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THE FRACTAL GEOMETRY OF THE BOUNDARY OF DRAGON CURVES

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ABSTRACT

This article discusses the fractal geometry of the boundary of the dragon curves by analyzing the boundary's structure, calculating its Hausdorff dimension, and computing the area enclosed within the boundary. The algorithms for generating computer graphics for the boundary curves have also been included.

Introduction

Fractals are everywhere in this world. From clouds, coastlines, leaves, to biological organs, many scientists can use fractal geometry to describe complex natural objects and processes [1-3]. Opening several new research areas in mathematics, geometry, and computer graphics, fractal geometry seems to describe natural shapes and forms more gracefully and succinctly than does Euclidean geometry.

The concept of scale invariance or self-similarity is an underlying theme in all fractal structures, with many examples going back to the classical mathematicians.

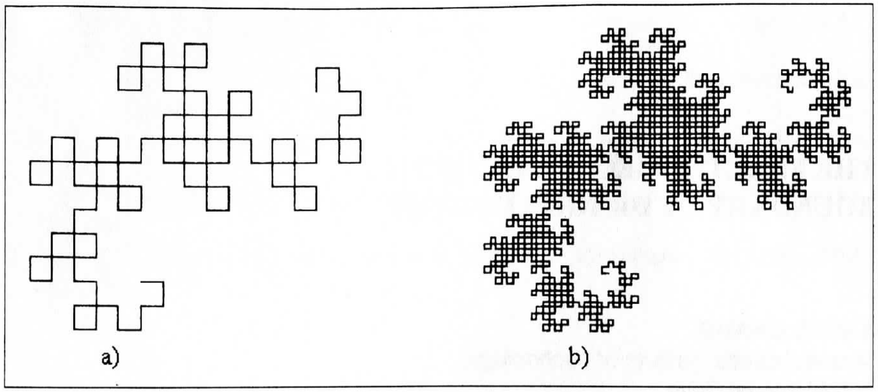

Figure 1. The dragon curve: a) after seven folds; b) after eleven folds.

What we know so well as the Cantor set, the Koch curve, and the Hilbert curve, are among these classical irregular curves, along with the dragon curve.

The dragon curve is a space filling curve with dimension 2 which originally came from the repeated folding of a long strip of paper in the same direction [4]. After pleating the paper, it is then unfolded with each adjacent segments of paper formed into a right angle. The dragon curve emerges slowly with striking complexity and beauty. As shown in Figure 1, it is a construction that fills a finite area in a plane but has a fractal boundary. This same curve can also be constructed by a computer algorithm using IFS (Iterated Function Systems) or the chaos game [3].

Although the dragon curve has already been investigated in several articles, its boundary has remained unexplored. To alleviate this neglect, it is the purpose of this article to study the boundary of the dragon curve. We first partition the boundary into several pieces to find its IFS structure. From the IFS structure we then calculated its Hausdorff dimension. Included in this article are also several algorithms to generate the boundary using a computer. Finally, we computed the area enclosed by the border of the dragon curve.

IFS Structure of the Boundary

While it is evident that the boundary of the dragon curve is a fractal with recurring self-similarity, its exact structure remains elusive. To discover its hidden pattern, we partition the complete boundary of the dragon curve into four similar pieces, as shown in Figure 2. Figure 3 shows one section of the border to which we will now focus our attention. Upon observation, it is apparent that this piece can be constructed with an infinite number of self-similar copies of itself with different sizes and rotated at various angles.

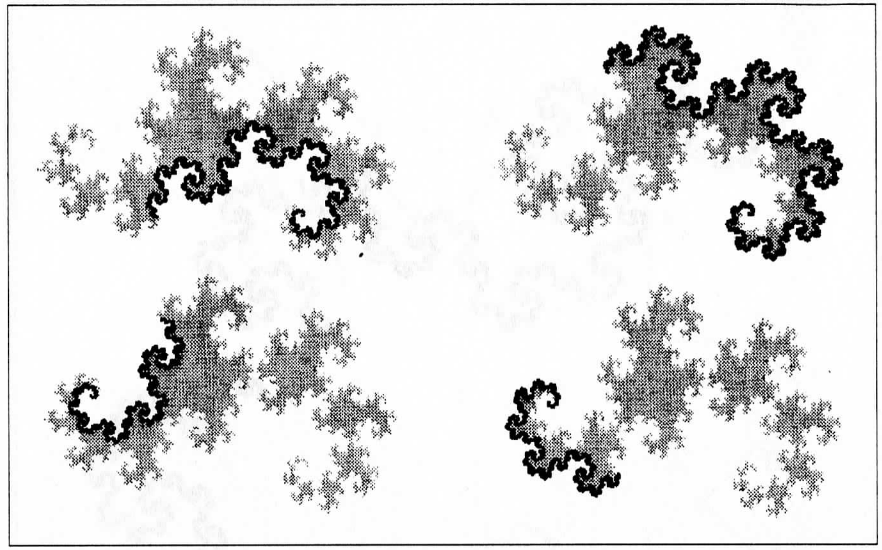

Figure 2. The four self-similar pieces of the boundary of the dragon curve.

