Apparent horizontal diffusion in stratified vertical shear flow

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ABSTRACT

An investigation of the horizontal turbulent diffusion in relation to environmental conditions, such as density stratification, vertical current shear, and current variability, is presented. Tracer experiments have been conducted in fjords, in coastal areas, and in offshore waters with a view to obtaining data for such a study. In addition, observations of wind velocity, salinity, temperature, and current have been carried out. A model is proposed assuming that the apparent horizontal diffusion is determined by the combined action of vertical diffusion and vertical current shear. The theoretical predictions are found to compare well with the experimental results.

Observations

In situ measurements and water samples have been used for the salinity and temperature determinations. The current is observed by means of Ekman current meters, and in a few cases moored current meters. The observations are made at intervals of 1–2 m, and they are carried out over an extended period in connection with the tracing. Mean values of density stratification and current shear are determined from the measurements. The data are given by Kullenberg (1968, 1971), and specifications of the experiments are given in Table 1.

The tracing technique is described by Kullenberg (1968). A density adjusted solution of rhodamine B is injected in a subsurface layer with initial thickness from 70 to 400 cm. The injection is either momentaneous or lasting 2–5 minutes (Table 1). The momentaneous injection is made with a barrel which is opened at the selected depth. In other cases the dye is injected through a 2–4 m long vertical diffuser, connected by a hose to the containers on board.

The dye concentration is recorded continuously by a submersible fluorometer towed behind the ship. The position of the ship is determined either relative to a parachute drogue drifting at the depth of injection and through a thin wire connected to a surface buoy, or by using DECCA navigation. The ship is cruising systematically over the area in order to obtain a full coverage of the rhodamine spot.

The experiments are performed in stratified water layers with a vertical current shear. In most cases the rhodamine after a short initial period is observed in layers with a thickness varying between 30 and 80 cm. The vertical concentration distribution in the layers is difficult to determine. Usually the layers are sharply defined with almost homogeneous concentration distribution. In the present investigation the vertical distribution is treated as on an average homogeneous. This is a permissible approximation.

The observed volume concentrations are plotted along the track of the ship. Whenever possible, crossings close to each other in time and space are combined in order to smooth the momentaneous values. Isolines are drawn in the plot, illustrating the distribution of the volume concentration. The corresponding surface concentration is found by multiplying with the thickness of the spot, which is assumed to be constant over the entire spot. The observed mass found by horizontal integration is between 20 and 50 % of the injected amount. The vertically diffused rhodamine is however usually not detected because of the influence of the shear. It can be estimated by using the observed vertical diffusion coefficient (Kullenberg, 1971), and the mass balance is found by adding this amount to the integrated mass. The mass balance varies between 50 and 100 %.

Table 1. *Area, depth, type of injection and duration of the experiments*

Date	Position/area		Depth (m) dye	of bottom	Type of injection	Duration hours
1965, 1.4	N 57°22.′5 E 12°02′	Outer part of fjord	17–19	20	Momentaneous	4
17.11	N 55°46.′4 E 12°45.′4	The Sound	12–14	15–27	Momentaneous	4
1966, 24.10	N 57°40.′7 E 11°35′	Coastal	10–14	30	Diffuser	8
1967, 16.8	N 57°40.′5 E 11°35′	Coastal	18–24	30	Diffuser	5.5
20.12	N 57°41.′5 E 11°34.′4	Coastal	15–20	30–35	Diffuser	15
21.12	N 57°43′ E 11°32.′5	Coastal	17–19	30–35	Diffuser	5
1968, 8.10	N 59°47.′8 E 09°32.′8	Inner fjord	9–11	100	Diffuser	7
27.11	N 57°40.′5 E 11°34.′5	Coastal	13–16	30	Diffuser	7
1969, 9.12	N 54°58.′5 E 14°00.′5	Open sea	22–28	45–50	Diffuser	17

The observed distributions used in the present calculations are given by Kullenberg (1971). They are more or less asymmetrical, usually elongated in the direction of the mean current shear and with skewed concentration distribution. Often the concentration is not smooth inside the patch. Mostly a good coverage of the spot has been achieved, and the distributions are regarded as fairly reliable.

Calculations of diffusion parameters

By means of the observed concentration distributions vertical and horizontal diffusion parameters are determined.

In studying the vertical diffusion two separate cases are treated, viz. strong and weak stratification. In both cases the diffusion equation

$$\frac{\partial c}{\partial t} = K \frac{\partial^2 c}{\partial z^2} \qquad (1)$$

is used. The diffusion coefficient is regarded as constant in each trial.

When the stratification is strong, thin, well-defined layers of dye are observed with virtually constant thickness for long periods of time. This configuration is interpreted as an effect of the vertical current shear in combination with the stratification. The decrease of the dye concentration in the layers is attributed to vertical diffusion only.

In the case of weak stratification the vertical exchange is more effective and an increase of the thickness of the layers can be detected.

The models and data are presented by Kullenberg (1968, 1971). The values of K compare reasonably well with other similarly obtained results. The values used in the present investigation are given in Table 3.

