotient inductive types (of the EAT's terms) in a dependently typed language.

GATs can be thought of as quotient inductive-inductive types [KK18, KKA19] (of the GAT's terms) in a dependently typed language.

6.1.2 Examples

- Category theory is an EAT, with as prior theory the theory of reflexive graphs, which is an instance of a presheaf theory and therefore an example of a MAT.
- Dependent type theory is a GAT and therefore an EAT, which needs to be constructed incrementally: a non-equational rule that mentions other rules, can only be written down when these other rules are part of the prior theory.

6.2 Monadic EATs

Definition 6.2. A **monadic EAT** with prior monadic MAT or EAT M_0 is a monad on $C := \mathsf{EM}(M_0)$.

Note that if the prior theory M_0 is the identity at Set^S, corresponding to the empty presentation, then $\mathsf{EM}(M_0) \cong \mathrm{Set}^S$, so an EAT over the empty MAT is just a MAT itself.

Definition 6.3. To an EAT presentation $\mathfrak{A}$ over $\mathfrak{P}$, we associate, if existent, the monad $M_{\mathfrak{A}}$ on $\mathsf{EM}(\mathfrak{P})$ which sends $X \in \mathsf{EM}(\mathfrak{P})$ to the initial $\mathfrak{A}$-algebra over X.

Proposition 6.4. The category $\mathsf{EM}(M_{\mathfrak{A}})$ of Eilenberg-Moore algebras of $M_{\mathfrak{A}}$ is isomorphic to the category $\mathsf{Model}(\mathfrak{A})$ of models of $\mathfrak{A}$, and the isomorphism commutes with the forgetful functors to C on the nose. □

6.2.1 The Infixed Monad is Uninteresting

Note that the monad M_0 decomposes as an adjunction $F_{M_0} \dashv U_{M_0}$ over $\mathsf{EM}(M_0)$ and similarly M decomposes as an adjunction $F_M \dashv U_M$ over $\mathsf{EM}(M)$. Decomposing these adjunctions yields a new monad $N = U_{M_0} U_M F_M F_{M_0} = U_{M_0} M F_{M_0}$ which again has an EM-category $\mathsf{EM}(N)$. It is natural to ask how all these relate.

It is well-known that the EM-adjunction is the final adjoint decomposition of any monad, hence we have a functor $H : \mathsf{EM}(M) \to \mathsf{EM}(N)$ such that $H F_M F_{M_0} = F_N$ and $U_N H = U_{M_0} U_M$. However, H is not in general an equivalence and moreover, neither the monad N nor its EM-category is a general object of interest.

Indeed, consider the example where $M_0 : \mathrm{Set}^2 \to \mathrm{Set}^2$ sends a pair of sets $X_0 = (N, E)$ to the free reflexive graph with generating nodes in N and generating edges in E. Note that the graph $M_0 X_0$ is completely unconnected: every non-reflexive edge has its own private source and target. Now $\mathsf{EM}(M_0)$ is the category of reflexive graphs; let $M : \mathsf{EM}(M_0) \to \mathsf{EM}(M_0)$ send a graph X to the free category over it. Then $\mathsf{EM}(M)$ is the category of categories.

Now the monad $N = U_{M_0} M F_{M_0}$ sends a pair of sets X_0 to the carrier of the free category over the free graph over X_0. But the free graph is unconnected and therefore has no candidates for composition. As such, $M F_{M_0} \cong F_{M_0}$ and $N \cong M_0$ and by consequence, $\mathsf{EM}(N) \cong \mathsf{EM}(M_0)$ is again just the category of reflexive graphs.

This is remarkable: what distinguishes an EAT from a MAT is perhaps that **its category of models is *not* the Eilenberg-Moore category of the 'free model' monad,** at least not over Set^S. The decomposition of the adjunction here seems crucial. From this I conclude that we should not *want* to define EATs (or GATs) in a single stroke.

6.2.2 Two-Stage EATs are Insufficient in Practice

Adámek et al.'s definition of an EAT [AHR99, nLa21] requires that the prior mode be a MAT. As explained above, this means that the category of models is an EM-category over an EM-category over Set^S.

We give an example of a theory that apparently needs to be defined in *three* stages: the theory of skeletal categories. A category is *skeletal* if every isomorphism is an identity. Let M_0 be the monadic MAT of reflexive graphs, M_1 over $\mathsf{EM}(M_0)$ the monadic EAT of categories, and M over $\mathsf{EM}(M_1)$ the monadic EAT of skeletal categories. We already know that infixing M_0 with M_1 just yields M_0 again. A similar thing is observed for M_1 and M: a category of the form $F_{M_1} X$, i.e. a free category over some reflexive graph, is automatically skeletal, so $M F_{M_1} \cong F_{M_1}$. What this shows is that it is *pointless* to consider skeletality without first knowing that a graph is a category, and *pointless* to consider categoryhood without first knowing that a pair of sets constitutes a graph. The theory of skeletal categories is a three-stage EAT and as such, two-stage EATs are not good enough for all intents and purposes.

Note that we are not making a formal claim. In particular, we have not proven that the category $\mathsf{EM}(M)$ of skeletal categories cannot be equivalently obtained as the EM-category $\mathsf{EM}(N)$ of some monadic EAT N over some EM-category $\mathsf{EM}(N_0)$ of some monadic MAT N_0 over Set^S; in fact it can as we will see in section 6.3.1. However, though such a construction technically possible, it is not entirely clear why we would want this: graphs and categories are excellent intermediate concepts between pairs of sets and skeletal categories. In general, one could draw up graphs with powers of sets at the roots, where each branch corresponds to an EAT (i.e. an added feature) and each non-root node is an EM-category. We can at least remark that there may be multiple paths from a given root to a given node (e.g. we can pass from pairs of sets to reflexive graphs directly or we can pass by ordinary graphs).

In this text, we treat finite-stage EATs; we have not considered whether there is ever a need for transfinite-stage EATs.

6.3 Lawverian EATs from Monadic EATs

Definition 6.5. A **Lawverian EAT** is a category $\mathcal{L}$ equipped with:

- a prior Lawverian MAT or EAT $\mathcal{S}$,
- a functor $\langle \sqcup \rangle : \mathcal{S} \to \mathcal{L}$ respecting the specific products/limits that $\mathcal{S}$ is equipped with (called **prior limits**),
- it has limits of the form $\lim_{(s,x)\in\int_{\mathcal{S}} X}\langle s\rangle$ for $X \in \mathsf{Model}(\mathcal{S})$ (called **current limits**), algebraically,
- every object is an (algebraically obtained) current limit up to a designated isomorphism (i.e. we have an operation to reconstruct any object as an algebraically obtained current limit up to a designated isomorphism).

A **morphism of Lawverian EATs** with prior theory $\mathcal{S}$ is a current limit, $\langle \sqcup \rangle$ and reconstruction preserving functor $\mathcal{L} \to \mathcal{L}'$. This constitutes a 2-category $\mathsf{Law}(\mathcal{S})$.

A **model** of $\mathcal{L}$ is a functor $\mathcal{L} \to \mathsf{Set}$ preserving current limits. The forgetful functor $U : \mathsf{Model}(\mathcal{L}) \to \mathsf{Model}(\mathcal{S})$ is defined by $U\,T = T \circ \langle \sqcup \rangle$.

Other authors [nLa21] instead simply consider categories that have (finite) limits. We force things a bit to make $\mathcal{L}^{\mathrm{op}}$ equivalent to the Kleisli category of the corresponding monadic EAT.

Proposition 6.6. Every monadic MAT M with prior monadic theory $M_0 : C_0 \to C_0$ gives rise to a Lawverian MAT $\mathcal{L}_M$ with corresponding prior monadic theory $\mathcal{S}$.

Construction. From section 5 (if M_0 is a MAT) or from the current section by induction on the length of the chain of prior theories (if M_0 is an EAT), we know that $\mathsf{EM}(M_0) \simeq \mathsf{Model}(\mathcal{S})$, which is a fully faithful subcategory of $\mathsf{Psh}(\mathcal{S}^{\mathrm{op}}) = \mathsf{Set}^{\mathcal{S}}$. Working in the full presheaf category, any presheaf X can be reconstructed as a colimit of representables:

$$X \cong \operatorname*{colim}_{(s,x)\in\int_{\mathcal{S}} X} \mathbf{y}(s).$$

Note also that $\mathbf{y}(s) = \mathrm{Hom}_{\mathcal{S}}(s, \sqcup) : \mathcal{S} \to \mathsf{Set}$ preserves all limits. Since the embedding is fully faithful, one can see that X also satisfies the universal property of the colimit in $\mathsf{Model}(\mathcal{S})$. For simplicity, we shall pretend that M is a monad on $\mathsf{Model}(\mathcal{S})$.

