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Studies on the General Development of Motion in a Two-Dimensional, Ideal Fluid

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Abstract

The purpose of this paper is to discuss qualitatively certain kinds of asymptotic motion in a two-dimensional, ideal fluid by help of methods of statistical mechanics. It is stressed that the final development of such a fluid cannot be adequately described by use of the ordinary equations of motion, but that a "coarse grain" representation should be used. In this representation, the development is characterized by the forming of a single, large vortex accompanied by a certain non-viscous dissipation. The final equilibrium is probably reached almost explosively after a *finite* time. Some experiments which are carried out seem to support this result. In the earlier stages of development we may expect to have some kind of a quasi-equilibrium motion. It is attempted to find conditions under which such a motion can exist, by studies of a point-vortex model.

As known, the hydrodynamic equations of motion cannot be solved analytically to give in closed form the time-development from any given initial state, but we must generally take resort here to some method of numerical integration. Besides obtaining precise numerical results in special cases, we are, however, also interested in getting a more general, qualitative description of the development, comprising features which are characteristic of all cases, or at least the major part of them, irrespective of their initial states.

Essential contribution to such a description is given by the modern theory of turbulence, as developed by Kolmogoroff and others, but the picture is still far from being complete. The intention of this paper is to make a further discussion of some general features with the aid of methods of statistical mechanics. Attempts in that direction have been given earlier by, for instance, BURGERS (1929), TOLLMIEN (1933) and ONSAGER (1949). However, it seems to be a general impression that no decisive advancement can be made along this line until the mathematical apparatus of statistical mechanics has been improved, so that it can effectively tackle continuous systems also. An undertaking of this task is now being made by HOPF (1951) and others, but certainly much remains to be done. Another way would be to abandon wholly the continuum concept and build up the theory from a molecular point of view, and some ideas in that direction will be presented in a later contribution. The general connection between the fields of kinetic theory of gases-hydrodynamics-theory of turbulence is pointed out already in this paper.

The author here wishes to express his thanks to Mr. G. Dahlquist for several stimulating discussions during the course of the investigation.

1. Basic principles of statistical mechanics

Since people working in the field of hydrodynamics are not generally familiar with the branch of statistical mechanics, it has been found advisable to start with a short discussion of its basic principles.

The theory of classical statistical mechanics is developed for conservative, Hamiltonian systems, which as known are basic in the classical mechanics. The equations of motion for such a system are

$$\left.\begin{aligned}\frac{dp_i}{dt}&=-\frac{\partial W}{\partial q_i}\\ &\qquad (i=1,2,\ldots n)\\ \frac{dq_i}{dt}&=\frac{\partial W}{\partial p_i}\end{aligned}\right\} \quad (1)$$

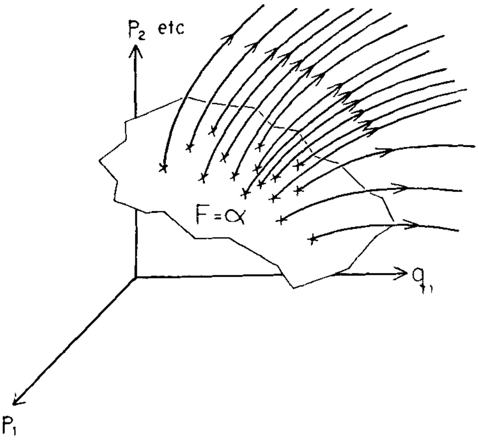

Fig. 1. Experiment "$F = \alpha$".

where $W(p_1,\ldots p_n, q_1,\ldots q_n)$ is the energy of the system, expressed in the coordinates $q_1,\ldots q_n$ and the corresponding "generalized momenta" $p_1,\ldots p_n$. The instantaneous state of a system can be represented as a point, the representative point, in a phase space (or ensemble space) of $2n$ dimensions with the p_i:s and q_i:s as rectangular coordinates. The time development of the system is then represented by a certain trajectory in this space.

Suppose now that we have, in an actual case, only an incomplete knowledge of the initial state of the system. As a specific example, let us assume that we can control at the initial time only a certain function of the initial phase space coordinates, $F(p_1,\ldots p_n, q_1,\ldots q_n)$. If we carry out a number of similar experiments, under steady outer conditions and with a fixed value of F, say α, the initial values p_{i_0} and q_{i_0} are in general found to vary from one experiment to the next due to non-controllable disturbances of our system. However, carrying out a great number of experiments we will find a certain statistical distribution of initial values, and corresponding to this a certain distribution of points on the hypersurface $F(p_1,\ldots p_n, q_1,\ldots q_n)=\alpha$ in the phase space. The whole statistical structure of the experiment "$F=\alpha$" is represented in the phase space by the bundle of trajectories starting at these points (Fig. 1).

Now, it is plausible to assume that the distribution of initial points found on $F=\alpha$ is invariantly connected with our physical system and so is representative also in other experiments with the system, say for instance the experiment "$F=\alpha$ and $G=\beta$". This requires of course that the non-controllable forces ultimately responsible for the statistical spreading of our initial points are not statistically affected by the special arrangement of our experiment. More generally we assume that to every part of the phase space there is related a universal a priori possibility density of initial points of our system. To determine this density distribution is one fundamental task in statistical mechanics.

So far, it has not been necessary to restrict ourselves only to Hamiltonian systems. Such systems, however, have a nice property, which immediately suggests a certain simple distribution of the a priori probability density. Introducing in the phase space the velocity vector $\mathbf{v}=\left(\frac{dp_1}{dt},\ldots\frac{dp_n}{dt},\frac{dq_1}{dt},\ldots\frac{dq_n}{dt}\right)$ we find

$$\nabla\cdot\mathbf{v}=\sum_{i=1}^{n}\left[\frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}\right)+\frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}\right)\right]=$$
$$=\sum_{i=1}^{n}\left[\frac{\partial}{\partial p_i}\left(-\frac{\partial W}{\partial q_i}\right)+\frac{\partial}{\partial q_i}\left(\frac{\partial W}{\partial p_i}\right)\right]=0$$

showing that the flow of representative points in the phase space is an incompressible one. Particularly, this means that a uniform distribution of points throughout the entire phase space is maintained permanently. Considering this result, one finds it natural to assume the existence of a constant a priori probability density in the phase space for all Hamiltonian systems. No rigorous proof of this has been given, but the hypothesis can be made very plausible also for other reasons, and so far it has never been contradicted by experiments. The hypothesis tells us how to construct our representative "ensemble of systems" in a case where we have only a partial knowledge of the state: the representative points should be distributed so as to all correspond to this partial knowledge, but otherwise they should be distributed uniformly in the phase space.

