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Counting families of generalized balancing numbers

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Abstract

Balancing numbers were introduced by Behera and Panda. Generalizations have been introduced by Panda (individually and with collaborators), the authors, and many others. There are numerous connections with entries of the Online Encyclopedia of Integer Sequences. Recently it was conjectured that the number of distinct families of $A(k,0)$-balancing numbers satisfied a simple formula. Using elementary algebraic number theory we settle this conjecture.

1 Introduction

Balancing numbers were introduced by Behera and Panda [3]. Generalizations have been introduced by Panda [12], Panda and Panda [13, 14], Panda and Rout [16, 17], Panda and Ray [15], the authors [1, 2], and many others [4, 5, 6, 8, 9, 10, 18, 20, 21, 22]. There are numerous connections with entries of [11] (e.g. A001109, A002203, A053141, A016278, A076293, A077259, A077443, A077446, A124124, A156066, A275797) and many related sequences do not yet appear in the encyclopedia. Recently it was conjectured that the number of distinct families of $A(k,0)$-balancing numbers k satisfied a simple formula in terms of the number of positive divisors of an integer [1]. Using elementary algebraic number theory we settle this conjecture.

Let k and w be integers with $k \geq 0$. A positive integer B is an almost k-gap balancing number with weight w, or an $A(k,w)$-balancing number, if $B \geq k$ and

$$1 + 2 + 3 + \cdots + (B-k) + w = (B+1) + (B+2) + \cdots + (B+r).$$

for some non-negative integer r. The integer r is called the $A(k,w)$-balancer for the $A(k,w)$-balancing number B. The integer w is considered a weight, and the integer k is the gap. The original class of balancing numbers has $w = 0$ and $k = 1$. Almost balancing numbers with $w = \pm 1$ were introduced by Panda and Panda [13]. The generalization to gaps $k \geq 2$ and general integer values of w is made in [2]. Dash, Ota, and Dash [6] give an alternate definition of gap balancing numbers. We find that our definition generalizes more easily.

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In this paper we consider the number of classes of $A(k,0)$-balancing numbers. Our main result is

Theorem 1.1. *For a given gap k the number of distinct families of $A(k,0)$-balancing numbers is the number of positive integer divisors of $2k^2-1$.*

This generalizes the result of Tekcan, Tayat and Özbek [22] which dealt with the case that $2k^2-1$ is an odd prime congruent to ± 1 modulo 8.

In the next section we use elementary algebraic number theory to count families of solutions to the Pell-like equation $y^2-2z^2=N$, where $N>1$ is an odd positive integer. In the third section we illustrate how this Pell-like equation arises in the study of generalized balancing numbers. From this connection we derive an upper bound on the number of distinct families of $A(k,0)$-balancing numbers. In the fourth section we show that this bound is tight by considering the action of the group of units in $\mathbb{Z}[\sqrt{2}]$ on solutions of the Pell-like equation $x^2-2y^2=2k^2-1$.

2 Classes of solutions to a Pell-like equation

Let R denote the ring $\mathbb{Z}[\sqrt{2}] \cong \mathbb{Z}[x]/\langle x^2-2\rangle$ and let p denote an odd integral prime. Because $R/p \cong \mathbb{Z}_p[x]/\langle x^2-2 \pmod{p}\rangle$ for an integer prime $p>2$, the diophantine equation

$$p = u^2 - 2v^2$$

is solvable in integers u and v if and only if $p \equiv \pm 1 \pmod 8$. Hence the prime elements of R up to associates, are

- $\sqrt{2}$
- $\alpha \in R$ such that $N(\alpha)=p$ is a prime $p \equiv 1,7 \pmod 8$
- integer primes $p \equiv 3,5 \pmod 8$.

So if $p \equiv \pm 1 \pmod 8$ it is not prime in R and therefore there are integers a, b so that $p = a^2 - 2b^2$. In other words, since R is a principal ideal domain [19], the ideal generated by p in R is uniquely factorable as the product of the prime ideals $\langle a + b\sqrt{2}\rangle$ and $\langle a - b\sqrt{2}\rangle$ in R.

