Tellus (1979), *31*, 487–492

On the kinematic properties of the −3 law for kinetic energy

By PHILIP E. MERILEES,[1] *International Meteorological Institute in Stockholm,*[2] *Arrhenius Laboratory, S-106 91 Stockholm, Sweden*

(Manuscript received April 4; in final form April 26, 1979)

ABSTRACT

It is shown that if the kinetic energy follows a −3 power law in terms of the two-dimensional index n then it will follow the same power law in term of the zonal wave number l and the north-south wave number j. The effects of truncation of representations on these power laws are documented to aid in the interpretation of data analyses and model results.

1. Introduction

In recent years the study of spectra and non-linear exchange in large-scale atmospheric flow has been cast in the framework of spherical geometry. Chen and Wiin-Nielsen (1978) have recently presented such results and briefly summarized the literature and some of the theoretical ideas behind the representations. Of particular interest is the observation that the kinetic energy spectra follow a power law of the *two-dimensional index n* which is close to −3 for a range of wavenumbers $n > 10$ at levels away from the lower boundary. This law is thought to be indicative of the existence of an inertial range in two-dimensional flow on a sphere and has theoretical as well as practical significance for the parameterization of non-linear exchange. However, the first observational evidence of the existence of a −3 power law came from one-dimensional spectra in terms of zonal wavenumber (Wiin-Nielsen, 1967). Results from general circulation model simulations also indicated −3 power laws of kinetic energy spectra in terms of zonal wavenumber (Wellck et al., 1971). For two-dimensional isotropic turbulence on a plane it is well known that if the kinetic energy spectrum follow a power law of the two-dimensional index then one-dimensional spectra will follow the same power law (Leith, 1971). Thus it is not surprising, although not obvious, that a similar result should hold for spherical geometry since the law applies for large values of n where the effects of spherical geometry should be small. In what follows we shall show that the above is indeed to be expected for isotropic turbulence on a sphere. Further, we shall use the results to indicate what will be the effects on one-dimensional spectra should the representations be truncated (as they inevitably must). In this way we should be in a better position to interpret one-dimensional spectra both for observations and model results.

We shall consider the spectrum of a single dynamical variable, the stream function, which we suppose is representative of the mid-troposphere. Both the kinetic energy of the rotational wind and the enstrophy spectra can readily be derived from this stream function. Further, the formalism is the same for such quantities as available potential energy and thus any conclusions can immediately be applied to the spectra of such other quantities.

2. Formulation

The stream function over the globe can be expressed as a series of spherical harmonics;

$$\psi(\lambda, \mu) = \sum_{n=1}^{\infty} \sum_{l=-n}^{n} \psi_n^l P_n^l(\mu) e^{il\lambda} \tag{1}$$

[1] On leave from the Canadian Climate Centre, 4905 Dufferin St., Downsview, Ontario, M3H 5T4, Canada.

[2] Contrib. No. 389.

0040-2826/79/060487-06$02.50/0

where P_n^l is the associated Legendre function, λ is longitude and μ is the sine of latitude. We have set the global average stream function (an arbitrary constant) to zero, i.e. $\psi_0^0 = 0$. Defining a bar operator as an areal average over a sphere we then find the variance of ψ, (V); the kinetic energy, (K); and the enstrophy, (E); have the following forms,

$$V = \overline{\psi^2}, \quad K = \tfrac{1}{2}\overline{\nabla\psi \cdot \nabla\psi}, \quad E = \tfrac{1}{2}\overline{\zeta^2} \tag{2}$$

where $\zeta = \nabla^2 \psi$. Substituting the expansion of the streamfunction (1) into (2) it can be shown that

$$V = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n} V_n^l \right] = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n} (2 - \delta_{0l}) |\psi_n^l|^2 \right] \tag{3}$$

$$2a^2 K = \sum_{n=1}^{\infty} K_n = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n} K_n^l \right] = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n} n(n+1) V_n^l \right] \tag{4}$$

$$2a^4 E = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n} E_n^l \right] = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n} n^2(n+1)^2 V_n^l \right] \tag{5}$$

where $\delta_{0l} = 1$ if $l = 0$ and is zero otherwise and a is the radius of the earth. V_n^l is the variance of the streamfunction belonging to the spherical harmonic mode characterized by the zonal index l and the two-dimensional index n. K_n^l is proportional to the kinetic energy and E_n^l to the enstrophy belonging to the same mode. As well, we can see that values of the complete set of any of V_n^l, K_n^l or E_n^l will permit the determination of the other two.

