

Congruences modulo high powers of 2 for Sloane’s box stacking function

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Abstract

We are given n boxes, labeled $1, 2, \dots, n$. Box i weighs i grams and can support a total weight of i grams. The number of different ways to build a single stack of boxes in which no box will be squashed by the weight of the boxes above it is denoted by $f(n)$. In a 2006 paper, the first author asked for “congruences for $f(n)$ modulo high powers of 2”. In this note, we accomplish this task by proving that, for $r \geq 5$ and all $n \geq 0$,

$$f(2^r n) - f(2^{r-1} n) \equiv 0 \pmod{2^r},$$

and that this result is “best possible”. Some additional complementary congruence results are also given.

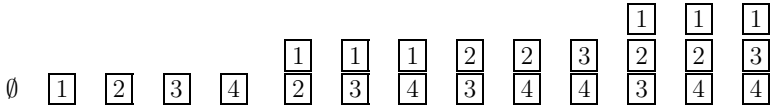
1 Introduction

In 2005, Sloane and Sellers [8] considered the following problem:

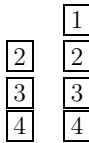
Suppose we have boxes with labels $1, 2, 3, \dots, n$. A box labeled i weighs i grams and can support a total weight of i grams. We wish to build single

stacks of boxes with distinct labels in such a way that no box will be squashed by the weight of the boxes above it. What is the number $f(n)$ of different ways to build such a single stack of boxes?

This function $f(n)$ is the function identified in the title as “Sloane’s Box Stacking Function” and is the primary object of interest in this work. By way of example, note that $f(4) = 14$ where the admissible stacks are the following:



The other two possible stacks



are excluded since $2 + 3 > 4$ and the box labeled 4 would collapse in both cases. The first few values of $f(n)$, for $n = 0, 1, 2, \dots$, are

$$1, 2, 4, 8, 14, 23, 36, 54, 78, 109, 149, 199, 262, 339, 434, 548, 686.$$

Additional values of $f(n)$ are given by Sloane [7, Sequence A089054].

Sloane and Sellers [8] extensively studied $f(n)$. One of their main results is that the generating function $F(q) = \sum_{n=0}^{\infty} f(n)q^n$ can be written as

$$F(q) = \frac{1}{(1-q)^2} \left(\frac{1}{1-q} - q + \sum_{i=1}^{\infty} \frac{q^{3 \cdot 2^{i-1}}}{\prod_{j=0}^i (1-q^{2^j})} \right). \tag{1}$$

In a more recent work, Rødseth [4] gave a simple proof of (1). At the end of his note, he asked for congruences for $f(n)$ modulo (high) powers of 2. In this work we satisfy that request by proving the following theorem.

Theorem 1 For $r \geq 5$ and all $n \geq 0$,

$$f(2^r n) - f(2^{r-1} n) \equiv 0 \pmod{2^r}.$$

Moreover, no higher power of 2 divides $f(2^r n) - f(2^{r-1} n)$ if n is odd.

Some complementary results are given at the end of Section 5; for example, we show that, for all $n \geq 0$,

$$f(2^3 n) - f(2^2 n) \equiv 0 \pmod{2^5}.$$

We note that Theorem 1 is similar to families of congruences satisfied by related functions. For example, Churchhouse [2] conjectured, and Rødseth [3] proved, that

the binary partition function (the function which counts the partitions of an integer n using only powers of 2 as parts) satisfies a similar set of congruences; also see [1, Section 10.2]. More recently, a similar family of results was proven for the function $b(n)$, originally defined by Sloane and Sellers [8], which counts the number of “non-squashing” partitions of n into distinct parts. See [6] for more details on these congruences satisfied by $b(n)$. Indeed, we will prove Theorem 1 using tools developed in [5, 6]. However, for $r < 5$, the difference $f(2^r n) - f(2^{r-1} n)$ behaves rather irregularly, and this creates some extra difficulties which we overcome below. These irregularities make the search for a result like Theorem 1 more difficult, especially as compared to the results in [2, 3] and [5, 6].

The structure of this paper is very straightforward. In Section 2 we introduce some auxiliaries which are necessary for the proof of Theorem 1. Theorem 1 is then proven in Section 3 with most of the technical details postponed for the sake of readability. Those necessary details are then given in Sections 4 and 5.

