Available Potential Energy and the Maintenance of the General Circulation

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Abstract

The available potential energy of the atmosphere may be defined as the difference between the total potential energy and the minimum total potential energy which could result from any adiabatic redistribution of mass. It vanishes if the density stratification is horizontal and statically stable everywhere, and is positive otherwise. It is measured approximately by a weighted vertical average of the horizontal variance of temperature. In magnitude it is generally about ten times the total kinetic energy, but less than one per cent of the total potential energy.

Under adiabatic flow the sum of the available potential energy and the kinetic energy is conserved, but large increases in available potential energy are usually accompanied by increases in kinetic energy, and therefore involve nonadiabatic effects.

Available potential energy may be partitioned into zonal and eddy energy by an analysis of variance of the temperature field. The zonal form may be converted into the eddy form by an eddy-transport of sensible heat toward colder latitudes, while each form may be converted into the corresponding form of kinetic energy. The general circulation is characterized by a conversion of zonal available potential energy, which is generated by low-latitude heating and high-latitude cooling, to eddy available potential energy, to eddy kinetic energy, to zonal kinetic energy.

1. The concept of available potential energy

The strengths of the cyclones, anticyclones, and other systems which form the weather pattern are often measured in terms of the kinetic energy which they possess. Intensifying and weakening systems are then regarded as those which are gaining or losing kinetic energy. When such gains or losses occur, the source or sink of kinetic energy is a matter of importance.

Under adiabatic motion, the total energy of the whole atmosphere would remain constant. The only sources or sinks for the kinetic energy of the whole atmosphere would then be potential energy and internal energy.

In general the motion of the atmosphere is not adiabatic. The only nonadiabatic process which directly alters kinetic energy is friction, which ordinarily generates internal energy while it destroys kinetic energy, but which may also, under suitable circumstances, change atmospheric kinetic energy into the kinetic and potential energy of ocean currents and ocean waves. The remaining nonadiabatic processes, including the release of latent energy, alter only the internal energy directly. Hence the only sources for the kinetic energy of the whole atmosphere are atmospheric potential energy and internal energy, while the environment may also act as a sink.

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It is easily shown (cf. HAURWITZ 1941) that the potential and internal energies within a column extending to the top of the atmosphere bear a constant ratio to each other, to the extent that hydrostatic equilibrium prevails. Hence, net gains of kinetic energy occur in general at the expense of both potential and internal energy, in this same ratio. It is therefore convenient to treat potential and internal energy as if they were a single form of energy. The sum of the potential and internal energy has been called *total potential energy* by Margules (1903).

Evidently the total potential energy is not a good measure of the amount of energy available for conversion into kinetic energy under adiabatic flow. Some simple cases will serve to illustrate this point. Consider first an atmosphere whose density stratification is everywhere horizontal. In this case, although total potential energy is plentiful, none at all is available for conversion into kinetic energy. Next suppose that a horizontally stratified atmosphere becomes heated in a restricted region. This heating adds total potential energy to the system, and also disturbs the stratification, thus creating horizontal pressure forces which may convert total potential energy into kinetic energy. But next suppose that a horizontally stratified atmosphere becomes cooled rather than heated. The cooling removes total potential energy from the system, but it still disturbs the stratification, thus creating horizontal pressure forces which may convert total potential energy into kinetic energy. Evidently removal of energy is sometimes as effective as addition of energy in making more energy available.

We therefore desire a quantity which measures the energy available for conversion into kinetic energy under adiabatic flow. A quantity of this sort was discussed by MARGULES (1903) in his famous paper concerning the energy of storms. Margules considered a closed system possessing a certain distribution of mass. Under adiabatic flow, the mass may become redistributed, with an accompanying change in total potential energy, and an equal and opposite change in kinetic energy. It the stratification becomes horizontal and statically stable, the total potential energy reaches its minimum possible value, and the gain of kinetic energy thus reaches its maximum. This maximum gain of kinetic energy equals the maximum amount of total potential energy available for conversion into kinetic energy under any adiabatic redistribution of mass, and as such may be called the *available potential energy*.[1]

Available potential energy in this sense can be defined only for a fixed mass of atmosphere which becomes redistributed within a fixed region. The storms with which Margules was primarily concerned do not consist of fixed masses within fixed regions, nor do any other systems having the approximate size of storms. It is perhaps for this reason that available potential energy has not become a more familiar quantity.

It is in considering the general circulation that we deal with a fixed mass within a fixed region—the whole atmosphere. It is thus possible to define the available potential energy of the whole atmosphere as the difference between the total potential energy of the whole atmosphere and the total potential energy which would exist if the mass were redistributed under conservation of potential temperature to yield a horizontal stable stratification.

The available potential energy so defined possesses these important properties:

(1) The sum of the available potential energy and the kinetic energy is conserved under adiabatic flow.

(2) The available potential energy is completely determined by the distribution of mass.

(3) The available potential energy is zero if the stratification is horizontal and statically stable.

