A Study of the Time-Dependent Wind-Driven Circulation in a Homogeneous, Rectangular Ocean[1]

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Abstract

A theoretical investigation of the ocean circulation induced by zonal winds whose amplitude varies harmonically with time is presented. The analysis is based on a perturbation scheme with respect to a small parameter which is essentially the frequency of the wind variation and the solutions are derived by boundary layer methods. Results are obtained for the variation of the intensity of the currents and their phase lag behind the wind, as well as for the deviation of the free surface from its equilibrium position.

Introduction

Much of the investigation, both theoretical and observational, in the field of oceanography has centered around the dynamics of ocean currents—including the mass transport of the Gulf Stream and the Kuroshio Current, and the general oceanic circulation. Recently interest has also developed regarding the response of the thermocline (the region of sharp vertical gradient of density) to a time-varying wind.

Since the time of EKMAN's (1905) first paper, a large number of articles dealing with various aspects of ocean currents have appeared in the literature. However, analytical investigations of the problem of general oceanic circulation have met with success only in recent years. In the past decade various interesting and meaningful mathematical models have been suggested by numerous investigators. SVERDRUP (1947), and REID (1948) proposed a fairly simple model which seems to give very good qualitative results for a region with only one north-south boundary. STOMMEL (1948) considered two linearized models with a simplified viscous term. His very important contribution to the overall problem is based on the difference between the results obtained with the two models. In one case, the Coriolis term was constant and the resulting streamline pattern is identical with the one in a model with no rotation. In the second case, the Coriolis term varied linearly with latitude and westward intensification resulted—a factor which was not present in the previous case. Since Stommel's paper all problems dealing with general circulation contain a varying Coriolis parameter. MUNK (1950) refined all the previous work and included the general viscous terms in the equations of motion. He solved the problem of a steady wind blowing over an enclosed ocean, taking account of many of the salient features which are present in the real ocean. Munk's work was extended by MUNK and CARRIER (1950) to include oceans of various geometrical shapes, viz., triangular and semi-circular. It was further extended by MUNK,

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GROVES, and CARRIER (1950) to include the non-linear terms by means of a perturbation procedure.

Along with the American publications, a number of papers have appeared in Japan. Notable among the Japanese authors is HIDAKA (1950 a, b, 1951) who published a series of articles covering many of the interesting phenomena of oceanographic problems. Among his contributions are a series of three papers on drift currents in an enclosed ocean and a contribution concerning the neglect of the non-linear terms in the solution of problems in dynamic oceanography (HIDAKA, MIYOSHI, 1949).

The general problem of oceanic circulation essentially consists of finding the dynamic pattern which results from a given distribution of winds acting on the ocean surface. The complete problem contains a large number of features, such as large-scale oceanic circulation, surface waves, upwelling, etc. To study all these features one would have to take into account the effects of wind, of density and temperature distributions, of the topography of the ocean bed, of the salinity variation, and of many other factors. Needless to say, a mathematical analysis including all these features is impossible. It is therefore necessary to decide what particular aspects of the problem one wishes to study. In this paper we shall confine our attention to the large-scale, wind-driven circulation.

Most of the existing work in this particular field is concerned with steady state motions. In the Atlantic Ocean these include the Gulf Stream and its counter-currents, the Sargasso Sea, etc. It is the purpose of this work to investigate the response of the large-scale motions to a prescribed time variation in the wind. This time-dependent problem has also been considered by ICHIYE (1951) and we shall discuss his work later on in this paper.

There is no doubt that the vertical variation of density plays an important role in the response of the ocean to a time varying wind and should therefore be taken into account. For purposes of clarity of exposition, we shall, however, reserve an examination of the effects of this density variation for a future paper and shall assume in the present paper that the ocean is homogeneous, and, moreover, that it is of uniform depth.

Formulation

We assume that the motion which we want to study can be defined mathematically by the Navier-Stokes equations of motion referred to a rotating sphere with the viscous terms replaced by terms arising from a macroscopic viscosity, viz., an eddy viscosity[1]. These equations are

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{q} =$$

$$= -\frac{1}{\varrho} \nabla p + \mathbf{F} + E(\mathbf{q}) \tag{1}$$

and the equation of conservation of mass is

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{q}) = 0 \tag{2}$$

where $\mathbf{q}$ denotes the relative velocity vector on the rotating sphere,
ϱ denotes the (constant) density,
p denotes the pressure,
$\mathbf{r}$ denotes the radius vector from the center of the sphere,
$\boldsymbol{\Omega}$ denotes the angular velocity vector representing the rotation of the earth,
$\mathbf{F}$ denotes the external forces per unit mass (in our case, gravitation),
$E(\mathbf{q})$ represents the eddy viscosity term which will be defined specifically later.

The full non-linear problem is too complicated to solve analytically. As a result, we shall make a series of simplifying assumptions which we shall list below. The reader is referred to VERONIS and MORGAN (1953) for a detailed study of some of these assumptions.

1. The fluid is assumed to be incompressible. The continuity equation (2), therefore, reduces to

$$\nabla \cdot \mathbf{q} = 0 \tag{2 a}$$

2. The equations on a rotating sphere are approximated by equations in a rectangular

[1] The eddy viscosity term approximately replaces the Reynolds' stresses arising from turbulence. The numerical value of the eddy viscosity coefficient is chosen so as to give a reasonable width to the Gulf Stream.

Cartesian system. The effect of the sphericity of the earth is retained by allowing the Coriolis parameter to depend on the latitude.

3. The vertical acceleration terms and the viscous terms are neglected in the equation of vertical motion so that, in effect, hydrostatic pressure is assumed, i.e., $\partial p/\partial z = -g\varrho$. We let η be the free surface height and require that $p = 0$ at $z = \eta$.