Now given an odd integer N which can be written as $N = a^2 - 2b^2$ by the fundamental theorem of arithmetic there are integral primes $p_1, p_2, \ldots, p_s, q_1, q_2, \ldots, q_t$ so that we can write N uniquely as $\prod_{i=1}^{s} p_i^{e_i} \prod_{j=1}^{t} q_j^{f_j}$ where each $p_i \equiv \pm 1 \pmod 8$, each $q_j \equiv \pm 3 \pmod 8$ and the integers e_i and f_j give the multiplicities with which the primes p_i and q_j respectively divide N. Further, the q_j's remain prime in R and each ideal generated by one of the p_i factors as $P_i\overline{P_i} = \langle a_i + b_i\sqrt{2}\rangle\langle a_i - b_i\sqrt{2}\rangle$ where the overline denotes conjugating the square root of 2.

Example 2.1. When $N = 161 = 7 \cdot 23$. In $\mathbb{Z}[\sqrt{2}]$,

$$\langle 161\rangle = \langle 3+\sqrt{2}\rangle\langle 3-\sqrt{2}\rangle\langle 5+\sqrt{2}\rangle\langle 5-\sqrt{2}\rangle.$$

Here we get four distinct class representatives of solutions to the equation $y^2 - 2z^2 = 161$. From $(3+\sqrt{2})(5+\sqrt{2}) = 17+8\sqrt{2}$ arises $y = 17, z = 8$, while $(3-\sqrt{2})(5-\sqrt{2})$ gives $y = 17, z = -8$. From $(3+\sqrt{2})(5-\sqrt{2}) = 13+2\sqrt{2}$ arises $y = 13, z = 2$. Lastly $(3-\sqrt{2})(5+\sqrt{2}) = 13-2\sqrt{2}$ yields $y = 13, z = -2$.

Theorem 2.1. *Let $N > 1$ be an odd integer with $N = a^2 - 2b^2$, for some integers a, b. Further write $N = AB$, where $A = \prod_{i=1}^{s} p_i^{e_i}$, where each $p_i \equiv \pm 1 \pmod 8$ and no prime divisor of $B = \prod_{j=1}^{t} q_j^{f_j}$ is congruent to $\pm 1 \pmod 8$. The number of distinct families of solutions to $y^2 - 2z^2 = N$ is the number of positive divisors of A.*

Proof. In R we can write the ideal generated by N, $\langle N\rangle$, as the product of two conjugate ideals $\langle a+b\sqrt{2}\rangle\langle a-b\sqrt{2}\rangle$ and so as a product of prime ideals in R if any prime divides $\langle a+b\sqrt{2}\rangle$ in R, its conjugate must divide $\langle a-b\sqrt{2}\rangle$ in R. Since all of the q_j's are self-conjugate, they must appear with even ramification in the factorization of $\langle N\rangle$ in R. Thus $f_j = 2g_j$ for some integer g_j for all j.

Furthermore if we write $\langle a+b\sqrt{2}\rangle$ as a product of primes in R, each time a prime P_i appears as a factor, its conjugate $\overline{P}_i$ must appear as a factor in the factorization of the ideal $\langle a-b\sqrt{2}\rangle$. So we can write $\langle a+b\sqrt{2}\rangle = \Pi_{i=1}^{s}\langle a_i+b_i\sqrt{2}\rangle^{x_i}\langle a_i-b_i\sqrt{2}\rangle^{e_i-x_i}\Pi_{j=1}^{t} q_j^{g_j}$. Here we can choose x_i to be any integer with $0 \le x_i \le e_i$. and thus there are e_i+1 choices for x_i.

