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The t-tone chromatic number of classes of sparse graphs

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Abstract

For a graph G and $t, k \in \mathbb{Z}^+$ a *t-tone k-coloring* of G is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(v) \cap f(w)| < d(v, w)$ for all distinct $v, w \in V(G)$. The *t-tone chromatic number* of G, denoted $\tau_t(G)$, is the minimum k such that G is t-tone k-colorable. For small values of t, we prove sharp or nearly sharp upper bounds on the t-tone chromatic number of various classes of sparse graphs. In particular, we determine $\tau_2(G)$ exactly when $\mathrm{mad}(G) < 12/5$ and bound $\tau_2(G)$, up to a small additive constant, when G is outerplanar. We also determine $\tau_t(C_n)$ exactly when $t \in \{3, 4, 5\}$.

1 Introduction

All of our graphs are finite and simple. We write $[k]$ to denote $\{1, \ldots, k\}$ and write $\binom{[k]}{t}$ to denote the collection of all subsets of $[k]$ of size t; we refer to elements of $\binom{[k]}{t}$ as *t-sets*. For a graph G and $v, w \in V(G)$, we write $d(v, w)$ for the distance (length of the shortest path) between v and w.

In 2009, Ping Zhang led N. Fonger, J. Goss, B. Phillips, and C. Segroves [11] in developing a new generalization of proper vertex coloring. They called it t-tone coloring.

Definition 1. For a graph G and $t, k \in \mathbb{Z}^+$ a *t-tone k-coloring* of G is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(v) \cap f(w)| < d(v, w)$ for all distinct $v, w \in V(G)$.

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A graph that has a t-tone k-coloring is *t-tone k-colorable*, and the *t-tone chromatic number* of G, denoted $\tau_t(G)$, is the minimum k such that G is t-tone k-colorable. We assume throughout the paper that always $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$; otherwise, computing $\tau_t(G)$ is trivial.

The most widely studied case of t-tone coloring is the case $t = 2$. Fonger et al. [11] calculated the 2-tone chromatic number for all trees. This includes stars, which often provide a good lower bound for $\tau_2(G)$; see Proposition 2.3. Bickle and Phillips [5] determined, among other results, the 2-tone chromatic number of cycles and the general t-tone chromatic number of paths; see Proposition 2.4. This problem has been studied for various graph classes [8, 14, 17, 16, 3, 9] with several papers investigating the t-tone chromatic number of graph products [4, 13, 6] and one studying t-tone coloring of random graphs [1].

The paper is organized as follows. In Section 2 we present our definitions, and collect some lemmas (proved elsewhere) that we will use in the rest of the paper.

In Section 3, we prove a sharp bound on $\tau_2(G)$ for all graphs G with $\text{mad}(G) < 12/5$ (which includes planar graphs with girth at least 12), and a nearly sharp bound on $\tau_2(G)$ for all outerplanar graphs. For all planar graphs G, we prove a new upper bound on $\tau_2(G)$, that is sharp up to a factor of $2/\sqrt{3} \approx 1.155$. We conclude the section with some challenging conjectures. Our results in Section 3 partially answer a question of West [15] about t-tone coloring of general planar graphs.

In Section 4 we determine $\tau_t(C_n)$ exactly, for all $t \in \{3, 4, 5\}$ and all $n \geq 3$; for each t, the value is constant when n is sufficiently large. The general case relies on a powerful lemma for combining t-tone colorings of subgraphs. And the stronger lower bounds needed for some exceptional cases are proved using integer linear programs. Again, we conclude the section with a challenging conjecture.

In Section 5, we study grid graphs $P_m \square P_n$. We determine exactly τ_3 and τ_4 and bound τ_5.

2 Definitions and Useful Lemmas

Let G be a graph and fix $v \in V(G)$. We denote by $N(v)$ the *neighborhood* of v, by $N^2(v)$ the *second neighborhood* of v (the set of vertices at distance 2 from v), by $d(v)$ the *degree* of v, and by $\Delta(G)$ the *maximum degree* of G. We write $H \subseteq G$ if H is a subgraph of G. We denote by $\overline{d}(G)$ the *average degree* of G and by $\text{mad}(G)$ the *maximum average degree* of G; recall that $\text{mad}(G) := \max_{\emptyset \neq H \subseteq G} 2|E(H)|/|V(H)|$. We let P_n, C_n, and $P_m \square P_n$ denote the path on n vertices, cycle on n vertices, and the $m \times n$ vertex grid graph (where $\square$ denotes the Cartesian product).

For a graph G and $t, k \in \mathbb{Z}^+$ a *partial t-tone k-coloring* of G is a function $f : V(G) \to \binom{[k]}{t} \cup \emptyset$ such that $|f(v) \cap f(w)| < d(v, w)$ for all distinct $v, w \in V(G)$. To construct a t-tone k-coloring of a graph G, we will often create a sequence of partial t-tone k-colorings, at each step choosing a nonempty label for an additional vertex whose label was previously empty.

Below we list a number of lemmas that we will use later. We generally omit formal proofs, but often include brief proof sketches. The reader should feel free to skip ahead to Section 3 and only return to this list as needed.

Lemma 2.1. *[11, Theorem 11] If H is a subgraph of G, then every t-tone coloring of G induces a t-tone coloring of H. In particular $\tau_t(H) \le \tau_t(G)$.*

Lemma 2.2. *[16, Theorem 1] $\tau_t(C_4) = 4t - 2$.*

Proof. The labels for each pair of non-adjacent vertices share at most one color. So $\tau_t(C_4) \ge t|V(C_4)| - 2(1)$. □

Lemma 2.3. *[11, Theorem 17] Every graph G satisfies $\left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil \le \tau_2(G)$.*

Proof. The star $K_{1,\Delta(G)}$ needs k colors with $\binom{k-2}{2} \ge \Delta(G)$. □

Lemma 2.4. *[5, Proposition 5] For all $t, n \ge 1$ we have*

$$\tau_t(P_n) = \sum_{i=0}^{n-1} \max\left\{0, t - \binom{i}{2}\right\}.$$

Proof. Color the path $v_1 \cdots v_n$ in order of increasing subscript. When vertex v_i is being colored, for each $j \in [i-1]$ there are j colors used on v_{i-j-1} that are unused on vertices closer to v_i. We use these colors on v_i until either (a) v_i has t colors or (b) we run out of vertices. In the latter case, we have used $\sum_{j=0}^{i-1} j = \binom{i}{2}$ colors from previous vertices, and need $t - \binom{i}{2}$ new colors. When $\binom{i}{2} \ge t$, no more new colors are needed. □

Lemma 2.5. *[7, Theorem 2.2] Every graph G satisfies $\tau_2(G) \le \lceil (2 + \sqrt{2})\Delta(G) \rceil$.*

Proof. We color greedily avoiding at most $2\Delta(G)$ colors on neighbors and at most $\Delta(G)(\Delta(G) - 1)$ 2-sets at distance 2. □

A graph is *k-degenerate* if each of its subgraphs contains a vertex of degree at most k.