As regards the horizontal diffusion this was studied by means of the theory of Joseph & Sendner (1958) in the previous work (Kullenberg, 1968). In this theory the tracer is assumed to be evenly dispersed in a layer which it does not leave, and only horizontal mixing is considered. Further, the observed horizontal distributions are transformed into rotationally symmetrical distributions. In the present experiments the diffusion is observed in rather thin subsurface layers undisturbed by solid boundaries, and it seems probable that the vertical diffusion in combination with the vertical current shear has an influence on the horizontal dispersion. The elongation of the

Table 2. *Observed values of the variances σ_x^2, σ_y^2, σ^2, the corresponding σ_{rc}^2, and the ratio σ^2/σ_{rc}^2*

Date	t (s)	σ_x^2 (cm²)	σ_y^2 (cm²)	σ^2 (cm²)	σ_{rc}^2 (cm²)	σ^2/σ_{rc}^2
1.4	4.2×10^3	4.4×10^6	1.7×10^6	6.1×10^6	1.3×10^6	4.7
17.11	7.2×10^3	1.8×10^7	2.4×10^6	2.0×10^7	3.0×10^6	6.7
17.11	1.3×10^4	2.3×10^7	1.0×10^7	3.3×10^7	1.7×10^7	1.9
24.10	1.5×10^4	4.1×10^8	2.0×10^7	4.3×10^8	5.1×10^7	8.4
16.8	1.1×10^4	1.7×10^8	1.7×10^7	1.9×10^8	4.7×10^7	4.0
20.12	2.9×10^4	2.5×10^8	5.0×10^7	3.0×10^8	9.7×10^7	3.1
20.12	4.9×10^4	8.9×10^8	2.8×10^8	1.2×10^9	9.5×10^8	1.3
21.12	1.2×10^4	2.8×10^8	1.0×10^7	2.9×10^8	1.2×10^8	2.4
8.10	1.2×10^4	1.4×10^7	1.9×10^6	1.6×10^7	5.6×10^6	2.9
27.11	1.4×10^4	1.6×10^8	1.8×10^7	1.8×10^8	3.0×10^7	6.0
9.12	3.8×10^4	3.7×10^9	3.0×10^7	3.7×10^9	4.3×10^8	8.6

distribution in the direction of the mean current shear supports this view.

A more appropriate analysis is to calculate the second-order central moments, or the variances, of the distributions, and to investigate the dependence of these values on the different conditions. The variance is defined by

$$\sigma^2(t) = \frac{\iiint_{-\infty}^{+\infty} (x^2+y^2)\, c(x,y,z,t)\, dx\, dy\, dz}{\iiint_{-\infty}^{+\infty} c(x,y,z,t)\, dx\, dy\, dz} \tag{2}$$

In the present case the integration is carried out only in the horizontal plane since the vertical concentration distribution is treated as homogeneous. It is noted that the actually observed mass is used in the denominator. The origin of the rectilinear coordinate system is placed in the region of the highest concentration with the x-axis pointing in the direction of the elongation and the z-axis positive downwards. The integration is carried out to the limit of detection. Only distributions regarded as good, with at least 3 concentration levels for the calculations, have been used. A consistency check is provided by the relations

$$\iint_{-\infty}^{+\infty} c(x,y,t)\, dx\, dy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} c\, dy = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} c\, dx$$

Usually these values differ by about 10%, and it is estimated that the possible error in σ^2 is $\pm 10\%$.

It is

$$\sigma_x^2 = \iint_{-\infty}^{+\infty} x^2 c\, dx\, dy \Big/ \iint_{-\infty}^{+\infty} c\, dx\, dy,$$

$$\sigma_y^2 = \iint_{-\infty}^{+\infty} y^2 c\, dx\, dy \Big/ \iint_{-\infty}^{+\infty} c\, dx\, dy \tag{3 a, b}$$

and $\sigma^2 = \sigma_x^2 + \sigma_y^2$.

The elongation is defined as the ratio

$$e = \sigma_y / \sigma_x \tag{4}$$

The values are given in Table 2. The elongation is very marked in some cases.

Despite the evidence that a symmetrical treatment of the data is not quite adequate, it is of interest to compare the variance derived by direct integration with the corresponding rotationally symmetrical variance.

The horizontal distribution can be transformed into the symmetrical form

$$c(r,t) = \frac{M}{2\pi\sigma_{rc}^2} e^{-r^2/2\sigma^2_{rc}} \tag{5}$$

Here r is the distance from the moving point of injection. It is determined as the radius of the circle having the same area as the area enclosed by the really observed isoline. The variance is found from the definition

$$\sigma_{rc}^2 = \frac{\int_0^{\infty} r^2 c(r,t)\, 2\pi r\, dr}{\int_0^{\infty} c(r,t)\, 2\pi r\, dr} \tag{6}$$

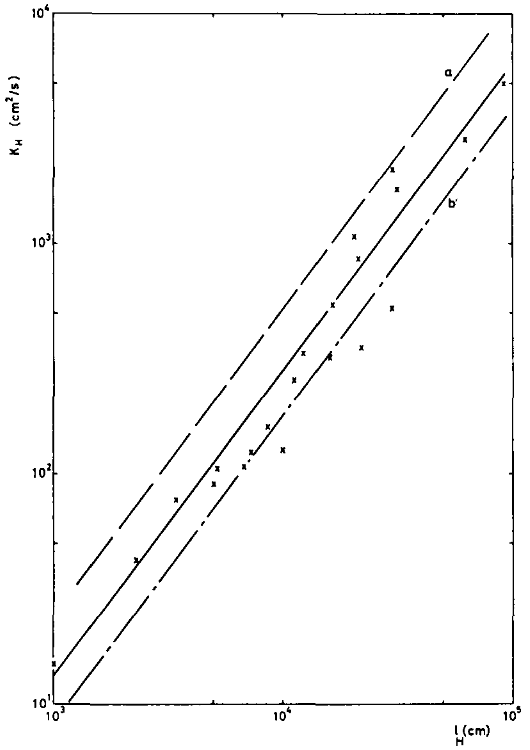

Fig. 1. Horizontal coefficient K_H vs. length scale l_H. *a, b* regression lines of Okubo & Ozmidov (1970); Present values yield the full drawn line $\propto l_H^{4/3}$.