Finally it is known that solutions in integers to $a^2 - 2b^2 = N$ come in infinite cyclic families where the elements in each family differ by a unit in R. This means that any two members from the same family must differ by a unit. So they are divisible by the same primes in R. Thus each family of solutions corresponds to a particular choice of the exponents x_i. So the number of families of solutions is the number of positive integer divisors of A. $\square$

Example 2.2. When $N = 3^2 \cdot 7 \cdot 23$, we have $A = 161$ as in Example 2.1. but now $B = 9$. In $\mathbb{Z}[\sqrt{2}]$,

$$\langle N\rangle = \langle 3\rangle^2\langle 3+\sqrt{2}\rangle\langle 3-\sqrt{2}\rangle\langle 5+\sqrt{2}\rangle\langle 5-\sqrt{2}\rangle.$$

Here we get four distinct class representatives of solutions to the equation $y^2 - 2z^2 = N$. From $3\cdot(3+\sqrt{2})(5+\sqrt{2}) = 51+24\sqrt{2}$ arises $y = 51, z = 24$, while $3\cdot(3-\sqrt{2})(5-\sqrt{2})$ gives $y = 51, z = -24$. From $3\cdot(3+\sqrt{2})(5-\sqrt{2}) = 39+6\sqrt{2}$ arises $y = 39, z = 6$. Finally, from $3\cdot(3-\sqrt{2})(5+\sqrt{2}) = 39-6\sqrt{2}$ we get $y = 39, z = -6$.

In greater generality if $R = \mathbb{Z}[\sqrt{d}]$ is a principal ideal domain and we have $y^2 - dz^2 = N$ where $N \not\equiv 0 \pmod d$, then the prime divisors of the left-hand side appear in conjugate pairs. We know that each prime ideal P in R contains exactly one prime ideal $\langle p\rangle$ with $p \in \mathbb{Z}$, so either $P = \langle p\rangle$, or there exist integers a, b so that $P = \langle a+b\sqrt{d}\rangle$, where $\langle p\rangle = \langle a+b\sqrt{d}\rangle\langle a-b\sqrt{d}\rangle$.

Recall that the Legendre symbol $(\frac{d}{p})$ is 1 if d is a quadratic residue mod p and -1 if d is a quadratic non-residue mod p. If we write $N = \Pi_{(\frac{d}{p_i})=1} p_i^{e_i}\Pi_{(\frac{d}{q_j})=-1} q_j^{2g_j}$

and $A = \Pi_{(\frac{d}{p_i})=1} p_i^{e_i}$, then the number of families of solutions to $y^2 - dz^2 = N$ is the number of positive integral divisors of A.

Again, members of the same family of solutions differ only by a unit in R. So if $d > 0$ there are infinitely many units and the families are infinite, cyclic and pairwise disjoint (by prime factorization). Meanwhile if $d < 0$ there are finitely many units in R and therefore finitely many solutions in integers to $y^2 - dz^2 = N$

Ireland and Rosen [7, p. 192] indicate that for $d > 0$ Gauss conjectured that there are infinitely many fields $\mathbb{Q}(\sqrt{d})$ for which the class number is one. This remains an open problem.

3 Upper bounds for the number of families of general balancing numbers

If $T(n) = n(n+1)/2$ is the nth triangular number, then B is an $A(k, w)$-balancing number if and only if

$$T(B - k) + T(B) + w = T(B + r). \tag{1}$$

Solving Equation (1) for r gives

$$r = \frac{-(2B+1) + \sqrt{8B^2 + 8(1-k)B + (2k-1)^2 + 8w}}{2}$$

So B is an $A(k, w)$-balancing number if and only if $8B^2+8(1-k)B+(2k-1)^2+8w$ is a perfect square. In this case the integer $C = \sqrt{8B^2 + 8(1-k)B + (2k-1)^2 + 8w}$ is the corresponding $A(k, w)$-Lucas-balancing number. Further the pair of numbers (B, C) is a solution (x, y) to the Pell-like equation $y^2 = 8x^2+8(1-k)x+(2k-1)^2+8w$, which after a change of variable $z = 2x + 1 - k$ takes the form

$$y^2 - 2z^2 = 2k^2 + 8w - 1 \tag{2}$$

which is called the $A(k, w)$-companion equation [2].