Lemma 2.6. *[7, Theorem 3.5] If G is k-degenerate, $k \ge 2$, and $\Delta(G) \le r$, then for every t we have $\tau_t(G) \le kt + kt^2\Delta(G)^{1-1/t}$.*

Proof. We color greedily with $c + kt$ colors. Neighbors forbid at most kt colors, and vertices at distance d, for each $d \in \{2, \ldots, t\}$, forbid at most $\binom{t}{d}\binom{c-d}{t-d} dk\Delta(G)(\Delta(G) - 1)^{d-2}$ sets of size t that share at least d elements. □

Lemma 2.7. *[2, Theorem 2] For every planar graph G there exists $v \in V(G)$ such that $d(v) \le 5$ and v has at most two neighbors with degree at least 11.*

Lemma 2.8. *[10, Theorem 5][12] For every outerplanar graph G there exists $xy \in E(G)$ with $d(x) = 1$, or $d(x) = 2$ and $d(y) \leq 4$.*

We conclude this section with a construction of a planar graph that improves the trivial lower bound, from Lemma 2.3, on colors needed to 2-tone color a planar graph of given maximum degree.

Lemma 2.9. *For each $t \geq 1$, we form H_t from K_3 by replacing each edge $vw \in E(K_3)$ with a copy of $K_{2,t}$, identifying the high degree vertices with v and w. For all t we have $\left\lceil \sqrt{3\Delta(H_t) + 0.25} + 0.5 \right\rceil \leq \tau_2(H_t) \leq \left\lceil \sqrt{3\Delta(H_t) + 30.25} + 0.5 \right\rceil$. (When $t \geq 27$ these two bounds differ by at most 1.)*

Proof. Fix a positive integer k to be determined later. We consider a 2-tone k-coloring of H_t. It is easy to check that $\tau_2(C_6) = 5$, so assume $t \geq 2$. Let x, y, and z denote the vertices of degree at least 4. For the lower bound, note that all $3\Delta(H_t)/2 = 3t$ vertices excluding x, y, z must get distinct 2-element subsets of $[k]$. The inequality $\binom{k}{2} \geq 3\Delta(H_t)/2$ is equivalent to the lower bound.

Now we prove the upper bound. Color x with $\{1, 2\}$; color y with $\{3, 4\}$; and color z with $\{5, 6\}$. Now we assign each remaining vertex of H_t a distinct element of $\binom{[k]}{2} \setminus \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. This requires that no vertex of degree 2 receive a label from $\binom{[6]}{2}$. Thus, we need $\binom{k}{2} - \binom{6}{2} \geq 3t$. This inequality is equivalent to the upper bound. We must also ensure that the coloring is proper, i.e., all labels including 1 or 2 (other than $\{1, 2\}$) be used on vertices non-adjacent to x, and similarly for $\{3, 4\}$ with y and for $\{5, 6\}$ with z. However, this is easy to ensure.

Finally, we show that the bounds differ by at most 1 when $t \geq 27$. For this conclusion, it suffices that $\sqrt{3\Delta(H_t) + 30.25} - \sqrt{3\Delta(H_t) + 0.25} \leq 1$. This inequality holds when $\Delta(H_t) \geq 70$, i.e., when $t \geq 35$. And it easy to check the remaining eight cases by hand. □

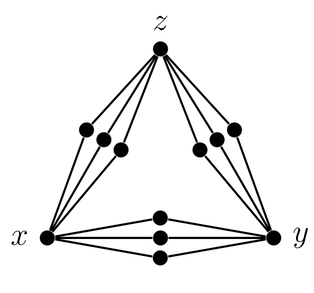

Figure 1: The graph H_3.

3 2-tone Coloring of Planar Graphs

In this section, we prove our first two main results. In Theorem 3.3 we determine $\tau_2(G)$ for all outerplanar graphs, up to a small additive constant. And in Theorem 3.5 we determine $\tau_2(G)$ for all graphs G with $\mathrm{mad}(G) < 12/5$ and $\Delta(G) \geq 11$. This includes planar graphs with girth at least 12. As a warm-up, in Theorem 3.2 we bound $\tau_2(G)$ for all planar graphs; as $\Delta(G)$ grows, our bound is sharp asymptotically up to a factor of $2/\sqrt{3} \approx 1.155$.

All our proofs in this section proceed by minimal counterexample. This approach requires extra care, since a 2-tone coloring of a subgraph H of G might fail to induce a 2-tone coloring of $G[V(H)]$. Specifically, if we delete a vertex v to form a subgraph H, we allow the possibility that neighbors of v in G will receive identical labels in H; of course, this is forbidden in a 2-tone coloring of G. To avoid this difficulty, rather than deleting vertices, we often instead contract edges, which never increases distances. However, this adds the potential issue of increasing the maximum degree. To avoid this pitfall, we typically contract an edge with one endpoint of degree at most 2. To extend a partial 2-tone coloring of a graph G, we will often use the following helpful lemma.

Lemma 3.1. *Let G be a graph and φ be a partial 2-tone k-coloring of G. For any uncolored vertex $v \in V(G)$, if $\binom{k-2|N(v)|}{2} > |N^2(v)|$, then φ can be extended to v.*

Proof. Let G, φ, and v be as in the lemma. To extend φ to v, we must avoid all colors used on $N(v)$, which forbids at most $2|N(v)|$ colors. We must also avoid all 2-sets used on $N^2(v)$, which forbids at most $|N^2(v)|$ 2-sets. Thus, it suffices to have $\binom{k-2|N(v)|}{2} > |N^2(v)|$. □

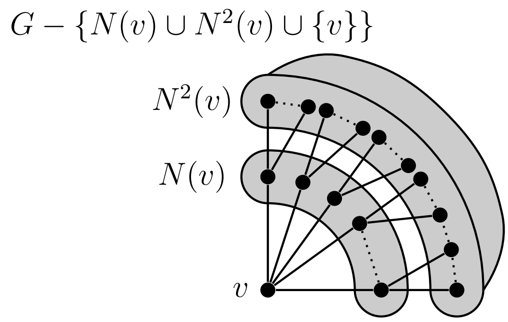

Figure 2: A vertex v with its neighbours and second neighbours.

We first prove an upper bound on $\tau_2(G)$ for every planar graph G, and then show how to strengthen it for two classes of "sparse" planar graphs. For a general planar graph G (with maximum degree $\Delta(G)$), our upper bound in the next theorem differs from the lower bound in Lemma 2.3 by a factor of approximately $\sqrt{2}$. However, for our construction H_t in Lemma 2.9 the present upper bound differs from the lower bound by only a factor of $2/\sqrt{3} \approx 1.155$.

Theorem 3.2. *If G is a planar graph, then $\tau_2(G) \le \left\lfloor \sqrt{4\Delta(G) + 50.25} + 31.1 \right\rfloor \le \left\lfloor \sqrt{4\Delta(G)} + 36.5 \right\rfloor$. Furthermore, $\tau_2(G) \le \max\left\{41, \left\lfloor \sqrt{4\Delta(G) + 50.25} + 11.5 \right\rfloor \right\}$.*

Proof. In the first statement, the second inequality is easy to verify, so we focus on the first. The second statement is clearly stronger when $\Delta(G)$ is sufficiently large, but we include the first to give a better bound when $\Delta(G)$ is small. We prove both statements simultaneously.