4. The non-linear terms in the equations of motion are neglected. MUNK, GROVES, and CARRIER (1950) have concluded that inclusion of the non-linear terms does not alter the qualitative behavior of the steady state solution. While, in the authors' opinion, this point deserves further careful investigation, we shall assume here that the conclusion of these authors is correct, and moreover, that it also holds in the non-steady case. It should be pointed out, however, that in view of the very drastic simplification made, the purpose of our analysis is the determination of the consequences of stipulating a particular, simple, theoretical model, rather than the investigation of a model which, on the basis of sound reasoning, may confidently be expected to resemble the real ocean.

5. The Coriolis parameter is linearized. In effect, we write $2\Omega \sin\left(\frac{y}{R}\right) \approx \beta y$ where $\beta = 2\Omega/R$. Here, R is the mean radius of the earth, Ω is the angular speed of rotation of the earth and y is the south-north coordinate measured positive northward.

With the above assumptions equations (1) and (2) in component form become

$$\frac{\partial u}{\partial t} - \beta y v = -g\frac{\partial \eta}{\partial x} + A\triangle u + \frac{\partial}{\partial z}\left(A_3 \frac{\partial u}{\partial z}\right) \quad (3)$$

$$\frac{\partial v}{\partial t} + \beta y u = -g\frac{\partial \eta}{\partial y} + A\triangle v + \frac{\partial}{\partial z}\left(A_3 \frac{\partial v}{\partial z}\right) \quad (4)$$

$$\frac{\partial p}{\partial z} = -g\varrho \quad (5)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6)$$

where the eddy viscosity terms

$$A\triangle + \frac{\partial}{\partial z}\left(A_3\frac{\partial}{\partial z}\right) \equiv A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(A_3\frac{\partial}{\partial z}\right)$$

are chosen to have the same mathematical form as the viscosity term in laminar flow, subject to the assumption, however, that the kinematic eddy viscosity associated with lateral shear may be different from that associated with vertical shear, the former, A, being taken as a constant, the latter, A_3, as an arbitrary function of position.

The equations (3), (4), and (6) together with the boundary conditions $u = v = w = 0$ on a land-water boundary and an appropriate free surface condition must be solved for u, v, w, η. The wind-stress enters as the value of the shear force at the free surface.

The problem as it has been defined thus far is an extremely difficult one to solve because of the presence of four independent variables x, y, z, t. Consequently, we propose to integrate the equations over the vertical coordinate, z, whereby information regarding the vertical distribution of the velocities is lost. As we are primarily concerned with general oceanic circulation and mass transport, however, and because the integration reduces the complexity of the equations considerably, the advantages gained more than balance the loss of information involved.

Therefore, we choose a constant depth $z = -D$ and integrate the equations from $z = -D$ to the free surface $z = \eta\ (x, y, t)$. Then the equations (3), (4), become

$$\frac{\partial \overline{U}}{\partial t} - \beta y \overline{V} = -g(D+\eta)\frac{\partial(\eta\varrho)}{\partial x} + A\triangle\overline{U} + A_3\frac{\partial u}{\partial z}\Big|_{z=-D}^{z=\eta} \quad (7)$$

$$\frac{\partial \overline{V}}{\partial t} + \beta y \overline{U} = -g(D+\eta)\frac{\partial(\eta\varrho)}{\partial y} + A\triangle\overline{V} + A_3\frac{\partial v}{\partial z}\Big|_{z=-D}^{z=\eta} \quad (8)$$

where

$$\overline{U} = \int_{-D}^{\eta} u\varrho\, dz, \qquad \overline{V} = \int_{-D}^{\eta} v\varrho\, dz$$

We have neglected the non-linear terms resulting from interchanging the derivatives and integrals of the velocity terms.

The vertical viscous terms $A_3\frac{\partial u}{\partial z}\Big|_{z=-D}^{z=\eta}$ and $A_3\frac{\partial v}{\partial z}\Big|_{z=-D}^{z=\eta}$ represent the wind-stress components τ_x, τ_y at the free surface, $z = \eta$, and the bottom friction at the depth, $z = -D$. We assume that the bottom friction terms are negligible and approximate $D + \eta$ by D. Then

$$\frac{\partial \bar{U}}{\partial t} - \beta y \bar{V} = -gD \frac{\partial(\eta\varrho)}{\partial x} + A \triangle \bar{U} + \tau_x \quad (9)$$

$$\frac{\partial \bar{V}}{\partial t} + \beta y \bar{U} = -gD \frac{\partial(\eta\varrho)}{\partial y} + A \triangle \bar{V} + \tau_y \quad (10)$$

If the continuity equation is integrated and the free surface condition (LAMB, 1932)

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}\right)(\eta - z) = 0$$

is used, the integrated continuity equation becomes

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = -\frac{\partial(\eta\varrho)}{\partial t} \quad (11)$$

The depth $z = -D$ represents either the depth of an ideal ocean of uniform depth or else a constant level above which the mass transports $\bar{U}$, $\bar{V}$ are to be found and at which w is negligible. In either case the analysis assumes that the shear forces at $z = -D$ are negligible.

We now consider a rectangular ocean bounded by land at $x = 0$, $x = r_1$, and by water at $y = 0$, $y = s$. Then the boundary conditions on $x = 0$, $x = r_1$ are that the mass transports $\bar{U}$, $\bar{V}$ vanish there.

The wind-stress is specified to be

$$\tau_x = -(W' + \Gamma' \sin \omega t) \cos ny, \quad \tau_y = 0 \quad (12)$$

where W', Γ' represent the magnitude of the mean wind-stress and the amplitude of the time variation of the wind-stress, respectively,

ω is the frequency of wind variation,

n is the wave number associated with the wind distribution.