It seems fairly obvious that the available potential energy so defined is the only quantity possessing these properties, although a rigorous proof would be somewhat involved. Moreover, it possesses the further property:

(4) The available potential energy is positive if the stratification is not both horizontal and statically stable.

It follows from property (1) that available potential energy is the only source for kinetic

[1] This quantity is called *available kinetic energy* by Margules, since it represents an amount of kinetic energy attainable. From the point of view of this discussion the term *available potential energy* is preferable, since it represents a part of the existing total potential energy.

energy. On the other hand, it is not the only sink. When friction destroys kinetic energy it creates internal energy, but in doing so it increases the minimum total potential energy as well as the existing total potential energy. Thus the loss of kinetic energy exceeds the gain of available potential energy.

There is no assurance in any individual case that all the available potential energy will be converted into kinetic energy. For example, if the flow is purely zonal, and the mass and momentum distributions are in dynamically stable equilibrium, no kinetic energy at all can be realized. It might seem desirable to redefine available potential energy, so that, in particular, it will be zero in the above example. But the available potential energy so defined would depend upon both the mass and momentum distributions. If it is desired to define available potential energy as a quantity determined by the mass distribution, the definition already introduced must be retained.

2. Analytic expressions and approximations

- Θ : potential temperature
- p : pressure
- T : temperature
- z : elevation
- $\mathbf{v}$: horizontal velocity vector
- g : acceleration of gravity
- c_v, c_p : specific heats of air at constant volume and constant pressure
- R : gas constant for air, equal to $c_p - c_v$
- λ : ratio of specific heats, c_p/c_v, approximately $^7/_5$
- $\varkappa$: ratio R/c_p, equal to λ-1/λ, approximately $^2/_7$

Under adiabatic flow, the statistical distribution of Θ is conserved. More precisely, if $g(\Theta_1)d\Theta$ is the probability that a unit mass of atmosphere selected at random has a value of Θ between Θ_1 and $\Theta_1 + d\Theta$, the probability function $g(\Theta)$ is conserved. If

$$\bar{p}(\Theta_1) = \bar{p}_0 \int_{\Theta_1}^{\infty} g(\Theta)\, d\Theta \tag{1}$$

where $\bar{p}_0$ is the average value of the pressure p_0 at the earth's surface, regarded as horizontal, $\bar{p}(\Theta)$ is also conserved under adiabatic flow. If the particles for which $\Theta = \Theta_1$ form a continuous surface which intersects every vertical column exactly once, $\bar{p}(\Theta_1)$ is the average pressure on the isentropic surface $\Theta = \Theta_1$, with respect to the area of the horizontal projection of this surface. Equation (1) also defines the average pressure over isentropic surfaces which intersect the ground, or which lie entirely "underground", if along each vertical we define $p(\Theta) = p_0$ if $\Theta < \Theta_0$, where Θ_0 is the value of Θ at the earth's surface.

To express the minimum total potential energy in terms of the invariant $\bar{p}$ (Θ) it is sufficient to express the total potential energy in terms of $p(\Theta)$. The potential and internal energies per unit mass are gz and c_vT, respectively. Since, as mentioned previously, the potential and internal energies P and I of a vertical column above a unit area bear the ratio $P/I = (c_p - c_v)/c_v$, and since an element of mass per unit area is $g^{-1}\, dp$,

$$P + I = c_p g^{-1} \int_{0}^{p_0} T dp \tag{2}$$

Upon substituting $T = \Theta p^{\varkappa} p_{00}^{-\varkappa}$, where $p_{00} = 1{,}000$ mb, and integrating by parts, we find that

$$P + I = (1 + \varkappa)^{-1} c_p g^{-1} p_{00}^{-\varkappa} \int_{0}^{\infty} p^{1+\varkappa} d\Theta \tag{3}$$

The minimum total potential energy which can result from adiabatic rearrangement occurs when $p = \bar{p}$ everywhere, and is obtained by setting $p = \bar{p}$ in (3). Thus the average available potential energy per unit area of the earth's surface is

$$\overline{A} = (1 + \varkappa)^{-1} c_p g^{-1} p_{00}^{-\varkappa} \int_{0}^{\infty} (\overline{p^{1+\varkappa}} - \bar{p}^{1+\varkappa}) d\Theta \tag{4}$$

where the bar over $p^{1+\varkappa}$ again denotes an average over an isentropic surface.

Since p is always positive and $1 + \varkappa > 1$, it is readily shown that $\overline{p^{1+\varkappa}} - \bar{p}^{1+\varkappa} > 0$ unless $p \equiv \bar{p}$. The precise magnitude of $\overline{A}$, particularly as compared to the average total potential energy per unit area, $\overline{P} + \overline{I}$, is less obvious. Expansion in a power series will aid the comparison. Thus, if $p = \bar{p} + p'$, it follows from the binomial theorem, applied to $p^{1+\varkappa}$, that

$$\overline{A} = (1+\varkappa)^{-1} c_p g^{-1} p_{00}^{-\varkappa} \int_0^\infty \overline{p}^{1+\varkappa} \cdot \left[\frac{\varkappa(1+\varkappa)}{2!}\overline{\left(\frac{p'}{\overline{p}}\right)^2} - \frac{(1-\varkappa)\varkappa(1+\varkappa)}{3!}\overline{\left(\frac{p'}{\overline{p}}\right)^3} + \ldots\right] d\Theta \qquad (5)$$

The ratio of $\overline{A}$ to $\overline{P}+\overline{I}$ is a suitable mean value of the quantity enclosed in square brackets.