By use of the theorem of the non-divergent flow and the hypothesis of equal a priori probability density in the phase space one can derive a fundamental theorem, the so-called H-theorem. This says that a certain quantity

$$H = \underset{\text{all space}}{\int \cdots \int} P \log P \, dV$$

where P is the probability density of the representative points and dV stands for the volume element $dp_1 \, dp_2 \ldots dp_n \, dq_1 \, dq_2 \ldots dq_n$, will decrease in the run of time towards an absolute minimum value. For the derivation of this theorem reference is given to textbooks (see f.i. TOLMAN: *The Principles of Statistical Mechanics*). For molecular systems, H is proportional to the negative of the entropy, and the H-theorem thus expresses the second law of thermodynamics.

The H-theorem enables us to determine the state of statistical equilibrium of our system, this being one main task undertaken by the statistical mechanics. The minimum condition of H is in variational form

$$\delta H = \underset{\text{all space}}{\int \cdots \int} (\log P + 1) \, \delta P dV = 0 \qquad (2)$$

and to this we should add the subsidiary condition

$$\underset{\text{all space}}{\int \cdots \int} P \, dV = 1$$

Other subsidiary conditions are given by our partial knowledge of the system. A common case is that we know in the equilibrium state the ensemble mean value of some functions $F_1, F_2, \ldots$ depending on the phase space coordinates,

$$\underset{\text{all space}}{\int \cdots \int} F_i P \, dV = \alpha_i, \quad (i = 1, 2, \ldots)$$

Handling these subsidiary conditions by help of Lagrangian multipliers (a_i), we then get for the final probability density the expression

$$P = e^{-(1 + a_0 + a_1 F_1 + a_2 F_2 \ldots)} \qquad (3)$$

where the constants a_i are determined by reinsertion of (3) into the above conditions. Another frequently occurring case is that some integral to the equations is prescribed, restricting the motion of the systems to a certain hypersurface in the phase space. Most important is here the energy integral, $W(p_1, \ldots p_n, q_1, \ldots q_n) = \text{constant}$.

Regarding the non-equilibrium states, we have no such general method to fall back on, and it is only in the special branch of kinetic theory of rarefied gases that a complete theory has been worked out. It would be of some interest to discuss here some principal aspects of this theory, which in fact forms the bridge between the statistical mechanics of discrete systems and the hydrodynamic theory of continua.

Starting from the previous phase space representation, the first step in developing the kinetic theory is to introduce the probability distribution function for a single molecule, independent of the states of the other molecules:

$$f(p_1, p_2, p_3, q_1, q_2, q_3) = \\ = \underset{\text{all space}}{\int \cdots \int} P dp_4 \ldots dp_n dq_4 \ldots dq_n$$

where q_1, q_2, q_3 and p_1, p_2, p_3 are the cordinates and momenta, respectively, of the molecule (all molecules regarded as mass-points). For f we can set up a continuity equation, the so-called Boltzmann equation, by equating the net number of molecules disappearing per second in a specified element $dp_1 \, dp_2 \, dp_3 \, dq_1 \, dq_2 \, dq_3$ to the net number of molecules

a. leaving the q-element due to rectilinear motion.

b. leaving the p-element due to the action of outer forces.

c. leaving the p-element due to molecular collisions.

The contributions according to a. and b. depend directly on f, while the contribution according to c. depends on the joint probability distribution for molecular pairs. To get an equation for f only, we must here introduce an additional hypothesis, namely that the velocities of the colliding molecules are non-correlated (molecular chaos). If the gas is sufficiently rarefied this hypothesis seems to be plausible enough, however, no rigorous proof has been given.

The second step is to introduce the hydrodynamic variables and stresses as certain mean values computed in the f-distribution: the density ϱ is defined as the mean mass per unit volume, the mass velocity $\mathbf{v}$ as the mean molecular velocity, the normal stress component p_{xx} as the quadratic mean of the difference between the x-components of molecular and mass velocity, etc. Making use of the laws of conservation of mass and momentum in the molecular collisions, we may then derive the hydrodynamic equations of motion. However, these contain as unknowns also the stresses. To get the ordinary Navier-Stokes equation, containing only the hydrodynamic variables ϱ, v and p (where p is defined as the negative mean of the three normal stress components p_{xx}, p_{yy} and p_{zz}), further simplifications have to be made: the molecular velocity distribution must be assumed to deviate only little from the Maxwellian form, and also an additional statistical hypothesis must be introduced (this hypothesis enters the computation very implicitly and cannot be described in simple physical terms).

Examining these two steps we find that they are very similar in nature: in each of the steps the equations are simplified by introducing new dependent variables, defined as certain mean values of the old ones, and at the same time a statistical assumption is being made. From the probability distribution P in the phase space we went over to the single-molecule distribution function f and introduced the statistical hypothesis of non-correlated collisions, and from this we went over to the hydrodynamic variables and introduced again a certain statistical assumption. One can easily understand that the introduction of some hypothesis in each step *is* necessary. As an example, let us assume that we want to compute the further development according to the Boltzmann equation from a given initial field f_0. Now, corresponding to this f_0-field a great number of different distributions P_0 may be constructed. The further developments of these, controlled directly by the Hamiltonian equations, will in general differ, and so will also the corresponding developments of f. To get a *unique* development of f it is then necessary to introduce some restrictive hypothesis, bringing about a direct coupling between the f- and the P fields, and, in fact, such a coupling is yielded by our hypothesis of non-correlated collisions.

Finally, it seems natural to consider the forming of the hydromechanical equations for the mean motion in a turbulent field as the next step in the same sequence. Then a question immediately rises: which statistical hypothesis should be introduced now? This question, which is certainly one of the most fundamental ones in the theory of turbulence, has not yet been answered. However, the recent work by Kolmogoroff and others regarding the quasi-equilibrium state of a turbulent field may eventually provide the basis for a fresh attack on the problem.

2. Development towards a final equilibrium of a two-dimensional, ideal fluid

We consider a two-dimensional, ideal fluid inside a closed and energetically isolated region. At a certain initial time we put the fluid in some arbitrary motion and leave it. How does the motion develop in the long run of time?

First we observe that our system is conservative so that no energy can escape, and thus *some* non-zero asymptotic motion should be found. However, it seems hard to think of a development towards a real equilibrium state. To be able to speak here of an equilibrium approach, we must of course have some irreversible feature in the motion, but our hydrodynamic equations cannot at all distinguish between the two time directions, as is seen by making the transformation $t \to -t$, $\mathbf{v} \to -\mathbf{v}$. In this case every proof, based only on the hydrodynamic equations, that the fluid in the long run approaches some specific equilibrium state can easily be changed to a proof that the fluid instead *departs* from the equilibrium, just by making the above transformation everywhere in the proof.