One can consider the above form for the wind-stress as a typical term in a Fourier series for a more general wind distribution. The numerical results in this paper are based on a value of ω corresponding to a period of one year, and n is set equal to $2\pi/s$, where s is the north-south length of the ocean, Fig. 1.

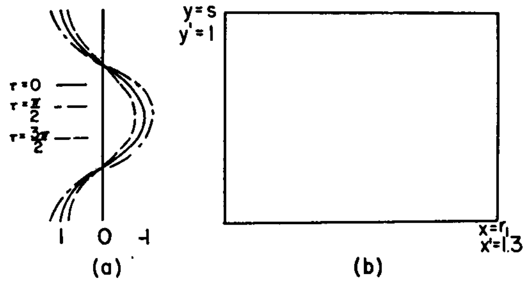

Fig. 1. (a) The wind-stress distribution vs. y coordinate. (b) The rectangular ocean with dimensions in both the dimensional and non dimensional coordinate systems.

The problem to be solved now consists of finding $\bar{U}$, $\bar{V}$, η from equations (9), (10), and (11) together with the boundary conditions $\bar{U} = \bar{V} = 0$ on $x = 0$, r_1, and the specified wind-stress components (12). Since the y derivative of the wind-stress vanishes at $y = 0$, s, plausible boundary conditions at $y = 0$, s are that these lines be streamlines and that there be no lateral shear, i.e., $\partial \bar{U}/\partial y = 0$.

Solution

The problem as defined in the previous paragraph can be solved by boundary layer methods. In order to apply boundary layer technique it is convenient to non-dimensionalize equations (9), (10), and (11). The choice[1] of the non-dimensional quantities suggested by an inspection of the equations is

$$x' = \frac{x}{s} \quad y' = \frac{y}{s} \quad nW' = W \quad n\Gamma' = \Gamma$$

$$\alpha = \frac{\Gamma'}{W'} = \frac{\Gamma}{W} \quad \tau = \omega t \quad V = \frac{\beta \bar{V}}{W} \quad U = \frac{\beta \bar{U}}{W}$$

$$\varepsilon = \frac{A}{\beta s^3} \quad \delta = \frac{\omega}{\beta s} \quad \Theta = \frac{ngD}{\beta^2 s^3} \quad H = \frac{\varrho \eta \beta^2 s^2}{W}$$

The extent of the rectangular ocean in dimensionless coordinates is given by

$$0 \leq x' \leq \frac{r_1}{s} = r \quad 0 \leq y' \leq 1$$

[1] See VERONIS, MORGAN (1953) for details.

Then (9), (10), and (11) become

$$ns\delta\frac{\partial U}{\partial\tau} - nsy\,V + \Theta\frac{\partial H}{\partial x} = ns\varepsilon\,\triangle\,U -$$
$$-(1+\alpha\sin\tau)\cos nsy \qquad (13)$$

$$ns\delta\frac{\partial V}{\partial\tau} + nsy\,U + \Theta\frac{\partial H}{\partial y} = ns\varepsilon\,\triangle\,V \qquad (14)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\delta\frac{\partial H}{\partial\tau} \qquad (15)$$

where we have dropped the primes from x' and y' and will work in the dimensionless coordinate system henceforth.

Attempts to solve equations (13) to (15) in closed form were unsuccessful. We therefore resort to seeking solutions by a perturbation expansion in the parameter δ. This implies "small" δ and hence »small» frequency. We shall learn later just what this means. Let

$$U = U_0 + \delta U_1 + \delta^2 U_2 + \dots$$
$$V = V_0 + \delta V_1 + \delta^2 V_2 + \dots$$
$$H = H_0 + \delta H_1 + \delta^2 H_2 + \dots$$

Our formal procedure is to regard the coefficients U_0, U_1, etc. as coefficients in a power series in δ. If these expansions be substituted into equations (13) to (15), then a series of independent equations will result by equating the coefficients of equal powers of δ on the two sides of the equations.

A. *The zero-order equations.* We have

$$-nsy\,V_0 + \Theta\frac{\partial H_0}{\partial x} = ns\varepsilon\triangle U_0 -$$
$$-(1+\alpha\sin\tau)\cos nsy \qquad (16)$$

$$nsy\,U_0 + \Theta\frac{\partial H_0}{\partial y} = ns\varepsilon\triangle V_0 \qquad (17)$$

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} = 0 \qquad (18)$$

Eliminating H_0 between (16) and (17) we obtain

$$\varepsilon\left[V_{0xxx} + V_{0xyy} - U_{0xxy} - U_{0yyy}\right] - V_0 =$$
$$= (1+\alpha\sin\tau)\sin nsy \qquad (19)$$

A stream function ψ may be defined by

$$V_0 = \frac{\partial\psi}{\partial x} \quad U_0 = -\frac{\partial\psi}{\partial y} \qquad (20)$$

so that (18) is satisfied identically. Then (19) becomes

$$\varepsilon\triangle\triangle\psi - \psi_x = (1+\alpha\sin\tau)\sin nsy \qquad (21)$$

where $\triangle\triangle \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^2}$ is the biharmonic operator and subscripts x, y, τ denote differentiation.