Suppose the theorem is false and let G be a counterexample that minimizes $|V(G)|$. If $\Delta(G) \le 12$, then Lemma 2.5 gives

$$\tau_2(G) \le \left\lceil (2+\sqrt{2})\Delta(G) \right\rceil \le \lfloor \sqrt{4\Delta(G)+50.25}+31.1 \rfloor \le 41.$$

So we assume that $\Delta(G) \ge 13$. By Lemma 2.7, there exists $v \in V(G)$ such that $d(v) \le 5$ and v has at most two neighbors of degree at least 11. If $d(v) \ge 3$, then pick $w \in N(v)$ with $d(w) \le 10$; otherwise let w be an arbitrary neighbor of v. Form H from G by contracting vw. Since $|V(H)| < |V(G)|$ and $\Delta(H) \le \max\{\Delta(G), 5+10-2\} = \Delta(G)$, by induction

$$\begin{aligned}\tau_2(H) &\le \max\left\{41, \left\lfloor \sqrt{4\Delta(H)+50.25}+11.5 \right\rfloor\right\}\\ &\le \max\left\{41, \left\lfloor \sqrt{4\Delta(G)+50.25}+11.5 \right\rfloor\right\}.\end{aligned}$$

Similarly, $\tau_2(H) \le \left\lfloor \sqrt{4\Delta(H)+50.25}+31.1 \right\rfloor \le \left\lfloor \sqrt{4\Delta(G)+50.25}+31.1 \right\rfloor$. This 2-tone coloring of H induces a partial 2-tone coloring of G, with v uncolored. Now $N_G(v)$ forbids at most $2|N_G(v)| \le 10$ colors from use on v. Further, vertices in $N_G^2(v)$ forbid at most $2(\Delta(G)-1)+3(9) = 2\Delta(G)+25$ distinct 2-sets from use on v. By Lemma 3.1 we can extend any partial 2-tone k-coloring of G (with v uncolored) to a 2-tone k-coloring of G whenever $\Delta(G) \ge 13$ and

$$\binom{k-10}{2} > 2\Delta(G)+25.$$

This inequality is easy to verify when $k = \left\lfloor \sqrt{4\Delta(G)+50.25}+11.5 \right\rfloor$, which completes the proof of both statements. □

In the next two theorems, we consider special classes of planar graphs that are in a sense "tree-like". For these graphs, we improve the leading coefficient in the bound of Theorem 3.2 by a factor of approximately $\sqrt{2}$, so that it matches that in the lower bound given by Lemma 2.3.

Theorem 3.3. *If G is outerplanar, then*

$$\tau_2(G) \le \left\lfloor \sqrt{2\Delta(G)+4.25}+5.5 \right\rfloor \le \left\lfloor \sqrt{2\Delta(G)}+6.6 \right\rfloor.$$

Proof. The second inequality is easily verified by algebra, so we focus on the first. Suppose the theorem is false and let G be a counterexample minimizing $|V(G)|$. Note that the class of outerplanar graphs is closed under edge contraction.

By Lemma 2.8 there exists $vw \in E(G)$ such that $d(v) = 1$, or $d(v) = 2$ and $d(w) \le 4$. In either case, form H by contracting vw (restricting to the underlying simple graph if we create a pair of parallel edges). Note that $|H| < |G|$ and $\Delta(H) \leq \Delta(G)$. By the minimality of G,

$$\tau_2(H) \leq \left\lfloor \sqrt{2\Delta(H) + 4.25} + 5.5 \right\rfloor \leq \left\lfloor \sqrt{2\Delta(G) + 4.25} + 5.5 \right\rfloor.$$

The vertices in $N_G(v)$ forbid at most $2|N(v)| \leq 4$ colors from use on v. Further, the vertices in $N_G^2(v)$ forbid at most $\Delta(G) - 1 + (4 - 1) = \Delta(G) + 2$ distinct 2-sets from use on v. By Lemma 3.1 we can extend any 2-tone k-coloring of H to G when

$$\binom{k-4}{2} > \Delta(G) + 2.$$

This inequality is easy to verify when $k = \left\lfloor \sqrt{2\Delta(G) + 4.25} + 5.5 \right\rfloor$. □

Lemma 3.4 is a structural result that we will use to prove Theorem 3.5. As a special case, that theorem will exactly determine τ_2 for planar graphs with sufficiently large girth and max degree.

We will also need some new definitions. A d^+-vertex, d^--vertex, or d-vertex is, respectively, a vertex of degree at least d, at most d, and exactly d. An *ℓ-thread* in a graph G is a trail of length $\ell + 1$ in G whose ℓ internal vertices have degree 2 in G. We refer to the non-internal vertices of an ℓ-thread as *endpoints*. So an ℓ-thread has two endpoints, not necessarily distinct. For Lemma 3.4 and Theorem 3.5 we present the proofs as if each ℓ-thread has two distinct endpoints, but all arguments remain valid if the endpoints are not distinct.

Lemma 3.4. *Let G be a graph with $\delta(G) \geq 2$. If mad$(G) < 12/5$, then G contains at least one of the following:*

(a) a 4-thread,

(b) a 3-thread with a 5^--vertex as an endpoint, or

(c) a 2-thread with a 3^--vertex and a 5^--vertex as endpoints.

Proof. Let G be a graph with $\delta(G) \geq 2$ and $\text{mad}(G) < 12/5$. Assume for contradiction that G has no threads of type (a), (b), and (c). If G contains a 2-regular component, then it contains an instance of (c); so assume no component of G is 2-regular. Thus, every 2-vertex appears in a unique maximal thread, and the endpoints of that thread are 3^+-vertices. We give each vertex v initial charge $d(v)$. To redistribute charge, each maximal thread takes charge $1 - 12/(5d(v))$ from each of its endpoints. Each thread redistributes its charge equally to its internal vertices. Below we show that each vertex ends with charge at least $12/5$, contradicting that $\text{mad}(G) < 12/5$.

Since G has no 4-thread, each maximal thread has at most 3 internal vertices. If a thread t has a vertex v as an endpoint, then the charge that t receives from v is:

$1 - 12/(3(5)) = 1/5$ if $d(v) = 3$; and at least $1 - 12/(4(5)) = 2/5$ if $d(v) \geq 4$; and at least $1 - 12/(6(5)) = 3/5$ if $d(v) \geq 6$.

Each 1-thread gains at least $1/5$ from each endpoint, so finishes with at least $12/5$.

Each 2-thread cannot be an instance of (c), so either (i) both of its endpoints are 4^+-vertices or (ii) it has a 6^+-vertex as an endpoint. So a 2-thread gains either (i) at least $2/5$ from each endpoint or (ii) at least $3/5$ from the endpoint that is a 6^+-vertex and at least $1/5$ from the other endpoint. Thus, each 2-thread finishes with at least $2(2) + 4/5 = 2(12/5)$.