Now, the same kind of problem has been met earlier in the statistical mechanics, where the basic equations (1) are also time-symmetric. In this case it was found possible to introduce an irreversible feature in the picture, becoming apparent in the famous H-theorem, by considering not the development of a single system but the development of a whole ensemble of systems starting from various initial states. It should be possible to go the same way in our case, seeing apart from the mathematical difficulties of handling an ensemble of continuous systems. However, we will see that the technique used in the derivation of the *H*-theorem can in fact be utilized here directly for a discussion of the development of the *single system.*

To begin with, let us consider the following simple but illustrative experiment, similar to one discussed by Gibbs in his work *Principles of Statistical Mechanics.* The fluid is initially divided up into a number of surface elements, which are coloured black and white (for instance in a chess-board pattern). The colour should follow the motion materially without any essential diffusion. When setting the fluid into motion the elements are deformed, and after some time they are drawn out in the form of long and thin bands, as shown in Fig. 2.

We now want to investigate the conservation of colour density during the motion. If the black colour is assigned the density 1 and the white colour is assigned the density 0, then the total colour mass of the fluid is $\int_F \varrho \, dF = F_b$, where F is the total area of the fluid, F_b the total area of the black elements. Also the total square density mass is $\int_F \varrho^2 dF = F_b$. Both these quantities are accordingly conservative, since the area F_b is preserved during the motion. However, regarding the total square density mass, the above result is true only if each integration element is either wholly black or wholly white. If we prescribe in advance the size of these elements, however small, we must always ultimately arrive at the situation, where the widths of the bands are found to be so small that the integration elements are each crossed by several bands of different colours. If, as usual, the colour density in an element is defined as the ratio of colour matter and area, then the square density in dF is now $\varrho^{\star 2} = \left[\frac{1}{dF}\int_{dF} \varrho \, \delta F\right]^2 = \left(\frac{dF_b}{dF}\right)^2$, where δF is a "superdifferential" element, and for the total square density mass we have now

$$\int_F \varrho^{\star 2} dF = \int_F \left(\frac{dF_b}{dF}\right)^2 dF = \int_F \left(\frac{dF_b}{dF}\right) dF_b \quad < F_b$$

In this sense, the quantity is not conservative, but decreases. With terms taken from the statistical mechanics, we might call the density $\varrho^\star$ of the fixed integration elements the "coarse grain density", to be distinguished from the "fine grain density" ϱ of the super-differential elements δF, which always are supposed to be small as compared to the widths of the bands.

The argument is now that what we observe actually is not the fine-grain density but the coarse-grain density, the size of the integration elements being always finite due to the existence of an observational limit scale. In the discussed experiment, we have by a direct visual observation a limit scale of some hundredths of a millimeter, and when the widths of the bands have become smaller than this, the "coarse-grain colour" has changed from black or white to grey.

We might expect that our final equilibrium should be characterized by a minimum total square density mass, $\delta \int_F \varrho^{\star 2} dF = 0$, with the subsidiary conditions $\int_F dF =$ constant and $\int_F \varrho^\star dF =$ = constant, giving a uniform (coarse grain) density distribution. The minimum value is here obviously $\frac{F_b}{F} \cdot F_b$. To our ordinary feeling it seems self-evident that a sufficient stirring would ultimately lead to a uniform density distribution, but this feeling may be false. There certainly exist processes which are ordinarily recognized as stirring ones but which do not lead to the uniform density, and it has not yet been proved rigorously that the natural motion of our fluid cannot be such a process. Instead, the ultimate uniformity must be introduced as a hypothesis. However, as far as the two-dimensional and ideal fluid is concerned, we can find support to this hypothesis. The vortex elements of such a fluid form a Hamiltonian system, as will be seen later, and the phase space of this system is identical with the multi-dimensional configuration space. The hypothesis of equal a priori probability density thus says here that each part of the configuration space should be