We define $\mathcal{L}_M := \mathsf{Kl}(M)^{\mathrm{op}}$, the opposite Kleisli category of M:

- We define $\langle s \rangle = \overline{\mathbf{y}(s)}$.
- Limits are given by colimits in $\mathsf{Kl}(M)$, which are in turn given by colimits in $\mathsf{Model}(\mathcal{S})$ (since $\langle \sqcup \rangle$ factors over J_M), which need not exist. However, as discussed above, the ones we need to have current limits in $\mathcal{L}_M$ do exist.
- The object $\overline{X}$ can be written as

$$\overline{X} \cong \overline{\operatorname*{colim}_{(s,x)\in\int_{\mathcal{S}} X} \mathbf{y}(s)} = \lim_{(s,x)\in\int_{\mathcal{S}} X} \overline{\mathbf{y}(s)} = \lim_{(s,x)\in\int_{\mathcal{S}} X} \langle s \rangle. \qquad \square$$

Let us consider what this means. By the universal property of the limit, the full Hom-structure of $\mathcal{L}_M$ is given by the Hom-sets with codomain in the image of $\langle \sqcup \rangle$. (We need to include the functorial action of Hom and $\langle \sqcup \rangle$, i.e. we need to consider the family of Hom-sets of a given domain as a presheaf over $\mathcal{S}$.) We have:

$$\mathrm{Hom}_{\mathcal{L}_M}\left(\lim_{(r,x)\in\int_{\mathcal{S}} X} \langle r\rangle, \langle s\rangle\right) \cong \mathrm{Hom}_{\mathcal{L}_M}\left(\overline{X}, \langle s\rangle\right) = \mathrm{Hom}_{\mathsf{Kl}(M)}\left(\overline{\mathrm{y}s}, \overline{X}\right) = \mathrm{Hom}_{C}(\mathrm{y}s, MX) \cong MX\, s,$$

which we can think of as the set of terms of (generalized) sort[3] s with r-sorted variables in $X\, r$, modulo equality. More generally, we have

$$\mathrm{Hom}_{\mathcal{L}_M}\left(\lim_{(r,y)\in\int_{\mathcal{S}} Y} \langle r\rangle, \lim_{(s,x)\in\int_{\mathcal{S}} X} \langle s\rangle\right) \cong \lim_{(s,x)\in\int_{\mathcal{S}} X} (MY\, s) \cong \mathrm{Hom}_C(X, MY).$$

The identity corresponds to the the tuple of variables variables[4], and composition amounts to substitution. The functor $J_M^{\mathrm{op}} : C^{\mathrm{op}} \to \mathcal{L}_M$ creates *pure* morphisms (i.e. morphisms without effects in the current monadic EAT M but perhaps with effects in M_0), which are renamings **and substitutions with terms of the prior theory**.

Proposition 6.7. The category $\mathsf{Model}(\mathcal{L}_M)$ of models of $\mathcal{L}_M$ is equivalent to the category $\mathsf{EM}(M)$ of EM-algebras of M, and the equivalence commutes with the forgetful functors to C.

Proof. We continue to pretend that M is a monad on $\mathsf{Model}(\mathcal{S})$.

$\boxed{\to}$ Given a model $T \in \mathsf{Model}(\mathcal{L}_M)$, i.e. a current-limit-preserving functor $\mathcal{L}_M \to \mathrm{Set}$, we build an EM-algebra $A \in \mathsf{EM}(M)$. For the carrier, we (must) take $A := T \circ \langle \sqcup \rangle : \mathcal{S} \to \mathrm{Set}$ so that

$$T\left(\lim_{(s,x)\in\int_{\mathcal{S}} X} \langle s\rangle\right) \cong \lim_{(s,x)\in\int_{\mathcal{S}} X} T\langle s\rangle = \lim_{(s,x)\in\int_{\mathcal{S}} X} A\, s \cong \mathrm{Hom}_C(X, A).$$

Recall that

$$\iota : \mathrm{Hom}_C(X, MY) \cong \mathrm{Hom}_{\mathcal{L}}\left(\lim_{(r,y)\in\int_{\mathcal{S}} Y} \langle r\rangle, \lim_{(s,x)\in\int_{\mathcal{S}} X} \langle s\rangle\right) : f \mapsto \overline{f}$$

so that the functorial action of T creates an operation

$$\ggg\!\!=^A : \mathrm{Hom}_C(X, MY) \to \mathrm{Hom}_C(Y, A) \to \mathrm{Hom}_C(X, A),$$

i.e. $T\,\overline{f}\,g = f \ggg\!\!=^A g$. By functoriality and coproduct preservation w.r.t. X, $\ggg\!\!=^A$ is a componentwise application of some 'A-specific bind' operation (where $\forall$ denotes an end)

$$\gg\!=^A : \forall(s \in S).MY\, s \to \mathrm{Hom}_C(Y, A) \to A\, s,$$

i.e. $(f \ggg\!\!=^A g)(s, x) = f_s(x) \gg\!=^A_s g$.

The rest of the reasoning could be copied verbatim from proposition 5.9.

$\boxed{\leftarrow}$ Given an EM-algebra $A \in \mathsf{EM}(M)$, we build a current-limit-preserving functor $T : \mathcal{L}_M \to \mathrm{Set}$. We define $T\left(\lim_{(s,x)\in\int_{\mathcal{S}} X}\langle s\rangle\right) := \mathrm{Hom}_C(X, A)$.

The reasoning to make this functorial could be copied verbatim from proposition 5.9.

To see that this constitutes a functor $\mathsf{EM}(M) \to \mathsf{Model}(\mathcal{L}_M)$, assume an algebra morphism $f : \mathrm{Hom}_C(A, A')$. We shall construct a natural transformation $\theta : T \to T'$. On objects, we set $\theta_{\lim_{(s,x)\in\int_{\mathcal{S}} X}\langle s\rangle} = (f \circ \sqcup) : \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, A')$. On morphisms, we have

$$f \circ (g \ggg\!\!=^A h) = (f \circ \alpha \circ Mh \circ g) = (\alpha' \circ M(f \circ h) \circ g)(mx) = g \ggg\!\!=^{A'} f \circ h.$$

$\boxed{\simeq}$ It is straightforward to check that the two functors constructed above constitute an equivalence of categories $\mathsf{Model}(\mathcal{L}_M) \simeq \mathsf{EM}(M)$. $\square$

[3] Here, s is really an object of the prior theory. If that theory is a MAT, then s is a product of sorts. If it is an EAT, then s is a limit of generalized sorts.

[4] These are to generalized sorts what variables are to sorts.

6.3.1 Two-stage EATs are Sufficient in Theory

From proposition 6.7, we know that the EM-category $\mathsf{EM}(M)$ of any monadic EAT M, defined in any number of stages, is equivalent to the category $\mathsf{Model}(\mathcal{L}_M)$ of current-limit-preserving functors $\mathcal{L}_M \to \mathsf{Set}$.

Now $\mathsf{Psh}(\mathcal{L}_M^{\mathrm{op}}) = \mathsf{Set}^{\mathcal{L}_M}$ is the EM-category of the free presheaf monad M_0 on $\mathsf{Set}^{\mathsf{Obj}(\mathcal{L}_M)}$, which is a monadic MAT. There is an EAT on $\mathsf{EM}(M_0) \cong \mathsf{Psh}(\mathcal{L}_M^{\mathrm{op}})$ which enforces preservation of current limits by adding an operation for creating compatible tuples in the images of current-limit-objects in $\mathcal{L}_M$ and enforing β- and η-rules. This gives rise to an (idempotent) monad M_1 on $\mathsf{EM}(M_0)$ such that $\mathsf{EM}(M_1) \cong \mathsf{Model}(\mathcal{L}_M) \simeq \mathsf{EM}(M)$.

Thus, we have decomposed M as a two-stage EAT. The fact that this is possible in general is doubtlessly useful in the mathematical analysis of the concept of an EAT, but seems less useful in practice. Indeed, the intermediate concept of presheaves over $\mathcal{L}_M^{\mathrm{op}} \simeq \mathsf{Kl}(M)$ does not seem very interesting. Objects of $\mathcal{L}_M$ can be seen as arities-with-conditions, and a presheaf over $\mathcal{L}_M^{\mathrm{op}}$ that does *not* preserve limits is essentially a model of M which has not only a set of terms for every sort, but also for every arity-with-conditions, with the flaw that neither existence nor uniqueness of compatible tuple objects at these arities is guaranteed. Moreover, this two-stage decomposition is not even canonical: if we apply the algorithm described above to its own output, we arrive at a more complicated decomposition.

6.4 Monadic EATs from Lawverian EATs

Proposition 6.8. The functor $\mathsf{Monad}(\mathsf{Model}(\mathcal{S})) \to \mathsf{Law}(\mathcal{S}) : M \mapsto \mathcal{L}_M$ is a 2-equivalence.

Proof. We continue to pretend that M is a monad on $\mathsf{Model}(\mathcal{S})$.

$\boxed{\leftarrow}$ To a Lawverian EAT $\mathcal{L}$, we associate a monad $M_{\mathcal{L}}$ on $C = \mathsf{Model}(\mathcal{S})$ given by

- $M_{\mathcal{L}} X\, s := \mathrm{Hom}_{\mathcal{L}}\left(\lim_{(r,x) \in \int_{\mathcal{S}} X} \langle r \rangle, \langle s \rangle\right)$,
- $(M_{\mathcal{L}} f)_s(mx) := mx \circ (\pi_{(r, f_r(x))})_{(r,x)}$
- $\eta_s(x) := \pi_{(s,x)}$,
- $\mu_s(mmx) := mmx \circ (mx)_{(r,mx)}$.