These values (Table 2) are always less than the variances found by direct integration which, however, is to be expected (Okubo, 1968). It is estimated that the error in σ_{rc}^2 is ± 15 %.

The present values can be compared with other reported results. Okubo & Ozmidov (1970) used every available set of data to investigate the relation between the horizontal turbulent diffusion coefficient K_H and the scale l_H, applying the following definitions

$$K_H = \sigma_{rc}^2/4t,\ l_H = 3\sigma_{rc}$$

The coefficient K_H is the apparent horizontal diffusivity.

All σ_{rc}^2-values determined by the present technique (Kullenberg, 1971), have been used to calculate K_H and l_H by the same relations (Fig. 1). The best fit is given by the equation (K_H in cm² sec⁻¹ and l_H in cm)

$$K_H = 1.3 \times 10^{-3} \times l_H^{1.31}$$

The value of the exponent is very close to the value 1.33 given by Okubo & Ozmidov. Okubo (1971) on the other hand gives the exponent 1.15. The scatter in Fig. 1 is considerable. Using only values of l_H larger than 6×10^3 cm the best fit exponent is found to be 1.5. Thus an error of the order ± 15 % is expected in the exponent.

Okubo & Ozmidov (1970) give one regression line for small scales (*a*, Fig. 1) and one for large scales (*b*, Fig. 1). The line *b* has here been extrapolated to smaller scales than in the original paper. The present values are slightly lower than those of Okubo & Ozmidov. This is attributed to the fact that they are determined for subsurface layers in stratified waters, where one should expect the turbulent mixing to be less intense than in the near surface zone from which most of Okubo's data come.

Influence of the shear on the apparent horizontal diffusion

The elongation of the distributions suggests that the mean shear in the layer exerts an influence on the dispersion. Kullenberg (1968) found that the horizontal diffusion velocity P, as defined by the Joseph-Sendner theory, was proportional to the parameter $H\,|dq/dz|$, where H is the initial thickness of the spot and $|dq/dz|$ the vertical gradient of the current vector. This empirical relationship cannot be expected to hold true for large time- and length-scales, but it does establish the importance of the shear.

The shear-effect, i.e. the dispersion of a vertical column caused by the combined effect of the vertical shear and the vertical diffusion, has been investigated by several authors. Below some of the work pertinent to the present investigation is discussed.

Taylor (1953, 1954) introduced the idea of shear-diffusion in studying the dispersion in a tube, both in laminar and turbulent flow, and clearly demonstrated the importance of the effect. Bowles et al. (1958) used the idea to explain the elongation of a diffusing patch in shallow waters.

Novikov (1958) developed a general theory which describes the dispersion from an momentaneous point source in a horizontal stream with vertical shears.

The velocity components are written in the form

$$u = u'(t) + A(t)\,z$$

$$v = v'(t) + B(t)\,z$$

Horizontal and vertical turbulent diffusion is included by means of time-dependent diffusion coefficients. After giving the general solution Novikov considers the special case of a steady stream, placing the coordinate system so that $B = 0$. He finds that the shear-effect will dominate over the purely horizontal diffusion after an initial period. Then the peak concentration decreases as $t^{-2.5}$ instead of as $t^{-1.5}$ in the absence of the shear. The concentration isolines in the x–z plane are ellipses strongly elongated in the direction of the shear.

Elder (1959) discusses the effect of the lateral velocity variation on the diffusion in a steady open channel flow by extending the analyses of Taylor (1954). Elder shows that the longitudinal dispersion is determined by the combined action of lateral diffusion and advection by the mean flow. This means that a patch of dye injected in the channel will become elongated in the direction of the mean flow. It is further shown experimentally that the concentration distributions are Gaussian, and so the diffusion may be described by effective diffusion coefficients.

Bowden (1965) extending the discussion to estuaries and coastal regions, formulates a model for both an oscillating and a steady current in the longitudinal direction. Transverse variations are neglected. He studies the case of a vertically bounded flow, with a depth-dependent mean velocity, and finds the dispersion to be inversely proportional to the vertical diffusion. The oscillating current consists of a single harmonic constituent, having negligible variations in phase with depth. Mean conditions over one or several tidal periods are investigated, and the importance of the shear-effect in this case is demonstrated.

Carter & Okubo (1965) discuss a model with the velocity spectrum consisting of two parts, viz. large and small scale motions. The large scale motions are responsible for the non-uniformities of the mean velocity field, appearing as steady shears in both vertical and horizontal directions. They treat a case with a mean velocity in the x-direction only, made up of a purely time-dependent part, a transverse and a vertical shear component, i.e.

$$U = U_0(t) - y\frac{dU}{dy} - z\frac{dU}{dz}$$

$$V = W = 0$$

Here $U_0(t)$ is the time-dependent velocity in the plane $z = 0$.

The small scale motions are incorporated in the model by constant eddy diffusion coefficients K_x, K_y and K_z in the x-, y-, and z-directions. The coordinate system is taken as moving with the velocity $U_0(t)$, and the diffusion equation is given and solved by Fourier transform technique. The isolines of the theoretical distribution form ellipsoids. The degree of elongation is a function of the time, the shears, and the diffusion coefficients. During an initial period the shear-effect is not very important, but a considerable time after release the elongation is very marked in the direction of the mean current.