To analyze the curve in Figure 3, let its size be represented by the length of line segment A_1A_∞ as shown in Figure 4. Placed in opposite directions, the first two copies and their sizes can be represented by the line segments A_1B_1 and B_1A_2 with

$$A_1B_1 = B_1A_2 = 2^{-3/2} A_1A_\infty \tag{1}$$

The rest of the infinite number of copies can be grouped into pairs with each pair containing two exact duplicates placed in the opposite directions. All pairs form an infinite sequence such that each element in the sequence is reduced by a factor of $2^1/2$, and rotated by an angle of –45° with respect to the previous one, i.e.,

$$A_{n+1}B_{n+1} = B_{n+1}A_{n+2} = 2^{-1/2} A_nB_n \quad \text{for} \quad n = 1, 2, \ldots, \infty \tag{2}$$

Hausdorff Dimension of the Boundary

The fractal dimension d of a self-similar curve constructed by N similar pieces with reduction factors $C_1, C_2, \ldots, C_N$ can be calculated by solving the equation [5]:

$$C_1^d + C_2^d + \ldots + C_N^d = 1 . \tag{3}$$

For the curve as shown in Figure 3, it is easy to show that

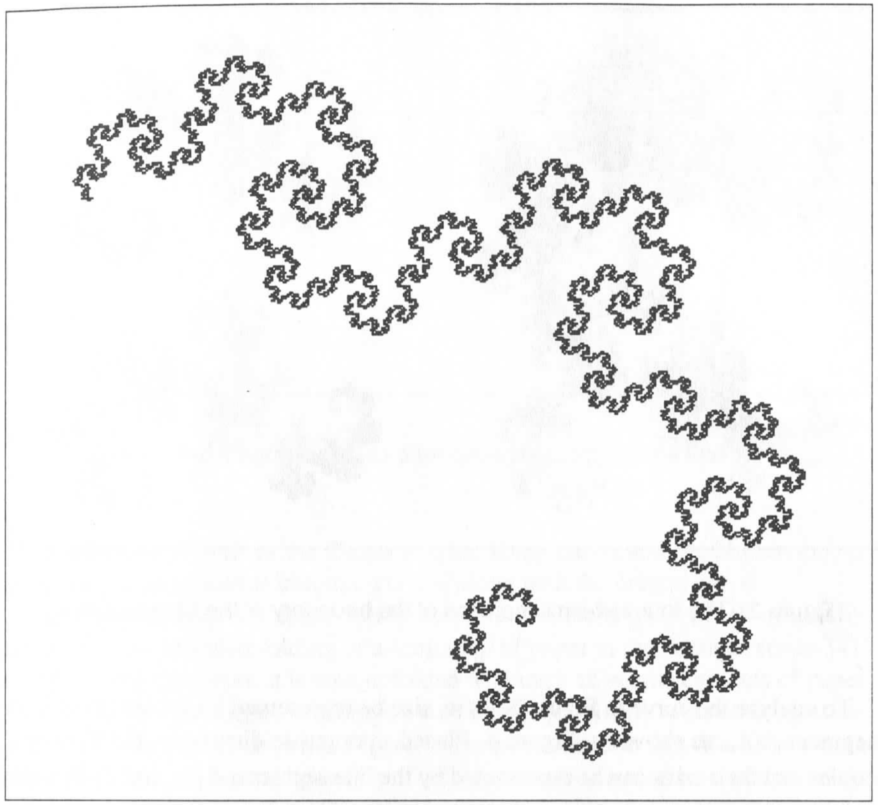

Figure 3. One section of the boundary of the dragon curve.

$$C_1 = C_2 = 2^{(-3/2)} \quad (4)$$

and

$$C_{2n+1} = C_{2n+2} = 2^{(-1/2)}\, C_{2n-1} \quad \text{for} \quad n = 1, 2, \ldots, \infty \quad (5)$$

By substituting (4) and (5) into (3), the equation for d can be rewritten as

$$2 \times 2^{(-3/2)d}\,(1 + 2^{(-1/2)d} + (2^{(-1/2)d})^2 + \ldots + (2^{(-1/2)d})^N) = 1 \quad (6)$$