The checking of naturality and monad laws could be copied verbatim from proposition 5.10.

$\boxed{\mathsf{Law}(\mathcal{S})}$ We can now observe that $\mathcal{L} \simeq \mathsf{Kl}(M_{\mathcal{L}})^{\mathrm{op}} = \mathcal{L}_{M_{\mathcal{L}}}$. Indeed, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{L}}\left(\lim_{(r,y) \in \int_{\mathcal{S}} Y} \langle r \rangle, \lim_{(s,x) \in \int_{\mathcal{S}} X} \langle s \rangle\right) &\cong \lim_{(s,x) \in \int_{\mathcal{S}} X} \mathrm{Hom}_{\mathcal{L}}\left(\lim_{(r,y) \in \int_{\mathcal{S}} Y} \langle r \rangle, \langle s \rangle\right) \\ &= \lim_{(s,x) \in \int_{\mathcal{S}} X} M_{\mathcal{L}} Y\, s \cong \mathrm{Hom}_C(X, M_{\mathcal{L}} Y).\end{aligned}$$

The argument that this isomorphism respects identity and composition could be copied verbatim from proposition 5.10. The same holds for the argument that this isomorphism is natural w.r.t. $\mathcal{L}$.

$\boxed{\mathsf{Monad}(\mathsf{Model}(\mathcal{S}))}$ For any monadic EAT M, we have an isomorphism of monads $M \cong M_{\mathcal{L}_M}$. Indeed, we have

$$\iota_s : M_{\mathcal{L}_M} X\, s = \mathrm{Hom}_{\mathcal{L}_M}\left(\lim_{(r,x) \in \int_{\mathcal{S}} X} \langle r \rangle, \langle s \rangle\right) = \mathrm{Hom}_{\mathsf{Kl}(M)}\left(\mathbf{y}s, \operatorname*{colim}_{(r,x) \in \int_{\mathcal{S}} X} \mathbf{y}r\right) \cong \mathrm{Hom}_{\mathsf{Kl}(M)}(\mathbf{y}s, X) \cong MX\, s.$$

The argument that this isomorphism respects the monad operations and is natural w.r.t. M could be copied verbatim from proposition 5.10. $\square$

6.5 The KEML-diagram for EATs

We can summarize our findings about EATs in a diagram. Assume that a monadic EAT M with prior theory M_0 and a Lawverian EAT $\mathcal{L}$ with corresponding prior theory $\mathcal{S}$ correspond. Then we obtain a commuting diagram as follows, featuring in the upper left triangle a part of the KEML-diagram of the prior theory:

$$\begin{tikzcd}
\mathsf{Kl}(M_0) \arrow[rr, dotted, "\overline{X}\mapsto(M_0X,\mu_0)"] \arrow[dd, dotted, dash, "\simeq"'] & & \mathsf{EM}(M_0) \arrow[r, "M"] \arrow[dl, dash, "\simeq"'] \arrow[dd, "J_M"] & \mathsf{EM}(M_0) \arrow[dr, dash, "\simeq"] & \\
& \mathsf{Model}(\mathcal{S}) \arrow[dd, "X\mapsto\lim_{(s,x)\in\int_{\mathcal{S}}X}\langle s\rangle"] & & & \mathsf{Model}(\mathcal{S}) \\
\mathcal{S}^{\mathrm{op}} \arrow[ur, dotted, "y=\mathrm{Hom}_{\mathcal{S}}"] \arrow[dr, dotted, "\langle\sqcup\rangle"'] & & \mathsf{Kl}(M) \arrow[r, "\overline{X}\mapsto(MX,\mu)"] \arrow[dl, dash, "\simeq"'] & \mathsf{EM}(M) \arrow[uu, "U"] \arrow[dr, dash, "\simeq"] & \\
& \mathcal{L}^{\mathrm{op}} \arrow[rrr, "\mathrm{Hom}_{\mathcal{L}}"'] & & & \mathsf{Model}(\mathcal{L}) \arrow[uu, "U:T\mapsto T\circ\langle\sqcup\rangle"']
\end{tikzcd}$$

The existence of each of the arrows is clear; in the case of the equivalence $\mathcal{L}^{\mathrm{op}} \simeq \mathsf{Kl}(M)$ it follows from the correspondence of $\mathcal{L}$ and M (propositions 6.6 and 6.8). The analogous equivalences for $\mathcal{S}$ and M_0 follow from the KEML-diagram of the prior theory. Commutation of the triangle out of $\mathcal{S}^{\mathrm{op}}$ and the parallelogram right next to it, is clear by construction of $\mathcal{L}_M$ (proposition 6.6). Commutation of the upper square is a general theorem about monads. Commutation of the right parallelogram follows from proposition 6.7, and commutation of the lower trapezium follows from its proof.

6.6 Discussion

6.6.1 The Empty EAT presentation

An EAT presentation with zero operators other than those in the prior theory gives rise to the identity monad $M = \mathrm{Id}$ on $\mathcal{C} = \mathsf{EM}(M_0)$.

- Then $J_{\mathrm{Id}} : \mathsf{EM}(M_0) \cong \mathsf{Kl}(M)$ is an isomorphism,
- In the Lawverian EAT, *all* morphisms are obtained from projections, tupling and prior operations, so that $X \mapsto \lim_{(s,x)\in\int_{\mathcal{S}}X}\langle s\rangle : \mathsf{EM}(M_0) \simeq \mathcal{L}^{\mathrm{op}}$ is an equivalence,
- Being an EM-algebra for $M = \mathrm{Id}$ is void, so $U : \mathsf{EM}(M) \cong \mathsf{EM}(M_0)$ is an isomorphism,
- Current limit preservation determines $T \in \mathsf{Model}(\mathcal{L})$ fully from $T \circ \langle\sqcup\rangle$ so $U : \mathsf{Model}(\mathcal{L}) \simeq \mathsf{EM}(M_0) : T \mapsto \lambda s.T\langle s\rangle$ is an equivalence,
- Then $\mathrm{Hom}_{\mathcal{L}} : \mathcal{L}^{\mathrm{op}} \simeq \mathsf{Model}(\mathcal{L})$ is also an equivalence.

6.6.2 Self-Prior Lawverian EATs

Especially if we look at Lawverian EATs, it is tempting to ask if we can set $\langle\sqcup\rangle = \mathrm{Id} : \mathcal{S} \to \mathcal{L}$ so that we can take limits involving new operations, perhaps accommodating EATs which allow rules to refer to one another cyclically. However this approach does not work: if $\mathcal{S} = \mathcal{L}$, then the current EAT is vacuous: J_M is a fully faithful embedding but also a factor of the identity, so it must be an equivalence itself (implying that $M \cong \mathrm{Id}$) and the other factor must be an equivalence, implying that $\mathsf{Kl}(M_0) \simeq \mathsf{EM}(M_0)$, i.e. all M_0-algebras are free (e.g. M_0 could be the Maybe-monad).

7 Extending MATs

EATs as defined in section 6 allow us to extend a prior theory $\mathfrak{P}$ – be it a MAT or an EAT – with additional operations, yielding an EAT $\mathfrak{A}$. These additional operations may require their arguments to satisfy conditions expressed using prior operations from $\mathfrak{P}$. If $\mathfrak{P}$ is a MAT and the new operations have no conditions at all, or if $\mathfrak{P}$ is an EAT with prior theory $\mathfrak{Q}$ and the conditions of the new operations use only operations from $\mathfrak{Q}$, then $\mathfrak{A}$ is effectively an *extension* of $\mathfrak{P}$: it could then have been directly defined as a MAT (if $\mathfrak{P}$ is a MAT) or as an EAT over $\mathfrak{Q}$.

In this section, we consider only extensions of MATs, which we will use in section 8.

7.1 Extending MAT Presentations

Definition 7.1. An extension $\mathfrak{B}$ (called the superMAT) of a MAT presentation $\mathfrak{A}$ (called the subMAT) is a MAT presentation whose operations and axioms form supersets of the subMAT's.

It is clear that every model of $\mathfrak{B}$ is a model of $\mathfrak{A}$ and hence there is a morphism of $\mathfrak{A}$-models from the syntax of $\mathfrak{A}$ to the syntax of $\mathfrak{B}$.

Proposition 7.2. For any MAT presentation extension $\mathfrak{A} \subseteq \mathfrak{B}$, the MAT $\mathfrak{B}$ can be regarded as an EAT presentation with prior theory $\mathfrak{A}$. □

7.2 Extending Monadic MATs

Definition 7.3. An extension of a monadic MAT $M \in \mathsf{Monad}(\mathsf{Set}^S)$ is a coslice under M, i.e. another monad $N \in \mathsf{Monad}(\mathsf{Set}^S)$ with a monad morphism $\zeta : M \to N$.

By reasoning based on that in section 5.2, it is clear that every MAT presentation extension $\mathfrak{A} \subseteq \mathfrak{B}$ gives rise to a monadic MAT extension $\zeta : M_{\mathfrak{A}} \to M_{\mathfrak{B}}$.