The longitudinal variance is a sum of 3 terms,

$$\sigma_x^2 = 2K_x t + \frac{2}{3}K_y\left(\frac{dU}{dy}\right)^2 t^3 + \frac{1}{6}K_z\left(\frac{dU}{dz}\right)^2 t^3$$

while the transverse variance is given by

$$\sigma_y^2 = 2K_y t$$

The dispersion of the contaminant is determined by the combined effect of diffusion due to small scale motions and of advection due to large scale motions. The shear of the large scales accelerates the diffusion. Pritchard et al. (1966) apply the Carter–Okubo model to observations in the surface layer. They find that both the observed decrease of the peak concentration and increase of the rotationally symmetrical variance conform with the theoretical predictions.

Okubo (1967) studies the case of an oscillatory current $U_t(z)$ with a single harmonic constituent having a phase independent of depth and superimposed on a steady current $U_s(z)$ in the same direction. No transverse processes are included, and a purely two-dimensional model in the x–z plane with vertical and horizontal diffusion is investigated.

The current field is specified as

$$U_s(z) = U_0(1 - z/d)$$

$$U_t(z) = V_0(1 - z/d') \sin \omega t$$

where U_0 is the steady current at $z = 0$, and V_0 is the amplitude of the oscillating component at $z = 0$; d and d' are length scales associated with the respective motions. Okubo formulates the diffusion equation and studies the dispersion in both a bounded and an unbounded sea.

In the case of an unbounded sea the equation is transformed by introducing a moment-generating function, and solved, yielding an expression for the variance in the direction of the current. The variance is determined by the sum of the purely horizontal diffusion and the dispersion due to the shears. The latter part is proportional to the vertical diffusion and consists of the effect of the steady flow, of the oscillatory flow, and in addition a combined action of these flows. For diffusion times large compared to the period of the oscillation the influence of the steady shear dominates.

The steady state model of Carter & Okubo (1965) is re-discussed by Okubo (1968), and the results are compared with the rotationally symmetrical theories of Joseph & Sendner (1958) of Ozmidov (1958), and others. It is shown that the time-dependence of the mean horizontal variance, defined as $\sigma_x \cdot \sigma_y$, is the same for the two entirely different approaches. However, from experimental results Okubo concludes that the shear diffusion models must be regarded as considerably more realistic than the symmetrical models.

Mork (1970) studies the salt flux in a tidal channel. The salt balance is maintained by longitudinal dispersion of salt induced by vertical shears in steady and periodic currents with a compensating net flow of less saline water introduced by river discharge. The joint equations governing the salt diffusion and the dynamics of the system are considered. With constant coefficients of eddy viscosity and diffusivity, velocity and salt profiles in the x–z plane are derived.

The results are consistent with observations.

In these investigations the longitudinal dispersion due to the shear-effect is considered.

Below an advection-diffusion model is formulated for the case of a vertically unbounded region including both a longitudinal and a transverse shear-effect, and aiming at a numerical comparison with the experimental results.

An advection-diffusion model

The apparent horizontal diffusion is interpreted as an effect of the combined action of vertical diffusion and advection due to the mean flow. The spectrum of the motion is regarded as consisting of two separate parts of small and large scales respectively. The small scales effect the diffusion and the large scales the advection. The current field is specified as a mean current varying with z only, and a superimposed oscillating current component, where the amplitude varies with z but the period is independent of z. Only one single harmonic constituent is included, and for simplicity depth dependence of the phase is neglected. The curent field is written in the form

$$u = U_0(z) + u_0(z) \cos \omega t$$

$$v = V_0 + v_0(z) \sin \omega t$$

The coordinate system is defined with the x-axis positive in the direction of the constant shear, and with z positive downwards. The plane $z = 0$ is placed in the center of the layer, and the system is moving with the mean velocity at $z = 0$. The current at the level z is

$$u = z \frac{dU_0}{dz} + z \frac{du_0}{dz} \cos \omega t = -(a_0 + a \cos \omega t) z$$

$$v = z \frac{dv_0}{dz} \sin wt = -bz \sin \omega t$$

The velocity gradients are treated as constants in the layer. Neglecting the purely horizontal diffusion, an approximation which seems justified by previous results (Kullenberg, 1968), the diffusion equation becomes

$$\frac{\partial c}{\partial t} - z(a_0 + a \cos \omega t) \frac{\partial c}{\partial x} - zb \sin \omega t \frac{\partial c}{\partial y} = K \frac{\partial^2 c}{\partial z^2} \quad (7)$$

This equation is transformed by using the technique of Okubo (1967). The moment-generating function

$$\Gamma(t, l, m, n) = \iiint_{-\infty}^{+\infty} e^{lx + my + nz} c(t, x, y, z) \, dx \, dy \, dz$$

is introduced in (7). It is assumed that $c \to 0$ sufficiently rapid to secure convergence. The transformed equation is

$$\frac{\partial \Gamma}{\partial t} + l(a_0 + a\cos\omega t)\frac{\partial \Gamma}{\partial n} + mb\sin\omega t\frac{\partial \Gamma}{\partial n} = Kn^2\Gamma \quad (8)$$

At time $t=0$ the diffusing substance is assumed to be evenly dispersed along a unit length line source parallel to the y-direction at $x=0$ and $z=z_0$. Thus the initial condition is

$$c(0, x, y, t) = M_0\,\delta(x)\,\delta(z - z_0) \quad (9)$$

where $\delta(x)$ and $\delta(z)$ is the Dirac delta function. Since the tracer actually occupies a finite volume at time $t=0$, a line source is preferred to a point source. We will return to the effect of the initial thickness later.