Each 3-thread has a 6^+-vertex for each endpoint, otherwise G contains (b). So a 3-thread gains at least $3/5$ from each endpoint. Thus, each 3-thread finishes with at least $3(2) + 6/5 = 3(12/5)$. If v is an endpoint of a thread, then v sees at most $d(v)$ threads. Thus, v has final charge $d(v) - d(v)(1 - 12/(5d(v)) = 12/5$. This implies that $\overline{d}(G) \geq 12/5$; which contradicts the hypothesis $\text{mad}(G) < 12/5$. $\square$

Theorem 3.5. *If G is a graph with mad$(G) < 12/5$, then*

$$\tau_2(G) \leq \max\left\{7, \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil\right\}.$$

Further, if G is planar with girth at least 12 and $\Delta(G) \geq 7$, then

$$\tau_2(G) = \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil.$$

Proof. The second statement follows from the first since a planar graph G with girth at least 12 has $\text{mad}(G) < 2(12)/(12 - 2) = 12/5$ and Lemma 2.3 implies that if $\Delta(G) \geq 7$, then $\tau_2(G) \geq 7$. We now prove the first statement.

Suppose the theorem is false and let G be a counterexample minimizing $|V(G)|$. If there exists v with $d(v) \leq 1$, then by minimality

$$\tau_2(G - v) \leq \max\left\{7, \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil\right\}.$$

And by Lemma 3.1 we get $\tau_2(G) \leq \max\left\{7, \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil\right\}$. Thus, we assume $\delta(G) \geq 2$.

By Lemma 3.4 we know G contains configuration (a), (b), or (c) in that lemma. We will show that none of these configurations can appear in our minimal counterexample G. To do so, we form a subgraph H by deleting some vertices of G, color H by minimality, and extend our coloring of H to the deleted vertices of G, to contradict that G was a counterexample. Let $k_G = \max\left\{7, \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil\right\}$. For an arbitrary subgraph H of G (which will be clear from context), let $k_H = \max\left\{7, \left\lceil \sqrt{2\Delta(H) + 0.25} + 2.5 \right\rceil\right\}$.

Case 1: G contains a 4-thread, as shown in Figure 3. Form H from G by deleting v_2 and v_3. Note that $|H| < |G|$ and $\Delta(H) \leq \Delta(G)$. By the minimality of

G, we have $\tau_2(H) \leq k_H \leq k_G$. Let φ be a 2-tone k_G-coloring of H. By Lemma 3.1, since $k_G \geq 7$ we can extend φ to v_2 followed by v_3, a contradiction.

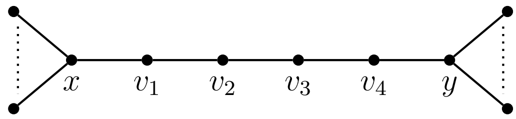

Figure 3: A 4-thread with endpoints x and y.

Case 2: G contains a 3-thread, as shown in Figure 4. Form H by deleting v_2 and v_3. Note that $|H| < |G|$ and $\Delta(H) \leq \Delta(G)$. By the minimality of G, we have $\tau_2(H) \leq k_H \leq k_G$. Let φ be a 2-tone k_G-coloring of H. By Lemma 3.1, since $k_G \geq 7$ we can extend φ to v_3 followed by v_2, a contradiction. In particular, since y forbids 2 colors from use on v_3 and the vertices at distance 2 (in G) from v_3 forbid at most 5 distinct 2-sets from use on v_3, since $k_G \geq 7$ we have at least $\binom{5}{2} - 5 = 5$ remaining 2-sets available for v_3. Afterwards, it is easy to color v_2. This finishes the extension of φ to a 2-tone k_G-coloring of G, which is a contradiction.

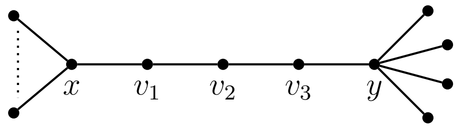

Figure 4: A 3-thread with endpoints x and y, where $d(y) \leq 5$.

Case 3: G contains a 2-thread, as shown in Figure 5. Form H from G by deleting v_1. Note that $|H| < |G|$ and $\Delta(H) = \Delta(G)$. By the minimality of G, we have $\tau_2(H) \leq k_H \leq k_G$. Let φ be a 2-tone k_G-coloring of H. Note that φ might fail to induce a partial 2-tone coloring of G since it is possible that $\varphi(v_2) = \varphi(x)$, which creates a problem since $d(v_2, x) = 2$. To avoid this issue we can simply recolor v_2, since $d(v_2) = 2$. In this case, v_2 is a leaf of H, so its neighbor forbids 2 colors from use on v_2; furthermore, the vertices at distance 2 from v_2 (in G) forbid at most 5 distinct 2-sets from use on v_2. So we can recolor v_2 with another 2-set, since $\binom{7-2}{2} > 4 + 1$; in fact, we have at least 5 choices of label for v_2. Thus, we assume that φ induces a proper 2-tone coloring of G. Finally, we consider coloring v_1. Its two neighbors forbid at most $2(2) = 4$ colors. And the three vertices at distance two forbid at most an additional three 2-sets. If $k_G \geq 8$, then we have a 2-set available to use on v_1. So assume instead that $k_G = 7$. If no 2-sets are available to use on v_1, then the two 2-sets used on its neighbors are disjoint. Further, the three 2-sets used on vertices at distance two are distinct, and they are all disjoint from the set of colors used on its neighbors. But now to escape this situation we can recolor v_2 with one of the other 4 possible 2-sets we had to choose from. Afterward, we can extend the 2-tone 7-coloring to G, a contradiction. □

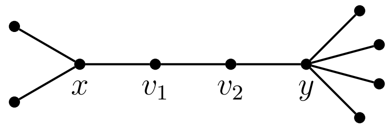

Figure 5: A 2-thread with endpoints x and y, where $d(x) = 3$ and $d(y) \le 5$.

We conclude this section with a few conjectures.

Conjecture 3.6. *There exists a constant C such that all planar G satisfy $\tau_2(G) \le \sqrt{3\Delta(G)} + C$.*

Perhaps the following stronger statement holds. It is essentially best possible, due to Lemma 2.9.

Conjecture 3.7. *If G is planar with $\Delta(G)$ sufficiently large, then*

$$\tau_2(G) \le \left\lceil \sqrt{3\Delta(G) + 30.25} + 0.25 \right\rceil.$$

We also believe that for planar graphs the girth requirement in Theorem 3.5 can be significantly weakened.

Conjecture 3.8. *There exists a constant C such that every planar graph G with girth at least 5 satisfies $\tau_2(G) \le \sqrt{2\Delta(G)} + C$.*

It is interesting to note the following. For every integer $t \ge 2$ there exists a girth g_t and a maximum degree Δ_t such the maximum value of $\tau_t(G)$, taken over all planar graphs G with girth at least g_t and $\Delta(G) \ge \Delta_t$, is achieved by a tree. Cranston, Kim, and Kinnersley [7, Theorem 2] showed that this maximum (for trees) is bounded by $c_t\sqrt{\Delta(G)}$ for some constant c_t; and this is asymptotically sharp. We briefly sketch the extension to planar graphs with sufficiently large girth and maximum degree. Following an approach similar to (but simpler than) the proof of Lemma 3.4, we can prove that if G has sufficiently low maximum average degree, then it contains either a 1^--vertex or a $3t$-thread. Every 1^--vertex can be handled inductively (by coloring greedily). For a $3t$-thread, we delete the middle t vertices and color the smaller graph by induction. We choose Δ_t large enough that $\tau_t(K_{1,\Delta_t}) \ge 3\tau_t(P_t)$. (Recall that $\tau_t(K_{1,\Delta_t}) \ge \tau_2(K_{1,\Delta_t}) \ge \sqrt{2\Delta_t}$, by Lemma 2.3.) Now the number of colors forbidden on all of the uncolored vertices (taken together) is at most $2\tau_t(P_t)$. Thus, we have at least $\tau_t(P_t)$ colors that are available for use on all of the uncolored vertices. So we can extend the coloring.