The boundary conditions are

$$\left.\begin{array}{lllll}\psi = \psi_x & = 0 & \text{on} & x = 0 & x = r\\ \psi = \psi_{yy} & = 0 & \text{on} & y = 0 & y = 1\end{array}\right\} \qquad (22)$$

The problem defined by (21) and (22) with $\alpha = 0$ is the one solved by MUNK (1950). Because ε is small a boundary layer[1] analysis at the boundaries $x = 0$, r is convenient. The solution is found to be

$$\psi = (1+\alpha\sin\tau)\sin nsy\Big\{-x + r - \varepsilon^{1/3} +$$
$$+\varepsilon^{1/3}e^{(x-r)\varepsilon^{-1/3}} + \Big[(\varepsilon^{1/3} - r)\cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) +$$
$$+\Big(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\Big)\sin\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big)\Big]e^{-\frac{x\varepsilon^{-1/3}}{2}}\Big\} \qquad (23)$$

From (20) U_0 and V_0 are found to be

$$U_0 = -ns(1+\alpha\sin\tau)\cos nsy\Big\{-x + r - \varepsilon^{1/3} +$$
$$+\varepsilon^{1/3}e^{(x-r)\varepsilon^{-1/3}} + \Big[(\varepsilon^{1/3} - r)\cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) +$$
$$+\Big(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\Big)\sin\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big)\Big]e^{-\frac{x\varepsilon^{-1/3}}{2}}\Big\} \qquad (24)$$

$$V_0 = (1+\alpha\sin\tau)\sin nsy\Big\{-1 + e^{(x-r)\varepsilon^{-1/3}} +$$
$$+\Big[\cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) +$$

[1] MUNK and CARRIER (1950) used this method for solving the steady problem in a triangular ocean. A detailed discussion of the solution to the problem defined by (21) and (22) is presented in Appendix 5 of VERONIS, MORGAN (1953).

$$+\left(\frac{2r\varepsilon^{-1/3}}{\sqrt{3}}-\sqrt{3}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\Bigg]\,e^{-\frac{x\varepsilon^{-1/3}}{2}}\Bigg\} \quad (25)$$

Solving for H_0, we have from (16) and (17) (neglecting terms of order ε),

$$\Theta H_0 = (1+\alpha\sin\tau)\,(\cos nsy + nsy\sin nsy)\cdot$$
$$\cdot\,(-x + r - \varepsilon^{1/3})$$
$$+\,(1+\alpha\sin\tau)\,nsy\sin nsy\Bigg\{\varepsilon^{1/3}e^{(x-r)\varepsilon^{-1/3}}$$
$$+\Bigg[(\varepsilon^{1/3}-r)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)+$$
$$+\left(\sqrt{3}\,\varepsilon^{1/3}-\frac{r}{\sqrt{3}}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\Bigg]\,e^{-\frac{x\varepsilon^{-1/3}}{2}}\Bigg\} \quad (26)$$

B. *The first-order equations.* These are

$$nsU_{0\tau} - nsy\,V_1 + \Theta H_{1x} = ns\varepsilon\,\Delta U_1 \quad (27)$$

$$nsV_{0\tau} + nsy\,U_1 + \Theta H_{1y} = ns\varepsilon\,\Delta V_1 \quad (28)$$

$$U_{1x} + V_{1y} = -H_{0\tau} \quad (29)$$

Eliminating H_1 from (27) and (28), we find

$$\varepsilon\,[V_{1xxx} + V_{1xyy} - U_{1xxy} - U_{1yyy}] - V_1 =$$
$$= [V_{0x} - U_{0y} - yH_0]_\tau \quad (30)$$

The boundary conditions are $U_1 = V_1 = 0$ on $x = 0$, $x = r$.

Approximate solutions of eq. (29) and (30) may again be found by a boundary layer analysis. The solutions are

$$V_1 = \frac{\alpha\cos\tau}{\Theta}(-x + r - \varepsilon^{1/3})\cdot$$
$$\cdot\,[(y^2ns + \Theta n^2s^2)\sin nsy + y\cos nsy] +$$
$$+\,\alpha\cos\tau\sin nsy\,\varepsilon^{-2/3}\Bigg\{\left(\left(\frac{1-r\varepsilon^{-1/3}}{3}x+\right.\right.$$
$$\left.\left.+\,C_2(y)\right)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)+\left(\frac{3-r\varepsilon^{-1/3}}{3\sqrt{3}}x+\right.\right.$$
$$\left.+\,C_3(y)\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\Bigg\}\,e^{-\frac{x\varepsilon^{-1/3}}{2}}+$$
$$+\,\alpha\cos\tau\sin nsy\,\varepsilon^{-1/3}\Bigg\{\frac{x-r}{3}\varepsilon^{-1/3}+$$
$$+\,A_1(y)\Bigg\}\,e^{(x-r)\varepsilon^{-1/3}} \quad (31)$$

$$U_1 = -\frac{\alpha\cos\tau}{\Theta}\,[2nsy\sin nsy + (y^2n^2s^2 + 2 +$$
$$+\,\Theta n^3s^3)\cos nsy]\left[-\frac{x^2}{2} + x(r-\varepsilon^{1/3})\right]+$$
$$+\,C_1(y) - \frac{\alpha\cos\tau}{\Theta}\,nsy\sin nsy\,\varepsilon^{2/3}e^{(x-r)\varepsilon^{-1/3}} -$$
$$-\,\alpha\cos\tau\,ns\cos nsy\Bigg[A_1(y) - \frac{1}{3} +$$
$$+\,\frac{(x-r)\varepsilon^{-1/3}}{3}\Bigg]\,e^{(x-r)\varepsilon^{-1/3}} -$$
$$-\,\alpha\cos\tau\sin nsy\,\frac{dA_1}{dy}\,e^{(x-r)\varepsilon^{-1/3}} -$$
$$-\,\frac{\alpha\cos\tau}{\Theta}\,nsy\sin nsy\,\cdot$$
$$\cdot\Bigg[(r\varepsilon^{1/3} - 2\varepsilon^{2/3})\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)-$$
$$-\,\frac{r\varepsilon^{1/3}}{\sqrt{3}}\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\Bigg]\,e^{-\frac{x\varepsilon^{-1/3}}{2}} -$$
$$-\,\alpha\cos\tau\,ns\cos nsy\varepsilon^{-2/3}\Bigg[\left(\left(-\frac{2x\varepsilon^{1/3}}{3} - \frac{\varepsilon^{2/3}}{3}+\right.\right.$$
$$\left.+\,\frac{rx}{3}\right)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)+$$
$$+\left(\frac{\varepsilon^{2/3}}{\sqrt{3}} - \frac{rx}{3\sqrt{3}} - \frac{2r\varepsilon^{1/3}}{3\sqrt{3}}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\Bigg]\,e^{-\frac{x\varepsilon^{-1/3}}{2}}+$$
$$+\,\alpha\cos\tau\,\frac{\varepsilon^{-1/3}}{2}\cdot$$
$$\cdot\,\frac{\partial}{\partial y}\Bigg\{(C_2 + \sqrt{3}\,C_3)\sin nsy\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)+$$
$$+\,(C_3 - \sqrt{3}\,C_2)\sin nsy\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\Bigg\}\,e^{-\frac{x\varepsilon^{-1/3}}{2}}$$
$$(32)$$

