Restriction of the Underflow in a Transition

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Abstract

Bernoulli's Theorem is applied to the problem of the control of salinity in an estuary by a transition.

STOMMEL and FARMER (1953) have developed a theoretical reason to explain why the mouth of a vertically stratified estuary should act as a check on the amount of salt water available for mixing in the estuary. The criterion for "overmixing" was obtained from the equation for a stationary interfacial wave, the equation of continuity, and the equation for the conservation of salt. However, the physical reason why the inflow of salt water cannot rise beyond a certain limit, in spite of a very thorough mixing, is, that an increased inflow of salt water will cause a decrease of the density difference between the ocean and the estuary and, consequently, a decrease of the forces driving the double flow. We shall therefore study the double flow through a transition by applying Bernoulli's Theorem.

We can consider an estuary connecting a river to the ocean through a narrow transition. We assume the mixing to be complete in the estuary as well as in the ocean, at least down to the level occupied by the bottom of the transition, which implies that neither the estuary nor the ocean are vertically stratified above the level referred to. Density and salinity in the estuary are denoted ϱ_1 and s_1, in the ocean ϱ_2 and s_2. The depths of the upper and lower layers in the transition are respectively y and $D - y$, D being the depth of the transition. The width of the transition, $B = B(z)$, is a function of the depth below the surface. We consider no other forces in the transition than the horizontal pressure gradients due to the higher level of the free surface in the estuary and the density difference between the ocean and the estuary. When ϱ_1 and ϱ_2 occur as factors, we make them equal to one.

In the transition the current velocity as well as the pressure gradient are zero in the interface. At the level z we have a pressure difference $g\,(\varrho_2 - \varrho_1)\,(y - z)$ between both ends of the transition. The flows, Q_1 and Q_2, in the upper and lower layers are

$$Q_1 = \int_0^y \sqrt{2\,g\,(\varrho_2 - \varrho_1)}\, B(z)\,(y - z)^{1/2}\, dz \qquad (1)$$

$$Q_2 = \int_y^D \sqrt{2\,g\,(\varrho_2 - \varrho_1)}\, B(z)\,(z - y)^{1/2}\, dz \qquad (2)$$

If the discharge, Q_0, of the river exceeds a certain value, Q, there is no double flow in the transition. We find Q by making $y = D$ in (1).

$$Q=\int_0^D \sqrt{2\,g\,(\varrho_2-\varrho_0)}\,B(z)\,(D-z)^{1/2}\,dz \qquad (3)$$

where ϱ_0 denotes the density of the fresh water. We have, further

$$Q_1-Q_2=Q_0 \qquad (4)$$

$$Q_1 s_1-Q_2 s_2=0 \qquad (5)$$

Putting $\varrho_2-\varrho_1=a\,(s_2-s_1)$ and $\varrho_2-\varrho_0= = as_2$ we find $\varrho_2-\varrho_1=as_2\,Q_0:Q_1$ and

$$Q=\int_0^D \sqrt{2\,gas_2}\,B(z)\,(D-z)^{1/2}\,dz=A \qquad (6)$$

$$Q_1^{3/2}\,Q_0^{-1/2}=\int_0^y \sqrt{2\,gas_2}\,B(z)\,(y-z)^{1/2}\,dz=A_1 \qquad (7)$$

$$Q_1^{1/2}\,Q_0^{-1/2}\,(Q_1-Q_0)= =\int_y^D \sqrt{2\,gas_2}\,B(z)\,(z-y)^{1/2}\,dz=A_2 \qquad (8)$$

Taking the quotients of (7) and (6), and (8) and (6), and denoting by a_1, a_2, q_0, and q_1 the quotients $A_1:A$, $A_2:A$, $Q_0:Q$, and $Q_1:Q$ respectively, we have

$$q_1^{3/2}\,q_0^{-1/2}=a_1 \qquad (9)$$

$$q_1^{1/2}\,q_0^{-1/2}\,(q_1-q_0)=a_2 \qquad (10)$$

The quantities a_1 and a_2 are determined by the geometric dimensions of the transition, and consequently the inflow of salt water, $q_2=q_1-q_0$, is influenced by the discharge of the river and the geometrical dimensions of the transition but is not dependent on the density of the sea water, when the critical discharge Q is used as a unit. The equations (9) and (10) enable us to determine q_0, q_1, and q_2.

$$q_0=\frac{a_1-a_2}{a_1}\sqrt{a_1\,(a_1-a_2)} \qquad (11)$$

$$q_1=\sqrt{a_1\,(a_1-a_2)} \qquad (12)$$

$$q_2=\frac{a_2}{a_1}\sqrt{a_1\,(a_1-a_2)} \qquad (13)$$

Applying this procedure to *a transition with a rectangular section*, and denoting $y:D=\eta$, we have

$$a_1=\eta^{3/2}, \quad a_2=(1-\eta)^{3/2} \qquad (14)$$

Elimination of η, using (9) and (10), leads to the equation

$$q_1+q_1^{1/3}\,(q_1-q_0)^{2/3}-q_0^{1/3}=0 \qquad (15)$$

The equation is valid for values of q_0 between 0 and 1. Though there is nothing to prevent q_0 from rising above 1, the present problem would not exist in that case, as there is no double flow if $q_0 \geq 1$. For a numerical computation it is convenient to use (11), (12), and (13); q_1 and q_2 are given in Fig. 1 as functions of q_0. When q_0 is small we have