When $N \rightarrow \infty$, formula (6) results in

$$\frac{2 \times 2^{(-3/2)d}}{1 - 2^{(-1/2)d}} = 1\,, \quad (7)$$

or

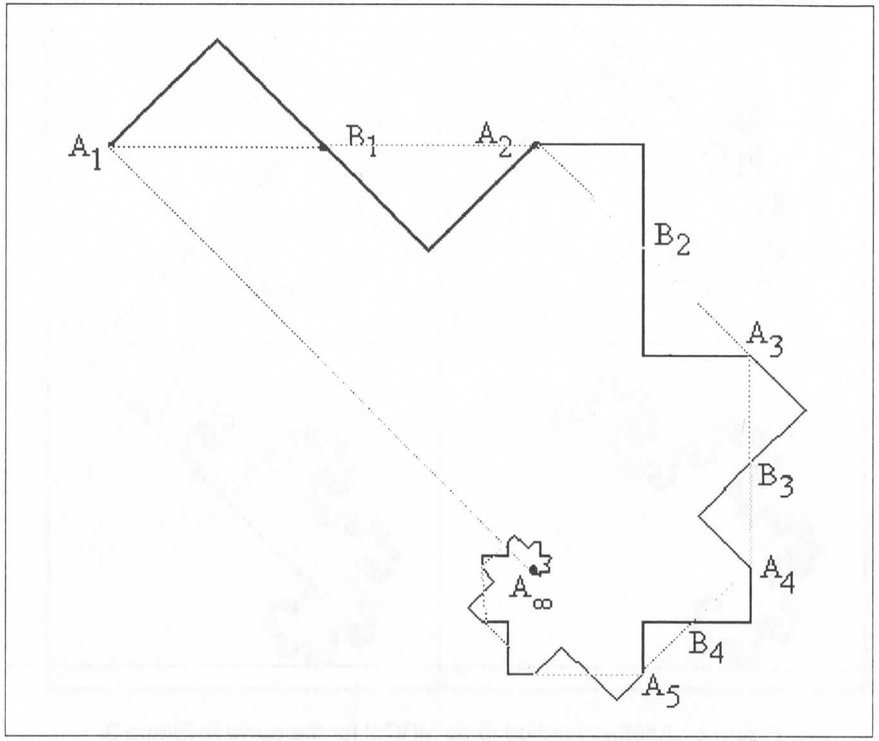

Figure 4. The size of the curve in Figure 3 can be represented by the line segment A_1A_∞. The total curve can be partitioned to the pairs of the reduced copies.

$$x^3 - x^2 - 2 = 0 \quad \text{where} \quad x = 2^{d/2}\,. \quad (8)$$

Solving equation (8) for x yields a real solution and two conjugated complex solutions. From the real solution, the fractal dimension d of the curve in Figure 3 can be expressed as

$$d = \frac{2\ln(\,D^{1/3} + \frac{1}{9}D^{-1/3} + \frac{1}{3}\,)}{\ln 2}\,, \quad (9)$$

where $D = \frac{28}{27} + \frac{1}{9}\sqrt{29*3}\,,$

the numerical value of d is calculated to be approximately 1.523627085.

An alternative method to computer d is to partition the curve in Figure 3 into three similar pieces, as shown in Figure 5. Then the equation for d is:

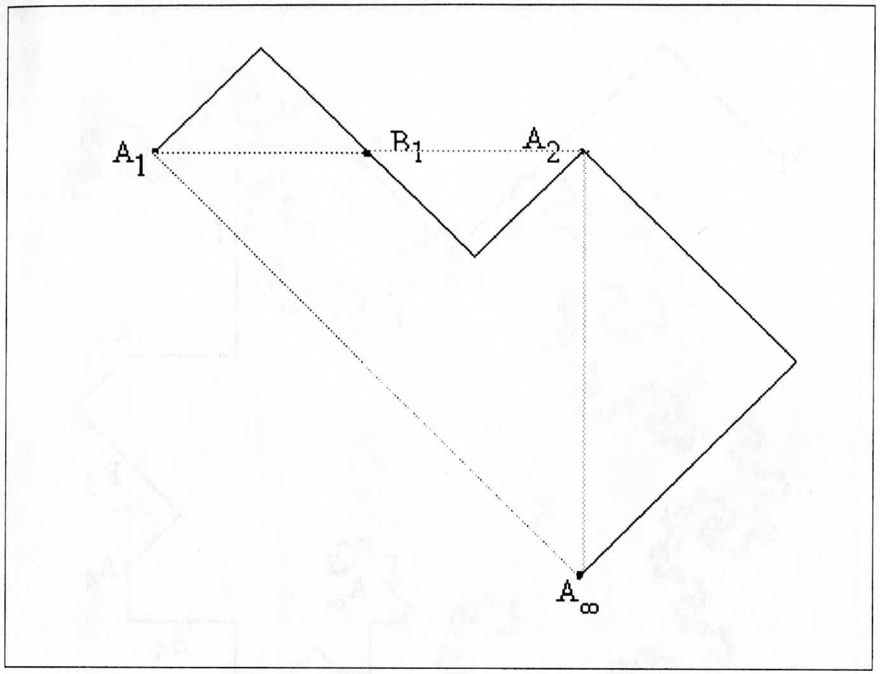

Figure 5. Another method to do MRCM for the curve in Figure 3.

$$2 \times 2^{(-3/2)d} + 2^{(-1/2)d} = 1, \tag{10}$$

which can be reduced to the same algebraic equation as (8) to yield the same solution for the fractal dimension d.

