Nonlinear interaction of waves and zonal current in a two-layer baroclinic model[1]

By T. J. SIMONS, *Canada Centre for Inland Waters, Burlington, Ontario, and* DESIRAJU B. RAO, *University of Wisconsin, Milwaukee, Wisconsin*

(Manuscript received September 2, 1971)

ABSTRACT

Nonlinear interactions between a wave and a zonal current in a two-layer quasi-geostrophic model on a β-plane are considered. Initially there is no horizontal shear in the basic flow. Subsequently, the non-linear effects will introduce horizontal shears in the basic flow, thus introducing a barotropic energy exchange. Modification produced in the purely baroclinic solution by barotropic effects is presented.

1. Introduction

According to linear theory a uni-directional two-layer-flow is unstable with respect to disturbances of the scale of atmospheric cyclones if the vertical shear between the two layers exceeds a certain value. Since such an unstable eddy will grow exponentially, it will soon become so large that the linear theory breaks down and the basic current will be modified by the nonlinear self-interaction of the wave. This in turn will alter the stability character of the basic flow and consequently the rate of growth of the wave. Phillips (1954) computed the second-order changes in the basic current resulting from an unstable wave superimposed on this current in a two-layer model. Baer (1970, 1971) included the feed-back of the wave to the basic flow and presented an exact solution for the nonlinear interaction between the zonal flow and a finite-amplitude wave in a two-layer purely baroclinic system. This solution is periodic for all values of the basic state parameters and for all wavelengths. Thus the growth of the wave will be reduced and subsequently halted when the perturbation amplitude becomes sufficiently large, and finally the amplitude will become at least as small as its initial value. Precisely the same conclusions were recently reached by Pedlosky (1970) who adopted the same physical model but a different mathematical approach.

Since it might be anticipated that the character of Baer's (loc. cit.) solution would be altered if the constraint of purely baroclinic flow was relaxed, we have in the past carried out a number of computations to study this effect. In these experiments the basic flow had vertical but no horizontal shear at the initial time such that the perturbation could grow only as a result of baroclinic energy conversions. From then on the flow was allowed to develop without the purely baroclinic restriction. Thus the nonlinear effects will result in horizontal shears in the basic current and consequently barotropic energy exchange processes will come into play. The computations showed indeed a considerable modification of the baroclinic solution due to the barotropic effects. The modifying effects were found to be very consistent, both in spherical models (Baer's solution) and in so-called beta-plane models (Phillips, 1954; Pedlosky, 1970). It is considered of interest to present here a typical example of these solutions. To facilitate comparison with Phillips' (loc. cit.) and Pedlosky's (loc. sit.) solutions we have chosen the more familiar beta-plane model.

2. Basic equations and solution

Consider an adiabatic, inviscid, quasi-static, and quasi-geostrophic flow on a beta-plane.

[1] This work was performed at the Department of Atmospheric Science, Colorado State University, Fort Collins, Colorado, and was supported through funds provided by N.S.F. Grant GA-11637.

A flow of this type is completely described by the quasi-geostrophic form of the vorticity equation and the thermodynamic equation. For the present two-layer model the latter may be combined into the so-called potential-vorticity equations for the upper and lower layer respectively (see, e.g., Phillips, 1954)

$$\frac{\partial \bar{q}}{\partial t} + J(\bar{\psi}, \bar{q}) = 0, \quad \frac{\partial \bar{p}}{\partial t} + J(\bar{\phi}, \bar{p}) = 0 \qquad (1)$$

$$\bar{q} \equiv \nabla^2 \bar{\psi} + S(\bar{\phi} - \bar{\psi}) + \beta y, \quad \bar{p} \equiv \nabla^2 \bar{\phi} + S(\bar{\psi} - \bar{\phi}) + \beta y \qquad (2)$$

Here $\bar{\psi}$ and $\bar{\phi}$ are the stream functions for the upper and lower layer, respectively, $\bar{q}$ and $\bar{p}$ are the potential vorticities of the upper and lower layer, and S is a measure of the static stability of the flow which compares with the constant λ^2 defined in Phillips' paper (loc. cit). The remaining symbols have their customary meaning, thus, t is time, y is the northward coordinate, $\beta \equiv df/dy$ is the latitudinal variation of the Coriolis parameter (assumed constant), ∇^2 is the horizontal Laplacian and J is the Jacobian operator