4 3-Tone, 4-Tone, and 5-Tone Coloring of Cycles

We can easily prove that $\tau_t(C_n) = O(t^{3/2})$, as follows. Let $f(t) := \tau_t(P_t)$. By Lemma 2.4, there exists a constant c such that $\tau_t(P_t) \le ct^{3/2}$ for all t. Further, $\tau_t(P_n) = \tau_t(P_t)$ for all $n \ge t$. Whenever $n \ge 2t + 2$, to prove $\tau_t(C_n) \le 2f(t)$ we simply color the first $t+1$ vertices with one set of $f(t)$ colors and the remaining vertices with a disjoint set of $f(t)$ colors. But is it true that $\tau_t(C_n) = \tau_t(P_n)$ for all n sufficiently large (as a function of t)? Bickle and Phillips [5, Theorem 18] showed that $\tau_2(C_n) = 6$ when $n \in \{3, 4, 7\}$ and otherwise $\tau_2(C_n) = \tau_2(P_n) = 5$. We generalize their approach to prove analogous results for τ_3, τ_4, and τ_5. Our next lemma plays a key role in these proofs.

Lemma 4.1. *Fix $t, k, n \in \mathbb{Z}^+$. Let $\mathcal{C}$ be a set of positive integers, each at least t. If n can be written as an integer linear combination of elements in $\mathcal{C}$ (with nonnegative coefficients), then $\tau_t(C_n) \le k$ provided that the following two properties hold:*

(1) For each $\ell \in \mathcal{C}$, there exist a t-tone k-coloring φ_ℓ of C_ℓ; and

(2) For each ordered pair $(\ell_1, \ell_2) \in \mathcal{C} \times \mathcal{C}$ (allowing $\ell_1 = \ell_2$), we get a t-tone k-coloring of C_{2t} if we color its first t vertices as vertices $\ell_1 - t + 1, \ldots, \ell_1$ of C_{ℓ_1} under φ_{ℓ_1} and we color its last t vertices as vertices $1, \ldots, t$ of C_{ℓ_2} under φ_{ℓ_2}.

Proof. Fix t, k, and $\mathcal{C}$ satisfying the hypotheses. We prove the stronger statement that if n satisfies the hypotheses, then C_n has a t-tone k-coloring in which its vertices are partitioned into copies of P_{ℓ_i}, with each $\ell_i \in \mathcal{C}$, and each copy of P_{ℓ_i} colored by φ_{ℓ_i}. Our proof is by induction on the sum of the coefficients in the integer linear combination representation of n.

Assume, by symmetry, that ℓ_1 has a positive coefficient, and let $n' := n - \ell_1$. By hypothesis, we have the desired t-tone k-coloring $\varphi_{n'}$ of $C_{n'}$. We insert a path on ℓ_1 vertices between the "first" and "last" vertex of the cycle $C_{n'}$ to get C_n. Note that $\varphi_{n'}$ induces a partial t-tone k-coloring of C_n, with these ℓ_1 successive vertices uncolored. To extend this partial coloring, we color the uncolored vertices using φ_{ℓ_1}. By properties (1) and (2), this yields a t-tone k-coloring of C_n, as desired. □

Note that Property (2) holds trivially if each t-tone coloring φ_{ℓ_i} agrees on (is identical on) its first t vertices. For example, as illustrated in Figure 6, we can use Lemma 4.1 with $\mathcal{C} = \{4, 5\}$ to show $\tau_3(C_{13}) \le 10$ since $13 = 2(4) + 1(5)$ and the 3-tone 10-colorings of C_4 and C_5 agree in the first 3 vertices. We use Lemma 4.1 to prove our next three theorems, which show that $\tau_t(C_n) = \tau_t(P_n)$ for all $t \in \{3, 4, 5\}$, for all but a small (finite) number of values of n.

Theorem 4.2.

$$\tau_3(C_n) = \begin{cases} 10 & \text{if } n \in \{4, 5\} \\ 9 & \text{if } n \in \{3, 7, 10, 13\} \\ 8 & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that $\tau_3(P_3) = 8$. So $\tau_3(C_n) \ge \tau_3(P_3) = 8$ for all $n \ge 3$. Pan and Wu [16] showed that $\tau_3(C_n) = 9$ when $n \in \{3, 7\}$ and that $\tau_3(C_n) = 10$ when $n \in \{4, 5\}$. So we assume below that $n = 6$ or $n \ge 8$. The case $n \in \{10, 13\}$ is

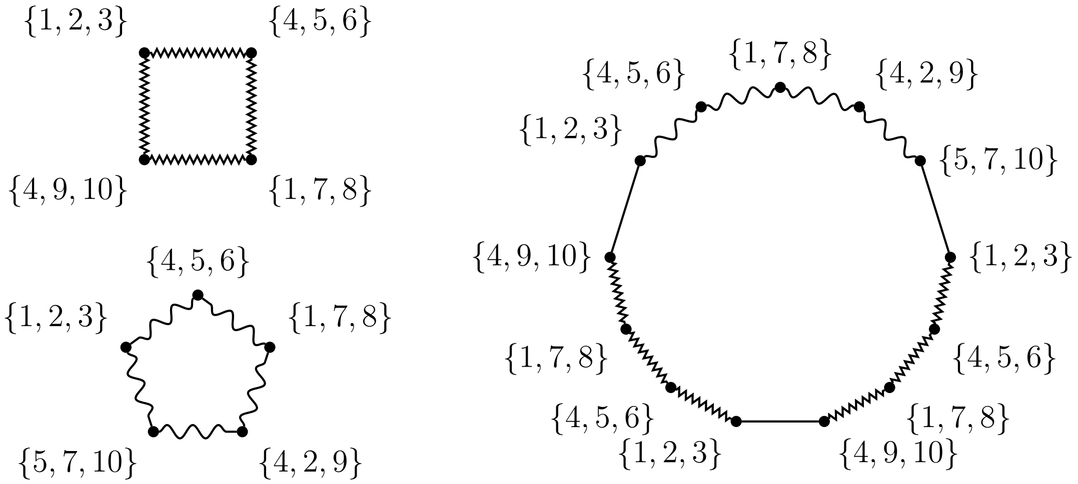

Figure 6: Using 3-tone 10-colorings of C_4 and C_5 to show $\tau_3(C_{13}) \leq 10$.

exceptional, so we defer it briefly to handle the general case. In Lemma 4.1, we let $\mathcal{C} = \{6, 8, 9, 11\}$ and take φ_k as described below.