Since the boundary of the dragon curve is composed of four copies of the curve we just examined, the fractal dimension of the entire boundary is also d = 1.523627085.

Algorithms to Generate the Boundary of the Dragon Curve

In accordance with the two different methods for determining the dimension of the boundary of the dragon curve, we present two different algorithms for generating the four sections making up the boundary. Like the first method for calculating d, Algorithm 1 uses the idea of separating the boundary into an infinite number of copies, while Algorithm 2 corresponds with the second method by viewing the boundary as three similar pieces. From Figures 6 and 7, we see that Algorithm 1 is quicker to converge and aesthetically more pleasing than Algorithm 2.

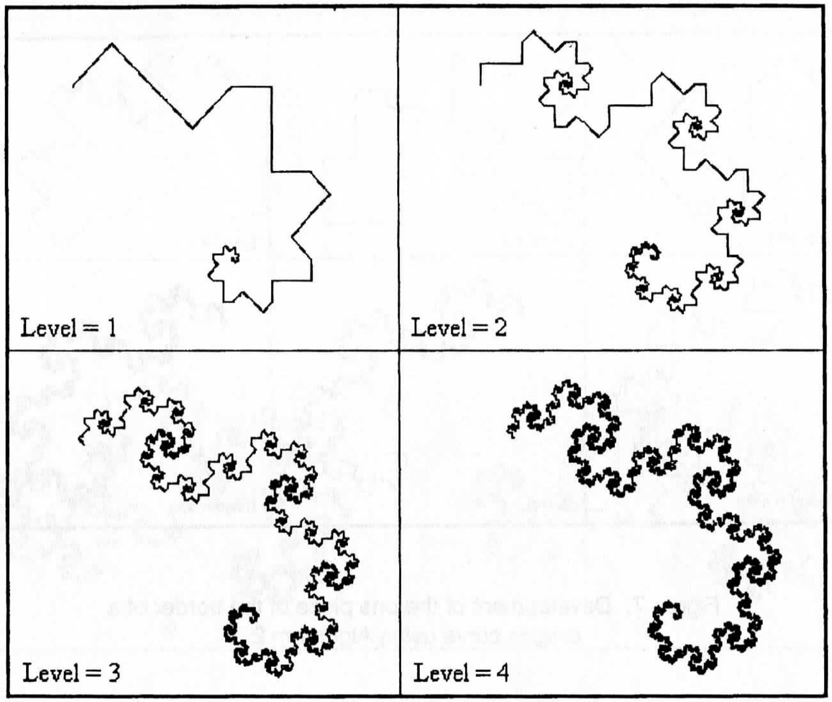

Figure 6. Development of a piece of the boundary of the dragon curve using Algorithm 1.

Algorithm 1:

```
PROCEDURE DragonBorder1(Start; n; Degree; Level);
BEGIN
   If n > 0.25 Then
   Begin
      If (level > 1) Then
         DragonBorder1(Start,n/(2*SQRT(2)),Degree+45,Level-1)
      Else DrawTriangle(Start,n,Angle);
      a.x:=Start.x+Cos(Degree)*2*n;
      a.y:=Start.y-Sin(Degree)*2*n;
      Degree:=Degree+180;
      If Level > 1 Then
         DragonBorder1(a,n/(2*SQRT(2)),Degree,Level-1)
      Else DrawTriangle(Start,n,Angle)
      Degree:=Degree+135;
      DragonBorder1(a,n/SQRT(2),Degree,Level)
   End
   Else PutPixel(Start.x,Start.y,12);
END;
```