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

Our purpose is to study the nonlinear dynamics of baroclinic waves superimposed on a zonal current. In particular we consider a flow consisting initially of the zonal flow (denoted by a capital letter) plus a wave of one particular zonal wave number k. If x denotes the zonal coordinate, the total stream functions may be written

$$\bar{\psi} = \Psi(t, y) + 2 \,\mathrm{Re}\, \psi(t, y)\, e^{ikx}$$
$$\bar{\phi} = \Phi(t, y) + 2 \,\mathrm{Re}\, \phi(t, y) e^{ikx} \qquad (3)$$

where $i \equiv \sqrt{-1}$ and Re denotes the real part of a complex quantity. Similarly then we will have

$$\bar{q} = Q(t, y) + 2 \,\mathrm{Re}\, q(t, y) e^{ikx}$$
$$\bar{p} = P(t, y) + 2 \,\mathrm{Re}\, p(t, y) e^{ikx} \qquad (4)$$

where Q, q, P, and p are obtained immediately in terms of Ψ, ψ, Φ, and ϕ, by substituting (3) and (4) into (2). Furthermore, by substituting (3) and (4) into (1) we obtain the following equations for the wave

$$\frac{\partial q}{\partial t} = ik\left(\frac{\partial \Psi}{\partial y} q - \frac{\partial Q}{\partial y}\psi\right), \quad \frac{\partial p}{\partial t} = ik\left(\frac{\partial \Phi}{\partial y} p - \frac{\partial P}{\partial y}\phi\right) \qquad (5)$$

and for the zonal flow

$$\frac{\partial Q}{\partial t} = ik\frac{\partial}{\partial y}(\psi^* q - \psi q^*), \quad \frac{\partial P}{\partial t} = ik\frac{\partial}{\partial y}(\phi^* p - \phi p^*) \qquad (6)$$

where the asterisk denotes the complex conjugate. It should be noted here that the nonlinear self-interaction of the wave not only modifies the zonal flow but also generates a wave of wave number $2k$. This paper is concerned with the former process only. The latter effect has been incorporated in a more general study where it was found to be small (within the span of the time scale considered) to the extent that it did not alter the typical character of the solution to be presented here.

Let the flow be constrained in lateral direction by two vertical walls at $y = 0$ and $y = W$ where W is the width of the channel. Since the normal component of the velocity should vanish at these walls, i.e., $\partial\bar{\phi}/\partial x = \partial\bar{\psi}/\partial x = 0$, it follows from (3) that $\psi = \phi = 0$ at $y = 0$, W. We may then write the general solution for the wave as follows

$$\psi = \sum_n \psi_n(t) \sin\frac{n\pi y}{W}, \quad \phi = \sum_n \phi_n(t) \sin\frac{n\pi y}{W} \qquad (7)$$

The quasi-geostrophic boundary conditions do not impose any restrictions on the zonal flow. As shown by Phillips (loc. cit.) we may derive from the zonal momentum equation the condition

$$\frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial t}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial t}\right) = 0 \quad \text{at } y = 0, W \qquad (8)$$

The effect of the wave on the zonal flow may then be evaluated by substituting (7) into the right-hand sides of (6) and solving the resulting non-homogeneous differential equations for $\partial\Psi/\partial t$ and $\partial\Phi/\partial t$ such that the boundary conditions (8) are satisfied.

The present study is concerned with a zonal flow which is purely baroclinic at the initial time, i.e., a basic flow with vertical shear only. Let U and V be the constant zonal velocitities of the basic current in the upper and lower layer, respectively, then $\Psi = -Uy$ and $\Phi = -Vy$ at the initial time. Upon substituting (7) into the right-hand sides of (6) the resulting non-homogeneous corrections to the zonal stream functions are clearly of a form similar to (7). In order to satisfy (8) we must therefore add the homogeneous solutions of (6), thus arriving at the following general expressions for the zonal stream-functions.

$$\Psi = -Uy + H(t,y) + \sum_n \Psi_n(t) \sin \frac{n\pi y}{W}$$

$$\Phi = -Vy + G(t,y) + \sum_n \Phi_n(t) \sin \frac{n\pi y}{W} \qquad (9)$$

where $H(t, y)$ and $G(t, y)$ are the homogeneous solutions. The latter can be found most easily by adding and subtracting the zonal equations (6). The sum-equation appears to allow for a homogeneous solution linear in y. However, an evaluation of its right-hand side and comparison with the zonal momentum equation—which was also referred to in order to establish the conditions (8)—show that the homogeneous solution must be discarded, which simply reflects the fact that the present flow conserves its total zonal momentum. As a result then the homogeneous solution for Φ is equal but of opposite sign to the homogeneous solution for Ψ. By subtracting the zonal equations (6) we find then

$$H = -G = \Psi_0(t) \cosh \sqrt{2s}(y - \tfrac{1}{2}W) + \Phi_0(t) \sinh \sqrt{2s}(y - \tfrac{1}{2}W) \qquad (10)$$