φ_6: $-123 - 456 - 178 - 234 - 156 - 478-$
φ_8: $-123 - 456 - 178 - 234 - 568 - 127 - 345 - 678-$
φ_9: $-123 - 456 - 178 - 234 - 568 - 174 - 238 - 156 - 478-$
φ_{11}: $-123 - 456 - 178 - 234 - 568 - 127 - 634 - 578 - 126 - 345 - 678-$

So it remains to show that n can be written as an integer linear combination of elements of $\mathcal{C}$ whenever $n \geq 3$ and $n \notin \{3, 4, 5, 7, 10, 13\}$. To see this, we consider the integer linear combinations, $6, 8, 9, 11, 6+6, 6+8, 6+9, 8+8, 8+9, 9+9, 8+11$ and note that every larger integer can be written as one of the final 6, plus some multiple of 6.

Now assume $n \in \{10, 13\}$. To see that $\tau_3(C_n) \leq 9$, consider the two following 3-tone 9-colorings.

$$\begin{aligned} \text{3-tone 9-coloring of } C_{10} :& -123 - 456 - 178 - 369 - 458 - 279 - 368 - 245- \\ & -169 - 578- \\ \text{3-tone 9-coloring of } C_{13} :& -123 - 456 - 178 - 369 - 458 - 279 - 368 - 459- \\ & -278 - 369 - 245 - 168 - 579- \end{aligned}$$

Finally, we show for each $n \in \{10, 13\}$ that $\tau_3(C_n) > 8$. Assume the contrary, let φ be a 3-tone 8-coloring of C_n, and let c_i denote the number of vertices receiving color i under φ for each $i \in [8]$. Let $s := (n-1)/3$. It is straightforward to check that, for at least $(c_i - s)2$ pairs of vertices at distance 2, both vertices receive color i. Note that $\sum_{i=1}^{8} c_i = 3n = 9s+3$. Further, $\sum_{i=1}^{8}(c_i - s)2 = 18s+6-16s = 2s+6$. Observe that C_n has precisely $n = 3s+1$ pairs of vertices at distance 2. Since $n \in \{10, 13\}$, we have $s \in \{3, 4\}$, so $2s+6 > 3s+1$. Thus, by Pigeonhole some pair of vertices at distance 2 receive two common colors under φ, a contradiction. $\square$

Theorem 4.3.

$$\tau_4(C_n) = \begin{cases} 15 & \textit{if } n = 5 \\ 14 & \textit{if } n = 4 \\ 13 & \textit{if } n = 7 \\ 12 & \textit{otherwise.} \end{cases}$$

Proof. We have $\tau_4(C_n) \geq \tau_4(P_n) = 12$. Using results from [16], we have $\tau_4(C_3) = 12$, $\tau_4(C_4) = 14$, $\tau_4(C_5) = 15$, and $\tau_4(C_7) = 13$. We let $\mathcal{C} = \{6, 8, 9, 10, 11, 12, 13\}$ and take φ_k as described below.

φ_6: $-1,2,3,4-5,6,7,8-1,9,10,11-2,3,5,12-4,6,7,9-8,10,11,12-$
φ_8: $-1,2,3,4-5,6,7,8-1,9,10,11-2,3,5,12-4,7,8,11-1,3,6,10-$
$-2,5,8,9-7,10,11,12-$
φ_9: $-1,2,3,4-5,6,7,8-1,9,10,11-2,3,5,12-4,7,8,11-3,6,9,10-$
$-1,4,5,12-2,7,8,10-6,9,11,12-$
φ_{10}: $-1,2,3,4-5,6,7,8-1,9,10,11-2,3,5,12-4,7,8,11-6,9,10,12-$
$-1,3,5,11-2,4,8,12-3,6,7,10-5,9,11,12$
φ_{11}: $-1,2,3,4-5,6,7,8-1,9,10,11-2,3,5,12-1,4,6,7-5,8,9,10-$
$-2,3,7,11-4,6,8,12-1,3,5,10-2,6,7,9-8,10,11,12-$
φ_{13}: $-1,2,3,4-5,6,7,8-1,9,10,11-2,3,5,12-4,7,8,11-6,9,10,12-$
$-1,3,5,11-2,7,8,12-4,9,10,11-3,5,6,12-1,2,8,11-4,6,7,10-$
$-5,9,11,12-$

So it remains to show that n can be written as an integer linear combination of elements of $\mathcal{C}$ whenever $n \geq 3$ and $n \notin \{3, 4, 5, 7\}$. To see this, we consider the integer linear combinations, $6, 8, 9, 10, 11, 6+6, 13$ and note that every larger integer can be written as one of the final 6, plus some multiple of 6. $\square$

Theorem 4.4.

$$\tau_5(C_n) = \begin{cases} 20 & \textit{if } n = 5 \\ 18 & \textit{if } n \in \{4, 6\} \\ 17 & \textit{if } n \in \{7, 9\} \\ 15 & \textit{if } n = 3 \\ 16 & \textit{otherwise.} \end{cases}$$

Proof. We have $\tau_5(C_n) \geq \tau_5(P_n) = 16$ when $n \geq 4$. Using results from [16], we have $\tau_5(C_3) = 15$, $\tau_5(C_4) = 18$, $\tau_5(C_5) = 20$, $\tau_5(C_6) = 18$, and $\tau_5(C_7) = 17$. We let

$\mathcal{C} = \{8, 10, 11, 12, 13, 14, 15, 17\}$ and take φ_k as described below.

φ_8: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-4,5,9,10,14-$
$-1,3,7,8,13-2,6,10,11,12-9,13,14,15,16-$

φ_{10}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-4,5,9,10,14-$
$-7,8,12,13,16-1,5,6,11,15-2,3,9,10,16-4,7,8,11,14-6,12,13,15,16-$

φ_{11}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-7,8,4,5,11-$
$-1,6,9,10,14-7,12,13,15,16-2,3,5,8,14-1,4,7,10,11-6,2,9,12,13-$
$-8,11,14,15,16-$

φ_{12}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-1,7,8,4,5-$
$-6,11,12,9,10-1,2,3,13,14-6,7,8,15,16-1,11,12,4,5-6,2,3,9,10-$
$-1,7,8,13,14-6,11,12,15,16-$

φ_{13}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-4,5,9,10,13-$
$-1,7,8,11,15-2,6,10,12,14-3,4,7,13,16-5,9,10,11,15-1,2,8,12,16-$
$-4,5,6,7,14-3,8,10,11,13-9,12,14,15,16-$

φ_{14}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-4,5,9,10,13-$
$-1,7,8,11,15-2,6,10,12,14-3,4,7,13,16-5,9,10,11,15-1,2,8,12,16-$
$-3,5,6,13,14-1,4,7,10,15-2,8,9,11,14-6,12,13,15,16$

φ_{15}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-4,5,9,10,14-$
$-7,8,12,13,16-1,6,11,14,15-2,3,9,10,16-4,5,12,13,15-7,8,11,14,16-$
$-1,6,9,10,15-2,3,12,13,16-4,5,8,10,14-1,7,9,11,13-6,12,14,15,16-$

φ_{17}: $-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-6,2,3,15,16-4,5,9,10,13-$
$-1,7,8,11,15-2,6,10,12,14-3,4,7,13,16-5,9,10,11,15-1,2,8,12,16-$
$-3,5,6,13,14-1,4,7,10,15-3,8,9,11,16-2,5,12,14,15-1,3,6,10,13-$
$-4,7,9,11,14-8,12,13,15,16-$

So it remains to show that n can be written as an integer linear combination of elements of $\mathcal{C}$ whenever $n \geq 3$ and $n \neq 9$. To see this, we consider the integer linear combinations, $8, 10, 11, 12, 13, 14, 15, 8+8, 17, 8+10, 8+11, 10+10, 10+11, 11+11, 8+15, 8+8+8, 10+15$ and note that every larger integer can be written as one of the final 8, plus some multiple of 8.

