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SHORT CONTRIBUTION

A note on the relation between average age and average transit time in natural reservoirs

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ABSTRACT

A general relation between the average age and the transit time distribution of material in a reservoir at steady state is derived. Given the average transit time, it is shown that the average age increases with increasing standard deviation in the transit time distribution.

In a previous note (Bolin and Rodhe, 1973), the concepts of "transit-time" and "age" and distribution functions for these were defined in a general way for material in a reservoir. In a recent paper by Nir and Lewis (1975), the use of tracer theory in geophysical systems is discussed in similar terms. In the present note, some further relations between these quantities will be discussed. For formal definitions of the basic concepts, reference is made to the note by Bolin and Rodhe (1973).

It was shown by these authors that *turn-over time* and *average transit time* always have the same value for a reservoir at steady state. It was also shown that this value is equal to the *average age* in the case of a so-called *well-mixed reservoir*, but examples were given to illustrate that average transit time needs not generally be the same as the average age. There must, however, be some relation between the two. If the statistical distribution of the transit time of material entering the reservoir is known, and if the reservoir is known to be at steady state, the average age of material at any one moment present in the reservoir should also be fixed. The purpose of the present note is to demonstrate a general relation of this kind.

Consider two reservoirs, both at steady state and both having the same average transit time τ_0. In one

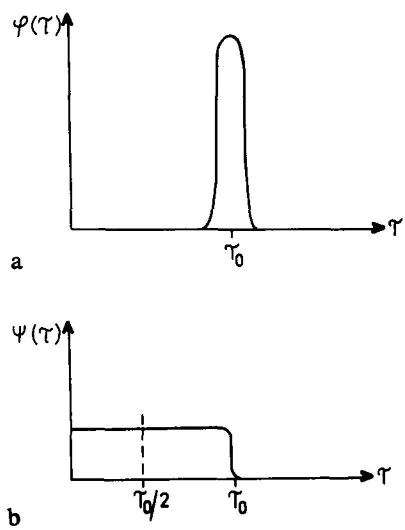

Fig. 1. (a) Example of a peak-shaped transit time distribution function, centred at τ_0. (b) The steady-state age distribution function corresponding to the transit time distribution in (a).

case, we assume that the material that enters the reservoir stays there for a time close to τ_0, after which it leaves the reservoir. The distribution function φ of the transit time will then consist of a peak around the value τ_0, and be negligibly small for all other values (see Fig. 1a). Measuring the age

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of material elements present in such a reservoir, one would obtain an almost uniform age distribution ψ, centred around $\tau_0/2$ which is then the average age, as shown in Fig. 1b.

In the second case, let the transit times of the mass elements have the same average value τ_0, but let there be a considerable spread around this value, as shown in Fig. 2a. Since some material spends a considerably longer time than τ_0 inside the reservoir, the age distribution function ψ will be different from zero for $\tau > \tau_0$, while in the previous case it was equal to zero (see Fig. 2b).

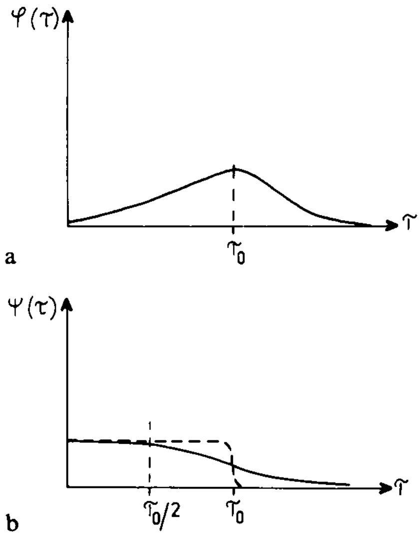

Fig. 2. (a) Example of a transit time distribution function, centred at τ_0 and with a larger standard deviation than in Fig. 1. (b) The steady-state age distribution function corresponding to the transit time distribution in (a) (solid line). For comparison, the age distribution in Fig. 1(b) is indicated (dashed line).