The power series (5) must converge if $p' < \overline{p}$ everywere, but the rapidity with which it converges depends upon typical values of $p'/\overline{p}$. The distribution of p' is in general far from normal, since tropospheric isentropic surfaces tend to be nearly horizontal in the tropics, so that p' is close to its maximum value over about half the area of the earth. Suppose that on a particular isentropic surface p = 1,000 mb over half the area, and p decreases linearly from 1,000 mb to 300 mb over the remaining half. In this case $\overline{p}$ = = 825 mb, $\overline{p'^2}/\overline{p}^2 = 0.075$, and $\overline{p'^3}/\overline{p}^3 =$ — 0.019. The ratio of the second to the first term in the power series is therefore—0.06, so that even in this rather extreme case, the power series is well represented by its leading term.

Therefore, approximately

$$\overline{A} = \frac{1}{2}\varkappa c_p g^{-1} p_{00}^{\varkappa} \int_0^\infty \overline{p}^{1+\varkappa} \overline{\left(\frac{p'}{\overline{p}}\right)^2} d\Theta \qquad (6)$$

and $\overline{A}$ depends upon the *variance* of pressure over the isentropic surfaces.

This variance is closely related to the variance of potential temperature on an isobaric surface, which in turn resembles the variance of temperature on an isobaric or horizontal surface. If $\overline{\Theta}$ and $\overline{T}$ are the average values of Θ and T on a isobaric surface, and Θ' and T' are the departures of Θ and T from $\overline{\Theta}$ and $\overline{T}$, the function $\overline{\Theta}(p)$ is not completely determined by the function $\overline{p}(\Theta)$, but approximately $p = \overline{p}\left(\overline{\Theta}(p)\right)$, so that

$$p' = \overline{p}(\Theta - \Theta') - \overline{p}(\Theta) \sim -\Theta' \partial\overline{p}/\partial\Theta \qquad (7)$$

Thus

$$\overline{A} = \frac{1}{2}\varkappa c_p g^{-1} p_{00}^{-\varkappa} \int_0^{p_0} \overline{\Theta}^2 \overline{p}^{-(1-\varkappa)} \left(-\frac{\partial\overline{\Theta}}{\partial p}\right)^{-1} \cdot \overline{\left(\frac{\Theta'}{\overline{\Theta}}\right)^2} dp \qquad (8)$$

From the hydrostatic equation, it follows that

$$\partial\Theta/\partial p = -\varkappa\Theta p^{-1}(\Gamma_d - \Gamma)\Gamma_d^{-1} \qquad (9)$$

where $\Gamma = -\partial T/\partial z$ is the lapse rate of temperature and $\Gamma_d = g c_p^{-1}$ is the dry-adiabatic lapse rate. Since $\Theta'/\Theta = T'/T$,

$$\overline{A} = \frac{1}{2}\int_0^\infty \overline{T}(\Gamma_d - \overline{\Gamma})^{-1} \overline{\left(\frac{T'}{\overline{T}}\right)^2} dp \qquad (10)$$

Expression (10) is suitable for estimating the ratio $A/(P+I)$. This ratio evidently equals suitable average value of $\frac{1}{2}\Gamma_d(\Gamma_d - \Gamma)^{-1}\overline{T'^2}\overline{T}^{-2}$. The maximum values of $(\Gamma_d - \overline{\Gamma})^{-1}$ and probably also of $\overline{T'^2}$ occur in the troposphere. If $\overline{\Gamma} = \frac{2}{3}\Gamma_d$ and $\overline{T'^2} = (15°)^2$ are taken as typical values,

$$\overline{A}/(\overline{P}+\overline{I}) \sim 1/200$$

Hence less than one per cent of the total potential energy is generally available for conversion into kinetic energy.

3. Available potential energy and kinetic energy

It is a familiar observation that the total potential energy of the atmosphere greatly exceeds the kinetic energy. In considering the possible release of kinetic energy, however, we should compare the kinetic energy with the available potential energy.

The average kinetic energy per unit area of the earth's surface is approximately

$$\overline{K} = \frac{1}{2}g^{-1}\int_0^{\overline{p}_0} \overline{V^2} dp \qquad (11)$$

In this expression we have neglected horizontal variations of p_0. From (2) it follows that

$$\overline{P} + \overline{I} = \frac{1}{\lambda - 1} g^{-1} \int_0^{\overline{p_0}} \overline{c^2}\, dp \qquad (12)$$

where $c^2 = \lambda RT$ is the square of the speed of sound. If we assume that a typical average wind speed is $^1/_{20}$ of the speed of sound, which lies between 300 m sec^{-1} and 350 m sec^{-1}, we find that

$$\overline{K}/(\overline{P} + \overline{I}) \sim 1/2000$$

If we also assume that our result

$$\overline{A}/(\overline{P} + \overline{I}) \sim 1/200$$

is typical, we find that

$$\overline{K}/\overline{A} \sim 1/10$$

Evidently, if kinetic energy is not released, it is not because a supply of available potential energy is lacking.