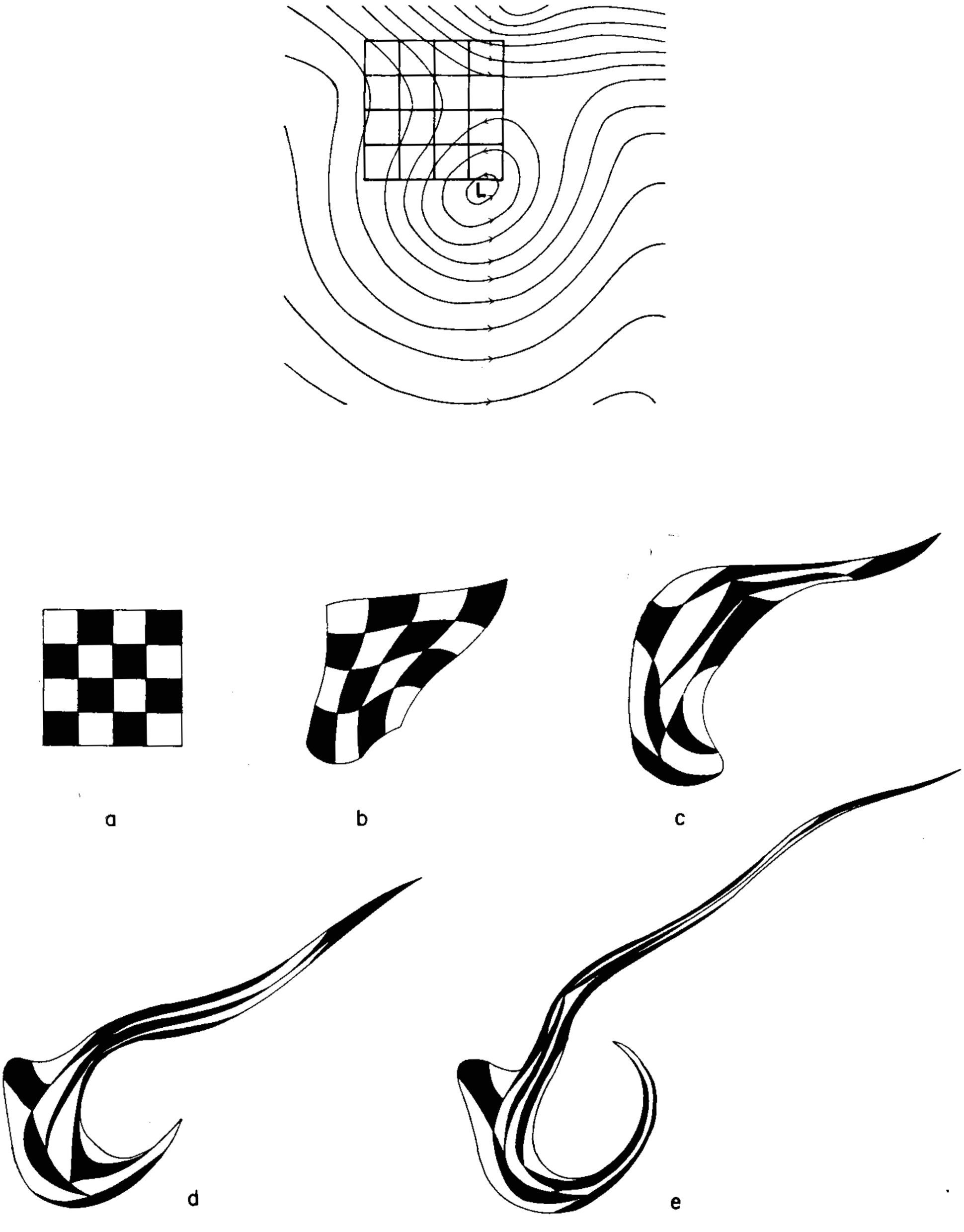

Fig. 2. Deformation of a fluid surface.

The picture shows the deformation of an air layer at the 500 mb level in the atmosphere after 6 h, 12 h, 24 h and 36 h, respectively. The deformation is computed by means of the barotropic model, assuming the layer to behave like a two-dimensional, ideal fluid. The initial stream-line pattern is seen at the top. The sides of the square elements are 300 km. For numerical estimates of the deformation in atmospheric flows see also SAUCIER (1953) and VUORELA (1953).

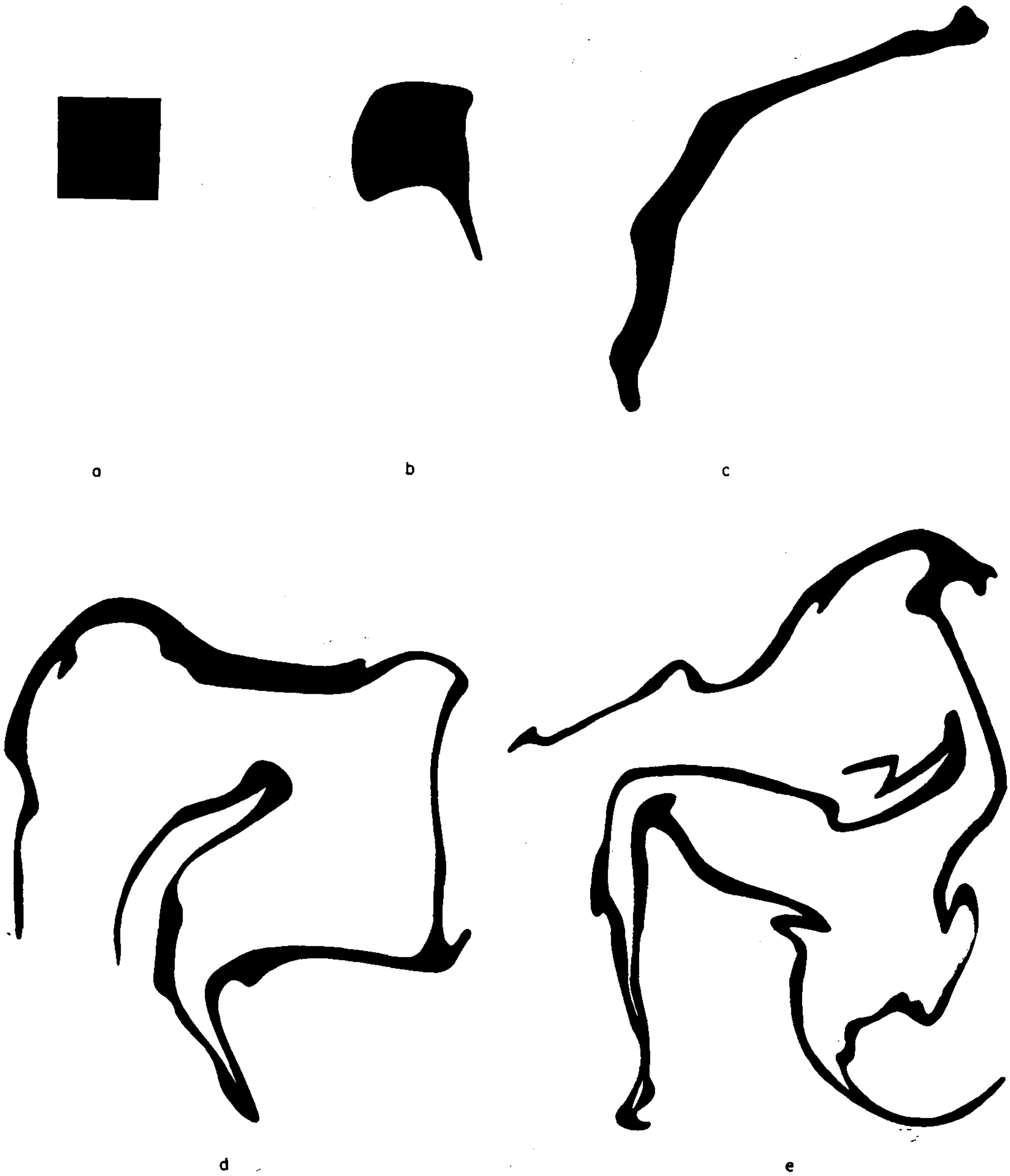

Fig. 3. Observed deformation of a fluid element.

The picture shows the observed deformation of a small, coloured square element of a fluid surface. A rectangular vessel of dimensions 50 × 30 × 30 cm filled with water to half the depth was used for the experiment. On the water surface was put a film of butanol, which was divided into square elements by means of a metal grid. One or several of these elements were coloured with methyl-red and the water was set into horizontal motion. The grid was then quickly taken away and the fluid was left to move undisturbed. To keep the motion two-dimensional, the whole fluid mass was set into a slow basic rotation before the initial disturbance was created.

f

g

Fig. 3.

given equal statistical weight and this speaks much for the existence of the ultimate uniformity.

Our previous discussion was concentrated on a very special example, but the technique is about the same when considering a more general case. Assuming an arbitrary colour distribution, we just divide the fluid into elements *dF* so small that we can put initially the density constant within each of these elements. The total coarse grain and fine grain square densities are the same at this time, but at a later time the former quantity is again being diminished, while the latter one is constant. This is seen immediately by inspecting the expressions $\int_F \overline{\varrho^2}^{dF} dF$ and $\int_F (\bar{\varrho}^{dF})^2 dF$, representing these two quantities, respectively (the bar here indicates a mean taken over *dF*).