2 Auxiliaries

The power series in this paper will be elements of $\mathbb{Z}[[q]]$, the ring of formal power series in q with coefficients in \mathbb{Z} . We define a \mathbb{Z} -linear operator U , acting on $\mathbb{Z}[[q]]$, by

$$U \sum_n a(n)q^n = \sum_n a(2n)q^n.$$

Notice that if $A(q), C(q) \in \mathbb{Z}[[q]]$, then

$$U(A(q)C(q^2)) = (UA(q))C(q). \tag{2}$$

In the following, a power series $A(q) \in \mathbb{Z}[[q]]$ will often be written as A . If the argument is not q , then we will, of course, include the argument in the notation.

We shall use below the following identity for binomial coefficients:

$$\binom{2n+r-1}{r} = \sum_{i=0}^{\lfloor r/2 \rfloor} (-1)^i 2^{r-2i} \binom{r-i}{i} \binom{n+r-i-1}{r-i}. \tag{3}$$

The truth of this relation follows by expanding both sides of the identity

$$(1+q)^{-2n} = (1+q(2+q))^{-n}$$

and comparing the coefficient of q^r on each side of the equation.

Let

$$h_i = \frac{q}{(1-q)^{i+1}}, \quad i \geq 0.$$

Then

$$h_i = \sum_{n=1}^{\infty} \binom{n+i-1}{i} q^n, \tag{4}$$

so that

$$Uh_r = \sum_{n=1}^{\infty} \binom{2n+r-1}{r} q^n.$$

It follows from (3) and (4) that

$$Uh_r = \sum_{i=0}^{\lfloor r/2 \rfloor} (-1)^i 2^{r-2i} \binom{r-i}{i} h_{r-i} \tag{5}$$

for $r \geq 0$. Simple calculations show that

$$Uh_0 = h_0, \tag{6}$$

$$Uh_1 = 2h_1, \tag{7}$$

$$Uh_2 = 4h_2 - h_1. \tag{8}$$

Also notice that

$$U \frac{1}{1-q} = \frac{1}{1-q}. \tag{9}$$

We recursively define functions L_r and M_r for $r \geq 2$ by

$$L_2 = 2^2 h_2 \quad \text{and} \quad L_{r+1} = U \left(\frac{1}{1-q} L_r \right) \tag{10}$$

and

$$M_2 = 2h_2 \quad \text{and} \quad M_{r+1} = L_{r+1} - (UL_r) \frac{1}{1-q} + UM_r. \tag{11}$$

The motivation for these definitions will become clear in the following section.

3 Proof of Theorem 1

In this proof we will use the auxillary function

$$B(q) = \frac{1}{1-q} + \sum_{i=1}^{\infty} \frac{q^{3 \cdot 2^{i-1}}}{\prod_{j=0}^i (1-q^{2^j})}. \tag{12}$$

This function is the generating function of $b(n)$, mentioned in the Introduction. It was introduced and studied by Sloane and Sellers [8] in connection with their study of $f(n)$. In particular, they showed that $B(q)$ satisfies the functional equation

$$B^*(q) = \frac{1}{1-q} B^*(q^2) + \frac{q}{1-q^2}, \tag{13}$$

where $B^* = B - 1$.

By (1) and (12), we have

$$F(q) = \frac{1}{(1-q)^2} B^*(q) + \frac{1}{1-q},$$

which we can write as

$$F = \left(h_1 + \frac{1}{1-q} \right) B^*(q) + \frac{1}{1-q}. \tag{14}$$

Using (13), we get

$$F = \left(h_2 + h_1 + \frac{1}{1-q} \right) B^*(q^2) + h_1 \frac{1}{1-q^2} + \frac{1}{1-q}.$$

Next, we apply the operator U . Using the linearity of U along with (2) and (7)–(9), we obtain

$$UF = \left(4h_2 + h_1 + \frac{1}{1-q} \right) B^* + 2h_2 + \frac{1}{1-q},$$

so that, by (14),

$$UF - F = 4h_2B^* + 2h_2. \tag{15}$$

Recalling the definitions of L_r and M_r given in (10) and (11), we now prove by induction on r that, for all $r \geq 2$,

$$U^{r-1}F - U^{r-2}F = L_rB^* + M_r. \tag{16}$$

By (15), we know that (16) is true for $r = 2$. Suppose that (16) holds for some $r \geq 2$. By (13), we then have

$$\begin{aligned} U^{r-1}F - U^{r-2}F &= L_r \left(\frac{1}{1-q} B^*(q^2) + \frac{q}{1-q^2} \right) + M_r \\ &= \left(\frac{1}{1-q} L_r \right) B^*(q^2) + \frac{1}{1-q} L_r - L_r \frac{1}{1-q^2} + M_r, \end{aligned}$$

and applying U and using (2), we get by (10) and (11),

$$\begin{aligned} U^rF - U^{r-1}F &= L_{r+1}B^* + L_{r+1} - (UL_r) \frac{1}{1-q} + UM_r \\ &= L_{r+1}B^* + M_{r+1}. \end{aligned}$$

Thus (16) holds for all $r \geq 2$.