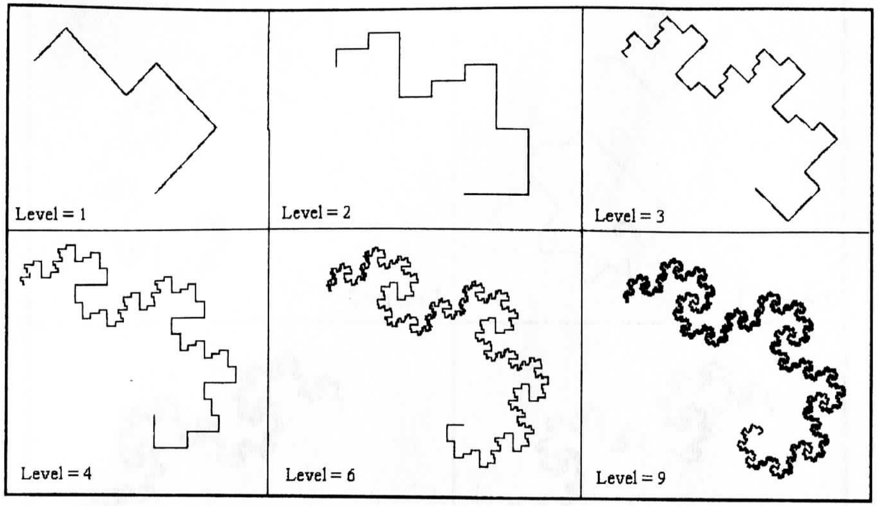

Figure 7. Development of the one piece of the border of a dragon curve using Algorithm 2.

Algorithm 2:

```
PROCEDURE DragonBorder2(Start; n; Degree; Level);
BEGIN
   If Level > 0 Then
   Begin
      Degree:=Degree+45;
      DragonBorder2(Start,n/(2*SQRT(2)),Degree,Level-1);
      NewPoint(n/SQRT(2),Degree,Start,a);
      a.x:=Start.x+Cos(Degree)*n/SQRT(2);
      a.y:=Start.y-Sin(Degree)*n/SQRT(2);
      Degree:=Degree+180;
      DragonBorder2(a,n/(2*SQRT(2)),Degree,Level-1);
      Degree:=Degree+90;
      DragonBorder2(a,n/SQRT(2),Degree,Level-1)
   End
   Else DrawTriangle(Start,n,Angle)
END;
```

To Produce the entire boundary of the dragon curve, four similar pieces of the border, generated by the above algorithms, have to be arranged as shown in Figure 8. With a suitable scaling of n and the orientation of the curve, the pieces should fit together to form the boundary dragon curve.

The boundary of the dragon curve can also be generated using the chaos game. For the generation of the boundary of the dragon curve using the chaos game, we

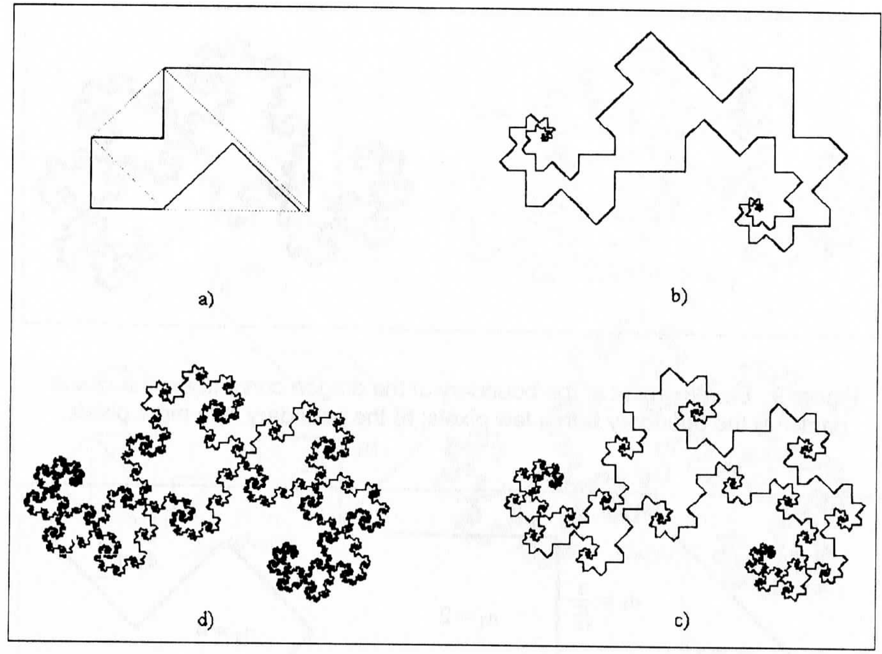

Figure 8. Computer generated boundary of the dragon curve.

utilize the IFS structure that we had analyzed earlier in Figure 5. According to the theory of the chaos game [2], the boundary as shown in Figure 5 can be generated by a random iterated system which corresponds to these three linear affine transformations:

$$T_1(x,y) = ((x-y)/4-1/2,(x+y)/4+1)$$
$$T_2(x,y) = ((y-x)/4+1/2,-(x+y)/4+1)$$
$$T_3(x,y) = ((x+y)/2+1/2,(y-x)/2)$$

Because the complete boundary is pieced together from four similar fractals, the computer is required to "play" four similar chaos games at the same time to generate the entire border. Figure 9 shows the progress of the boundary of the dragon curve as generated by the chaos game.