From this we turn to the question of the final velocity field. This field is completely determined by the vorticity field of the fluid, and, since the relation between velocity and vorticity is a linear one, the coarse grain velocity field is determined by the coarse grain vorticity field in a corresponding way. Now, the vorticity is in a two-dimensional, ideal fluid a conservative property, which follows the motion of the fluid just in the same way as the colour matter, and the previous conclusions hold as well if we replace the colour density ϱ by the vorticity η. Accordingly, if the stirring is complete, we will ultimately get a state of constant coarse-grain vorticity. The coarse-grain flow is then of the type indicated in Fig. 4, forming a single large vortex.

The total vorticity is conserved here, while a certain amount of square vorticity is lost. This loss is obviously proportional to the mean square fluctuations of the fine grain vorticity in the initial state,

$$-\triangle\left(\int_F \eta^2 dF\right) = \int_F \eta^2 dF - \left[\frac{1}{F}\int \eta\, dF\right]^2 \cdot F =$$

$$= \int_F \left[\eta^2 - (\bar{\eta}^F)^2\right] dF = \int_F (\eta - \bar{\eta}^F)^2 dF =$$

$$= \overline{(\eta - \bar{\eta}^F)^2}^F \cdot F$$

Corresponding to this we also have a certain loss of kinetic energy, disappearing into "turbulent heat".

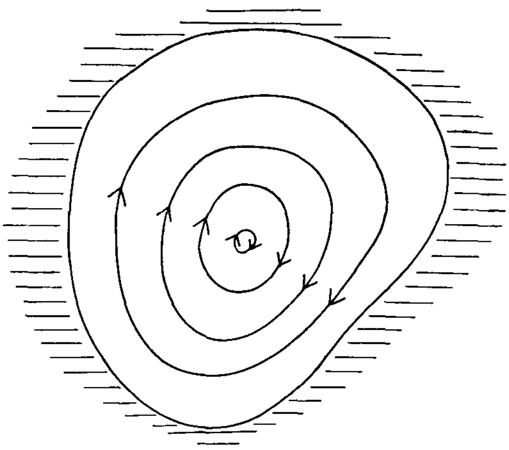

Fig. 4. The *final velocity field.*

Now, let us stop and try to make the situation clear for us again. In terms of the ordinary fine-grain quantities we cannot describe any final equilibrium state, firstly because of the reversibility which is inherent in the equations of motion, and secondly because of the fact that the velocity field will approach in the run of time a discontinuous state. However, in terms of the coarse-grain quantities an equilibrium is likely to exist, and this equilibrium is here found from the condition of "complete mixing", which is the same as the condition that the mean square vorticity (or the mean square of any other individually conserved quantity) is at the absolute minimum. Two subsidiary conditions hold, namely that the total area and the total vorticity of the fluid is conserved, but these are also the only conservation theorems in the coarse-grain representation. There is one exception, namely when the boundary is exactly circular so that the pressure reactions from the boundary can give no moment around the centre. The angular momentum of the flund is then also conserved with respect to the centre, and since the vorticity of the fluid enters linearly in the angular momentum, this is also conserved in the coarse-grain representation. However, it is to be supposed that the slightest deformation of the boundary will completely change the character of the motion and make it break down again into a complete mixing. We could compare here with the case when a mass-point is moving in a circular region, bounded by a perfectly reflecting wall. If the boundary is *perfectly* circular, the angular momentum (with respect to the centre) is conserved and the mass-point is restricted to move all the time in a certain zone of the region. However, the slightest deformation of the wall somewhere changes the picture completely. The mass-point is now free to move into any other state compatible with the actual energy value, and in fact it is possible to show that it *will* in the run of time take on almost all of those states[1]. In special, we will after a sufficiently long time certainly observe a change of sign in the angular momentum!

We also want to say something about the time necessary to establish the equilibrium. This time is of course directly dependent on the rate at which the bands are stretched out, and thus we are going to investigate in more detail the mechanism of this stretching. To begin with, it may be good to have a qualitative description of the stretching of a typical band as observed in the experiments (Fig. 3).

Starting from a square element with a side small compared to the scale of the initial motion, we observe first a uniform shearing deformation of the element together with a translation and rotation. Later on the variations of the deformation field over the element become important and the element is deformed more irregularly. After some time a long band is formed, following mainly the streamlines of the fluid, and this is steadily stretched out in the run of time. When the band lies at right angle to the streamlines, folds are found to develop at the points of maximum velocity, also folds appear when the velocity gradient is nearly opposite to the velocity direction. One gets the general impression that the stretching is the natural, stable type of development, while all kinds of shrinkings are unstable, the band immediately bending aside to form a fold which can maintain the stretching tendency. Finally, there have formed a great number of simple and multiple folds and the development then proceeds very quickly to a chaotic state where no detailed features of the band can be observed.

In the theoretical discussion, it is natural to make the assumption that the rate of stretching of a typical band is proportional to the length l of the band itself, the constant of proportionality depending on the features of the velocity field at the actual time,

$$\frac{dl}{dt} \sim \alpha(t)\, l$$

According to the experimental evidence, α should be a positive quantity.

If we have a stationary turbulence field, α should be regarded as a constant and we should find an exponential stretching,

$$l \sim l_0\, e^{\alpha(t-t_0)}$$

This case is considered earlier by BATCHELOR (1952). However, in the case we are discussing, namely the development of the turbulent field from an initial large-scale motion, α will not be constant but increases all the time, and it seems as if no stationary motion of Batchelor's type exists at all. It seems difficult to make here any precise statement, but we will at least make likely that α increases so rapidly that the bands are stretched out to infinite length and the whole turbulent break-down is completed within a *finite* time.

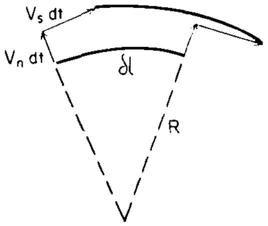

Fig. 5. Stretching of a line element.