Now assume that $n = 9$. To see that $\tau_5(C_9) \leq 17$, consider the following 5-tone 17-coloring.

$$\begin{aligned}\text{5-tone 17-coloring of } C_9 : &-1,2,3,4,5-6,7,8,9,10-1,11,12,13,14-\\ &-6,2,3,15,16-4,5,7,9,12-1,8,10,11,15-\\ &-2,4,6,13,14-3,7,8,12,16-9,11,13,15,17-\end{aligned}$$

Finally, we will prove that $\tau_5(C_9) \geq 17$. Assume, to the contrary, that C_9 has a 5-tone 16-coloring. Note that each color appears on at most 4 vertices. Each color must appear on at least one vertex, since $\tau_5(C_9) \geq \tau_5(P_4) = 16$. For each $i \in [4]$, let s_i denote the number of colors used on exactly i vertices. So we have $\sum_{i=1}^{4} s_i = 16$ and $\sum_{i=1}^{4} i s_i = 9(5) = 45$. Further, let s_3' denote the number of colors used on exactly 3 vertices, where some pair is at distance 2, and let s_3'' denote the number of colors used on exactly 3 vertices, where each pair is distance 3. Note that each color used on 4 vertices is used on 3 pairs of vertices at distance 2. Since C_9 has 9 pairs of vertices at distance 2, and each pair can share at most 1 common color, we get $3s_4 + s_3' \leq 9$. Similarly, by considering vertex pairs with a common color that are at distance 3, we get $s_3' + 3s_3'' \leq 18$. Multiplying the first inequality by 2, adding it to the second inequality, and dividing by 3 (recalling $s_3' + s_3'' = s_3$) gives

$$2s_4 + s_3 \leq 12. \tag{$*$}$$

Recall that $\sum_{i=1}^{4} s_i = 16$ and $\sum_{i=1}^{4} is_i = 9(5) = 45$. Multiplying the first equation by 3 and subtracting the second gives $2s_1 + s_2 - s_4 = 3$. Adding this to $(*)$ gives $2s_1 + s_2 + s_3 + s_4 \leq 12 + 3 = 15$. Since $s_1 \geq 0$, this contradicts the first equation, and this contradiction finishes the proof. □

We conclude this section with a bold conjecture.

Conjecture 4.5. *For each $t \geq 2$ there exists $N \in \mathbb{N}$ such that $\tau_t(C_n) = \tau_t(P_n)$ for all $n \geq N$.*

5 3-Tone, 4-Tone, and 5-Tone Coloring of Grid Graphs

In this section we will consider the t-tone chromatic number of grid graphs for each $t \in \{3, 4, 5\}$.

Bickle [4, Proposition 32] (also Cooper and Wash [6, Theorem 5]) showed that $\tau_2(P_n \square P_m) = 6$ for all $n, m \geq 2$. It is useful in their proof, and in the following three theorems, to imagine the grid graph as being drawn in the first quadrant of the xy-plane with vertices as integer points. Now their proof can be viewed as coloring lines of slope 1 by cycling through the colors 1, 2, 3 and coloring lines of slope -1 by cycling through the colors 4, 5, 6. Each vertex v needs two colors; it takes one color from the line through it of slope 1 and takes the other color from the line through it of slope -1.

For Theorem 5.1, the proof can be viewed as coloring the lines of slope 1 and slope -1 as above, but also coloring lines of slope 2. This theorem improves a result in [6, Theorem 8]. For Theorem 5.2, the proof can be viewed as coloring the lines of slope 1, slope -1, and slope 2 as in Theorem 5.1, but further coloring lines of slope $-\frac{1}{2}$. Finally, for Theorem 5.3, the proof can also be viewed as coloring the lines of slope 1, slope -1, slope 2, and slope $-\frac{1}{2}$ as in Theorem 5.2, but adding colors to lines of slope 1.

For the following three theorems we consider the vertices of $P_m \square P_n$ as integer points on the xy-plane where a vertex (x_i, y_j) is denoted by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. For all vertices (i_1, j_1) and (i_2, j_2) in $V(P_m \square P_n)$, note that the distance between them is exactly $|i_1 - i_2| + |j_1 - j_2|$.

Theorem 5.1. *$\tau_3(P_m \square P_n) = 10$ for all integers m and n with $2 \leq m \leq n$.*

Proof. Lemmas 2.1 and 2.2 imply that $10 = \tau_3(C_4) \leq \tau_3(P_m \square P_n)$. So it suffices to construct a 3-tone 10-coloring of $P_m \square P_n$. Let $f : V(P_m \square P_n) \to \binom{[10]}{3}$ where we write

$f((i,j))$ as $f(i,j)$ and we let $f(i,j) := \{f_1(i,j), f_2(i,j), f_3(i,j)\}$, where

$$\begin{aligned} f_1(i,j) &:= (i-j) \bmod 3 \\ f_2(i,j) &:= ((i+j) \bmod 3) + 3 \\ f_3(i,j) &:= ((2i+j) \bmod 4) + 6. \end{aligned}$$

Denote v by (i_1, j_1) and w by (i_2, j_2). It suffices to prove the following three claims.

Claim 1: If $|f(v) \cap f(w)| = 3$, then $d(v,w) \geq 4$.

If $|f(v) \cap f(w)| = 3$, then $f_i(v) = f_i(w)$ for all $i \in [3]$. So $(i_1 - j_1) \equiv (i_2 - j_2) \bmod 3$ and $(i_1 + j_1) \equiv (i_2 + j_2) \mod 3$. Thus $i_1 \equiv i_2 \mod 3$ and $j_1 \equiv j_2 \mod 3$. If $d(v,w) \leq 3$ and $v \neq w$, then $i_1 \equiv i_2 \pm 3$ and $j_1 = j_2$ or else $i_1 = i_2$ and $j_1 = j_2 \pm 3$. But now $(2i_1 + j_1) \not\equiv (2i_2 + j_2) \mod 4$.

Claim 2: If $|f(v) \cap f(w)| = 2$, then $d(v,w) \geq 3$.

Assume $|f(v) \cap f(w)| = 2$. If $\{f_1(v), f_2(v)\} = \{f_1(w), f_2(w)\}$, then the argument in Claim 1 still holds. Instead we assume $f_3(v) = f_3(w)$ and $d(v,w) \leq 2$. Thus $i_1 = i_2 \pm 2$ and $j_1 = j_2$, but now $f_1(v) \neq f_2(v)$ and $f_2(v) \neq f_2(w)$, a contradiction.