Any monad morphism $\zeta : M \to N$ yields a fully faithful functor $\mathsf{EM}(\zeta) : \mathsf{EM}(N) \to \mathsf{EM}(M)$; in the case of a monad morphism arising from a presentation extension $\mathfrak{A} \subseteq \mathfrak{B}$, this fully faithful functor corresponds to the inclusion of $\mathfrak{B}$-models in $\mathfrak{A}$-models mentioned above.

On the other hand, every monad morphism $\zeta : M \to N$ also yields a functor $\mathsf{Kl}(\zeta) : \mathsf{Kl}(M) \to \mathsf{Kl}(N)$ (note the opposite variance for the Kleisli and the EM category).

Proposition 7.4. For any monadic MAT extension $\zeta : M \to N$, if $Z := \mathsf{EM}(\zeta) : \mathsf{EM}(N) \to \mathsf{EM}(M)$ has a left adjoint W, then we obtain a monad $O = ZW$ on $\mathsf{EM}(M)$, i.e. an EAT with prior theory M. Moreover, if $\zeta = U_M \eta^{Z \dashv W} F_M$,[5] then $Z \dashv W$ is the EM-decomposition of O, i.e. $\mathsf{EM}(N) \cong \mathsf{EM}(O)$.

Proof. That we have a monad O is obvious.

We show that $Z \dashv W$ is the EM-decomposition of O. By finality, we certainly have a morphism G of adjoint decompositions:

$$\begin{tikzcd}
 & & \mathsf{EM}(N) \arrow[dr, "Z"] \arrow[dd, "G"] \arrow[drr, bend left, "U_N"] & & \\
\mathsf{Set}^S \arrow[r, "F_M"] \arrow[urr, bend left, "F_N"] & \mathsf{EM}(M) \arrow[ur, "W"] \arrow[rr, "O"] \arrow[dr, "F_O"'] & & \mathsf{EM}(M) \arrow[r, "U_M"] & \mathsf{Set}^S \\
 & & \mathsf{EM}(O) \arrow[ur, "U_O"'] & &
\end{tikzcd}$$

Here, the triangles with O commute by definition of $O = ZW$ and by definition of the EM-category. The fact that $U_N = U_M Z$ is obvious by construction of $Z = \mathsf{EM}(\zeta)$ and the fact that $F_N \cong W F_M$ follows by uniqueness of the adjoint.

[5] As argued in the proof, this seems like a fairly mild condition, although I have not been able to prove that it is automatic.

Note that, as a property of composite adjunctions, we then know that $\eta^N = U_M \eta^{Z \dashv W} F_M \circ \eta^M$. On the other hand, sinze ζ is a monad morphism, we also know that $\eta^N = \zeta \circ \eta^M$. Thus, the condition in the theorem is $\zeta = U_M \eta^{Z \dashv W} F_M$ seems fairly mild.

The action of G is given by

$$G(A, \alpha^N) = (Z(A, \alpha^N), Z\varepsilon^{W \dashv Z}) = ((A, \alpha^N \circ \zeta_A), Z\varepsilon^{W \dashv Z})$$

where $\varepsilon^{W \dashv Z} : WZ \to \mathrm{Id}$ is the co-unit of $W \dashv Z$.

We construct an inverse H to G. Let $H((A, \alpha^M), \alpha^O) = (A, \alpha^N)$ where

$$\alpha^N : NA = U_M U_O F_O F_M U_M U_O((A, \alpha^M), \alpha^O) \to U_M U_O((A, \alpha^M), \alpha^O) = A$$
$$\alpha^N = U_M U_O\Big(\varepsilon^O \circ F_O \varepsilon^M U_O\Big) = U_M \alpha^O \circ U_M O \varepsilon^M U_O = U_M \alpha^O \circ U_M O(\alpha^M, \_)$$

where ε^M and ε^O denote the co-units of the EM-adjunctions and where $\_$ denotes the proof that α^M is an M-algebra morphism.

To see that $HG = \mathrm{Id}$, we plug in $\alpha^O = Z\varepsilon^{W \dashv Z}$ and $\alpha^M = \alpha^N \circ \zeta_A$ and obtain

$$\begin{aligned}
\alpha^N &\overset{?}{=} U_M Z \varepsilon^{W \dashv Z} \circ U_M O(\alpha^N \circ \zeta_A, _) \\
&= U_N\Big(\varepsilon^{W \dashv Z} \circ W(\alpha^N \circ \zeta_A, _)\Big) && U_M Z = U_N \text{ and } U_M O = U_N W, \\
&= U_N\Big(\varepsilon^{W \dashv Z} \circ W(\alpha^N, _) \circ W(\zeta_A, _)\Big) && \alpha^N \text{ and } \zeta_A \text{ are } M\text{-algebra morphisms,} \\
&= U_N\Big(\varepsilon^{W \dashv Z} \circ WZ(\alpha^N, _) \circ W(\zeta_A, _)\Big) && \alpha^N \text{ is an } N\text{-algebra morphism,} \\
&= U_N\Big((\alpha^N, _) \circ \varepsilon^{W \dashv Z} \circ W(\zeta_A, _)\Big) \\
&= \alpha^N \circ U_N\Big(\varepsilon^{W \dashv Z} \circ W(\zeta_A, _)\Big) \\
&= \alpha^N \circ U_N\Big(\varepsilon^{W \dashv Z} \circ W \eta^{Z \dashv W} F_M\Big) && \zeta = U_M \eta^{Z \dashv W} F_M \text{ so } (\zeta, _) = \eta^{Z \dashv W} F_M, = \alpha^N \quad \text{adjunction law.}
\end{aligned}$$

The fact that $GH = \mathrm{Id}$ is automatic by finality of the Eilenberg-Moore decomposition. □

7.3 Lawverian MAT Extensions from Monadic MAT Extensions

Definition 7.5. An extension of a Lawverian MAT $\mathcal{L} \in \mathsf{Law}(S)$ is a coslice under $\mathcal{L}$, i.e. another Lawverian MAT $\mathcal{K} \in \mathsf{Monad}(\mathsf{Set}^S)$ with a Lawverian MAT morphism $L : \mathcal{L} \to \mathcal{K}$.

By functoriality of the construction in section 5.3, it is clear that every monadic MAT extension $\zeta : M \to N$ (and hence every MAT presentation extension $\mathfrak{A} \subseteq \mathfrak{B}$) gives rise to a Lawverian MAT extension $L_\zeta : \mathcal{L}_M \to \mathcal{L}_N$ and that the composition functor $\mathsf{Model}(\mathcal{L}_N) \to \mathsf{Model}(\mathcal{L}_M)$ corresponds to the functor $\mathsf{EM}(\zeta)$.

7.4 Monadic MAT Extensions from Lawverian MAT Extensions

The functorial part of proposition 5.10 asserts that there is in fact a correspondence between monadic and Lawverian MAT extensions.

7.5 Factorizing the KEML-diagram for MAT extensions

We summarize our findings about MATs in a diagram. Assume that $\zeta : M \to N$ and $L : \mathcal{L} \to \mathcal{K}$ are corresponding MAT extensions (hence M corresponds to $\mathcal{L}$ and N to $\mathcal{K}$). Then we obtain the following commuting diagram, which contains the full KEML-diagram of N and $\mathcal{K}$ but not that of M and $\mathcal{L}$: the dotted arrows are *not* part from the KEML-diagram of M and $\mathcal{L}$:

$$\begin{tikzcd}
 & \mathsf{Set}^S \arrow[rrr, dotted, "N"] \arrow[ddl, "X\mapsto\prod_{s\in S}\langle s\rangle^{X\,s}"'] \arrow[dr, "J_M"] \arrow[dddr, "J_N"'] & & & \mathsf{Set}^S & \\
 & & \mathsf{Kl}(M) \arrow[r, dotted, "\overline{X}\mapsto(NX,\mu^N_X\circ\zeta_{NX})"] \arrow[dd, "\mathsf{Kl}(\zeta)"] \arrow[dll, dash, "\simeq"] & \mathsf{EM}(M) \arrow[ur, "U"] \arrow[drr, dash, "\simeq"] & & \\
\mathcal{L}^{\mathrm{op}} \arrow[rrrrr, dotted, "\mathrm{Hom}_{\mathcal{K}}(L\sqcup,L\sqcup)"'] \arrow[dd, "L^{\mathrm{op}}"'] & & & & & \mathsf{Model}(\mathcal{L}) \arrow[uul, "U:T\mapsto\lambda s.T\langle s\rangle"'] \\
 & & \mathsf{Kl}(N) \arrow[r, "\overline{X}\mapsto(NX,\mu^N)"] \arrow[dll, dash, "\simeq"] & \mathsf{EM}(N) \arrow[uu, "\mathsf{EM}(\zeta)"] \arrow[uuur, "U"'] \arrow[drr, dash, "\simeq"] & & \\
\mathcal{K}^{\mathrm{op}} \arrow[rrrrr, "\mathrm{Hom}_{\mathcal{K}}"'] & & & & & \mathsf{Model}(\mathcal{K}) \arrow[uu, "\sqcup\circ L"']
\end{tikzcd}$$

8 Higher-Order Algebraic Theories (HOATs)

In section 8.2, we define λHOATs, which are a completely different take on HOATs from Arkor and McDermott's [AM20]. In order to disambiguate, we will call their notion **closed HOATs (CHOATs)**.