Consider a small line element of length δl and with a radius of curvature R. During a time dt the length of this element is changed to

$$\delta l + d(\delta l) = \delta l + \frac{\partial v_s}{\partial s}\,\delta l\, dt + \frac{v_n}{R}\,\delta l\, dt$$

where v_s and v_n are the tangential and normal velocity components, respectively (Fig. 5). Thus,

$$\frac{d}{dt}(\delta l) = \left(\frac{\partial v_s}{\partial s} + \frac{v_n}{R}\right)\delta l$$

and integrating over a band from one end-point P_0 to the other P_1, we find for the rate of change of its total length l

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[1] The precise statement that can be made here is difficult to express in non-mathematical terms, and the interested readers are referred to text-books on ergodic theory.

$$\frac{dl}{dt} = \int_{P_0}^{P_1} \left(\frac{\partial v_s}{\partial s} + \frac{v_n}{R} \right) ds = v_{s_1} - v_{s_0} + \int_{P_0}^{P_1} \frac{v_n}{R} ds \quad (4)$$

Considering now the experimental results (Fig. 3) we see clearly that the curvature term is large only at certain folds of the band. On the other hand, just at the folds the large curvature term must be approximately balanced by a tangential shrinking,

$$\int_{P_1}^{P_1'} \frac{v_n}{R} ds \sim -(v'_{s_1} - v_{s_1})$$

$$\int_{P_2}^{P_2'} \frac{v_n}{R} ds \sim -(v'_{s_2} - v_{s_2})$$

etc.

(see Fig. 6).

If the curvature term could be neglected except just at the folds, we should then have

$$\frac{dl}{dt} \sim v_0 + 2v_1 + 2v_2 \ldots$$

where v_1, v_2, etc. are the velocity magnitudes at the respective folds. Thus, if V is the characteristic velocity of the fluid, and ν is the actual number of (simple) folds, the order of magnitude of $\frac{dl}{dt}$ is

$$\frac{dl}{dt} \sim 2\nu V$$

Of course, the foregoing estimate is not very satisfactory if the velocity field varies relatively little between the neighbouring folds. Accordingly, folds formed by the small-scale motion of the fluid should not be counted until they have been diffused so far away from each other that they can move independently.

Further support to the assumed mechanism is obtained from a numerical study of the experimental results. Except for a shorter initial period, it is found that about 70—80 % of the stretching is here due to the pure motion of the folds as described above.

Now, $\frac{d\nu}{dt}$, the number of folds formed per second, should obviously be directly proportional to the total length of the band. Of course, the effective length of the band available for folding is reduced by the existence of earlier folds, but this reduction is precisely balanced by the increased chances for multiple folds. The factor of proportionality should in its turn be inversely proportional to the characteristic length-scale λ of the motion and to a time-constant T, giving the characteristic time required for the developing of a fold, when the conditions for such a development are at hand. T will be something of the order of $\frac{\lambda}{V}$, and thus

$$\frac{d\nu}{dt} \sim \frac{lV}{\lambda^2}$$

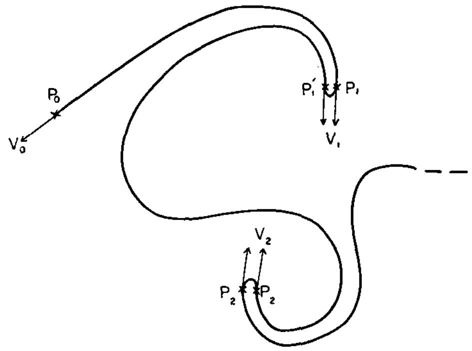

Fig. 6. Stretching of a band.

Here V could be identified with the root-mean square velocity of the fluid, which is conserved during the motion, but λ must be supposed to decrease at the same rate as the width of the band, that is inversely as the length of the band,

$$\lambda \sim \lambda_0 \cdot \frac{l_0}{l}$$

We then find $\frac{d\nu}{dt}$ proportional to l^3,

$$\frac{d\nu}{dt} \sim C^2 V l^3$$

where

$$C = \frac{1}{\lambda_0 l_0}.$$

Now,

$$\frac{d\nu}{dl} = \frac{d\nu}{dt} : \frac{dl}{dt} \sim \frac{1}{2} C^2 \frac{l^3}{\nu}, \text{ or}$$

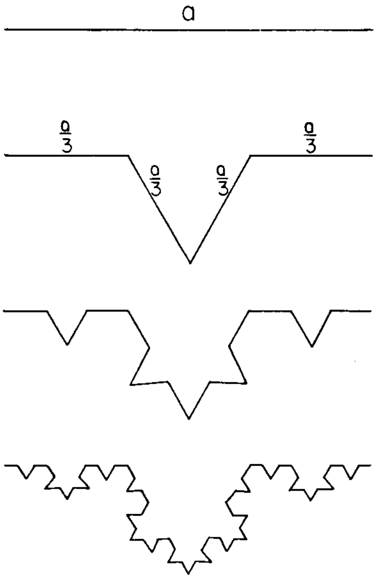

Fig. 7. Construction of the snow-flake curve.

$$\frac{d(v^2)}{dl} \sim C^2 l^3$$

and integrating we have

$$v \sim \frac{C}{2} l^2$$

The number of folds thus should be proportional to l^2.

We get

$$\frac{dl}{dt} \sim C V l^2$$

so that

$$\alpha(t) = C V \cdot l$$

and integrating

$$l \sim \frac{l_0}{1 - C_1(t - t_0)} \qquad (5)$$

where

$$C_1 = C l_0 V = \frac{V}{\lambda_0}$$

Accordingly, we arrive at the interesting result that the bands are drawn out to infinite length within a *finite* time, which is of the order of magnitude of $\frac{\lambda_0}{V}$, where λ_0 is the characteristic length-scale of the initial motion and V is the characteristic velocity. One might perhaps find the above result unreasonable, since the velocity of the fluid must always remain finite. However, it is certainly possible to deform a finite line-element into a curve of infinite length during a finite time and with finite velocities of all points on the curve. To take a simple example, we consider the steps of displacements shown in Fig. 7, leading to the so-called snowflake-curve.

If the initial length of the line is a, and we maximate all velocities to a value v, then the first step can be made during a time $\Delta t_1 < \frac{1}{3}\frac{a}{v}$, the second during a time $\Delta t_2 < \frac{1}{3^2}\frac{a}{v}$ etc., and the total time to perform any number of steps can be made smaller than $\sum_{k=1}^{\infty} \frac{1}{3^k}\frac{a}{v} = \frac{1}{2}\frac{a}{v}$, while on the other hand the length of the curve after n steps is $a_n = \left(\frac{4}{3}\right)^n a$ and so goes to infinity with n. However, in this example as well as in our fluid case, some space derivatives of the velocity must necessarily go to infinity, making the final velocity field wholly discontinuous.

Of course, the final explosive stretching of the fluid elements can never be observed in reality due to viscous effects, but the above result may be of interest for instance in connection with numerical forecasting, where one tries to predict the development by use of some idealized hydrodynamic model (in fact, indications of a break-down of the motion has been observed in several numerical forecasts with a barotropic model, but it has not yet been made clear whether this is caused by computational errors or not).

It will be stressed here again that our considerations only concern with the *general* (turbulent) development. Of course, we can select *special* cases where the results do not hold, as for instance the cases of stationary motion. The possibility of drawing any general conclusions regarding the development is wholly due to the fact that we exclude certain special and relatively improbable situations.