Corollary 3.1. *For a given gap k the number of distinct families of $A(k, 0)$-balancing numbers is at most the number of positive divisors of $2k^2 - 1$.*

Proof. Apply Theorem 2.1 with $N = 2k^2 - 1 = (1 + k\sqrt{2})(-1 + k\sqrt{2})$. If $q_j \equiv \pm 3 \pmod 8$ divides N, by Euclid's Lemma it either divides $1 + k\sqrt{2}$ or $1 - k\sqrt{2}$. But this is impossible since we then need $\pm 1/q_j \in \mathbb{Z}$. Thus the quantity B from Theorem 2.1 is an empty product. Since any balancing pair (x, y) gives a solution (y, z) to the companion equation, which in turn gives a prime factorization of N in R, the number of families of $A(k, 0)$-balancing numbers is at most the number of positive divisors of $N = A$. $\square$

For a fixed positive integer k, a k-circular balancing number satisfies the Diophantine equation

$$(k+1)+(k+2)+\cdots+(n-1)=(n+1)+(n+2)+\cdots+m+(1+2+\cdots+(k-1))$$

for some natural number m [14]. One can show that the companion equation for this type of balancing number is $y^2-8z^2=1-8k^2$.

Corollary 3.2. *For a positive integer k the number of distinct families of k-circular balancing numbers is at most the number of positive divisors of $8k^2-1$.*

4 Distinct families of $A(k,0)$-balancing numbers

4.1 The action of units on solutions to the companion equation

In this section $R=\mathbb{Z}[\sqrt{2}]$. The group of units $G\cong\mathbb{Z}_2\times\mathbb{Z}$ for R is generated by $\langle -1, w=1+\sqrt{2}\rangle$. The argument is two-fold: First, reduce to the case where $a+b\sqrt{2}\in G$, with $b\geq 0$. Second, induct on b.

We are concerned with the action of G on solutions of the companion equation $y^2-2z^2=N$, where $N>1$ is an integer. In particular we want to determine how multiplication by an element of G acts on a solution $y+z\sqrt{2}$ to the companion equation.

Two conjugate hyperbolas, see figure 4.1 arise here $H: y^2-2z^2=N$, and $\overline{H}: y^2-2z^2=-N$. Further $\overline{H}$ has an upper branch $\overline{H}_u$ and a bottom branch $\overline{H}_b$, and H has a right branch H_r and a left branch H_l.

The group G acts transitively on lattice points of these branches by multiplication. This action induces an action on the four branches $H_r, H_l, \overline{H}_u$, and $\overline{H}_b$.

The map $-$, which corresponds to multiplication by -1, takes a point $y+z\sqrt{2}$ to the point $-(y+z\sqrt{2})=-y-z\sqrt{2}$ and leaves the norm y^2-2z^2 invariant. So the action of $-$ on the branches is synopsized in cycle notation as $(H_r,H_l)(\overline{H}_u,\overline{H}_b)$.

The map w, which corresponds to multiplication by $w=1+\sqrt{2}$ takes a point $y+z\sqrt{2}$ to the point $y'+z'\sqrt{2}=(y+2z)+(y+z)\sqrt{2}$, but if $y^2-2z^2=N$, then $(y')^2-2(z')^2=-N$. Thus the upper part of H_r $(y,z>0)$ is mapped to the right part of $\overline{H}_u$ and vice versa. As the map is clearly continuous we must have $w(H_r)=\overline{H}_u$ and vice versa. Similarly with the lower part of H_l $(y,z<0)$ maps to the left part of $\overline{H}_b$. So the action of w on the branches is $(H_r,\overline{H}_u)(H_l,\overline{H}_b)$ in cycle notation.

Finally, the map $\overline{w}$ which corresponds to multiplication by $\overline{w}=1-\sqrt{2}$ sends $y+z\sqrt{2}$ to $y'+z'\sqrt{2}=(y-2z)+(z-y)\sqrt{2}$ and if $y^2-2z^2=N$, then $(y')^2-2(z')^2=-N$. Thus the lower part of H_r $(y>0, z<0)$ is mapped to the right part of $\overline{H}_b$, while the upper part of H_l $(y<0,z>0)$ is mapped to the left part of $\overline{H}_u$. So the action of $\overline{w}$ on the branches is $(H_r,\overline{H}_b)(H_l,\overline{H}_u)$ by continuity.