The Area within the Boundary

We now turn our attention to the area enclosed by the boundary of the dragon curve. To determine the area, let us consider the generation of the dragon curve by iteration as shown in Figure 10.

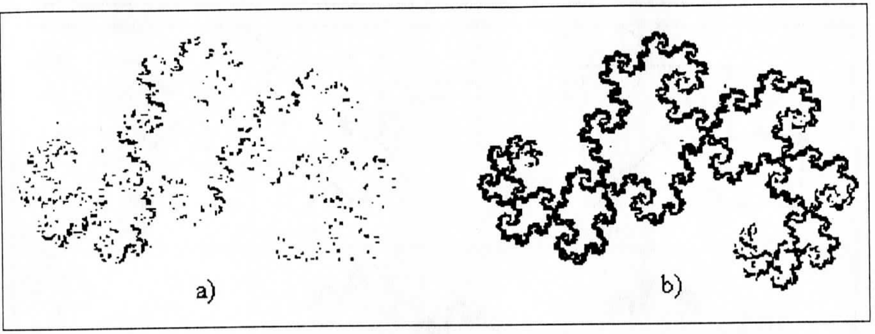

Figure 9. Development of the boundary of the dragon curve using the chaos game: a) the boundary with a few pixels; b) the boundary with more pixels.

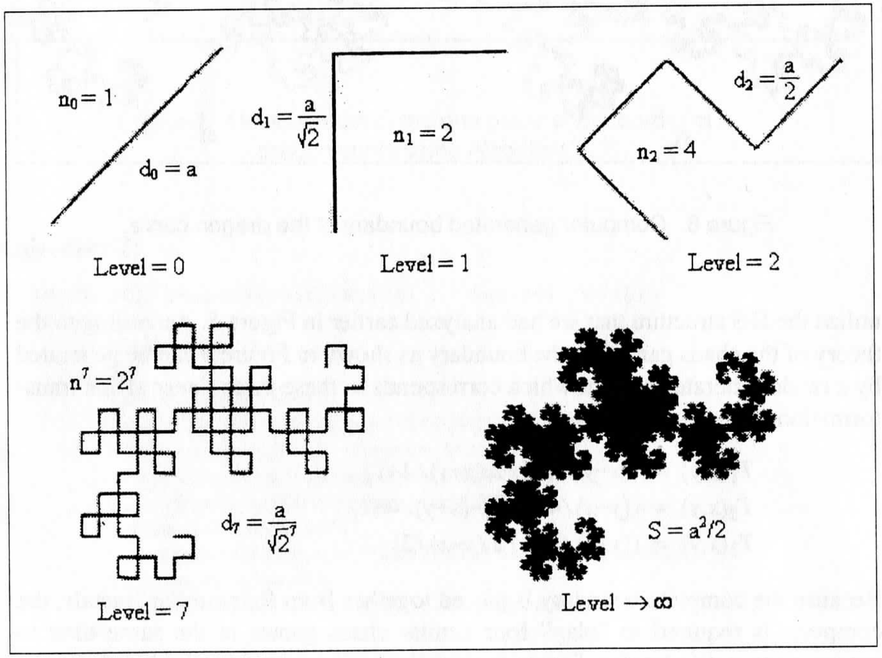

Figure 10. The generation of the dragon curve for different levels. The area at each level can be calculated by the summation of the areas of squares with a side of length $d_i = a/\sqrt{2}^i$

The area at any level i can be approximated by adding up the number of squares that are enclosed by the segments. Because each square is formed by four segments and each segment is shared by two squares, the number of squares is equal