To make a first crude test of the above theory, the mean stretching of the elements in the described experiment has been measured

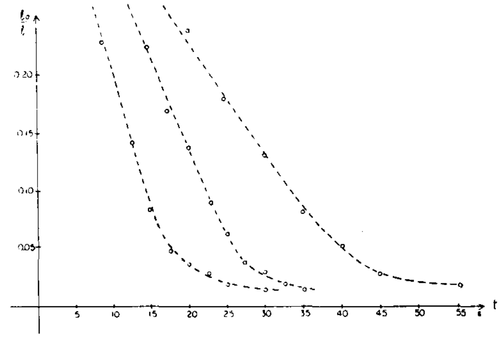

Fig. 8. Observed mean stretching of fluid elements.

(three cases) and plotted inversely against time (Fig. 8). One should not rely too much upon the numerical results, since the three-dimensional and viscous effects disturb the motion in the experiment considerably. However, it is interesting to see how well the points in each case fit a straight line as predicted by (5), at least during the first "inertial period", when the inertia forces still dominate over the viscous forces.

We will end this chapter by saying also a few words about the case of a three-dimensional ideal fluid. The development is here much the same as in the two-dimensional case, but in one respect there is a fundamental difference: the vorticity vectors must no longer be parallel but can turn freely. It has sometimes been argued that since the vorticity vectors are expected to arrive ultimately at a uniform directional distribution, we should find a complete non-viscous dissipation of the motion. However, this needs not to be true, because the magnitude of the vorticity vector and also the anomalies of this will all the time increase due to the stretching of the vortex tubes and the corresponding shrinking of their cross sections. Even if the directional distribution in the run of time becomes uniform, the anomalies may well increase at such a rate that a non-vanishing (coarse grain) velocity field is maintained. Obviously the question cannot be settled before some quantitative analysis has been performed.

3. Quasi-equilibrium motion

The precise development of motion in the foregoing experiment will of course depend on the special velocity field that is set up initially. However, we expect that the details of the initial field should be important only for the motion during a shorter initial period, and that the motion later on should develop into an asymptotic state, the form of which is essentially determined by such over-all quantities as the total energy, the characteristic scale of the motion, etc. Furthermore, it is likely that this asymptotic state could be characterized as a gradually changing quasi-equilibrium. To get a picture of the state, we can imagine the fluid as made up of a number of characteristic, separate fluid elements. The neighbouring elements interact with each other and try to arrive at some mutual equilibrium. On the other hand, they can preserve their individuality only for a shorter time, and are soon broken down into some new elements. However, the time necessary to establish their equilibrium is probably often short as compared to the life-time of the elements and to the time necessary for an essential over-all change of state, and then some kind of a quasi-equilibrium motion is obviously set up.

One can here make a comparison with the kinetic theory of gases. There the development of an initially given molecular distribution has been discussed, and it has been demonstrated that the distribution quickly settles down to a quasi-Maxwellian state, the further changes of which depend only on certain over-all micro-quantities (the hydrodynamic and thermodynamic variables). The time necessary to establish the Maxwellian equilibrium is here not more than some billionths of a second (at normal temperature and pressure), which is much less than the time characterizing an ordinary over-all change of state in the gas.

Concerning the existence of a quasi-equilibrium state in fluid developments, there have been many investigations in the statistical theory of turbulence on the basis of the Kolmogoroff Similarity Hypothesis (see f. i., BATCHELOR 1953), and both from these investigations and from experiments one finds that the quasi-equilibrium concept has a very general applicability. In the statistical theory of turbulence the reasoning centers around the Fourier spectrum and the correlation functions of the energy and velocity fields, and several precise results are obtained, which will not be repeated here. On the other hand it should be valuable to supplement this essentially statistical line of investigation with a more physical one, discussing by help of the methods of statistical mechanics some simplified mechanical models of the fluid. As a start, we shall here take up to discussion the simple point-vortex model, discussed earlier by Onsager, and try to find out under which conditions this model permits a quasi-equilibrium state to exist.

We consider a number (n) of vortices of strengths $\mu_1, \ldots \mu_n$, moving in an $x-y$-plane. These vortices generate a flow characterized by a stream-function

$$\psi^\star = \sum_{i=1}^{n} \mu_i G(x, y, x_i, y_i)$$

where G is Green's function appropriate to the problem. For an unlimited fluid, which is the only case considered here, we have

$$G(x, y, x_i. y_i) = \frac{1}{2\pi} \log \sqrt{(x-x_i)^2 + (y-y_i)^2}$$

The point-vortices should follow the motion of the fluid materially, thus their velocities are

$$u_k = \frac{dx_k}{dt} = -\left(\frac{\partial \psi^\star}{\partial y}\right)_{\substack{x=x_k \\ y=y_n}}$$

$$(k = 1, 2, \ldots n) \qquad (6)$$

$$v_k = \frac{dy_k}{dt} = \left(\frac{\partial \psi^\star}{\partial x}\right)_{\substack{x=x_k \\ y=y_k}}$$

We have

$$\left(\frac{\partial \psi^\star}{\partial x}\right)_{\substack{x=x_k \\ y=y_k}} = \frac{\partial}{\partial x_k}\left[\sum_{i=1}^{n} \mu_i G(x_k, y_k, x_i, y_i,)\right]$$

and similarly for $\left(\frac{\partial \psi^\star}{\partial y}\right)_{\substack{x=x_k \\ y=y_k}}$. These last expressions, however, include a (constant) singular contribution representing the infinite self-potential of the vortex-point, and this must be subtracted (rigorously this can be justified by starting from some small vortex-plates and letting their areas tend to zero in the final result). Introducing the energy function

$$W(x_1, y_1, \ldots x_n, y_n) = \sum_{\substack{i, j=1 \\ i>j}}^{n} \mu_i \mu_j G(x_i, y_i, x_j, y_j)$$

we find then that the system (6) can be written in the form

$$\mu_k \frac{dx_k}{dt} = -\frac{\partial W}{\partial y_k}$$

$$\mu_k \frac{dy_k}{dt} = \frac{\partial W}{\partial x_k}$$

Finally, introducing the variables

$$p_k = \sqrt{\mu_k}\, x_k$$

$$q_k = \sqrt{\mu_k}\, y_k$$

this becomes

$$\frac{dp_k}{dt} = -\frac{\partial W}{\partial q_k}$$

$$\frac{dq_k}{dt} = \frac{\partial W}{\partial p_k}$$

The system is thus Hamiltonian and conservative, since W does not depend explicitly on time. In fact ϱW equals the kinetic energy T of the associated fluid, apart from the singular contribution. Accordingly, we have here a case where the methods of statistical mechanics, as described in the first chapter, should apply. There are, however, some special questions to be considered. To begin with, we cannot have a statistically uniform distribution of vortex-points throughout the whole plane, since this would give us infinite velocities. On the other hand, if we consider a local cloud of vortex-points, we will expect that in the run of time the cloud spreads out and at least part of the points escape to infinity. The escape could of course be avoided by introducing a solid boundary around the cloud, but we prefer here to put a restriction on the characteristic scale λ of the system, prescribing its ensemble mean value. The characteristic scale could for instance be the mean distance between two vortex-points picked out at random.