For $r \geq 6$, we have, by Lemma 1 in Section 4,

$$L_r \equiv 0 \pmod{2^{r+1}},$$

and by Lemma 3 in Section 4,

$$M_r \equiv 2^{r-1}h_1 \pmod{2^r}.$$

By (16), we thus have

$$U^{r-1}F - U^{r-2}F \equiv 2^{r-1}h_1 \pmod{2^r},$$

and the proof of Theorem 1 is easily completed.

4 Three Technical Lemmas

We have chosen to collect the necessary technical lemmas in this section (to make the first portion of the paper more readable). First, we have a lemma involving L_r .

Lemma 1 *For $r \geq 3$ and $0 \leq i \leq r - 2$, there exist $\lambda_r(i) \in \mathbb{Z}$ such that*

$$L_r = \sum_{i=0}^{r-2} \lambda_r(i) h_{r-i}, \tag{17}$$

where

$$\lambda_r(i) \equiv 0 \pmod{2^{2r-i-1}}. \tag{18}$$

Proof. This follows from [5, Lemma 1], by observing that $K_r = 2L_r$ for the K_r defined in [5]. □

Second, we have a lemma concerning UL_r .

Lemma 2 *For $r \geq 5$ and $0 \leq i \leq r - 1$, there exist $\delta_r(i) \in \mathbb{Z}$ such that*

$$UL_r = \sum_{i=0}^{r-1} \delta_r(i) h_{r-i}, \tag{19}$$

where

$$\delta_r(i) \equiv 0 \pmod{2^{2r-i+1}}. \tag{20}$$

Proof. This follows from [5, Lemma 2]. □

Third, we prove a crucial lemma about M_r .

Lemma 3 *For $r \geq 6$, there exist $\mu_r(i) \in \mathbb{Z}$ such that*

$$M_r = \sum_{i=0}^{r-1} \mu_r(i) h_{r-i}, \tag{21}$$

where

$$\mu_r(i) \equiv 0 \pmod{2^{2r-i-1}} \quad \text{for } 0 \leq i \leq r - 2, \tag{22}$$

and

$$\mu_r(r - 1) \equiv 2^{r-1} \pmod{2^r}. \tag{23}$$

Proof. We use induction on r . By the expression for M_6 in Section 5, we see that the lemma is true for $r = 6$.

Suppose that for some $r \geq 7$ there are integers $\mu_{r-1}(j)$ such that

$$M_{r-1} = \sum_{j=0}^{r-2} \mu_{r-1}(j) h_{r-1-j}, \tag{24}$$

where

$$\mu_{r-1}(j) \equiv 0 \pmod{2^{2r-j-3}} \quad \text{for } 0 \leq j \leq r-3, \tag{25}$$

and

$$\mu_{r-1}(r-2) \equiv 2^{r-2} \pmod{2^{r-1}}. \tag{26}$$

Then, by (24) and (5),

$$\begin{aligned} UM_{r-1} &= \sum_{j=0}^{r-2} \mu_{r-1}(j) U h_{r-1-j} \\ &= \sum_{j=0}^{r-2} \mu_{r-1}(j) \sum_{k=0}^{\lfloor (r-1-j)/2 \rfloor} (-1)^k 2^{r-1-j-2k} \binom{r-1-j-k}{k} h_{r-1-j-k} \\ &= \sum_{i=1}^{r-1} \sum_{j=\max(0, 2i-r-1)}^{i-1} (-1)^{i-j-1} 2^{r+j-2i+1} \binom{r-i}{i-j-1} \mu_{r-1}(j) h_{r-i}. \end{aligned}$$

Moreover, by (11), (17), and (19),

$$\begin{aligned} M_r &= L_r - (UL_{r-1}) \frac{1}{1-q} + UM_{r-1} \\ &= \sum_{i=0}^{r-2} \lambda_r(i) h_{r-i} - \sum_{i=0}^{r-2} \delta_{r-1}(i) h_{r-i} + UM_{r-1}, \end{aligned}$$

so that (21) holds with

$$\begin{aligned} \mu_r(i) &= \lambda_r(i) - \delta_{r-1}(i) \\ &\quad + \sum_{j=\max(0, 2i-r-1)}^{i-1} (-1)^{i-j-1} 2^{r+j-2i+1} \binom{r-i}{i-j-1} \mu_{r-1}(j) \end{aligned} \tag{27}$$

for $0 \leq i \leq r-2$, and

$$\mu_r(r-1) = -\mu_{r-1}(r-3) + 2\mu_{r-1}(r-2). \tag{28}$$

It follows that all the $\mu_r(i)$ are integers. Furthermore, by (27), (18), (20) and (25), we get (22). Finally, (23) follows from (28), (25) with $j = r-3$, and (26). \square