We define λHOATs quite simply as extensions of the STLC, based on the well-known fact that the simply-typed lambda-calculus (STLC) is *the* internal language of a cartesian closed category (theorem 8.4). This is already a powerful concept but is a special case of what we already know from the previous section, unlike CHOATs. Moreover, λHOATs will be helpful when trying to understand CHOATs and their KEML-diagram (section 8.4), which is at least superficially very different from the ones we have seen so far.

I have not compared the practical usability of λHOATs and CHOATs in applied problems, and I can imagine that the more native support for binding available in CHOATs is often an advantage.

I would say that the most notable difference between both concepts is that the Lawvere theory of a λHOAT is essentially the category of judgements and derivations, whereas the Lawvere theory of a CHOAT is the category of arities/contexts/types and substitutions/terms. By consequence, the Lawvere theory of a CHOAT is cartesian closed, whereas the Lawvere theory of a λHOAT will merely be preclosed (definition 8.2).

8.1 The Simply-Typed Lambda-Calculus (STLC) as a MAT

Definition 8.1. The **simply-typed lambda-calculus** (STLC) with set of base types $\mathbb{B}$ is the MAT presented as follows:

- The set of sorts is:

$$\{\mathrm{Sub}(\Gamma,\Delta)\mid\Gamma,\Delta\in\mathrm{Ctx}\}\uplus\{\mathrm{Tm}(\Gamma,T)\mid\Gamma\in\mathrm{Ctx},T\in\mathrm{Ty}\}.$$

  where $\mathrm{Ctx}=\mathsf{List}(\mathrm{Ty})$ and types are given by the syntax:

$$\mathrm{Ty}\ni T::=(T\to T)|(T\times T)|\top|B_{b\in\mathbb{B}}$$

- There are each time introduction and elimination operators together with β- and η-axioms such that for any model A:

$$\begin{aligned}
A_{\mathrm{Sub}(\Gamma,())}&\cong\top,\\
A_{\mathrm{Sub}(\Gamma,(\Delta.T))}&\cong A_{\mathrm{Sub}(\Gamma,\Delta)}\times A_{\mathrm{Tm}(\Gamma,T)},\\
A_{\mathrm{Tm}(\Gamma,\top)}&\cong\top,\\
A_{\mathrm{Tm}(\Gamma,S\times T)}&\cong A_{\mathrm{Tm}(\Gamma,S)}\times A_{\mathrm{Tm}(\Gamma,T)},\\
A_{\mathrm{Tm}(\Gamma,S\to T)}&\cong A_{\mathrm{Tm}((\Gamma.S),T)}.
\end{aligned}$$

- For every context Γ, there is an identity substitution $\mathrm{id}_\Gamma \in O_{\mathrm{STLC}(\mathbb{B})}([], \mathrm{Sub}(\Gamma, \Gamma))$.
- There are *substitution* operations such that for any model A:

$$A_{\mathrm{Sub}(\Gamma,\Delta)} \times A_{\mathrm{Sub}(\Delta,\Theta)} \to A_{\mathrm{Sub}(\Gamma,\Theta)},$$
$$A_{\mathrm{Sub}(\Gamma,\Delta)} \times A_{\mathrm{Tm}(\Delta,T)} \to A_{\mathrm{Tm}(\Gamma,T)},$$

  and axioms that make all operations natural in Γ w.r.t. substitution.

The following definition is non-standard:

Definition 8.2. We call a category C with products **preclosed** if it is equipped with a functor $E : C^{\mathrm{op}} \times C \to C$ such that

- $E(\top, \sqcup) \cong \mathrm{Id}$,
- for every $x \in C$ (naturally in x), the functor $x/E := \mathrm{Hom}_C(x, E(\sqcup, \sqcup)) : C^{\mathrm{op}} \times C \to \mathrm{Set}$ is an alternative Hom-functor for $\mathrm{Obj}(C)$ with the same products, that does make C cartesian closed with exponentials given by E. This implies that E preserves products in its second argument.

Proposition 8.3. The Lawverian MAT of the STLC with base types from $\mathbb{B}$ is equivalent to the free cartesian preclosed category $\mathrm{CPC}(\mathbb{B})$ over $\mathbb{B}$.

Proof. It is sufficient to show that the Lawverian MAT $\mathcal{L}$ is cartesian preclosed and that the canonical map $\mathrm{CPC}(\mathbb{B}) \to \mathcal{L}$ sending b to $\langle \mathrm{Tm}((), B_b)\rangle$ has (up to natural isomorphism) a section (which is a preclosed cartesian functor) $F : \mathcal{L} \to \mathrm{CPC}(\mathbb{B})$.

To see cartesian preclosedness, write $\Sigma : \mathrm{Ctx} \to \mathrm{Ty}$ for the pairing up of a context:

$$\Sigma() = \top, \qquad \Sigma(\Gamma.T) = \Sigma\Gamma \times T.$$

We then have $\langle \mathrm{Tm}(\Gamma, T)\rangle \cong \langle \mathrm{Tm}((), \Sigma\Gamma \to T)\rangle$ and $\langle \mathrm{Sub}(\Gamma, \Delta)\rangle \cong \langle \mathrm{Tm}((), \Sigma\Gamma \to \Sigma\Delta)\rangle$. Then we can construct the cartesian product as $\langle \mathrm{Tm}((), S)\rangle \times \langle \mathrm{Tm}((), T)\rangle = \langle \mathrm{Tm}((), S \times T)\rangle$, and a preclosed structure as $E(\langle \mathrm{Tm}((), S)\rangle, \langle \mathrm{Tm}((), T)\rangle) := \langle \mathrm{Tm}((), S \to T)\rangle$.

- This satisfies $E(\top, \sqcup) \cong \mathrm{Id}$ because

$$E(\top, \mathrm{Tm}((), T)) \cong E(\mathrm{Tm}((), \top), \mathrm{Tm}((), T)) = \mathrm{Tm}((), \top \to T) \cong \mathrm{Tm}((), T),$$

- This allows for identities and composition,
- Functoriality of E follows from the fact that any derived rule (morphism in $\mathcal{L}$) of the form $\mathrm{Hom}(\mathrm{Tm}((), X), \mathrm{Tm}((), Y))$ can be weakened to $\mathrm{Hom}(\mathrm{Tm}(\Gamma, X), \mathrm{Tm}(\Gamma, Y))$: given morphsims $f \in \mathrm{Hom}(\mathrm{Tm}((), R), \mathrm{Tm}((), S))$ and $g \in \mathrm{Hom}(\mathrm{Tm}((), T), \mathrm{Tm}((), U))$, we can do

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{r : R \vdash r : R}}{r : R \vdash f(r) : S}\,f \qquad \vdash h : S \to T}{r : R \vdash h(f(r)) \vdash T}}{r : R \vdash g(h(f(r))) : U}\,g}{\vdash \lambda r.g(h(f(r))) : R \to U}.$$

We first define the action of $F : \mathcal{L} \to \mathrm{CPC}(\mathbb{B})$ on objects (again only considering sort objects of the form $\langle \mathrm{Tm}((), T)\rangle$):

$$\begin{aligned} F\langle \mathrm{Tm}((), T)\rangle &= F_{\mathrm{Ty}}(T), \\ F_{\mathrm{Ty}}(S \to T) &= E(F_{\mathrm{Ty}}S, F_{\mathrm{Ty}}T), \\ F_{\mathrm{Ty}}(S \times T) &= F_{\mathrm{Ty}}S \times F_{\mathrm{Ty}}T, \\ F_{\mathrm{Ty}}(\top) &= \top, \\ F_{\mathrm{Ty}}(B_b) &= b. \end{aligned}$$

It is evident that this action on objects is (up to natural isomorphism) a section of the action on objects of the canonical functor $\mathsf{CPC}(\mathbb{B}) \to \mathcal{L}$.

It is sufficient to define the functorial action on operation morphisms. Substitution amounts to x/E-composition. Everything else is automatic, e.g. for context extension we need to prove

$$F\langle \mathrm{Sub}(\Gamma, \Delta.T)\rangle \cong F\langle \mathrm{Sub}(\Gamma, \Delta)\rangle \times F\langle \mathrm{Ty}(\Gamma, T)\rangle$$

but both hands compute to

$$E(F_{\mathrm{Ty}}\Sigma\Gamma, F_{\mathrm{Ty}}\Sigma\Delta \times F_{\mathrm{Ty}}T) \cong E(F_{\mathrm{Ty}}\Sigma\Gamma, F_{\mathrm{Ty}}\Sigma\Delta) \times E(F_{\mathrm{Ty}}\Sigma\Gamma, F_{\mathrm{Ty}}T)$$

so this is just product preservation of E. One can verify, using the equational theory of the STLC, that this is a section of the morphism part of the canonical functor $\mathsf{CPC}(\mathbb{B}) \to \mathcal{L}$. □

If $\mathsf{CPC}(\mathbb{B})$ is the Lawvere theory of the STLC, then $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$ is its category of models. A model $T \in \mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$ is a product-preserving functor $T : \mathsf{CPC}(\mathbb{B}) \to \mathsf{Set}$. It is not required to preserve the preclosed structure in any particular way, but of course we do get a functor $T(E(\sqcup, \sqcup)) : \mathsf{CPC}(\mathbb{B})^{\mathsf{op}} \times \mathsf{CPC}(\mathbb{B}) \to \mathsf{Set}$ and in fact product-preservation does preserve the usability of morphisms such as $\circ : \mathrm{Hom}_{\mathsf{CPC}(\mathbb{B})}(E(y,z) \times E(x,y), E(x,z))$, i.e. substitution and function application are still possible in a model.