The arbitrary functions of y can be evaluated by means of the boundary conditions $U_1 = V_1 = 0$ on $x = 0, r$. We have

$$\sin nsy\, C_2 = \frac{\varepsilon - r\varepsilon^{2/3}}{\Theta}\left[(nsy^2 + \Theta n^2 s^2)\sin nsy + y\cos nsy\right] \quad (33)$$

$$\sin nsy\, A_1 = \frac{\varepsilon^{2/3}}{\Theta}\left[(nsy^2 + \Theta n^2 s^2)\sin nsy + y\cos nsy\right] \quad (34)$$

$$C_1 = \frac{\alpha\cos\tau}{\Theta}\cdot \left\{\left[2nsy\sin nsy + (y^2n^2s^2 + 2 + \Theta n^3 s^3)\cdot\cos nsy\right]\cdot \left[\frac{r^2}{2} - r\varepsilon^{1/3} + \varepsilon^{2/3}\right] - \varepsilon^{2/3}\left(\frac{\Theta ns}{3} + 1\right)\cos nsy\right\} \quad (35)$$

$$\sin nsy\, C_3 = \frac{1}{\sqrt{3}\,\Theta}\cdot \left\{\left[2\varepsilon^{1/3}\left(y^2 ns + \frac{2}{ns} + \Theta n^2 s^2\right)(r\varepsilon^{1/3} - r^2 - \varepsilon^{2/3}) + \frac{2\varepsilon}{3} - \frac{2\Theta\varepsilon^{1/3}}{3}\right]\sin nsy + \left(5y\cos nsy - \frac{4}{ns}\sin nsy\right)r\varepsilon^{2/3} - \left(9y\cos nsy - \frac{10}{ns}\sin nsy\right)\varepsilon^{2/3} + (r\varepsilon^{2/3} - \varepsilon)(y^2 ns + \Theta n^2 s^2)\sin nsy\right\} \quad (36)$$

The first-order contribution to H can be found from equations (27), (28). It is

$$H_1 = -\frac{ns\,\alpha\cos\tau}{\Theta^2}\cdot \left\{\left[(\Theta ns + y^2)\cos nsy + (y^3 ns + y\Theta n^2 s^2)\sin nsy\right]\cdot \left[\frac{1}{2}(x^2 + r^2) + (\varepsilon^{1/3} - r)(x + \varepsilon^{1/3})\right] + \frac{\Theta}{ns}\cos nsy + \varepsilon^{2/3}\left(\frac{\Theta ns}{3} + 1\right)\left(\frac{y\sin nsy}{ns} + \frac{\cos nsy}{n^2 s^2}\right)\right\} + \frac{\alpha\cos\tau}{\Theta}nsy\sin nsy\,\varepsilon^{-2/3}\cdot$$

$$\cdot\left\{\left[\frac{\varepsilon^{2/3}}{\sqrt{3}} - \frac{2r\varepsilon^{1/3}}{3\sqrt{3}} - \frac{rx}{3\sqrt{3}}\right]\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left[\left(\frac{r}{3} - \frac{2\varepsilon^{1/3}}{3}\right)x + \frac{\varepsilon^{2/3}}{3}\right]\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) - \frac{\varepsilon^{1/3}}{2}\cdot \left[(C_2 + \sqrt{3}\,C_3)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + (C_3 - \sqrt{3}\,C_2)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right]\right\}\cdot e^{-\frac{x\varepsilon^{-1/3}}{2}} + \frac{\alpha\cos\tau}{\Theta}nsy\sin nsy\cdot \left\{\frac{1}{3}(x - r - \varepsilon^{1/3})\varepsilon^{-1/3} + A_1\right\}e^{(x-r)\varepsilon^{-1/3}} \quad (37)$$

The terms U_1 and V_1 do not satisfy the boundary conditions $V_1 = \partial U_1/\partial y = 0$ on $y = 0.1$. We must recall that these boundary conditions were chosen rather arbitrarily as being plausible ones for the type of wind distribution specified, and the y dependence of the zero-order solution was accordingly chosen as $\sin nsy$. We cannot expect such a y dependence to satisfy the conditions for each set of equations. The fact that U_1 and V_1 do not satisfy these boundary conditions does not seem to be very serious since we do not really know what conditions are appropriate.