After substituting the general solutions (7) and (9) into (5) and (6) and applying the familiar orthogonality relationships we arrive at a set of "spectral prediction equations" for the variables ψ_n, ϕ_n, Ψ_n, Φ_n, $n = 1, 2, 3, \ldots$ The derivation of the spectral equations is straight-forward and will be dispensed with. Given the spectral prediction equations, the time-dependent expansion coefficients may be extrapolated in time by an appropriate finite-difference scheme. Finally, the variables Ψ_0 and Φ_0 are obtained at each time step by subtracting the two equations (9), substituting (10), and requiring that the conditions (8) be satisfied.

3. Results

Solutions to the above equations have been obtained for various initial conditions and flow parameters. As mentioned before, similar computations were also performed for a spherical model. No attempt will be made to list the results but rather we will discuss one solution the character of which is typical for all solutions of the present system. The results are most easily presented in terms of energy conversions. It is well known that the present model conserves the sum of kinetic and "available potential energy" (e.g., Phillips, loc. cit.). Thus the wave kinetic energy can change only as a result of two processes (i) the conversion of zonal kinetic energy into wave kinetic energy which we will denote by $C(\overline{K}, K)$ and (ii) the conversion of potential energy into kinetic energy $C(P, K)$. The first process may be associated with barotropic instability, the second with baroclinic instability. The appropriate expressions for the energy conversions are easily obtained and will be left out here. All energy quantities are averaged over the width of the channel, over the length of the wave, and over the two layers.

For comparison with Phillips' paper we adopt the parameters $W = 60°$ of latitude, $S = 1.1 \times 10^{-12}$ m^{-2}. Specifically we present the solution for $U = 35$ m/sec, $V = 10$ m/sec, a wavelength $L = 6\,000$ km, and an initial lateral wave structure given by $\psi = \psi_1 \sin(\pi y/W)$, $\phi = \phi_1 \sin(\pi y/W)$. For these values of the parameters the linear theory (Phillips, loc. cit.) shows that the wave is baroclinically unstable with a "growth rate" $\sigma_i = 0.447$ per day [wave amplitude is proportional to exp. $(\sigma_i t)$]. Furthermore we take the initial wave structure corresponding to the unstable mode which is found to be $\psi_1/\phi_1 = 1.17852 + 1.24274i$, and the initial wave kinetic energy equal to 10 % of the zonal kinetic energy, the latter being 331 m^2/sec^2. Fig. 1 shows the results of the time integration of our system of equations for this case. Presented are only the baroclinic conversion $C(P, K)$, the barotropic conversion $C(\overline{K}, K)$, and the "growth rate" of the wave. The latter is defined as the rate of change of wave kinetic energy divided by twice the wave kinetic energy, which becomes equal to the above defined growth rate for the unstable wave in the linear model.

Various truncations of the series (7) and (9) are denoted by N, i.e., the number of terms retained. It may be verified easily that the present initial perturbation can only generate wave corrections of odd lateral wave numbers