Theorem 8.4. The category $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$ of models of the STLC with base types from $\mathbb{B}$ is equivalent to the category of cartesian *closed* categories C with the same objects and same products as $\mathsf{CPC}(\mathbb{B})$ and exponentials given by E.[6]

Trivially, any such model C yields a cartesian preclosed identity-on-objects functor $\mathsf{CPC}(\mathbb{B}) \to C$.

Proof. Let $T : \mathsf{CPC}(\mathbb{B}) \to \mathsf{Set}$ be a product-preserving functor. We reorganize T into a CCC C with $\mathsf{Obj}(C) = \mathsf{Obj}(\mathsf{CPC}(\mathbb{B}))$.

- Define $\mathrm{Hom}_C(x, y) := T(E(x, y)) \in \mathsf{Set}$,
- Since T preserves products and the terminal object, this yields identity and composition operations for C,
- Since T preserves products and E preserves products in its second argument, we know that Hom_C preserves products in its second argument, i.e. the products of $\mathsf{CPC}(S)$ are products of C,
- We need to show that $E(y, x) \in \mathsf{Obj}(C)$ is an exponential. We have

$$\mathrm{Hom}_C(z, E(y, x)) = T(E(z, E(y, x))) \cong T(E(z \times y, x)) = \mathrm{Hom}_C(z \times y, x).$$

Conversely, given a CCC C with the same objects as $\mathsf{CPC}(S)$, we build a model T:

- $Tx = \mathrm{Hom}_C(\top, x)$. Note that we have to do this if we want to obtain an inverse to the converse action:

$$T\langle \mathrm{Tm}((), S)\rangle \cong T(E(\top, \langle \mathrm{Tm}((), S)\rangle)),$$

  Now products are preserved.

It is easy to check that these operations constitute an equivalence. □

8.2 λHOATs from the STLC

Definition 8.5. A (Lawverian / monadic / presentation of a) **λ-higher order algebraic theory** with set of sorts $\mathbb{B}$ is a (Lawverian / monadic / presentation of a) MAT extension of the STLC with set of base types $\mathbb{B}$.

In particular, a Lawverian λHOAT with sorts $\mathbb{B}$ is a coslice under $\mathsf{CPC}(\mathbb{B})$ in the category of Lawvere theories with the appropriate set of sorts.

[6] The requirement that the objects of the CCC are precisely those of $\mathsf{CPC}(\mathbb{B})$ may be unexpected but can be attributed to the fact that we are treating the STLC as a MAT. Should we treat it as a GAT with contexts and types as part of the syntax, then we would have some freedom in modelling contexts and types.

8.3 Intermezzo: Relative Monads

Before we compare to Arkor and McDermott's approach [AM20], we revise relative monads.

Definition 8.6. [ACU15, nLa22b] Given a diagram of functors (there is no opportunity for asking commutativity)

$$\begin{tikzcd} \mathcal{A} \arrow[d, "L"'] \arrow[dr, "I" near start] & \mathcal{B} \arrow[d, "R"] \arrow[dl, "J"' near start] \\ \mathcal{C} & \mathcal{D} \end{tikzcd}$$

we are interested in the case where there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(Lx, Jy) \cong \mathrm{Hom}_{\mathcal{D}}(Ix, Ry) : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$.

If $\mathcal{B} = \mathcal{C}$ and $J = \mathrm{Id}$, then we call this a (monadic) **relative adjunction** and we have $\mathrm{Hom}_{\mathcal{C}}(Lx, y) \cong \mathrm{Hom}_{\mathcal{D}}(Ix, Ry)$:

$$\begin{tikzcd} \mathcal{A} \arrow[d, "L"'] \arrow[dr, "I"] & \\ \mathcal{C} \arrow[r, "R"'] & \mathcal{D} \end{tikzcd}$$

In that case, RL is a relative monad over I.[7]

A **relative monad** over a functor $I : \mathcal{A} \to \mathcal{D}$ is a functor $M : \mathcal{A} \to \mathcal{D}$ equipped with natural transformations $\eta : I \to M$ and $\mathsf{bind} : \mathrm{Hom}_{\mathcal{D}}(Ix, My) \to \mathrm{Hom}_{\mathcal{D}}(Mx, My)$, such that $\mathsf{bind}(\eta) = \mathrm{id}$ and $\mathsf{bind}(\varphi) \circ \eta = \varphi$ and $\mathsf{bind}(\chi) \circ \mathsf{bind}(\varphi) = \mathsf{bind}(\mathsf{bind}(\chi) \circ \varphi)$.

The **Kleisli category** of $M : \mathcal{A} \to \mathcal{D}$ is the category $\mathsf{Kl}(M)$ with objects $\overline{x}$ where $x \in \mathcal{A}$ and morphisms $\overline{\varphi} \in \mathrm{Hom}_{\mathsf{Kl}(M)}(\overline{x}, \overline{y})$ where $\varphi \in \mathrm{Hom}_{\mathcal{D}}(Ix, My)$. The identity is $\overline{\eta}$ and composition is given by $\overline{\chi} \circ \overline{\varphi} = \overline{\mathsf{bind}(\chi) \circ \varphi}$. There is a functor $J_M : \mathcal{A} \to \mathsf{Kl}(M)$ sending x to $\overline{x}$ and φ to $\overline{\eta \circ I\varphi} = \overline{M\varphi \circ \eta}$, and a functor $P_M : \mathsf{Kl}(M) \to \mathcal{D}$ sending $\overline{x}$ to Mx and $\overline{\varphi}$ to $\mathsf{bind}(\varphi)$. We have a relative adjunction $\mathrm{Hom}_{\mathsf{Kl}(M)}(J_M x, \overline{y}) \cong \mathrm{Hom}_{\mathcal{D}}(Ix, P_M \overline{y})$ that composes to $M = P_M J_M$.

The **Eilenberg-Moore category** of $M : \mathcal{A} \to \mathcal{D}$ is the category $\mathsf{EM}(M)$ of EM-algebras of M. An **EM-algebra** is an object $a \in \mathcal{D}$ equipped with a natural isomorphism $\mathsf{bind}^a : \mathrm{Hom}_{\mathcal{D}}(Ix, a) \to \mathrm{Hom}_{\mathcal{D}}(Mx, a)$ such that $\mathsf{bind}^a(\varphi) \circ \eta = \varphi$ and $\mathsf{bind}^a(\chi) \circ \mathsf{bind}(\varphi) = \mathsf{bind}^a(\mathsf{bind}^a(\chi) \circ \varphi)$. A morphism of EM-algebras is a morphism $\zeta : a \to b$ such that $\zeta \circ \mathsf{bind}^a(\varphi) = \mathsf{bind}^b(\zeta \circ \varphi)$. There is a functor $F_M : \mathcal{A} \to \mathsf{EM}(M) : x \mapsto (Mx, \mathsf{bind})$ and a functor $U_M : \mathsf{EM}(M) \to \mathcal{D} : (a, \mathsf{bind}^a) \mapsto a$. Again we have a relative adjunction $\mathrm{Hom}_{\mathsf{EM}(M)}(F_M x, (y, \mathsf{bind}^y)) \cong \mathrm{Hom}_{\mathcal{D}}(Ix, U_M(y, \mathsf{bind}^y))$ that composes to $M = U_M F_M$.

8.4 Comparison to Arkor and McDermott's Approach

We will now compare and relate our approach to λHOATs to an informal and simplified account of Arkor and McDermott's HOATs [AM20] which we will here call CHOATs (for closed HOATs).