If we next consider the equations resulting from equating the coefficients of δ^2 in equations (13) to (15), we obtain

$$\varepsilon\left[V_{2xxx} + V_{2xyy} - U_{2xxy} - U_{2yyy}\right] - V_2 = (V_{1x} - U_{1y} - yH_1)_\tau\, U_{2x} + V_{2y} = -H_{1\tau}$$

In the first-order solution the unknown V_1 is of the same order of magnitude as the largest driving term, $V_{0x\tau}$. In the present case, $V_{1x\tau}$ is of order ε^{-1} in the boundary layer and hence V_2 can be expected to be of order ε^{-1} since the equations are completely analogous. By a similar argument, we can expect V_3 to be of order $\varepsilon^{-4/3}$, V_4 to be of order $\varepsilon^{-5/3}$, etc. If we therefore write out the series

$$V = V_0 + \delta V_1 + \delta^2 V_2 + \delta^3 V_3 + \delta^4 V_4 + \ldots$$

we have (in terms of orders of magnitude) near $x = 0$,

$$V = 0\,(\varepsilon^{-1/3}) + \delta\varepsilon^{-1/3}\,0\,(\varepsilon^{-1/3}) + \delta^2\varepsilon^{-2/3}\,0\,(\varepsilon^{-1/3}) + \\ + \delta^3\varepsilon^{-1}\,0\,(\varepsilon^{-1/3}) + \ldots$$

or factoring out the $0\,(\varepsilon^{-1/3})$, we have

$$V = 0\,(\varepsilon^{-1/3})\,[1 + \delta\varepsilon^{-1/3} + (\delta\varepsilon^{-1/3})^2 + \ldots]$$

The perturbation scheme may be expected to be valid provided $\delta\varepsilon^{-1/3} < 1$. We can expect a fairly good approximation from only the first two terms provided the more stringent condition $\delta\varepsilon^{-1/3} \ll 1$ is imposed. If $\delta\varepsilon^{-1/3} = 1/5$, the error involved in neglecting the third term is no larger than 5 % of the first term.

For yearly variation of the wind, $\delta\varepsilon^{-1/3} \approx 1/6$ Hence we shall keep only the first two terms of the series. It should be noted that α determines the magnitude of the effect of the perturbation but it has no bearing on the validity of the expansion.

Discussion of Results

In order to discuss the above solutions, we shall prescribe numerical values for the constants of the problem. Let

$r_1 = 6.5 \times 10^8$ cm	$\beta = 2 \times 10^{-13}$ cm^{-1} sec^{-1}
$s = 5 \times 10^8$ cm	$D = 5 \times 10^4$ cm
$A = 5 \times 10^7$ cm^2 sec^{-1}	$\omega = 2 \times 10^{-7}$ sec^{-1}
$n = 2\pi/s$	$W' = 0.65$ gm cm^{-1} sec^{-2}

The magnitudes of r_1, s, A, D correspond roughly to the Atlantic Ocean parameters. The value of β is chosen so as to give the best approximation to the Coriolis parameter in the latitude of Cape Hatteras. The equality $n = 2\pi/s$ corresponds roughly to the east–west components of the trades and the westerlies. The value of ω corresponds to yearly frequency of the wind variation, and $W' = 0.65$ gm cm^{-1} sec^{-2} is the value used by MUNK (1950) for the wind-stress.

Then the dimensionless constants have the values

$$\delta = \frac{\omega}{\beta s} = 2 \cdot 10^{-3} \qquad ns = 2\pi$$

$$\varepsilon = \frac{A}{\beta s^3} = 2 \cdot 10^{-6} \qquad \Theta = \frac{ngD}{\beta^2 s^3} = 0.123$$

$$r = 1.3$$

Also Γ' has been chosen so that

$$\alpha = 0.2$$

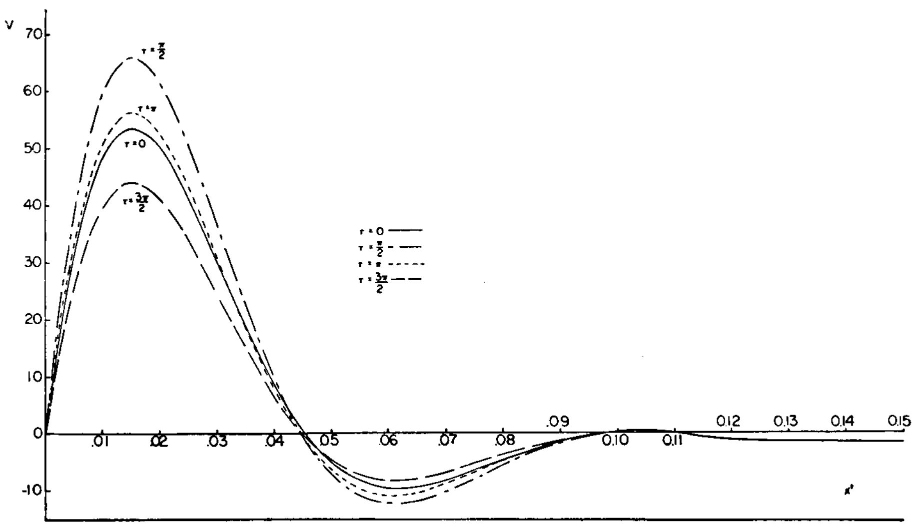

Fig. 2. The non-dimensional mass transport function V near $x' = 0$ (western boundary) showing Gulf Stream and counter current

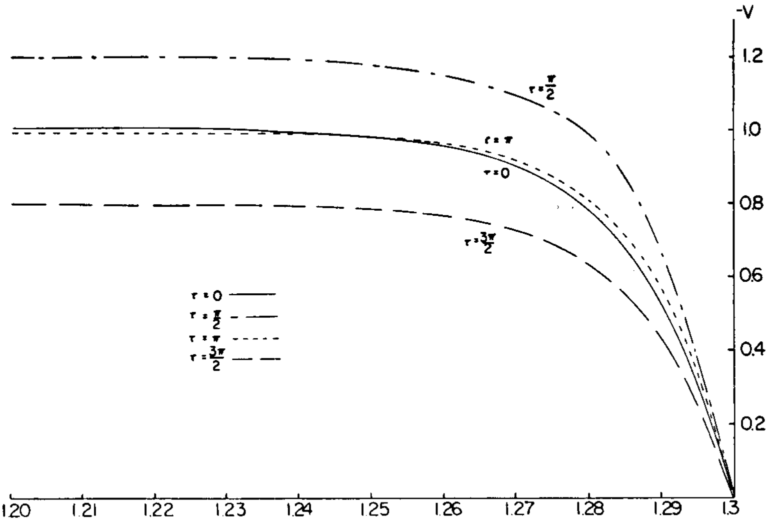

Fig. 3. Non-dimensional mass transport function V near $x' = r$ (eastern boundary)

The results for this numerical example are shown in Figs. 2—6.