Let us see next how $\overline{K}$ and $\overline{A}$ vary. From the equation of continuity

$$\nabla \cdot \mathbf{v} + \partial\omega/\partial p = 0$$

where $\omega = \dot{p} = dp/dt$ is the individual pressure change, determined in the free atmosphere primarily by the vertical speed, and the thermodynamic equation

$$\partial\Theta/\partial t + \mathbf{v} \cdot \Delta\, \Theta + \omega\partial\Theta/\partial p = c_p^{-1}\Theta T^{-1} Q \quad (14)$$

where Q is the rate of addition of heat, per unit mass, we find that

$$\frac{1}{2}\, \partial\overline{\Theta'^2}/\partial t = -\, \overline{\Theta\omega}\, \partial\overline{\Theta}/\partial p - \frac{1}{2}\, \partial\overline{\Theta'^2\omega}/\partial p +$$

$$+\, C_p^{-1}\overline{\Theta}\,\overline{T}^{-1}\overline{\Theta' Q'} \qquad (15)$$

The second term on the right of (15) involves the space average of the product of three quantities, each of which is itself a departure from a space average. Such "triple correlations" are often negligible. In this case the term arises because another triple correlation, namely the term involving $\overline{p'^3}$ in (5), has been omitted in deriving expressions (6), (8),and (10) for $\overline{A}$ from (5). If we neglect the term involving $\overline{\Theta'^2\omega}$, we find, since $\Theta'/\overline{\Theta} = T'/\overline{T}$, that

$$\partial\overline{A}/\partial t = -C + G \qquad (16)$$

where

$$C = -Rg^{-1}\int_0^{\overline{p_0}} p^{-1}\,\overline{T\omega}\, dp = -\int_0^{\overline{p_0}} \overline{V \cdot \nabla z}\, dp \quad (17)$$

and

$$G = g^{-1}\int_0^{\overline{p_0}} \Gamma_d(\Gamma_d - \overline{\Gamma})^{-1}\overline{T}^{-1}\overline{T'Q'}\, dp \quad (18)$$

The latter integral in (17) is obtained from the former through the hydrostatic equation and the equation of continuity.

Likewise, from the equation of continuity and the equations of horizontal motion

$$\partial\mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} + \omega\,\partial\mathbf{v}/\partial p =$$

$$= 2\,\Omega \cdot \mathbf{v} - g\,\nabla z + \mathbf{F} \qquad (19)$$

where Ω is the vector angular velocity of the earth, and F is the horizontal force of friction, per unit mass, we find that

$$\partial\overline{K}/\partial t = C - D \qquad (20)$$

where

$$D = -g^{-1}\int_0^{\overline{p_0}} \overline{\mathbf{v} \cdot \mathbf{F}}\, dp \qquad (21)$$

Under adiabatic frictionless flow the *generation* G and the *dissipation* D vanish, so that the sum of $\overline{A}$ and $\overline{K}$ is conserved.

We have seen that the available potential energy depends upon the departure of the density stratification from horizontal. If the wind were exactly geostrophic everywhere, the kinetic energy of the whole atmosphere would be zero if and only if the available potential energy were zero. Since the actual wind tends to be nearly geostrophic throughout much of the atmosphere, it still follows that the kinetic energy is generally small or large according to whether the available potential energy is small or large. Large increases in available potential energy and kinetic energy should in general accompany each other.

We have seen, however, that under adiabatic flow increases in available potential energy and decreases in kinetic energy must accompany each other. It follows that when both forms of energy increase together, non-adiabatic effects are involved. Likewise, since

$\overline{K}$ is usually about one tenth of $\overline{A}$, any increase in $\overline{A}$ by more than about ten percent must involve nonadiabatic processes. Such processes are certainly responsible for the large increase in $\overline{A}$ from summer to winter, which is accompanied by a proportionately large increase in $\overline{K}$. This situation, which involves a decrease in $\overline{P}+\overline{I}$, has been discussed by SPAR (1949).