Before doing so, we put λHOATs in a different perspective. We defined them above as extensions of the STLC, but as argued in sections 7.1 and 7.2, extensions can be seen as a special case of EATs. As such, we will slightly generalize our notion of λHOATs here to include all EATs over the STLC with base types $\mathbb{B}$. The KEML diagram for such a theory (as a special case of section 6.5) looks like this, where M_0 is the

[7] If instead $I = \mathrm{Id}$ then we can dually consider comonadic relative adjunctions and relative comonads over J.

monadic MAT for the STLC and $\mathcal{L}$ and M are the monadic and Lawverian EAT for the specific λHOAT.

$$\begin{tikzcd}
\mathsf{Kl}(M_0) \arrow[rr, dotted, "\overline{X}\mapsto(M_0X,\mu_0)"] \arrow[dd, dotted, dash, "\simeq"'] & & \mathsf{EM}(M_0) \arrow[r, "M"] \arrow[dl, dash, "\simeq"'] \arrow[dd, "J_M"] & \mathsf{EM}(M_0) \arrow[dr, dash, "\simeq"] & \\
& \mathsf{Model}(\mathsf{CPC}(\mathbb{B})) \arrow[dd, "X\mapsto\lim_{(s,x)\in\int_{\mathsf{CPC}(\mathbb{B})}} X\langle s\rangle"] & & & \mathsf{Model}(\mathsf{CPC}(\mathbb{B})) \\
\mathsf{CPC}(\mathbb{B})^{\mathrm{op}} \arrow[ur, dotted, "\mathrm{y}=\mathrm{Hom}_{\mathsf{CPC}(\mathbb{B})}"] \arrow[dr, dotted, "\langle\sqcup\rangle"'] & & \mathsf{Kl}(M) \arrow[r, "\overline{X}\mapsto(MX,\mu)"] \arrow[dl, dash, "\simeq"'] & \mathsf{EM}(M) \arrow[uu, "U"] \arrow[dr, dash, "\simeq"] & \\
& \mathcal{L}^{\mathrm{op}} \arrow[rrr, "\mathrm{Hom}_{\mathcal{L}}"'] & & & \mathsf{Model}(\mathcal{L}) \arrow[uu, "U:T\mapsto T\circ\langle\sqcup\rangle"']
\end{tikzcd}$$

Let us start by handwaving away the distinction between $\mathsf{EM}(M_0)$ and $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$. Then we get the following subdiagram:

$$\begin{tikzcd}
& \mathsf{Model}(\mathsf{CPC}(\mathbb{B})) \arrow[rrr, "M"] \arrow[dd, "X\mapsto\lim_{(s,x)\in\int_{\mathsf{CPC}(\mathbb{B})}} X\langle s\rangle"'] \arrow[dr, "J_M"] & & & \mathsf{Model}(\mathsf{CPC}(\mathbb{B})) \\
\mathsf{CPC}(\mathbb{B})^{\mathrm{op}} \arrow[ur, dotted, "\mathrm{y}=\mathrm{Hom}_{\mathsf{CPC}(\mathbb{B})}"] \arrow[dr, dotted, "\langle\sqcup\rangle"'] & & \mathsf{Kl}(M) \arrow[r, "\overline{X}\mapsto(MX,\mu)"] \arrow[dl, dash, "\simeq"'] & \mathsf{EM}(M) \arrow[ur, "U"] \arrow[dr, dash, "\simeq"] & \\
& \mathcal{L}^{\mathrm{op}} \arrow[rrr, "\mathrm{Hom}_{\mathcal{L}}"'] & & & \mathsf{Model}(\mathcal{L}) \arrow[uu, "U:T\mapsto T\circ\langle\sqcup\rangle"']
\end{tikzcd}$$

Now M is a monad on $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$. If we precompose M with $\mathrm{y} : \mathsf{CPC}(\mathbb{B})^{\mathrm{op}} \to \mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$, then we obtain $M\mathrm{y}$, which is a relative monad over y, having its own Kleisli and EM-categories. This yields the following diagram:

$$\mathsf{CPC}(\mathbb{B})^{\mathrm{op}} \xrightarrow{\mathrm{y}} \mathsf{Model}(\mathsf{CPC}(\mathbb{B})) \tag{1}$$

$$\begin{tikzcd}
\mathsf{CPC}(\mathbb{B})^{\mathrm{op}} \arrow[rrr, "M\mathrm{y}"] \arrow[dr, "J_{M\mathrm{y}}"] \arrow[ddr, "J_M"'] \arrow[ddd, "\langle\sqcup\rangle"'] & & & \mathsf{Model}(\mathsf{CPC}(\mathbb{B})) \\
& \mathsf{Kl}(M\mathrm{y}) \arrow[r, "\overline{s}\mapsto(M\mathrm{y}s,\mathrm{bind})"] \arrow[d, "\overline{s}\mapsto\overline{\mathrm{y}s}"] & \mathsf{EM}(M\mathrm{y}) \arrow[ur, "U_{M\mathrm{y}}"] & \\
& \mathsf{Kl}(M) \arrow[r, "\overline{X}\mapsto(MX,\mu)"'] \arrow[dl, dash, "\simeq"'] & \mathsf{EM}(M) \arrow[u, "(A,\alpha)\mapsto(A,\alpha\circ M\sqcup)"] \arrow[uur, "U_M"'] \arrow[dr, dash, "\simeq"] & \\
\mathcal{L}^{\mathrm{op}} \arrow[rrr, "\mathrm{Hom}_{\mathcal{L}}"'] & & & \mathsf{Model}(\mathcal{L}) \arrow[uuu, "U:T\mapsto T\circ\langle\sqcup\rangle"']
\end{tikzcd}$$

Arkor and McDermott's work [AM20] does not explicitly feature a KEML-diagram. However, we reconstruct one here using their symbols, and will then explain node per node how it corresponds to ours. We remain informal, which allows us to make a few simplifications:

- We only consider the situation at order ω,
- We ignore order incrementation and decrementation functors.

$$\begin{tikzcd}
\mathbb{L}(S)^{\mathrm{op}} \arrow[r, "p : x \mapsto \mathsf{Kl}_{\mathbb{L}(S)}(x \times \sqcup)"] & \mathsf{HLaw}(S)
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
\mathbb{L}(S)^{\mathrm{op}} \arrow[rrr, "T_L : x \mapsto \mathsf{Kl}_{\mathcal{L}}(x \times \sqcup)"] \arrow[dr, "J_{T_L}"] \arrow[dd, "L^{\mathrm{op}}"'] & & & \mathsf{HLaw}(S) \\
& \mathsf{Kl}(T_L) \arrow[r, "\overline{x} \mapsto (T_L x, \mathrm{bind})"] & \mathsf{EM}(T_L) \arrow[ur, "U_{T_L}"] \arrow[dr, dash, "\simeq"] & \\
\mathcal{L}^{\mathrm{op}} \arrow[rrr, "x \mapsto (\mathsf{Kl}_{\mathcal{L}}(x \times \sqcup), J_{x \times \sqcup})"] \arrow[ur, dash, "\simeq"] & & & (\mathcal{L}, L)/\mathsf{HLaw}(S) \arrow[uu, "U"']
\end{tikzcd}$$

- As their is no confusion with STLC sorts, the set of sorts is calles S instead of $\mathbb{B}$.
- The category $\mathbb{L}(S)$ is (equivalent to) the category whose objects are types of the STLC (with base types from S) and whose morphisms are single-variable STLC terms. Equivalently, we could take contexts as objects and substitutions as morphisms. This is also equivalent to the free cartesian closed category $\mathsf{CCC}(S)$ over S. It is slightly different to what we had in the upper left node of the KEML-diagram: the free cartesian preclosed category $\mathsf{CPC}(\mathbb{B})$, which corresponded to the category of judgements and derivations of the STLC.
- We write $\mathsf{HLaw}(S)$ instead of $\mathsf{Law}(S)$ to avoid collision with definition 5.5. The category $\mathsf{HLaw}(S)$ is the category of Lawverian HOATs with sorts in S. Such a Lawverian HOAT is defined as a pair $(\mathcal{L}, L)$ of a category $\mathcal{L}$ such that $\mathrm{Obj}(\mathcal{L}) = \mathrm{Obj}(\mathbb{L}(S))$ and such that $L : \mathbb{L}(S) \to \mathcal{L}$ is a cartesian closed identity-on-objects functor.

  Note that we could have given analogous definitions, e.g. a Lawverian MAT could have been defined as a cartesian identity-on-objects functor from the free cartesian category over S.

  We remark that $\mathsf{HLaw}(S)$ is quite similar to $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$:

  - $\mathsf{HLaw}(S)$ is the category of cartesian closed identity-on-objects functors from $\mathbb{L}(S) \simeq \mathsf{CCC}(S)$,
  - $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$ is the category of cartesian preclosed identity-on-objects functors from $\mathsf{CPC}(\mathbb{B})$ to closed categories (theorem 8.4).
- Given their definition of Lawverian HOATs, it is no surprise that the functor $L : \mathbb{L}(S) \to \mathcal{L}$ corresponds to our $\langle \sqcup \rangle : \mathsf{CPC}(\mathbb{B}) \to \mathcal{L}$.
- Using the correspondence between $\mathsf{HLaw}(S)$ and $\mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$, we relate $\mathbf{y}$ and p.

  If $x \in \mathsf{CPC}(\mathbb{B})$, then $\mathbf{y}x \in \mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$ is a cartesian functor

  $$\mathbf{y}x : \mathsf{CPC}(\mathbb{B}) \to \mathrm{Set} : y \mapsto \mathrm{Hom}_{\mathsf{CPC}(\mathbb{B})}(x, y).$$

  The proof of theorem 8.4 allows us to view $\mathbf{y}x$ as a category C_x with the same objects as $\mathsf{CPC}(\mathbb{B})$ and where $\mathrm{Hom}_{C_x}(y, z) = \mathbf{y}x(E(y, z)) = \mathrm{Hom}_{\mathsf{CPC}(\mathbb{B})}(x, E(y, z)) \cong \mathrm{Hom}_{\mathsf{CPC}(\mathbb{B})}(x \times y, z)$. In other words, $\mathbf{y}x$ is essentially the Kleisli category of the comonad $x \times \sqcup$ on $\mathsf{CPC}(\mathbb{B})$.