In Fig. 2 the non-dimensional, south–north component, V, of the mass transport is plotted against x' near $x' = 0$ for the value $y' = 0.25$. The region of large V corresponds to the Gulf Stream and the section adjacent to the Gulf Stream, with negative V, corresponds to the off-shore counter-current.

Assuming that the complete solution is represented sufficiently accurately by the first two terms of the perturbation series, we see that the transport V is given by $V_0 + \delta V_1$ or $(1 + \alpha \sin \tau)\, Q - \delta\, (\alpha \cos \tau)\, L$ where Q and L are functions of the dimensionless coordinates x and y and of the parameters ε and Θ. Since the wind is given by $1 + \alpha \sin \tau$, this shows that there exists a phase lag, σ, between the wind and the transport such that $\tan \sigma = L\delta/Q$, which, since $\sigma \ll 1$, is approximately given by $\sigma = L\delta/Q$. In terms of dimensional quantities, if $\sigma = \omega t_0$, then, since $\delta = \omega/\beta s$, $t_0 = (L/Q) \cdot \cdot (1/\beta s)$ and t_0 gives the lag in the variation of the transport behind that of the wind. The lag is independent of frequency ω and the relative amplitude of the wind variation α. For the transport shown in Fig. 2, the lag is 3 days near the center of the Gulf Stream and $4^1/_2$ days near the center of the counter-current. The maximum transport is $Q(1 + \alpha) + \delta^2 \alpha L^2/2Q$ and it occurs at $\tau = \pi/2 + \sigma$. Within the accuracy of the present treatment we can approximate the maximum transport by $Q(1 + \alpha)$ since $\delta^2 \alpha L^2/2Q^2 = 0\,(\delta^2 \varepsilon^{-2/3})$, i.e., the correction is of second-order.

It is apparent from Fig. 2 that the out-of-phase effect is of relatively greater importance in the counter-current than in the main stream. The inertial effects introduced by the time dependency give a relatively larger out-of-phase correction and hence, as noted previously, a larger time lag in the counter-current. The graph shows the various effects only up to the eastern edge of the counter-current at $x' = 0.1$. For $x' > 0.1$ only the mean value of the transport is plotted since the deviations from this mean value are very small.

In Fig. 3 the transport V is shown near the eastern boundary of the ocean.

Figures 4, 5, and 6 show surface contours for the southern half of the rectangular ocean for $\tau = 0, \pi/2, \pi, 3\pi/2$. The contribution of δH_1 is very small throughout the ocean[1] and

[1] If for any of the variables the magnitude of the coefficient of δ in the perturbation solution is of the same order as that of the zero-order term, the coefficient $\delta = 0.002$ renders such a correction negligible. Throughout the present example, the only sizable contribution of the out-of-phase term is found in the north-south transport V in the boundary layer where the function V increases by order $\varepsilon^{-1/3}$. However, H_0 and H_1 have the same order of magnitude throughout the ocean so that the first-order correction H_1 can be neglected throughout.