5 Final Calculations

The calculations in this section were performed with the goal of finding a good M_r to shape the crucial Lemma 3, and also to give us a starting point for the induction proof. We had to go all the way up to M_6 before we found an M_r which satisfied our requirements.

Our calculations run as follows. We first use (5) to add a few more cases to (6)–(8).

$$\begin{aligned} Uh_3 &= 2^3h_3 - 2^2h_2, \\ Uh_4 &= 2^4h_4 - 2^2 \cdot 3h_3 + h_2, \\ Uh_5 &= 2^5h_5 - 2^5h_4 + 2 \cdot 3h_3, \\ Uh_6 &= 2^6h_6 - 2^4 \cdot 5h_5 + 2^3 \cdot 3h_4 - h_3 \end{aligned}$$

Next, we use (10) and the expressions for Uh_i to express L_r , $3 \leq r \leq 6$, in terms of h_i .

$$\begin{aligned} L_3 &= 2^5h_3 - 2^4h_2, \\ L_4 &= 2^9h_4 - 2^9h_3 + 2^5 \cdot 3h_2, \\ L_5 &= 2^{14}h_5 - 2^{13} \cdot 3h_4 + 2^8 \cdot 39h_3 - 2^7 \cdot 7h_2, \\ L_6 &= 2^{20}h_6 - 2^{21}h_5 + 2^{12} \cdot 327h_4 - 2^{12} \cdot 71h_3 - 2^8 \cdot 53h_2 \end{aligned}$$

Then we apply the U -operator to obtain the following (after elementary simplifications):

$$\begin{aligned} UL_2 &= 2^4h_2 - 2^2h_1, \\ UL_3 &= 2^8h_3 - 2^6 \cdot 3h_2 + 2^4h_1, \\ UL_4 &= 2^{13}h_4 - 2^{11} \cdot 5h_3 + 2^7 \cdot 23h_2 - 2^5 \cdot 3h_1, \\ UL_5 &= 2^{19}h_5 - 2^{17} \cdot 7h_4 + 2^{11} \cdot 231h_3 - 2^9 \cdot 133h_2 + 2^7 \cdot 7h_1 \end{aligned}$$

Finally, we use (11) and the results above to express M_r , $3 \leq r \leq 6$, in terms of h_i .

$$\begin{aligned} M_3 &= 2^4h_3 - 2^2h_2 - 2h_1, \\ M_4 &= 2^8h_4 - 2^6 \cdot 3h_3, \\ M_5 &= 2^{13}h_5 - 2^{11} \cdot 5h_4 + 2^7 \cdot 19h_3 + 2^5 \cdot 7h_2, \\ M_6 &= 2^{19}h_6 - 2^{17} \cdot 7h_5 + 2^{11} \cdot 215h_4 + 2^9 \cdot 61h_3 - 2^8 \cdot 25h_2 - 2^5 \cdot 7h_1 \end{aligned}$$

Now, M_6 has the properties we need to prove Lemma 3. The coefficients of the terms h_i are divisible by sufficiently high powers of 2 while the coefficient of h_1 is not divisible by an excessively high power of 2. This second fact allows us to prove the “best possible” part of Theorem 1.

From these calculations, we get the following complementary results mentioned in the Introduction for free. The first congruence below follows from (15) and (4). For the others, put $r = 3, 4, 5, 6$ in (16) and use the results on L_r and M_r obtained in the present section. This gives us the following congruences, valid for all $n \geq 0$:

$$\begin{aligned} f(2n) - f(n) &\equiv n(n + 1) && \pmod{2^2}, \\ f(2^2n) - f(2n) &\equiv -2n(n + 2) && \pmod{2^4}, \\ f(2^3n) - f(2^2n) &\equiv 0 && \pmod{2^5}, \\ f(2^4n) - f(2^3n) &\equiv -2^4n(n + 1) && \pmod{2^7}, \\ f(2^5n) - f(2^4n) &\equiv 2^5n && \pmod{2^8} \end{aligned}$$

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