For simplicity we take $z_0 = 0$. The condition implies that at time $t=0$

$$\Gamma = M_0 \quad (10)$$

Eq. (8) is now solved by the method of characteristics.

$$\frac{dn}{dt} = l(a_0 + a\cos\omega t) + mb\sin\omega t$$

$$\frac{d\Gamma}{\Gamma} = Kn^2\,dt$$

and the solution is

$$\Gamma = c_2\exp\left\{\frac{1}{3}l^2a_0^2t^3K + \frac{Kt}{2\omega^2}(l^2a^2 + m^2b^2) + \frac{K}{4\omega^3}(m^2b^2 - l^2a^2)\sin 2\,\omega t - 2l^2Kt \times \frac{a_0 a}{\omega^2}\cos\omega t + 2l^2K\frac{a_0 a}{\omega^3}\sin\omega t - 2lm\,Kt\frac{a_0 b}{\omega^2}\sin\omega t - 2lm\,K\frac{a_0 b}{\omega^3}\cos\omega t + lm\,K\frac{ab}{2\omega^3}\cos 2\omega t + c_1^2Kt + c_1 l a_0 Kt^2 - 2c_1 lK\frac{a}{\omega^2}\cos\omega t - 2c_1 mK\frac{b}{\omega^2}\sin\omega t\right\} \quad (11)$$

where c_1 and c_2 are constants of integration. They are determined by means of (10).

The moment-generating function is now used to derive expressions for the second-order central moments. This is accomplished by calculating

$$\sigma_x^2 = \frac{1}{M_0}\left[\frac{\partial^2\Gamma}{\partial l^2} - \left(\frac{\partial\Gamma}{\partial l}\right)^2\right]_{l=m=n=o}$$

$$\sigma_y^2 = \frac{1}{M_0}\left[\frac{\partial^2\Gamma}{\partial m^2} - \left(\frac{\partial\Gamma}{\partial m}\right)^2\right]_{l=m=n=o}$$

$$\sigma_z^2 = \frac{1}{M_0}\left[\frac{\partial^2\Gamma}{\partial n^2} - \left(\frac{\partial\Gamma}{\partial n}\right)^2\right]_{l=m=n=o} \quad (12\,\mathrm{a,b,c})$$

It is found that

$$\sigma_x^2 = \frac{2}{3}Ka_0^2t^3 + \frac{a^2}{\omega^2}Kt - \frac{a^2}{2\omega^3}K\sin 2\omega t - \frac{4a_0 a}{\omega^2}Kt\cos 2\omega t + \frac{4a_0 a}{\omega^3}K\sin\omega t$$

$$\sigma_y^2 = \frac{9}{8}\frac{w^2}{b^2}Kt + \frac{Ktb^2a_0}{a^2\omega^2}(2a_0 - a) + \frac{Kb^2}{2\omega^3}\sin 2\omega t - \frac{Kb^2(a - 4a_0)}{a\omega^3}\sin\omega t$$

$$\sigma_z^2 = 2Kt \quad (13\,\mathrm{a,b,c})$$

The leading term in σ_x^2 is due to the shear of the mean current, while the other terms are due to the oscillating component in the same direction as the mean current. The terms containing $a_0\,a$ express a coupling between the steady and the fluctuating shears. The present form of σ_x^2 is essentially the same as the relation derived by Okubo (1967).

For the transverse dispersion the oscillating current component is decisive, while it is of minor importance for the longitudinal dispersion.

The trigonometric terms are negligible relative to the leading terms. It is noted that large periods of an oscillating current system become increasingly important for increasing diffusion time.

The solution gives the dispersion from a short, momentaneous line source parallel to the y-direction. The real source has also a finite thickness, and to find the dispersion due to the initial thickness the solution (13) is integrated vertically. At time $t = 0$ the concentration M_0 per unit area is given in the layer

$$-H/2 < z < +H/2$$

Consider a volume element at $z = z_0$ with height dz_0 and unit surface area. The dispersion of the surface concentration is studied in the coordinate system ξ, η, defined by

$$\xi = x - a_0 z_0 t$$

$$\eta = z - z_0$$

According to the solution above we have

$$\sigma_\xi^2 = \frac{2}{3} K a_0^2 t^3 + \frac{a^2}{\omega^2} K t$$

$$\sigma_x'^2 = \frac{1}{M_0} \iint_{-\infty}^{+\infty} x^2 c \, dx \, dz = \sigma_\xi^2 + a_0^2 z_0^2 t^2 \quad (14)$$

for the surface element. Thus the variance for the layer $|z| < H/2$ is

$$\sigma_x^2 = \frac{1}{H} \int_{-H/2}^{+H/2} \left(\frac{2}{3} K a_0^2 t^3 + \frac{a^2}{\omega^2} K t + a_0^2 z_0^2 t^2 \right) dz_0$$

$$= \frac{2}{3} K a_0^2 t^3 + \frac{a^2}{\omega^2} K t + \frac{1}{12} a_0^2 H^2 t^2 \quad (15)$$

Due to the shear and the initial thickness H the volume element is dispersed in the direction of the mean current shear, but the volume and the volume concentration are not influenced.

The horizontal second-order central moment is then

$$\sigma^2 = \sigma_x^2 + \sigma_y^2 = \frac{2}{3} K a_0^2 t^3 + \frac{1}{12} H^2 a_0^2 t^3 + \frac{Kt}{\omega^2} \times \left[a^2 + \frac{9}{8} b^2 + \frac{b^2}{a^2} (2a_0^2 - a_0 a) \right] \quad (16)$$

The vertical mixing is of fundamental importance for the transverse dispersion. In an earlier investigation (Kullenberg, 1968) the results suggested that in the present areas the vertical mixing is inversely proportional to the density stratification. With reference to (16) this result implies that the horizontal dispersion may decrease with increasing stratification. On the other hand the shear can become stronger in the case of increased stratification. The effect of the increased shear may balance the decrease of the vertical diffusion or even be more important.