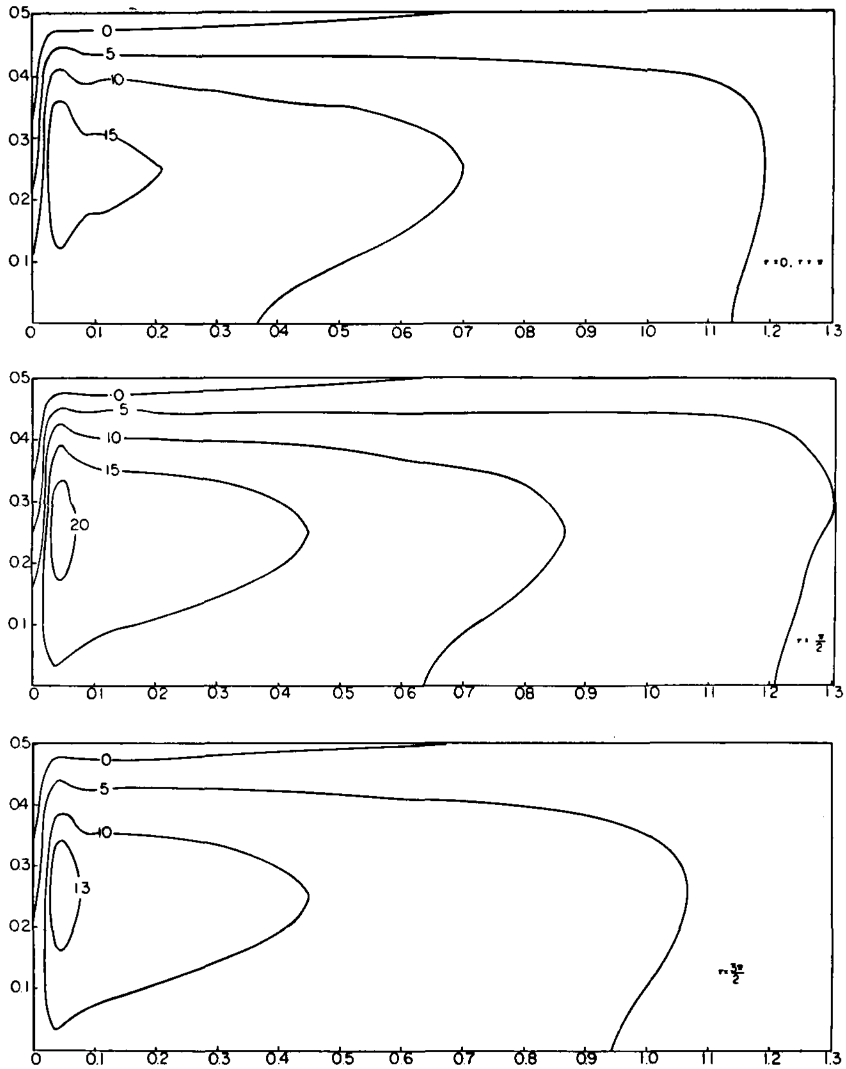

Fig. 4, 5 and 6 — Height in cms of free surface for southern half of rectangular ocean for different »times», $\tau = 0$, $\pi/2$, π, $3\pi/2$. The correction of the perturbation terms in negligible.

has therefore been neglected. Thus the graphs for $\tau = 0$ and $\tau = \pi$ coincide.

Our results are based on the assumption that the depth, D, is 500 meters. If we change the value of D, the first-order velocities are also changed since they are dependent on H_0 which is itself directly dependent on D. In particular, had we chosen, say, $D = 250$ meters rather than $D = 500$ meters, the value of H_0 would have been doubled and consequently the values of H_1, V_1, U_1 would have been increased. Their quantitative change must be found by numerical computation.

The above results appear to invalidate the solution of the problem as obtained by ICHIYE (1951). Ichiye neglected the contribution of the non-steady term in the integrated continuity equation. However, with the values of the parameters used in this section, the magnitude of this term in the interior of the ocean is as much as ten times that of the remaining non-steady terms which were retained in Ichiye's analysis. Even though we have neglected the contribution of these terms in the interior of the ocean, they are important for the purposes of matching with the boundary layer solution.

The steady mass transport of the Gulf Stream derived here for $\alpha = 0$ is $26.6 \cdot 10^6$ metric tons per second as compared to MUNK's value (1950) of $36 \cdot 10^6$ and the observed value of $72—80 \cdot \cdot 10^6$ metric tons per second. MUNK (1950) used the east—west component of an empirical wind system and the discrepancy is therefore due to the difference between the two wind systems. The maximum (minimum) transport is $1 + \alpha$ $(1 - \alpha)$ times the steady transport. In the counter-current the steady mass transport is $4.61 \cdot 10^6$ metric tons per second or approximately 17 % of the mass transport of the Gulf Stream. The relative value of 17 % agrees with observation, though the absolute value $4.61 \cdot 10^6$ is about one-third of the observed counter-current transport.

The difference between the computed and the observed values is not surprising when one considers the many idealizing assumptions made. Such features as the straight coast lines, the simplified theory of turbulence used, the neglect of the non-linear terms, and the idealized stress-effect of the wind on the water could easily combine to yield quantitative results which differ from the observed values by a factor of two or three.

The problem as stated and solved by the above method gives no sensible east—west variation in the position of the Gulf Stream, but a careful investigation of the eastern boundary of the Gulf Stream shows a ver small narrowing of the stream. How well su a result agrees with field evidence is uncer since our solution yields no inshore cou -current.

It would be interesting to ascertain how well our predicted results agree with observation; specifically, if the mass transport of the Gulf Stream responds as indicated to variations in the wind and if the lag of the transport is independent of the frequency.

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Bibliography

EKMAN, V. W., 1905: On the influence of the earth's rotation on ocean currents. *Arkiv för Matematik, Astronomi och Fysik*, **2**.

HIDAKA, K., 1950 a: Drift currents in an enclosed ocean. Part I. *Geophysical Notes (Geophysical Inst. Tokyo Univ.)*, **3**, No. 23.

— 1950 b: Drift currents in an enclosed ocean. Part II. *Geophysical Notes (Geophysical Inst. Tokyo Univ.)*, **3**, No. 38.

— 1951: Drift currents in an enclosed ocean. Part III. *Geophysical Notes (Geophysical Inst. Tokyo Univ.)*, **4**, No. 3.

HIDAKA, K., and MIYOSHI, H., 1949: On the neglect of the inertia terms in dynamic oceanography. *Geophysical Notes (Geophysical Inst. Tokyo Univ.)*, **2**, No. 22.

ICHIYE, T., 1951: On the variation of oceanic circulation, *The Oceanographic Magazine*, **3**, No. 2.

LAMB, H., 1932: *Hydrodynamics*, Dover, 6th edition, pp. 6—7.

MUNK, W. H., 1950: On the wind-driven ocean circulation. *J. Meteor.*, **7**, No. 2.

MUNK, W. H., and CARRIER, G. F., 1950: The wind-driven ocean circulation in ocean basins of various shapes. *Tellus*, **2**.

MUNK, W. H., GROVES, G. W., and CARRIER, G. F., 1950: Note on the dynamics of the Gulf Stream. *J. Marine Res.*, **9**, No. 3.

REID, R. O., 1948: The equatorial currents of the Eastern Pacific as maintained by the stress of the wind. *J. Marine Res.*, **7**, No. 2.

STOMMEL, H., 1948: The westward intensification of wind-driven ocean currents. *Trans. Amer. Geo. Union*, **29**, No. 2.

SVERDRUP, H. U., 1947: Wind-driven currents in a baroclinic ocean; with application to the equatorial currents of the Eastern Pacific. *Proc. Nat. Acad. Sci.*, **33**, No. 11.

VERONIS, G., and MORGAN, G. W., 1953: A study of the time-dependent wind-driven ocean circulation, Office of Naval Research, Report No. A11—101 under Contract N7onr-35801 (Brown University).