  Arkor and McDermott's functor p (neglecting an order change) also sends $x \in \mathbb{L}(S) \simeq \mathsf{CCC}(S)$ to the Kleisli category of the comonad $x \times \sqcup$ on $\mathbb{L}(S) \simeq \mathsf{CCC}(S)$.
- Similarly, we relate $M\mathbf{y}$ and T_L. Specifically, we assume that $M = M_{\mathcal{L}}$ arises from the Lawverian EAT $\mathcal{L}$ over the STLC.

  If $x \in \mathsf{CPC}(\mathbb{B})$, then $M_{\mathcal{L}}\mathbf{y}x \in \mathsf{Model}(\mathsf{CPC}(\mathbb{B}))$ is a cartesian functor

  $$M_{\mathcal{L}}\mathbf{y}x : \mathsf{CPC}(\mathbb{B}) \to \mathrm{Set}.$$

  From the proof of proposition 6.8, we know that

  $$M_{\mathcal{L}}\,(\mathbf{y}x)\,y = \mathrm{Hom}_{\mathcal{L}}\left(\lim_{(z,\varphi) \in \int_{\mathsf{CPC}(\mathbb{B})} \mathbf{y}x} \langle z \rangle, \langle y \rangle\right) \cong \mathrm{Hom}_{\mathcal{L}}(\langle x \rangle, \langle y \rangle).$$

The proof of theorem 8.4 allows us to view $M_{\mathcal{L}}\mathbf{y}x$ as a category $\mathcal{D}_x$ with the same objects as $\mathrm{CPC}(\mathbb{B})$ and where $\mathrm{Hom}_{\mathcal{D}_x}(y,z) = M_{\mathcal{L}}(\mathrm{y}x)(E(y,z)) = \mathrm{Hom}_{\mathcal{L}}(\langle x\rangle, \langle E(y,z)\rangle)$.

Translating this to Arkor and McDermott's framework and recalling that L is identity-on-objects and properly closed, we get $\mathrm{Hom}_{\mathcal{L}}(x, z^y) \cong \mathrm{Hom}_{\mathcal{L}}(x \times y, z)$. So in their framework, $\mathcal{D}_x$ is the Kleisli category of the comonad $x \times \sqcup$ on the Lawverian HOAT $\mathcal{L}$. This turns out to be a relative monad over p and is called T_L.

- It turns out that the Lawverian CHOAT $\mathcal{L}$ is now equivalent to the Kleisli category of the *relative* monad T_L, whereas for λHOATs it was equivalent to the Kleisli category of the ordinary monad.

- Arkor and McDermott define a category of (Set-valued) models of a Lawverian CHOAT as the category of cartesian closed functors $\mathcal{L} \to \mathrm{Set}$.

  However they have another model-like notion called term algebras. Rather than delving into what term algebras are, we take their observation for granted that the category of term algebras of $(\mathcal{L}, L)$ is equivalent to the category of coslices under $(\mathcal{L}, L)$.

  It turns out that $\mathsf{EM}(T_L)$ is isomorphic to the category of term algebras [AM20, thm. 8.2]. To understand this intuitively:

  – An EM-algebra for T_L is a lawvere theory $(\mathcal{K}, K)$ such that for any x (naturally in x) a morphism of Lawvere theories $\mathsf{Kl}_{\mathbb{L}(S)}(x \times \sqcup) \to \mathcal{K}$ lifts to $\mathsf{Kl}_{\mathcal{L}}(x \times \sqcup) \to \mathcal{K}$. Taking $x = \top$ already implies that $K : \mathbb{L}(S) \to \mathcal{K}$ lifts to $\mathcal{L} \to \mathcal{K}$, so we have produced a coslice under $(\mathcal{L}, L)$.

  – Conversely, given a coslice $F : (\mathcal{L}, L) \to (\mathcal{K}, K)$, we argue that any morphism of Lawvere theories $G : \mathsf{Kl}_{\mathbb{L}(S)}(x \times \sqcup) \to \mathcal{K}$ lifts to $\mathsf{Kl}_{\mathcal{L}}(x \times \sqcup) \to \mathcal{K}$. Note that any morphism in $\mathsf{Kl}_{\mathcal{L}}(x \times \sqcup)$, say $\overline{\varphi} : \overline{y} \to \overline{z}$, i.e. $\varphi \in \mathrm{Hom}_{\mathcal{L}}(x \times y, z)$, can be reconstructed from the morphism $\overline{\mathrm{id}} \in \mathrm{Hom}_{\mathsf{Kl}_{\mathbb{L}(S)}(x \times \sqcup)}(\overline{y}, \overline{x \times y})$ and φ. We can apply G to the former and F to the latter, yielding $F\varphi \circ G\overline{\mathrm{id}} \in \mathcal{K}(y,z)$. After verifying that this action is functorial and inverse to the previous one, we can conclude that term algebras correspond to EM-algebras.

8.5 Models and Term Algebras of the Trivial HOAT

In this section, we consider what are

- models of the empty λHOAT, i.e. the empty EAT over the STLC,
- models and term algebras of the trivial CHOAT, i.e. $(\mathbb{L}(S), \mathrm{Id})$.

8.5.1 Models of the Empty λHOAT

This is simply the category $\mathsf{Model}(\mathrm{CPC}(\mathbb{B}))$. As discussed in theorem 8.4, this is the category of CCCs with the same objects and products as $\mathrm{CPC}(\mathbb{B})$ and exponentials given by E, or differently put the category of CCCs equipped with a cartesian preclosed identity-on-objects functor from $\mathrm{CPC}(\mathbb{B})$.

Any CCC $\mathcal{D}$ with a map $\mathbb{B} \to \mathrm{Obj}(\mathcal{D})$ (yielding a preclosed functor $F : \mathrm{CPC}(\mathbb{B}) \to \mathcal{D}$) can be turned into such a model by defining a new CCC C with objects $\mathrm{Obj}(C) = \mathrm{Obj}(\mathrm{CPC}(\mathbb{B}))$ and morphisms $\mathrm{Hom}_C(x,y) = \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$.

As such, our framework really includes all models, *including* presheaf models. Indeed $\mathrm{CPC}(S)$ is not the category of contexts and morphisms (whence we would map to a presheaf category) but the category of judgements and derivations (whence we can map to Set and still accommodate a presheaf model, sending judgements to sets of presheaf morphisms).

8.5.2 Models of the Trivial CHOAT

For Arkor and McDermott, models of the Lawvere theory of a CHOAT are cartesian closed functors from its Lawvere theory $\mathcal{L}$ to Set (or another category).

Let us first consider only Set-valued models. These seem in a way less powerful than (Set-valued) models of λHOATs, because the insistence on preservation of exponentials rules out presheaf models and enforces the inclusion of exotic functions as semantic functions. Indeed, we cannot model types S and T as presheaves and then model the function type $S \to T$ as a presheaf exponential, because the framework insists that the semantic *terms* $\vdash f : S \to T$ are *all of the metatheoretic functions* sending semantic terms $\vdash s : S$ to semantic terms $\vdash f(s) : T$. There is no hope to prove results like parametricity, naturality w.r.t. isomorphisms (simply typed HoTT) or Löb induction this way. In other words, not every CCC is a Set-valued *model* of the STLC seen as a CHOAT. In the next subsection, we shall see that CCCs are instead *term algebras* of the STLC seen as a CHOAT.

However, unlike for λHOATs, it makes a lot of sense to consider presheaf-valued models of a CHOAT $(\mathcal{L}, L)$, i.e. closed functors $\mathcal{L} \to \mathsf{Psh}(\mathcal{W})$. Indeed, in the Lawvere theory of a λHOAT, objects are judgements and morphisms are derivations, and it seems bizarre to model judgements (with context!) as presheaves and derivations as presheaf morphisms.

However, CHOATs correspond to a Gentzen-style presentation instead of a presentation with contexts, so that the objects of the Lawvere theory are really just types and the morphisms are really just functions. Of course it is sensible to model types as presheaves and functions as presheaf morphisms, this is in fact exactly how presheaf models work.

A dependent (GAT-style) generalization of CHOATs that could cover dependent type theory seems difficult, because Gentzen-style dependent type theory contains dependent judgements (which are therefore not self-contained statements).

8.5.3 Term Algebras of the Trivial CHOAT

A term algebra of $(\mathcal{L}, L)$ is a slice under $(\mathcal{L}, L)$ in $\mathsf{HLaw}(S)$. In the case of the trivial CHOAT $(\mathbb{L}(S), \mathrm{Id})$, which is the initial object of $\mathsf{HLaw}(S)$, a term algebra is just a Lawverian CHOAT, which is basically a CCC with the same objects as $\mathbb{L}(S) \simeq \mathsf{CCC}(S)$, which are really also the same objects we find in $\mathsf{CPC}(S)$. So we could say that a model of the STLC as a λHOAT is the same as a term algebra of the STLC as a CHOAT.

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A Version History

A.1 v0.2

- Arkor and McDermott do have a theorem on EM-algebras [AM20, thm. 8.2].
- Adapt the abstract: considering prior theories is not novel.
- Adapt the abstract: our Lawvere theories for MATs and EATs diverge from the literature.
- Definition 6.1: fix the definition of the set of lists of conditions in the case of a MAT.
- Added section 6.3.1 on two-stage decomposition of EATs.
- Improved introductory text in section 8.

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