We may ask at this point whether any appreciable changes in the ratio $\overline{K}/\overline{A}$ are possible under adiabatic flow, if the winds remain nearly geostrophic. That such changes are possible may be seen from the approximate expressions

$$\overline{A} = \frac{1}{2} \overset{\kappa^{-1}}{\cancel{K^{-1}}} g R^{-1} \int_0^{\overline{p_0}} \Gamma_d (\Gamma_d - \overline{\Gamma})^{-1} \overline{T}^{-1} \cdot$$

$$\cdot\, p^2 \overline{(\partial z'/\partial p)^2}\, dp \qquad (22)$$

$$\overline{K} \sim \frac{1}{2} g \int_0^{\overline{p_0}} \overline{f^{-2} (\nabla z)^2}\, dp \qquad (23)\,.$$

obtained from (10) and (11) by substituting the hydrostatic and geostrophic relations, f being the Coriolis parameter. Both $\overline{A}$ and $\overline{K}$ depend upon the distribution of z' throughout the atmosphere, but, for a given variance of z', $\overline{K}$ is larger the more the fluctuations of z' in the horizontal, while $\overline{A}$ is larger the more the fluctuations of z' in the vertical. If the vertical variance spectrum of z' is relatively constant, the ratio $\overline{K}/\overline{A}$ will be larger when shorter wave lengths predominate in the horizontal variance spectrum of z'. It is thus possible for $\overline{K}$ and $\overline{A}$ to vary under adiabatic quasi-geostrophic flow. A familiar hypothetical example is the increase of kinetic energy which accompanies the exponential growth of short-wave pertubations superposed upon an unstable zonal current.

4. Zonal and eddy energy

An approach to the general circulation which has currently found much favor consists of resolving the field of motion into the mean zonal motion and the eddies superposed upon it. This resolution partitions the kinetic energy of the whole atmosphere into two types, which may be called *zonal kinetic energy* and *eddy kinetic energy*, and which represent the kinetic energies of the two types of motion. The maintenence of each type of kinetic energy is then considered. In addition to other forms of energy, each type of kinetic energy is a possible source or sink for the other type.

This partitioning of kinetic energy is essentially an analysis of variance of the wind field, and is possible because, aside from the contribution of the average wind, kinetic energy is the sum of the variances of the wind components. A similar analysis of variance of the temperature field is possible. To the extent that available potential energy is measured by the variance of temperature, this analysis partitions the available potential energy into two types, one due to the variance of zonally averaged temperature, and one due to the variance of temperature within latitude circles. These types may be called *zonal available potential energy* and *eddy available potential energy*. In addition to other forms of energy, each type of available potential energy may be a source or a sink for the other type.

Analytic expressions for $\overline{A_Z}$ and $\overline{A_E}$, the zonal and eddy available potential energies, and $\overline{K_Z}$ and $\overline{K_E}$, the zonal and eddy kinetic energies, per unit area, may be obtained from expressions (11) and (12) for $\overline{A}$ and $\overline{K}$ by replacing T and V by their zonal averages, or their departures from their zonal averages. Thus

$$\left.\begin{aligned}
\overline{A}_Z &= \frac{1}{2} \int_0^{\overline{p_0}} (\Gamma_d - \overline{\Gamma})^{-1} \overline{T}^{-1} \overline{[T]'^2}\, dp \\
\overline{A}_E &= \frac{1}{2} \int_0^{\overline{p_0}} (\Gamma_d - \overline{\Gamma})^{-1} \overline{T}^{-1} \overline{T^{*2}}\, dp \\
\overline{K}_Z &= \frac{1}{2} g^{-1} \int_0^{\overline{p_0}} \overline{[V]^2}\, dp \\
\overline{K}_E &= \frac{1}{2} g^{-1} \int_0^{\overline{p_0}} \overline{V^{*2}}\, dp
\end{aligned}\right\} \qquad (24)$$

Here square brackets denote an average with respect to longitude, at constant latitude and constant pressure, while an asterisk denotes a departure from an average denoted by square brackets.

The time derivatives of these quantities may by applying suitable averaging processes to the continuity equation (13), the thermodynamic equation (14), and the equations of motion (19). Thus, if we allow the same sort of approximations which we used in the expressions for $\partial\bar{A}/\partial t$ and $\partial\bar{K}/\partial t$,

$$\left.\begin{aligned} \partial\bar{A}_Z/\partial t &= -C_Z - C_A + G_Z \\ \partial\bar{A}_E/\partial t &= -C_E + C_A + G_E \\ \partial\bar{K}_Z/\partial t &= C_Z - C_K - D_Z \\ \partial\bar{K}_E/\partial t &= C_E + C_K - D_E \end{aligned}\right\} \quad (25)$$