Thus the subgroup K of G that leaves the set of lattice points of H_r $(y>0)$ invariant must be the subgroup of index 4 generated by $u=w^2=3+2\sqrt{2}$.

Consequently, it follows that:

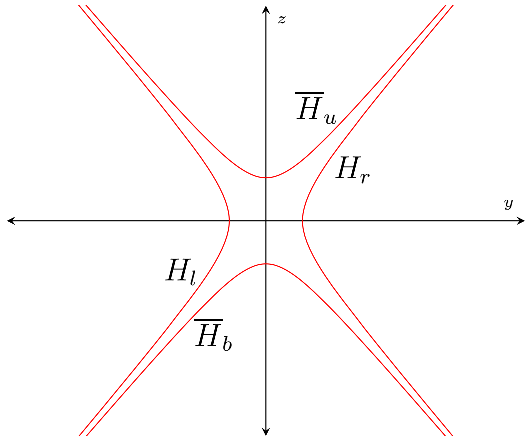

Figure 1: Hyperbolas H and $\overline{H}$

Theorem 4.1. *Two solutions $y + z\sqrt{2}$ and $y' + z'\sqrt{2}$ of the companion equation $y^2 - 2z^2 = N$ with $y > 0$ that lead to the same factorization of N into two conjugate factors over R must differ by a power of $3 + 2\sqrt{2} = u$, say u^e, where $e \in \mathbb{Z}$.*

4.2 Inverse integral maps for $A(k, 0)$-balancing numbers

For $A(k, 0)$-balancing numbers, the map P which sends a balancing pair (x, y) to a solution of the companion equation by (y, z) by $y \mapsto y$ and $x \mapsto z = 2x - k + 1$ clearly takes lattice points to lattice points. In fact the inverse map P^{-1} by $y \mapsto y$ and $z \mapsto x = (z + k - 1)/2$ does too.

To see this first realize that if (y, z) is a solution to the companion equation, then $y^2 = 2z^2 + 2k^2 - 1$ is odd. Thus $y = 2m + 1$ for some integer m. If we additionally assume $k = 2l + 1$ for some integer l, the companion equation can be solved for $z^2 = 2m^2 + 2m - 4l^2 - 4l$. Thus z^2 is even, so z has opposite parity from k. If we assume that $k = 2l$ instead, solving the companion equation for $z^2 = 2m^2 + 2m - 4l^2 - 1$ shows z^2 is odd. So in either case z and k have opposite parity. Hence P^{-1} maps lattice points to lattice points.

4.3 Decomposition of a map

As defined in [1] there is a map f which translates a balancing pair of $A(k, 0)$-balancing numbers in a particular family to the next member in the family by $(x_i, y_i) \mapsto (x_{i+1}, y_{i+1})$ where

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + (1 - k) \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

We state without proof the following lemma.

Lemma 4.2. *The map f decomposes as PAP^{-1} where the map A represents multiplication by u in R via matrix multiplication. The matrix for A is* $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$.

Proof of Theorem 1.1: By Corollary 3.1 we know that the number of distinct families of $A(k,0)$-balancing numbers is at most the number of positive divisors of $2k^2-1$. It remains to show that the number of distinct families is at least the number of positive divisors of $2k^2-1$.

Denote a complete set of representatives of distinct factorizations of $2k^2-1$ into two conjugate factors over R by S. If we start with a balancing pair (x,y) and move to a solution (y,z) to the companion equation, there is an integer e so that the factorization of $2k^2-1$ into two conjugate parts over R from the pair (y,z) is in the same class as arising from $y'+z'\sqrt{2} \in S$. That is $y+z\sqrt{2} = u^e(y'+z'\sqrt{2})$ for some integer e. Thus $P^{-1}(y',z') = (x',y') = f^e(x,y)$. Thus each factorization of $2k^2-1$ in $\mathbb{Z}[\sqrt{2}]$ into two conjugate factors produces at most one family of balancing pairs. □

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