On the other hand, some material now leaves the reservoir at an age less than τ_0, so the values of the age distribution function ψ in the interval up to $\tau = \tau_0$ will be reduced compared to the first case. It follows that the first moment of the function ψ, i.e. the *average age*, must be larger in this case.

The formal definition of the average age τ_a is:

$$\tau_a = \int_0^\infty \tau\psi(\tau)\,d\tau$$

We shall now derive a formal expression for this observation. The natural parameter, by which to measure the spread of any distribution function, is its variance σ^2, defined in the case of the transit-time distribution by the integral

$$\sigma_t^2 = \int_0^\infty (\tau - \tau_0)^2 \varphi(\tau)\,d\tau \tag{1}$$

and similarly for the age distribution function:

$$\sigma_a^2 = \int_0^\infty (\tau - \tau_a)^2 \psi(\tau)\,d\tau \tag{2}$$

where it is assumed that the two integrals converge.

We develop (1):

$$\sigma_t^2 = \int_0^\infty \tau^2 \varphi(\tau)\,d\tau - 2\tau_0 \int_0^\infty \tau\varphi(\tau)\,d\tau + \tau_0^2 \times \int_0^\infty \varphi(\tau)\,d\tau = \int_0^\infty \tau^2 \varphi(\tau)\,d\tau - \tau_0^2$$

where use has been made of the relations

$$\tau_0 = \int_0^\infty \tau\varphi(\tau)\,d\tau \quad \text{and} \quad 1 = \int_0^\infty \varphi(\tau)\,d\tau$$

Integrating by parts, we obtain

$$\int_0^\infty \tau^2 \varphi(\tau)\,d\tau = [\tau^2 \Phi(\tau)]_0^\infty - 2 \int_0^\infty \tau\Phi(\tau)\,d\tau \tag{3}$$

where Φ is a primitive function of φ. As shown by Bolin and Rodhe (1973), this primitive function is

$$\Phi(\tau) = -\,\tau_0\psi(\tau)$$

and we obtain

$$\int_0^\infty \tau^2 \varphi(\tau)\,d\tau = 2 \int_0^\infty \tau\tau_0 \psi(\tau)\,d\tau = 2\tau_0\,\tau_a$$

(the vanishing of the first term of the right side in (3) is a consequence of the convergence of (2)).

Combining, we obtain

$$\sigma_t^2 = 2\tau_a\tau_0 - \tau_0^2$$

which gives the result

$$\tau_a = \frac{\tau_0}{2} + \frac{\sigma_t^2}{2\tau_0} \tag{4}$$

Thus, if the distribution of the transit times is known (or, at least, the first and second moments of this distribution function), the formula (4) can be used to obtain the value of the average age τ_a of particles in a case of steady state.

Since the variance σ_t^2 is always a non-negative number, one consequence of (4) is

$\tau_a \geqslant \tau_0/2$

that is, the "peak" distribution of transit times shown in Fig. 1a represents an extreme case in the sense that is is the particular distribution which minimizes the average age of a steady-state distribution given the transit time τ_0. This minimum value of τ_a is equal to half the average transit time, and this situation occurs when every mass element that enters the reservoir stays there for precisely the time τ_0. Any spread of the transit time around this mean value thus has a tendency to increase the average age of material in the reservoir.

An interesting example of a natural reservoir where τ_a is substantially different from τ_0 has recently been given by Zimmerman (1976) in a study of water exchange in the Dutch Wadden Sea.