Claim 3: If $|f(v) \cap f(w)| = 1$, then $d(v,w) \geq 2$.

Assume that $d(v,w) = 1$. So either $i_1 = i_2$ and $j_1 - j_2 = \pm 1$ or else $j_1 = j_2$ and $i_1 - i_2 = \pm 1$. Now clearly $f_i(v) \neq f_i(w)$ for all $i \in [3]$, a contradiction. □

Theorem 5.2. *$\tau_4(P_m \square P_n) = 14$ for integers m and n with $2 \leq m \leq n$.*

Proof. Lemmas 2.1 and 2.2 imply that $14 = \tau_4(C_4) \leq \tau_4(P_m \square P_n)$. So it suffices to construct a 4-tone 14-coloring of $P_m \square P_n$. Let $f : V(P_m \square P_n) \to \binom{[14]}{4}$, where we write $f((i,j))$ as $f(i,j)$ and we let $f(i,j) := \{f_1(i,j), f_2(i,j), f_3(i,j), f_4(i,j)\}$, where

$$\begin{aligned} f_1(i,j) &:= (i-j) \bmod 3 \\ f_2(i,j) &:= ((i+j) \bmod 3) + 3 \\ f_3(i,j) &:= ((2i+j) \bmod 4) + 6 \\ f_4(i,j) &:= ((i+2j) \bmod 4) + 10. \end{aligned} \tag{$**$}$$

Denote v by (i_1, j_1) and w by (i_2, j_2). Assume $d(v,w) = 1$. It suffices to prove the following four claims.

Claim 1: If $|f(v) \cap f(w)| = 4$, then $d(v,w) \geq 5$.

Assume $|f(v) \cap f(w)| = 4$. So $f_i(v) = f_i(w)$ for all $i \in [4]$. Claim 1 in Theorem 5.1 implies $d(v,w) \geq 4$. Suppose $d(v,w) = 4$. Since $f_4(v) = f_4(w)$ we have $i_1 - i_2 \equiv 0 \mod 4$ and $j_1 = j_2$, or $j_1 - j_2 \equiv 0 \mod 4$ and $i_1 = i_2$. In either case this implies $f_k(v) \neq f_k(w)$ for each $k \in \{1, 2\}$, a contradiction.

Claim 2: If $|f(v) \cap f(w)| = 3$, then $d(v, w) \geq 4$.

Assume $|f(v) \cap f(w)| = 3$. Claim 1 in Theorem 5.1 implies $f_4(v) = f_4(w)$; and Claim 2 in Theorem 5.1 implies $d(v, w) \geq 3$. Suppose $d(v, w) = 3$. If $f_3(v) \neq f_3(w)$, then $i_1 \equiv i_2 \mod 3$ and $j_1 \equiv j_2 \mod 3$, but then $f_4(v) \neq f_4(w)$, a contradiction. If $f_3(v) = f_3(w)$, then $i_1 - i_2 \equiv j_1 - j_2 \mod 4$, which implies $f_1(v) \neq f_1(w)$ and $f_2(v) \neq f_2(w)$, contradicting $|f(v) \cap f(w)| = 3$.

Claim 3: If $|f(v) \cap f(w)| = 2$, then $d(v, w) \geq 3$.

Assume $|f(v) \cap f(w)| = 2$. If $f_4(v) \neq f_4(w)$, then by Claim 2 in Theorem 5.1 we know $d(v, w) \geq 3$. So we may assume $f_4(v) = f_4(w)$ and $f_k(v) = f_k(w)$ for some single $k \in [3]$. From Claim 3 in Theorem 5.1 we have that $d(v, w) \geq 2$. Suppose $d(v, w) = 2$. Since $f_4(v) = f_4(w)$ it must be that $i_1 = i_2$. So $j_1 - j_2 \equiv 2 \mod 4$; but now $f_k(v) \neq f_k(w)$ for all $k \in \{1, 2\}$, a contradiction.

Claim 4: If $|f(v) \cap f(w)| = 1$, then $d(v, w) \geq 2$.

Assume $|f(v) \cap f(w)| = 1$. If $f_4(v) \neq f_4(w)$, then Claim 3 in Theorem 5.1 implies $d(v, w) \geq 2$. So $f_4(v) = f_4(w)$, which implies $d(v, w) \geq 2$. □

Theorem 5.3. $20 \leq \tau_5(P_m \square P_n) \leq 22$ *for all* $2 \leq m < n$.

Proof. Using Lemma 2.2 when $t \geq 5$ implies $\tau_t(P_2 \square P_3) = 6t - 10$; in fact, an optimal t-tone coloring φ of $P_2 \square P_3$ is unique up to relabelling. This fact combined with Lemma 2.1 implies $20 = \tau_5(P_2 \square P_3) \leq \tau_5(P_m \square P_n)$.

It now suffices to show a 5-tone 22-coloring of $P_m \square P_n$. Let $f : V(P_m \square P_n) \to \binom{[22]}{5}$ where we will denote $f((i, j))$ as $f(i, j)$ and define $f(i, j) := \{f_1(i, j), f_2(i, j), f_3(i, j), f_4(i, j), f_5(i, j)\}$ where

$$\begin{aligned}
f_1(i, j) &:= (i - j) \bmod 3 \\
f_2(i, j) &:= ((i + j) \bmod 3) + 3 \\
f_3(i, j) &:= ((2i + j) \bmod 4) + 6 \\
f_4(i, j) &:= ((i + 2j) \bmod 4) + 10 \\
f_5(i, j) &:= ((i + 3j) \bmod 8) + 14.
\end{aligned}$$

Let $v = (i_1, j_1)$, $w = (i_2, j_2)$, and $q = |f(v) \cap f(w)|$. If $q \in \{0, \ldots, 4\}$ and $f_5(v) \neq f_5(v)$, then $(**)$ and the claims in Theorem 5.2 imply $d(v, w) \geq q + 1$. So we assume $f_5(v) = f_5(v)$. This implies $d(v, w) \geq 4$ since otherwise $((i_1 - i_2) + 3(j_1 - j_2)) \bmod 8 \neq 0$. So it suffices to prove the following two claims.

Claim 1: If $|f(v) \cap f(w)| = 4$, then $d(v, w) \geq 5$.

Assume $|f(v) \cap f(w)| = 4$. Suppose $d(v, w) = 4$. Since $f_5(v) = f_5(w)$, either: $i_1 - i_2 = \pm 1$ and $j_1 - j_2 = \mp 3$; or $i_1 - i_2 = \pm 2$ and $j_1 - j_2 = \pm 2$; or $i_1 - i_2 = \pm 3$ and $j_1 - j_2 = \mp 1$. In all cases $f_2(v) \neq f_2(w)$ and $f_3(v) \neq f_3(w)$, a contradiction to $|f(v) \cap f(w)| = 4$.

Claim 2: If $|f(v) \cap f(w)| = 5$, then $d(v, w) \geq 6$.

Assume $|f(v) \cap f(w)| = 5$. Claim 1 implies $d(v, w) \geq 5$. So $|i_1 - i_2| + |j_1 - j_2| = 5$. But now $f_5(v) \neq f_5(w)$, a contradiction. □

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