where

$$\left.\begin{aligned} C_Z &= -Rg^{-1}\int_0^{\bar{p}_0} P^{-1}\overline{[T]}\,\overline{[\omega]}\,dp = \\ &= -\int_0^{\bar{p}_0} \overline{[\mathbf{v}]\cdot\nabla[Z]}\,dp \\ C_E &= -Rg^{-1}\int_0^{\bar{p}_0} P^{-1}\overline{T^\star\omega^\star}\,dp = \\ &= -\int_0^{\bar{p}_0} \overline{\mathbf{v}^\star\cdot\nabla Z^\star}\,dp \\ C_A &= -\frac{R}{g}\int_0^{\bar{p}_0} \frac{\bar{\Theta}}{\bar{T}} \overline{\left([T^\star v^\star]\frac{\partial}{\partial y} + [T^\star\omega^\star]'\frac{\partial}{\partial p}\right)\cdot} \\ &\quad \overline{\cdot\left(\frac{\Gamma_d}{\Gamma_d - \bar{\Gamma}}\frac{1}{\bar{\Theta}}[T]'\right)}\,dp \\ C_K &= -\frac{1}{g}\int_0^{\bar{p}_0} \cos\phi\,\overline{\left([u^\star v^\star]\frac{\partial}{\partial y} + \right.} \\ &\quad \overline{\left. + [u^\star\omega^\star]\frac{\partial}{\partial p}\right)\left(\frac{[u]}{\cos\phi}\right)}\,dp \\ &\quad -\frac{1}{g}\int_0^{\bar{p}_0} \overline{\left([v^{\star 2}]\frac{\partial}{\partial y} - \sin\phi\,[V^{\star 2}] + \right.} \\ &\quad \overline{\left. + [v^\star\omega^\star]\frac{\partial}{\partial p}\right)[v]}\,dp \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} G_Z &= g^{-1}\int_0^{\bar{p}_0} \Gamma_d(\Gamma_d - \bar{\Gamma})^{-1}\bar{T}^{-1}\overline{[T]'[Q]'}\,dp \\ G_E &= g^{-1}\int_0^{\bar{p}_0} \Gamma_d(\Gamma_d - \bar{\Gamma})^{-1}\bar{T}^{-1}\overline{T^\star Q^\star}\,dp \\ D_E &= -g^{-1}\int_0^{\bar{p}_0} \overline{[\mathbf{v}]\cdot[\mathbf{F}]}\,dp \\ D_Z &= -g^{-1}\int_0^{\bar{p}_0} \overline{V^\star\cdot F^\star}\,dp \end{aligned}\right\} \quad (27)$$

In (26), u and v are the eastward and northward components of V, ϕ is latitude, and $\partial/\partial y$ is the derivative with respect to distance northward. The alternative forms for C_Z and C_E are analogous to the alternative forms (17) for C.

In equations (25) we observe that each of the quantities C_Z, C_E, C_A, and C_K occurs twice, with opposite signs. It is then tempting to say, for example, that C_K represents the rate of conversion from zonal to eddy kinetic energy, and to draw analogous conclusions about the other "C's", i.e., to interpret the C's as energy transformation functions, as described by MILLER (1950). We must note, therefore, that the C's are not uniquely defined by the time derivatives of the various forms of energy, since, for example, if all the C's were altered by the additions of the same quantity, equations (25) would still be valid. To justify the interpretation of the C's as conversions from one form of energy to another, we must examine the physical processes which they describe. The necessity for considering physical processes when interpreting energy equations has recently been emphasized by LETTAU (1954).

We note first the quantities G_Z and G_E, which may be called the *zonal generation* and the *eddy generation*, represent the generation (or destruction) of available potential energy by nonadiabatic processes, and do not involve conversion from one form of atmospheric energy to another. Similarly, the quantities D_Z and D_E, which may be called the *zonal dissipation* and the *eddy dissipation*, represent the dissipation of kinetic energy by friction, and do not involve conversions of energy. Here we have regarded friction as involving a simultaneous destruction of kinetic energy and generation of potential energy, rather than a process of conversion from kinetic to potential energy, since very little available potential energy is generated by frictional heating.

It follows that the sum of the C's in the change of any one form of energy must equal the sum of the conversions to that form of energy from all other forms.

We next note that the C's all involve horizontal or vertical transports of momentum or sensible heat. These transports may be resolved into separate modes of transport; for example, the vertical transport of sensible heat, represented by $[T\omega]$, may be resolved into a transport by meridional circulations, an eddy-transport whose value per unit area is independent of latitude, and an eddy-transport whose value vanishes when averaged over latitude. Each of these modes of transport enters only one of the terms in the relation

$$[T\omega] = [T][\omega] + [T^\star \omega^\star] + [T^\star \omega^\star]' \quad (28)$$

Let us agree to regard the separate modes of transport as separate physical processes.

We then observe the following situation: Horizontal eddy-transports of sensible heat, and vertical eddy-transports whose values vanish when averaged over latitude, enter the expression for C_A, but not C_K, C_Z, nor C_E. They therefore affect A_Z and A_E by altering the analysis of variance of temperature, but they do not affect K_Z, K_E, nor the sum $A_Z + A_E$.

Eddy-transports of momentum enter the expression for C_K, but not C_A, C_Z, nor C_E. They therefore affect K_Z and K_E by altering the analysis of variance of wind, but they do not affect A_Z, A_E, nor the sum $K_Z + K_E$.

Transports of sensible heat by meridional circulations, and accelerations due to horizontal displacements by meridional circulations, enter the equivalent expressions for C_Z, but not C_E, C_A, nor C_K. They therefore affect A_Z and K_Z by altering the variance of zonally averaged temperature and wind, but they do not affect A_E, K_E, nor the sum $A_Z + K_Z$.

Vertical eddy-transports of sensible heat whose value per unit area is independent of latitude, and accelerations due to horizontal displacements by eddies, enter the equivalent expressions for C_E, but not C_Z, C_A, nor C_K. They therefore affect A_E and K_E by altering the variance of temperature and wind within latitude circles, but they do not affect A_Z, K_Z, nor the sum $A_E + K_E$.