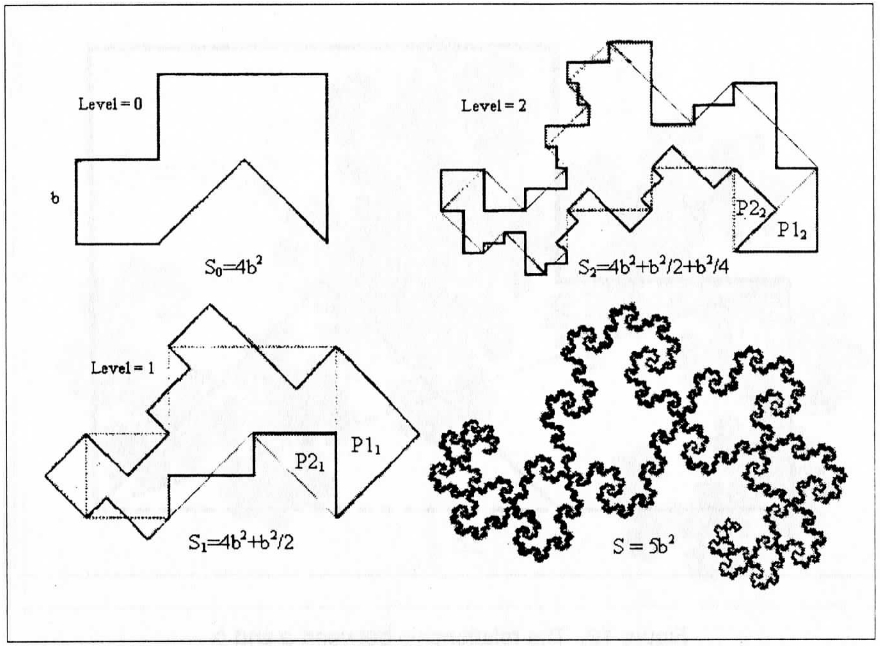

Figure 11. The area enclosed by the border of the dragon curve at different levels.

to the number of segments multiplied by 2/4. Then, at level i, the total area, S_i, can be represented by the following equation:

$$S_i = d_i^2 \times n_i/2 , \qquad (11)$$

where d_i is the length of the segment and n_i is the number of segments at level i.

If the initial segment length at level 0 is equal to a, then d_i and n_i can be expressed as follows:

$$d_i = \left(\frac{\sqrt{2}}{2}\right)^i a \quad \text{and} \quad n_i = 2^i . \qquad (12)$$

From equations (11) and (12) we can compute the area of the dragon curve, represented by S, the limit of S_i as i approaches infinity.

$$S = \lim_{i\to\infty} S_i = \lim_{i\to\infty}\left[\left(\frac{\sqrt{2}}{2}\right)^{2i} a^2 \frac{2^i}{2}\right] = \frac{a^2}{2} . \qquad (13)$$

Therefore, the area of the dragon curve is $a^2/2$.

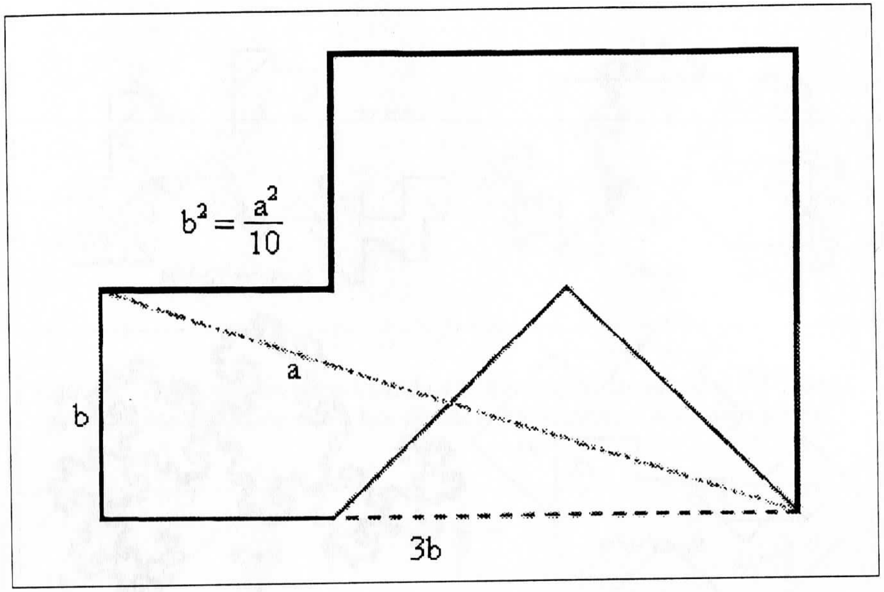

Figure 12. The relationship between *a* and *b*.

An alternate method to compute the area enclosed within the boundary of the dragon curve involves the second IFS boundary structure discussed. Letting b be the length shown in Figure 11 and S_i be the area enclosed by the border at any level i, we can calculate the area of the dragon curve by observing the changes in area from one level to the next. Notice that the starting area, S_0, is equal to $4b^2$.

From Figure 11 it can be seen that as the level of generation increases, the boundary of the dragon curve folds in and out, increasing the area at one point and decreasing at another. Upon closer observation we notice from one level to the next all these areas cancel with the exception of the areas of $P1_i$ and $P2_i$. Because $P1_i$ increases the areas of the dragon curve and $P2_i$ decreases it, the total area increases by $P1_i - P2_i$ from level i to level $i+1$. Since $P1_i = b^2/2^i$ and $P2_i = b^2/2^{i+1}$, the relation between S_{i+1} and S_i is as follows:

$$S_{i+1} = S_i + (P1_i - P2_i) = S_i + b^2/2^i - b^2/2^{i+1} = S_i + b^2/2^{i+1} . \qquad (14)$$

From this equation we realize that from level i to level $i + 1$, the area enclosed by the curve increases by $b^2/2^{i+1}$.