The elongation is given by $e = \sigma_y/\sigma_x$ and the distribution is elongated in the direction of the mean current shear.

The time dependence of the peak concentration $C_p \propto (\sigma_x \sigma_y \sigma_z)^{-1}$ in a 3-dimensional model is investigated by considering the product

$$\sigma_x \sigma_y \sigma_z = \frac{a_0 b}{\omega} K t^{2.5} \left[K \left(\frac{9}{4} + \frac{4a_0^2 - 2a_0 9}{a^2} \right) \times \left(\frac{2}{3} + \frac{H^2}{12Kt} + \frac{a^2}{a_0^2 \omega^2 t^2} \right) \right]^{\frac{1}{2}} \quad (17)$$

It is noted that the frequency of the oscillating current has a very significant influence.

Three different possibilities can be investigated.

(1) For large diffusion times, when

$$H^2 < 12Kt, \; a^2 < a_0^2 \omega^2 t^2$$

it is found that

$$\sigma_x \sigma_y \sigma_z \approx \frac{a_0 b}{\omega} K t^{2.5} \left[K \left(\frac{3}{2} + \frac{4(2a_0^2 - a_0 a)}{3a^2} \right) \right]^{\frac{1}{2}} \quad (18)$$

i.e. the peak concentration decreases at $t^{-2.5}$. Pritchard et al. (1966), applying the model of Carter & Okubo (1965), show that this time-dependence is in agreement with experimental results from the near surface zone for $t \geqslant 20$ hours.

(2) For the case

$$H^2 \gg 12Kt, \; a^2 \ll a_0^2 \omega^2 t^2$$

$$\sigma_x \sigma_y \sigma_z \approx \frac{a_0 b}{\omega} K H t^2 \left[\frac{1}{12} \left(\frac{9}{4} + \frac{4a_0^2 - 2a_0 a}{a^2} \right) \right]^{\frac{1}{2}} \quad (19)$$

i.e. the peak concentration decreases as t^{-2}. The significance of this result depends upon the intensity of the vertical diffusion and the initial thickness.

Table 3. *Parameters for the theoretical calculation of the variances*

Diffusion time t, thickness H, mixing coefficient K, longitudinal shears a_0 and a, transversal shear b, and frequency ω.

Date	t (s)	H (cm)	K (cm^2/s)	$a_0 \times 10^3$ (s^{-1})	$a \times 10^3$ (s^{-1})	$b \times 10^3$ (s^{-1})	$\omega \times 10^4$ (s^{-1})
1.4	4.2×10^3	70	0.08	20	15	37	10
17.11	7.2×10^3	80	0.08	20	24	55	7.5
17.11	1.3×10^4	80	0.08	13	24	55	7.5
24.10	1.5×10^4	300	0.2	15.5	6	6.6	2.9
16.8	1.1×10^4	400	0.35	12.5	—	—	—
20.12	2.9×10^4	400	0.7	3.4	12	25	5.9
20.12	4.9×10^4	400	0.6	3.4	12	25	2.9
21.12	1.2×10^4	400	1.5	8	—	—	—
8.10	1.2×10^4	200	0.08	5.7	7.5	25	7
27.11	1.4×10^4	200	0.05	17	—	—	—
9.12	3.8×10^4	300	0.13	14.5	25	32	4.4

(3) In the case of a very thin initial source and when

$$a^2 > a_0^2 \omega^2 t^2, \quad a_0 < a$$

we have

$$\sigma_x \sigma_y \sigma_z \approx \frac{3}{2} \frac{ab}{\omega^2} (Kt)^{3/2} \tag{20}$$

i.e. the peak concentration decreases as $t^{-1.5}$. Pritchard et al. (1966) show this time-dependence to be in agreement with results from point source experiments in the near surface zone for $t \leq 8$ hours.

The connection between the rotationally symmetrical variance σ_{rc}^2 and the variances σ_x^2, σ_y^2 is investigated. The original distribution is assumed to be Gaussian with the isolines forming ellipses. The x- and y-axes are placed in the directions of the major and minor axes, respectively. This orientation is in accordance with the one used above provided the elongation really is in the direction of the mean current shear. This is usually the case in the present experiments. Areas enclosed by corresponding isolines in the elliptical as well as the rotationally symmetrical distribution must be equal. This condition yields the relation

$$\sigma_{rc}^2 = \sigma_x \sigma_y \tag{21}$$

This case is discussed by Kullenberg (1971).

Comparison with experimental results

The predicted forms of the variances are subsequently compared with the experimental

Table 4. *Diffusion time t, theoretical values of the variances σ_x^2, σ_y^2, σ^2, and the ratio σ_y/σ_x*

Date	t (s)	σ_x^2 (cm^2)	σ_y^2 (cm^2)	σ^2 (cm^2)	Theoretical σ_y/σ_x —	Observed σ_y/σ_x —
1.4	4.2×10^3	4.5×10^6	1.6×10^6	6.1×10^6	0.6	0.6
17.11	7.2×10^3	1.9×10^7	5.1×10^6	2.4×10^7	0.5	0.35
17.11	1.3×10^4	3.5×10^7	6.7×10^6	4.2×10^7	0.45	0.65
24.10	1.5×10^4	5.1×10^8	1.8×10^7	5.3×10^8	0.2	0.2
16.8	1.1×10^4	3.0×10^8	—	—	—	0.3
20.12	2.9×10^4	2.5×10^8	3.8×10^7	2.9×10^8	0.4	0.45
20.12	4.9×10^4	1.0×10^9	2.2×10^8	1.2×10^9	0.5	0.55
21.12	1.2×10^4	2.4×10^8	—	—	—	0.2
8.10	1.2×10^4	1.8×10^7	1.2×10^6	1.9×10^7	0.3	0.4
27.11	1.4×10^4	2.2×10^8	—	—	—	0.35
9.12	3.8×10^4	3.3×10^9	3.2×10^7	3.3×10^9	0.1	0.09