It follows that C_A, C_K, C_Z, and C_E are energy transformation functions, which involve respectively only available potential energy, only kinetic energy, only zonal forms of energy, and only eddy forms of energy.

The energy transformation function C_K has appeared frequently in recent works. It is a modification of an expression derived by REYNOLDS (1894) in connection with turbulent flow. It has been presented in nearly the same form by VAN MIEGHEM (1952), while the first integral in expression (25) for C_K, which is the dominating term, has been discussed by KUO (1951) and STARR (1953).

The energy transformation function C_A bears nearly the same relation to temperature which C_K bears to wind. It depends upon the transport of sensible heat along the gradient of temperature in much the same way in which C_K depends upon the transport of angular momentum along the gradient of angular velocity.

The two possible remaining energy transformation functions—the conversions from A_Z to K_E and from A_E to K_Z, do not enter equations (25). Moreover, if we regard the separate modes of transport as separate physical processes, there is no process which affects both A_Z, and K_E, or both A_E and K_Z. These remaining energy transformation functions therefore vanish identically.

It must be remembered that this conclusion depends upon our regarding the separate modes of transport as separate physical processes. Without the distinction between the two modes of eddy transport, it would be impossible to say whether or not a direct conversion of zonal available potential energy to eddy kinetic energy is possible, although conversion from any form of available potential energy to eddy kinetic energy, which involves nonvanishing values of $T^\star$, would still require the *presence* of eddy available potential energy. The distinction between the modes of eddy-transport is probably as logical, if not as familiar, as the distinction between eddy transport and transport by meridional circulations. Without the latter distinction none of the energy transformation functions involving zonal or eddy available potential energy could be defined.

5. The maintenance of the energy of the general circulation

The zonal winds and the superposed eddy motions are not identical with the meridional pressure gradient and the superposed pressure perturbations, since the former are features of the distribution of momentum, which possesses kinetic energy, and the latter are features of the distribution of mass, which possesses available potential energy. It is therefore a legitimate problem to study exchanges of kinetic energy between the zonal winds and the eddies, without considering similar exchanges of available potential energy.

Nevertheless the zonal winds are often identified with the meridional pressure gradient, and cyclonic and anticyclonic circulations are often identified with the low and high pressure systems which almost always accompany them. Indeed the wind systems could not long maintain their identities without the accompanying pressure systems, and vice versa. It may therefore be possible to achieve a better understanding of the general circulation by regarding exchanges of kinetic energy and exchanges of available potential energy as features of a single problem.

The conversion C_K of zonal to eddy kinetic energy depends primarily upon the transport of angular momentum horizontally and vertically by eddies along the gradient of angular velocity. Recent computations by STARR (1953), based upon wind observations over the northern hemisphere (see STARR and WHITE 1954) confirm the earlier suspicions of some meteorologists that the horizontal transport is predominantly against the gradient of angular velocity, and yield the approximate value—10×10^{20} ergs per second for the integral of C_K over the northern hemisphere, so that the eddies appear to supply sufficient kinetic energy to the zonal flow to maintain it against frictional dissipation.

At this point we must become more specific and state just what sort of dissipation we are considering. The dissipation of zonal kinetic energy by molecular friction is probably very small. The principal "frictional" dissipation of zonal kinetic energy is instead due to small-scale turbulent eddies, and it is principally the kinetic energy of these eddies which is dissipated by molecular friction. It is therefore not correct to say that the eddies on the whole supply kinetic energy to the zonal flow, if all scales of eddies are included.

Instead, we must make a distinction between large-scale eddies, which, roughly speaking, are the eddies large enough to appear on synoptic weather maps, and the remaining small-scale eddies. It is then correct to say that the large-scale eddies supply enough kinetic energy to the zonal flow to maintain it against the dissipative effects of small-scale eddies. This important feature of the general circulation would be obscured if eddies of all scales were included in a single category.

Again, if only molecular friction and conduction were considered, the skin friction and surface heating would have nearly infinite values, per unit mass, throughout nearly infinitesimal depths. The generation and dissipation functions G_Z and D_Z would then depend largely upon the usually unmeasured temperatures and winds in a thin layer next to the ground. This difficulty is over-overcome if eddy viscosity and conductivity replace molecular viscosity and conductivity, so that the skin friction and surface heating have moderate values throughout moderate depths. Let us agree, therefore, to regard only the large-scale eddies as eddies, and to include the small-scale eddies in a category with molecular motions.

The conversion C_A of zonal to eddy available potential energy depends primarily upon the transport of sensible heat horizontally and vertically across the gradient of temperature T'. The studies of STARR and WHITE (1954) confirm the generally accepted idea that the horizontal transport is with the temperature gradient, and computations based upon the results of this study yield the approximate value 200×10^{20} ergs per second for the integral of C_A over the northern hemisphere, so that C_A is about twenty times as large as C_K.