Using (14), we get that the area enclosed by the boundary of the dragon curve is:

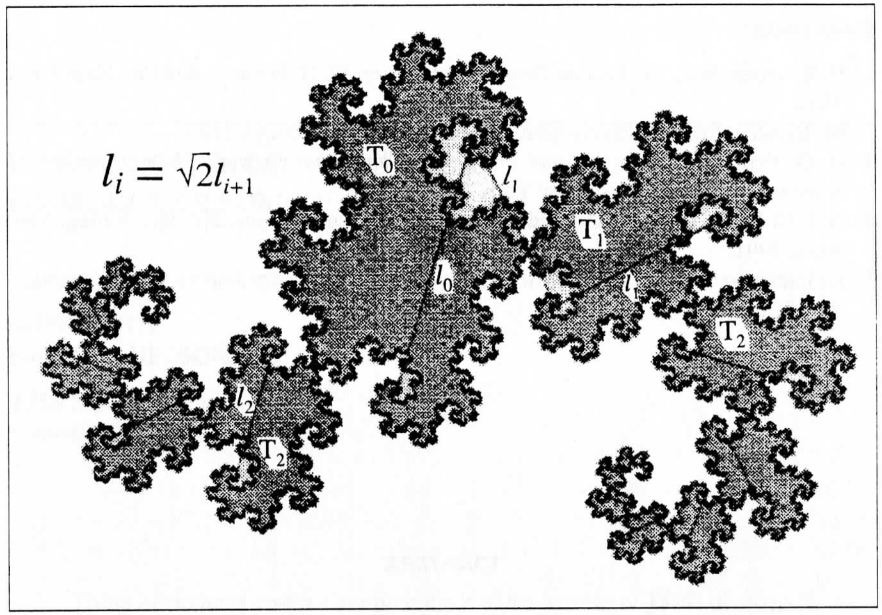

Figure 13. The relationship between the linear dimension of the shapes making up the dragon curve.

$$S = \lim_{i \to \infty} S_i = S_0 + b^2/2 + b^2/4 + b^2/8 + b^2/16 + \ldots = 5b^2 . \qquad (15)$$

To reconcile this answer for the area with our first solution in equation (13), recall that the first solution uses the length a as shown in Figure 12. From the Pythagorean Theorem we know that $a^2 = 9b^2 + b^2$ so that $b^2 = a^2/10$. From here it becomes apparent that the two solutions from equations (13) and (15) are one and the same, since $S = 5b^2 = a^2/2$.

In addition we can also calculate the area of each small shape of which the dragon curve is composed. Since it is self-evident that these shapes are similar, we can determine the relationship between their size as shown in Figure 13. Using the relationship between their linear dimension, we can determine the relationship between their areas and we find that $T_0 = 2^1 T_1 = 2^2 T_2 = 2^3 T_3 = \ldots = 2^i T_i$.

From Figure 13 it is apparent that

$$\begin{aligned} S &= T_0 + T_1 + 2(T_2 + T_3 + \ldots + T_\infty) \\ &= T_0 + \frac{T_0}{2} + 2\left(\frac{T_0}{4} + \frac{T_0}{8} + \ldots + \frac{T_0}{2^\infty}\right) = \frac{5}{2} T_0 . \end{aligned}$$

Solving for T_0, we get $T_0 = 0.4S$.

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COUNTING CERTAIN TRIANGLES OF MODULAR SUMS

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ABSTRACT

Using a computer, counts have been made of the number of TMSs of orders 2 through 15. TMSs, or triangles of modular sums, are related to TADs. TADs are triangles of absolute differences discussed by Martin Gardner in his book *Penrose Tiles to Trapdoor Ciphers.* TMS triangles contain N copies of each of the integers 0, 1, ..., $M - 1$ arranged in K rows $[M \times N = K(K+1)/2]$, where each integer above those in the bottom row is the sum (mod M) of the two integers below it. Among other items presented are the two primitive solutions for $N = 4$, $M = 30$, $K = 15$. The authors explain aspects of the counting procedure and some of the underlying mathematics.

Can the integers 1, 2, and 3 be arranged in a triangle (two on the top, one on the bottom) in such a way that the lower number is the (absolute) difference of the other two? Of course, and in only the four ways illustrated below, with some being reflections of others.

```
3   2      2   3      3   1      1   3
  1          1          2          2
```

These demonstrate a type of "pool ball" triangle or TAD (triangle of absolute differences). The order-2 (two rows) problem just considered is the beginning of an investigation suggested by Col. George Sicherman. In Martin Gardner's words [1, p. 128], one may ask "For what orders is it possible to form a TAD, where m is