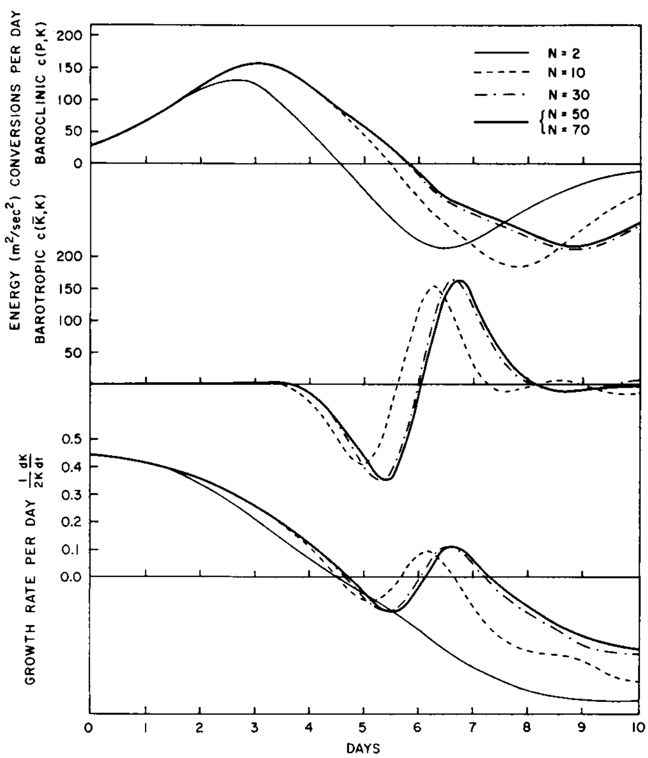

Fig. 1. Energy conversions and growth rate of wave superimposed on a baroclinic zonal flow in a two-layer model. Upper figure: baroclinic conversion of potential energy into kinetic energy of the wave; second figure: barotropic conversion of zonal kinetic energy to wave kinetic energy; lower figure: rate of growth of the wave defined as the time rate of change of wave kinetic energy divided by the actual wave kinetic energy. N denotes the lateral resolution of the spectral solution such that $N = 2$ corresponds to the purely baroclinic solution.

and zonal corrections of even lateral wave numbers, thus n = odd in (7) and n = even in (9). The purely baroclinic solution is then obtained for $N = 2$ which is shown by the thin solid line. Obviously the barotropic conversion is identically equal to zero for this case. The remaining energy quantities are periodic for the baroclinic case as shown by Baer (loc. cit.) and Pedlosky (loc. cit.). The present period is about 18 days but only the first 10 days are shown here since that part of the solution is sufficient to demonstrate the barotropic modification of this solution. The latter effect is shown by the dashed line, the dash-dot curve, and the heavy solid curve of Fig. 1, for increasing lateral resolution, i.e., for increasing barotropic degrees of freedom. The convergence of the solutions as a function of series truncation is found to be satisfactory. Time extrapolations have been performed with explicit and implicit finite difference schemes without noticeable differences.

The heavy solid curve of Fig. 1 shows the typical energy conversions which take place if a wave is superimposed on a zonal flow which initially is purely baroclinic, if barotropic processes are allowed to operate. Since our initial perturbation is baroclinically unstable the wave will grow due to baroclinic conversions $C(P, K)$. The initial rate of growth is equal to the growth rate of the unstable mode since the initial configuration was chosen accordingly. Due to the nonlinear effects the growth rate will decrease quite similarly to the purely baroclinic solutions discussed earlier ($N = 2$). However, just before the rate of growth becomes zero (maximum wave kinetic energy) the wave starts feeding large quantities of kinetic energy into the zonal flow [negative $C(\overline{K}, K)$]. Immediately thereafter this energy is returned to the wave which may cause the growth rate to become positive again but at least to increase. Linear analysis shows that the zonal flow at that moment is barotropically unstable. Subsequently, the development of the flow tends to become again baroclinic. This character of the nonlinear solution described above is typical for all solutions obtained so far. The magnitude of the barotropic energy exchange is found to be proportional to the shear of the zonal flow and the initial wave amplitude, both of which tend to increase the maximum amplitude attained by the wave. In all cases the maximum of the barotropic conversion is of the same order of magnitude as the maximum of the baroclinic conversion.

REFERENCES

Baer, F. 1970. Analytic solutions for low-order spectral systems. *Arch. Met. Geoph. Biokl.* (*Ser. A*) *19*, 255–282.

Baer, F. 1971. Energetics of low-order spectral systems. *Tellus 23*, 218–231.

Pedlosky, J. 1970. Finite-amplitude baroclinic waves. *J. Atmos. Sci. 27*, 15–30.

Phillips, N. A. 1954. Energy transformations and meridional circulations associated with simple baroclinic waves in a two-level, quasigeostrophic model. *Tellus 6*, 273–286.

НЕЛИНЕЙНОЕ ВЗАИМОДЕЙСТВИЕ ВОЛН И ЗОНАЛЬНОГО ПОТОКА В ДВУХУРОВЕННОЙ БАРОКЛИННОЙ МОДЕЛИ

Рассматриваются нелинейные взаимодействия волны и зонального потока в двухуровенной квазигеострофической модели на β-плоскости. В начальный момент основной поток не обладает горизонтальным сдвигом скорости. Затем нелинейные взаимодействия приводят к возникновению горизонтального сдвига в основном потоке, приводящему к баротропному обмену энергией. Найдены изменения в чисто бароклинном решении, обусловленные баротропными эффектами.