It follows that if the zonal winds and the meridional pressure gradient are regarded as separate manifestations of the same zonal pattern, and if the wind and pressure variations within latitude circles are regarded as separate manifestations of the same eddies, it is not possible to say that the eddies maintain the zonal circulation. All that can be said is that the zonal pressure field maintains the eddy pressure variations, but the eddy motion maintains the zonal motion.

To understand the maintenance of the energy of the general circulation, it is therefore not sufficient to know the exchange of energy between the zonal circulation and the eddies. A knowledge of all the energy transformation functions and the generation and dissipation functions is required.

The presence of net heating in low latitudes and net cooling in high latitudes is a familiar feature of the general circulation; it leads to a generally positive value of $\overline{[T]'[Q]'}$, so that the zonal generation G_Z is positive, and indeed seems to represent the primary source of the energy of the general circulation. Crude estimates of G_Z, based upon radiation-balance figures of Albrecht (see HAURWITZ 1941), and neglecting the release of latent energy, yield the approximate figure $G_Z = 200 \times 10^{20}$ ergs per second so that G_Z and C_A are about equal.

Less obvious is the sign of the eddy generation G_E, which depends upon $\overline{T^{\star}Q^{\star}}$, and hence upon the correlation between temperature and heating within latitude circles. Presumably it is negative, in view of the probable warming of cold air masses and cooling of warm air masses in middle latitudes, but the possible preference of warm longitudes for the release of latent energy may suppress this negative value.

The dissipation functions D_Z and D_E may safely be regarded as positive. We have just seen that C_K is negative, while C_A is positive. It follows by continuity that C_E must be positive, since it represents the only remaining source for eddy kinetic energy. This positive value must be associated with sinking of colder air and rising of warmer air at the same latitude. Hemispheric data for the direct computation of C_E are unfortunately not available.

The sign of C_Z cannot be inferred by continuity, but the magnitude seems to be small, in view of the failure of hemispheric wind observations to reveal strong meridional cells. A value of -2×10^{20} ergs per second for the integral of C_Z over the northern hemisphere has been estimated by STARR (1954) from the data available. The negative sign occurs because the middle-latitude indirect cell occupies the zone of maximum temperature gradient.

We are thus led to the following picture of the maintenance of the energy of the general circulation: The net heating of the atmosphere by its environment in low latitudes and the net cooling in high latitudes result in a continual generation of *zonal available potential energy*. Virtually all of this energy is converted into *eddy available potential energy* by the eddies. Some of this energy may be dissipated through heating of the colder portions of the eddies and cooling of the warmer portions; the remainder is converted into *eddy kinetic energy* by sinking of the colder portions of the eddies and rising of the warmer portions. Some of this energy is dissipated by friction; the remainder is converted into *zonal kinetic energy* by the eddies. Most of this energy is dissipated by friction; a small residual is converted into zonal available potential energy again by an indirect meridional circulation.

In conclusion, let us see how our picture of the energy transformation compares with earlier descriptions of the general circulation. Certainly it bears little resemblance to any theory which attributes the conversion of potential into kinetic energy to a general rising motion in low latitudes and sinking in high latitudes. However, discussions of some of the alternative processes which we have described have been appearing with increasing frequency in recent meteorological literature.

The idea that eddy kinetic energy is the immediate source of zonal kinetic energy is closely related to the idea expressed by ROSSBY (1947, 1949) that large-scale mixing processes (eddies) may account for the distribution of zonal winds. EADY (1950) has described the inequality of the mean zonal angular velocity as a result of turbulence (eddies). Computations of a positive rate of conversion of eddy into zonal kinetic energy, based upon wind observations, have been presented by KUO (1951) and STARR (1953).

An idealized quantitative model in which the motions associated with waves in the westerlies (eddies) are responsible for the generation of kinetic energy has been presented by MINTZ (1947). The importance of sinking cold air masses and rising warm air masses was pointed out by ROSSBY (1949). VAN MIEGHEM (1952) describes the conversion of potential energy into *eddy* kinetic energy as one of the two most important energy transformations on the scale of the general circulation, the other being the conversion of eddy into zonal kinetic energy.

The importance of a poleward eddy-transport of sensible heat, in conjunction with the excess of radiational heating in low latitudes and the deficit in high latitudes, has long been recognized (cf. HAURWITZ 1941), but the inevitable other effect of this transport—an increasing of the variance of temperature within latitude circles—seems to have been generally overlooked. Very recently STARR (1954) has described an idealized two-stage process of conversion of potential into kinetic energy, the first step consisting of a deformation of zonally oriented isotherms. This stage is essentially a conversion from zonal into eddy available potential energy. But regardless of whether one wishes to think in terms of available potential energy, a variance of temperature within latitude circles is a prerequisite for the conversion of potential energy into eddy kinetic energy, which involves a correlation within latitude circles between temperature and vertical motion (cf. VAN MIEGHEM 1952). If non-adiabatic heating creates primarily a cross-latitude variance of temperature, a poleward eddy-transport of sensible heat is necessary to maintain the variance of temperature within latitude circles. The conversion of zonal to eddy available potential energy may therefore be regarded as a third important energy transformation on the scale of the general circulation.

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