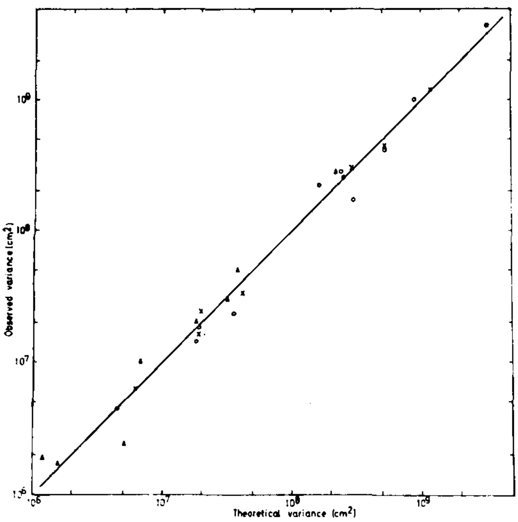

Fig. 2. Observed vs. theoretical variance. $\times = \sigma^2$; $\bigcirc = \sigma_x^2$; $\triangle = \sigma_y^2$.

results. All the parameters entering the relations can be determined from the observations.

The mean current direction and strength is calculated. The calculation is done for each depth over the period of observation. From the results the mean current for the depth interval as well as the mean shear are found. The dominating periods of the fluctuating part of the current field are then determined by plotting the transverse component vs. time for the observations at each depth. Over limited depth intervals the periods are nearly independent of depth. The amplitudes of the oscillations at each depth are determined and from the gradient of these the value of the transverse shear (b) is found. The corresponding procedure is carried out for the longitudinal component in order to find the longitudinal shear (a). In most cases the periods determined from the transverse and longitudinal components, respectively, are reasonably close.

In carrying out the calculations it is assumed that the current really can be divided in a steady and a time-dependent part. In order to check the justification for this assumption the gliding means of the observed current are calculated using the periods found as time intervals. The oscillations are then greatly reduced, and the steady part becomes pronounced. The calculations are performed whenever the series of observations are long enough, especially when moored current meters have been available (24.10, 20.12, 9.12). The results show that the separation of the current in a steady and an oscillating part is permissible. It is required that several harmonic constituents are included in the time-dependent part.

The initial thickness H of the rhodamine layer is found from observations shortly after the injection when the barrel type of injection is used, and from the length of the tube when the diffuser type of injection is used.

The vertical diffusion coefficient K is determined independently as described by Kullenberg (1968, 1971a).

The predicted variances are given in Table 4, and the values of the parameters used in the calculations are given in Table 3. From Fig. 2, where the observed and the theoretical variances are compared, it is evident that the agreement between observation and theory is satisfactory. The conclusion is that the model is supported by the observations. It is, however, not possible to establish the gradual change of relative importance of the different parameters. Nor is it feasible to investigate the various cases of time-dependence of the peak concentration. Long time series of observations are needed for that purpose. The decisive influence of the current variability and the vertical current structure in combination with the vertical mixing on the dispersion is clearly demonstrated by the present results.

There are, however, important observational deficiencies. The Ekman current observations are carried out from a single-moored ship. During the measurements the heading of the ship is noted. Usually this varies only slightly, and the influence of the oscillation of the ship on the observations is not considered as very serious except in some cases. Also, unfortunately, the current observations are not quite adequate for the determination of the periods of the oscillating components since in some cases only a few periods are covered by the observations. For these reasons the periods obtained from the Ekman current measurements are used with caution.

The observations indicate that the period, but not the phase, is approximately independent of depth over a limited depth interval. Although it is possible to include phase variations in the model this has not been considered worth while at present.

It is approximately correct to regard the shears as constants in the thin layers under consideration. This may not be the case when the current measurements are carried out on a smaller vertical scale than at present.

Discussion and conclusion

The proposed model emphasizes the influence on the apparent horizontal diffusion of the vertical mixing, the vertical current structure, and the variability of the current.

The interpretation of the current field as consisting of a steady and a time-dependent part has led to reasonable results. It is a deficiency that only one harmonic constituent is included in the model. For a proper description of the current field several constituents would be required. However, the dispersion of the rhodamine spot is a function of the scale. For small diffusion times, when the spot is small, it is the small-scale components of the motion which determine the dispersion. The large scales merely advect the whole spot. As the spot grows with increasing diffusion time the large-scale components of the motion successively become important for the dispersion. The influence of the large scale will then dominate over the influence of the small scale. The small-scale component will primarily influence the dispersion inside the spot. In the present formulation the steady part of the current includes all the components of the motion having scales so large that they only advect the spot. The combined action of the vertical mixing and the shear of the steady motion influences the longitudinal dispersion. The time-dependent part is interpreted as including all the smaller scales. The single harmonic constituent included in the model then represents the component of the motion having the dominating influence on the dispersion. The time-dependent motion determines, in combination with the vertical mixing, the transverse dispersion.

The results cover only a rather limited range of scales. The values of the diffusion parameters are consistent with other reported results, and the scale-dependence of the horizontal diffusion is in agreement with the general trend of other data. The effects of the genuine horizontal turbulence and the horizontal shears on the dispersion are not included in the model since it would obscure the model and make comparison with observations difficult. The result suggests that these effects may be considered as unimportant with the present scales.

To reveal how and when the horizontal turbulence and current structure enter the problem, experiments covering considerably larger scales are needed. The experiments must include long time series of current measurements in a horizontal network at several depths.