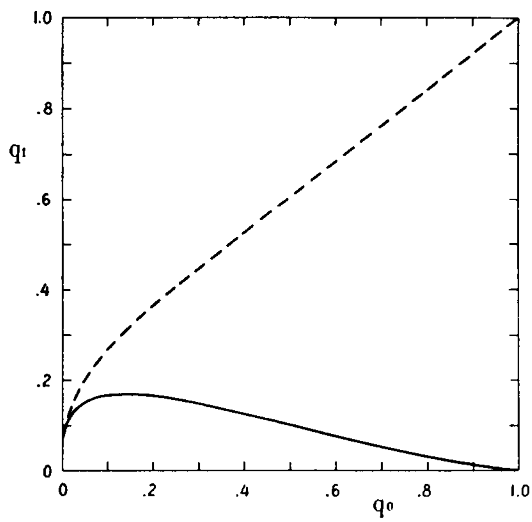

Fig. 1. Upper and lower flow (dotted line resp. full-drawn line) in a transition with a rectangular section, as functions of the discharge of the river. The discharge that is just sufficient to prevent a double flow in the transition is used as a unit.

approximately $a_1=a_2$, or $\eta=1/2$, whence, according to (9), $q_1=1/2\;q_0^{1/3}$. However, as q_0 rises, the relationship very soon becomes practically linear. The flow in the lower layer also rises very quickly with q_0, when q_0 is small. At $q_0=0.142$, q_2 has a maximum $=0.171$, and then q_2 is coming down to zero as q_0 increases to 1. The relationship between η and q_0 is approximately linear, as shown by Fig. 2. With $s_1:s_2=\nu$, then $\nu=q_2:q_1= =a_2:a_1$.

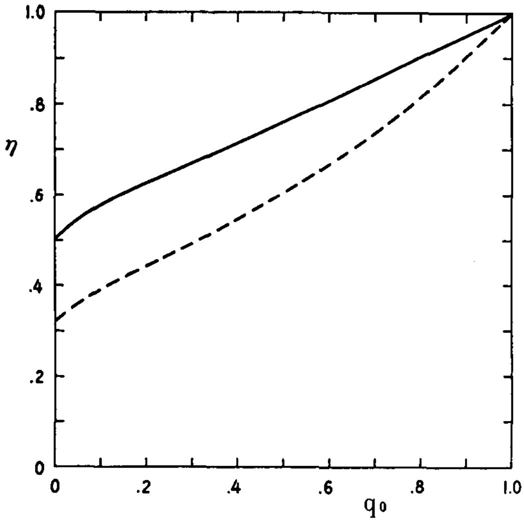

Fig. 2. Ratio of the depth of the upper flow and the total depth of the transition (η) as a function of the discharge of the river. Full-drawn line refers to transition with a rectangular section, dotted line to transition with a triangular section.

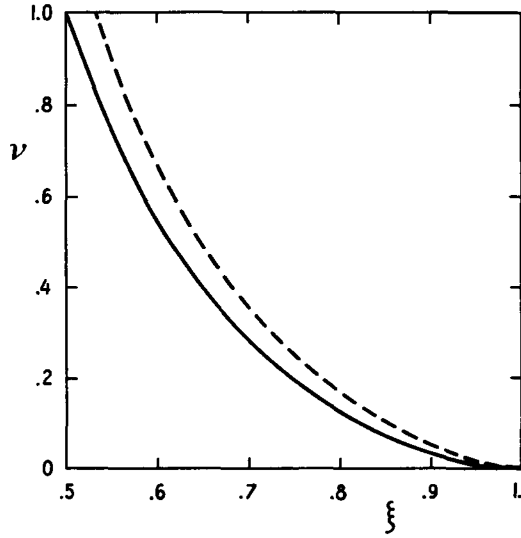

Fig. 3. The ratio (ν) of salinities of the upper and lower layers as a function of the ratio (ξ) of the area occupied by the upper flow and the total area of the transition. Full-drawn line refers to transition with a rectangular section, dotted line to transition with a triangular section.

$$\nu = \left(\frac{1-\eta}{\eta}\right)^{3/2} \tag{16}$$

Applying the same procedure to *a transition with a triangular section*, we have

$$a_1 = \frac{1}{3}(5-2\eta)\,\eta^{3/2} \tag{17}$$

$$a_2 = \frac{2}{3}(1-\eta)^{5/2} \tag{18}$$

It follows that the flow is independent of the shape of the triangle, if the critical discharge Q is used as a unit. The maximal inflow of salt water is smaller than in the case of a rectangular transition, and corresponds to a smaller discharge of the river. By making $a_1 = a_2$ we find that $q_0 = 0$ requires $\eta = 0.316$, which implies that the sectional area occupied by the upper layer is but slightly more than half the sectional area of the transition. Once again the relationship between η and q_0 is approximately linear (Fig. 2). When q_0 is small we have approximately $q_1 = 0.316\, q_0^{1/3}$. The quotient of the salinity in the estuary and that in the ocean is

$$\nu = \frac{2(1-\eta)^{5/2}}{(5-2\eta)\,\eta^{3/2}} \tag{19}$$

In Fig. 3 ν is plotted as a function of the quotient of the sectional area occupied by the upper flow and the total sectional area of the transition. It appears that there is no great difference between the two cases of a rectangular section and a triangular one. It follows that interjacent cases, such as transitions with a trapezium for a section, do not differ appreciably from the two cases treated above.

Any transition with an irregular section can be treated according to (11), (12), and (13).

We have found, among other things, that if the mixing is complete, so as to eliminate vertical stratification, the quotient of the inflow of salt water and the discharge of the river tends to become infinite when the discharge is coming down to zero. Though the case of a less thorough mixing is not treated in this paper, it should be pointed out that the same quotient preserves a finite value, when the discharge is coming down to zero, if the mixing is not complete.

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