In Zimmerman's model, water enters the Wadden Sea either as fresh water from Ijsselmeer or as sea-water from the North Sea. The main patterns of water exchange are a tidal flow to and from the North Sea and a flow of fresh water from Ijsselmeer to the North Sea, superposed on the tidal exchange. Zimmerman has defined and computed some time-scales of interest in connection to this exchange. He has also computed the age distribution and the transit-time distribution for each of the two types of water. A system of transfer equations is used to describe how the spread of a dissolvent in the Wadden Sea proceeds with time. By performing a time integration from a state where concentrations are zero everywhere except at the boundary to the North Sea, where it is kept equal to one, Zimmerman studies the successive penetration of North Sea water into the Wadden Sea and arrives at the cumulative age distribution for sea-water in the Wadden Sea. Starting instead from a state where concentrations are zero everywhere except at the boundary to Ijsselmeer, the corresponding age distribution for fresh water is derived. Examples of the results are shown in Fig. 3. In a similar way, the transit time distribution is computed.

For sea-water, the average age is found to be several times longer than the average transit time. The cause of this is that since sea-water enters the Wadden Sea near the same point as it leaves, most of the sea-water is washed out again very rapidly after entering the Wadden Sea. The bulk of sea-water present, therefore, is made up of only a small

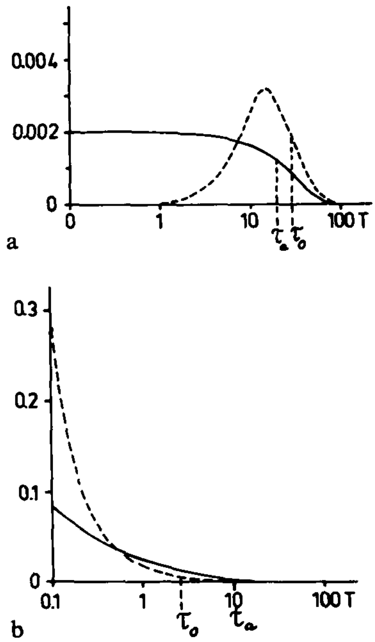

Fig. 3. Example of statistical distribution of age (solid line) and transit time (dashed line) of fresh water (a) and sea-water (b) in the western Dutch Wadden Sea as a fraction of the total volume. From Zimmerman (1976).

fraction of the total inflow, that has penetrated inside of the inlet area and come to stay in the Wadden Sea for a much longer time than the average sea-water parcel entering it.

For fresh water, on the other hand, the points of entrance are situated at larger distances from the outlets. Accordingly the average age is found to be considerably smaller than the average transit time. As Zimmerman points out, this is because the water parcels must travel a certain distance before they reach the outlet, and "the 'most probable' transit time is different from zero". In Table 1 are

Table 1. *Examples of average age and average transit time (hours) for fresh water and sea-water, as calculated by Zimmerman*

	Fresh water	Sea-water
Age	19.8	11.3
Transit time	29.1	2.7

summarized the values for average age and average transit time computed by Zimmerman. It can be noted that the average age for fresh water, although less than the average transit time, is larger than half this value. As shown in the present note, this is a necessity if the system is at steady state.

It is sometimes illustrative to introduce the relative variability of a stochastic quantity, defined as the ratio of the standard deviation to the average value. In the case of transit times, $\tau_t = \sigma_t/\tau_0$, and eq. (4) gives $\tau_a = \frac{1}{2}\tau_0 \cdot (1 + r_t^2)$ or $r_t = \sqrt{2\tau_a/\tau_0 - 1}$.

Inserting the values of τ_a and τ_0 given in Table 1, we find for the transit time for fresh water a value of $r_t = 60\%$ while for sea-water we find $r_t = 270\%$. These numbers indicate the difference in spread of transit times for water entering the Wadden Sea from Ijsselmeer compared to water from the North Sea.

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ЗАМЕТКА О СООТНОШЕНИИ МЕЖДУ СРЕДНИМ ВОЗРАСТОМ И СРЕДНИМ ВРЕМЕНЕМ ПРОХОЖДЕНИЯ В ЕСТЕСТВЕННЫХ РЕЗЕРВУАРАХ

Выводится общее соотношение между средним возрастом и распределением времени прохождения вещества в резервуаре в стационарном случае. Если время прохождения задано, то показано, что средний возраст увеличивается с увеличением стандартного отклонения в распределении времен прохождения.