The present experiments have been carried out in a variety of environmental conditions. Despite the observational deficiencies, it is concluded that the proposed model yields a reasonably good description of the apparent horizontal turbulent diffusion for the conditions and scales covered by the experiments.

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Notations

a_0	vertical gradient (shear) of mean current velocity
a, b	vertical gradients of longitudinal and transverse components of the oscillating current
c	concentration
e	elongation
H	initial thickness of rhodamine layer
K	vertical diffusion coefficient
K_H	apparent horizontal diffusion coefficient
K_x, K_y	horizontal diffusion coefficients in x-, and y-directions, respectively
l_H	horizontal length scale
M_0, M	injected and observed amount of diffusing substance, respectively
P	horizontal diffusion velocity
q	horizontal current vector
r	equivalent radius
t	time
u, v	current velocities in x-, and y-directions, respectively
U, V, W	mean current velocities in x-, y-, and z-directions, respectively

x, y, z	orthogonal coordinates	σ_{rc}^2	rotationally symmetrical variance
Γ	moment-generating function	σ_x^2, σ_y^2	variances in the x-, and y-directions, respectively
ω	frequency of oscillating current	σ_z^2	variance in the z-direction
σ^2	horizontal variance		

REFERENCES

Bowden, K. F. 1965. Horizontal mixing in the sea due to a shearing current *J. Fluid Mech.* *21* (*2*), 83–95.

Bowles, P., Burns, R., Hudswell, F. and Whipple R. 1958. Sea disposal of low activity effluent. *Proc. 2nd Conf. Peaceful Uses of Atomic Energy*, U. N., Geneva 18, pp. 376–389.

Carter, H. H. & Okubo, A. 1965. A study of the physical processes of movement and dispersion in the Cape Kennedy area. Chesapeake Bay Inst., The Johns Hopkins Univ., *Final Rept.* under U.S. AEC Contract AT (30–1)-2973 *Ref.* 65–2, 150 + 12 pp.

Elder, J. W. 1959. The dispersion of marked fluid in turbulent shear flow. *J. Fluid. Mech.* *5*(*4*), 544–560.

Joseph, J. & Sendner, H. 1958. Über die horizontale diffusion im Meere. *Dt. Hydrogr. Z.* *11* (*2*), 49–77.

Kullenberg, G. 1968. Measurements of horizontal and vertical diffusion in coastal waters. *Rep. No. 3*, Inst. Phys. Oceanogr., Univ. Copenh. 52 pp.

Kullenberg, G. 1971. Results of diffusion experiments in the upper region of the sea. *Rep. No. 12*, Inst. Phys. Oceanogr., Univ. of Copenh. 66 pp.

Kullenberg, G. 1971*a*. Vertical diffusion in shallow waters. *Tellus 23* (*2*), 129–135.

Mork, M. 1970. A note on longitudinal diffusion in a tidal channel. *Rep. No. 23*, Geophysical Inst., Univ. of Bergen, 10 pp.

Novikov, E. A. 1958. Concerning a turbulent diffusion in a stream with a transverse gradient of velocity. *Jour. Appl. Math. and Mech. 22* (*3*), 576–579 (translated from Russian).

Okubo, A. 1967. The effect of shear in an oscillatory current on horizontal diffusion from an instanteous source. *Int. J. Oceanol Limnol. 1* (*3*), 194–204.

Okubo, A. 1968. Some remarks on the importance of the "shear effect" on horizontal diffusion. *J. Oc. Soc. Jap.*, *24*, *2*, p. 20–29.

Okubo, A. 1971. Oceanic diffusion diagrams. *Deep-Sea Research 18* (*8*), 789–802.

Okubo, A. & Ozmidov, R. V. 1970. Empirical dependence of the coefficient of horizontal turbulent diffusion in the ocean on the scale of the phenomenon in question. *Izv. Atm. Oc. Phys.*, *6* (*5*), 534–536.

Ozmidov, R. V. 1958. On the calculation of the horizontal turbulent diffusion of the pollutant patches in the sea. *Doklady Akad. Nauk, SSSR*, *120*, 761–763.

Pritchard, D. W., Okubo, A. & Carter, H. H. 1966. Observations and theory of eddy movement and diffusion of an introduced tracer material in the surface layers of the sea. In *Disposal of radioactive wastes into seas, oceans and surface waters, I.A.E.A.*, Vienna, 397–424.

Taylor, G. I. 1953. Dispersion of soluble matter in solvent flowing slowly through a tube. *Proc. Roy. Soc. Lond. A 219*, 446–68.

Taylor, G. I. 1954. The dispersion of matter in turbulent flow through a pipe. *Proc. Roy. Soc. Lond. A 223*, 446–68 (also *Scientific Papers 2*, 466–488).

НАБЛЮДАЕМАЯ ДИФФУЗИЯ В СТРАТИФИЦИРОВАННОМ ПОТОКЕ С ВЕРТИКАЛЬНЫМ ГРАДИЕНТОМ СКОРОСТИ

Представлены результаты исследования горизонтальной турбулентной диффузии в связи с окружающими условиями, такими, как стратификация плотности, вертикальный градиент скорости и изменчивость течения. Проведены эксперименты с трассерами на фьордах, в прибрежных зонах и вдали от берега с целью получения данных для такого исследования. В дополнение были проведены измерения скорости ветра, солености, температуры и скорости течений. Предложена модель, основанная на предположении, что наблюдаемая диффузия определяется совместным действием вертикальной диффузии и вертикального градиента скорости потока. Найдено, что теоретические предсказания хорошо согласуются с экспериментальными результатами.