Moreover, some restriction should be put on the energy of the system. In most experiments, we study systems having energies in a fairly narrow range, and an ensemble lacking any restriction in this respect cannot at all be representative. In fact, such an ensemble generally puts an overwhelming weight to the very large energies. Thus we shall prescribe here at least the ensemble mean energy.

Introducing then the three subsidiary conditions

$$\mu_1 \mu_2 \dots \mu_n \underset{\text{all space}}{\int \dots \int} P\,dV = 1$$

$$\mu_1 \mu_2 \dots \mu_n \underset{\text{all space}}{\int \dots \int} \lambda P\,dV = \overline{\lambda} \qquad (7)$$

$$\mu_1 \mu_2 \dots \mu_n \underset{\text{all space}}{\int \dots \int} W P\,dV = \overline{W}$$

where dV stands for a configuration element $dx_1\,dx_2 \dots dx_n\,dy_1\,dy_2 \dots dy_n$, the minimization of the quantity H (cf. (2)) will lead to the following expression for the equilibrium distribution of probability density

$$P = C e^{-a\lambda} \underset{\substack{i,j=1 \\ i>j}}{\Pi} r_{ij}^{-\frac{\mu_i \mu_j}{2\pi\Theta}} \qquad (8)$$

C, a and Θ are constants to be determined from the subsidiary conditions (7). C and a are of course both positive, while Θ may be either positive or negative.

We recall here the significance of P: $P\mu_1\mu_2 \dots \mu_n\,dx_1\,dy_1 \dots dx_n\,dy_n$ is the probability to find a configuration state where the first vortex-point μ_1 lies inside a surface element $dx_1\,dy_1$ at the point x_1, y_1, and the second vortex-point μ_2 lies inside a surface element $dx_2\,dy_2$ at the point x_2, y_2, etc.

Now, to get any equlibrium at all we must require that all the previous integrals converge. At infinity the integrals certainly tend toward zero rapidly enough to secure convergence, but some trouble may arise at the point $r_{ij} = 0$. The most critical contribution is of the type

$$\underset{\text{at } r_{ij}=0}{\int \dots \int} \log r_{ij}\, r_{ij}^{-\frac{\mu_i \mu_j}{2\pi\Theta}} dx_i dy_i\, dx_j dy_j$$

Introducing here x_i, y_i as a new fixed origin, we may consider instead

$$\int_{r_{ij}=0}^{\varepsilon} \log r_{ij}\, r_{ij}^{-\frac{\mu_i \mu_j}{2\pi\Theta}} dx_i dy_i\, 2\pi r_{ij} dr_{ij} =$$

$$= 2\pi dx_i dy_i \int_0^{\varepsilon} \log r_{ij} r_{ij}^{1-\frac{\mu_i \mu_j}{2\pi\Theta}} dr_{ij}$$

and derive the convergence criterion

$$\mu_i \mu_j < 4\pi\Theta \quad (i, j = 1, 2, \dots n) \qquad (9)$$

Regarding the sign of Θ, which is of course of great importance, some information could be obtained from a qualitative discussion of the "structure function"

$$\phi(E) = \underset{W>E}{\int \dots \int} e^{-a\lambda} dV$$

the integration being taken over that part of the phase space where the energy W is less than a certain value E. The probability to find the system at an energy E, varied over a small range dE, is proportional to

$$e^{-\frac{E}{\Theta}} \frac{d\phi}{dE}(E)\,dE$$

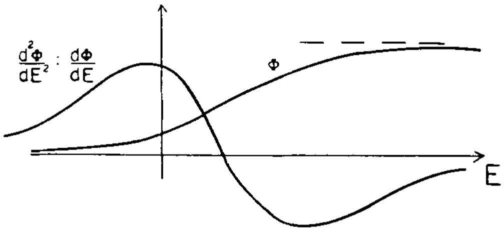

Fig. 9. Appearance of the structure function.

and the maximum probability occurs when

$$\frac{d^2\phi}{dE^2}(E) = \frac{1}{\Theta} \frac{d\phi}{dE}(E)$$

The energy value at which this maximum is attained is expected to lie in the neighbourhood of $\overline{W}$, and $\frac{d^2\phi}{dE^2}(\overline{W}) : \frac{d\phi}{dE}(\overline{W})$ may then be considered as a first approxmation to $\frac{1}{\Theta}$. The general form of ϕ and $\frac{d^2\phi}{dE^2} : \frac{d\phi}{dE}$ is indicated in Fig. 9.

Combining now this with the convergence criterion we can draw the following general conclusions:

1. For a given set of vortices, an equilibrium exists if the energy of the system is sufficiently high or sufficiently low.

2. For a given energy value, an equilibrium exists if the vortices are sufficiently weak.
3. In the non-equilibrium case, the vortices will combine into new vortices, which are intensified if the energy is high but weakened if the energy is low.

In the general case it seems difficult to give some quantitative results, since we cannot evaluate the ϕ-integral and determine the exact form of Θ as depending on E and the μ_i:s. On the other hand, it seems possible to get precise results of interest in some special cases. The discussion of these cases, which involve a lot of mathematical transformations will not be taken up here, however.

Finally we would like to make a remark concerning the physical significance of the model discussed. It is clear that our point-vortex model can represent a given fluid state to any degree of accuracy if the number of point-vortices is increased sufficiently. However, our intention is not to represent here the final equilibrium state of the continuous fluid (in fact, it is not possible to make a transition over the continuous state in our result (8) by making the number of point-vortices infinite!), but to represent the momentary equilibrium attained by some characteristic, finite lumps of fluid. It seems most natural to think of each point-vortex as representing a real, physically separated vortex of the fluid. As an example, in describing the large-scale atmospheric state by our model we should represent each cyclone and anticyclone by one point-vortex of appropriate strength. It cannot be denied that the interpretation is somewhat vague and subjective, but it seems hard to make here any precise statement. On the other hand, considering later on more refined models where the scales of the motion also enter, for instance a model built up of vortex-plates, one should be able to set up more precise connections between the model and the fluid.

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