From the above expression we now proceed to construct one dimensional spectra by summing over the distributions of V_n^l, K_n^l and E_n^l. The obvious one-dimensional spectra are obtained by summing over l to obtain a spectrum dependent on n and to sum over n to obtain a spectrum dependent on l. In these cases we obtain spectra as a function of the two-dimensional index n and the zonal wave number l respectively. We can also form a spectrum by summing over components such that $j = n - l$ is constant. Such a spectrum is considered to be a function of the north-south index j. We shall refer to these spectra as the n, l and j spectra respectively. These definitions are analogous to those considered in plane geometry where spectra are defined along orthogonal axes in wave number space, although the relationship between the indices is quite different.

Thus the one-dimensional spectra for variance are defined as

$$V_n = \sum_{l=0}^{n} V_n^l, \tag{6}$$

$$V_l = \sum_{n=l}^{\infty} V_n^l, \tag{7}$$

$$V_j = \sum_{l=0}^{\infty} V_{l+j}^l, \tag{8}$$

with similar expressions for the one-dimensional spectra of kinetic energy and enstrophy.

In order to make further progress in this simple treatment we shall suppose that the distribution of variance is isotropic. This means that V_n^l depends only on n. From the studies of Baer (1972, 1974) we know that there are serious discrepancies from such an ideal case. However, these are mainly concentrated on the lower values of n, l and j and we will be most interested in the higher values of n, l and j where we expect power laws to be valid.

3. Isotropic spectra

For an isotropic spectrum V_n^l depends only on the index n. In that case both the distributions of kinetic energy and enstrophy will also be isotropic. For if

$$V_n^l = G(n), \tag{9}$$

then

$$K_n^l = n(n+1)\, G(n) \tag{10}$$

and

$$E_n^l = n^2(n+1)^2\, G(n) \tag{11}$$

are functions of n alone. Further, the n spectra will be simply related to $G(n)$ because

$$V_n = \sum_{l=0}^{n} V_n^l = (n+1)\, G(n), \tag{12}$$

$$K_n = \sum_{l=0}^{n} n(n+1)\, V_n^l = n(n+1)^2 G(n), \tag{13}$$

$$E_n = \sum_{l=0}^{n} n^2(n+1)^2 V_n^l = n^2(n+1)^3\, G(n). \tag{14}$$

Now, the observational studies of Wiin-Nielsen (1972), Chen and Wiin-Nielsen (1978) as well as those of Baer (1972, 1974) indicate that the n spectrum of kinetic energy is well approximated by a power law for n greater than about 10. Thus we shall assume the following mathematical model for the n spectrum of kinetic energy,

$$\begin{aligned} K_n &= A, \quad n < n^*, \\ &= B(n+1)\, n^{-4}, \quad n \geqslant n^*. \end{aligned} \tag{15}$$

Using Chen and Wiin-Nielsen's expression for kinetic energy in the −3 range we find that a reasonable fit is obtained with $B = 8.0 \times 10^{21}$ joules, $A = 8.8 \times 10^{18}$ joules and $n^* = 10$.

The model given by (15) then permits the evaluation of $G(n)$ and subsequently the n spectra of variance and enstrophy, i.e.

$$\begin{aligned} G(n) &= \frac{K_n}{n(n+1)^2} = \frac{A}{n(n+1)^2}, \quad n < n^*, \\ &= \frac{Bn^{-5}}{n+1}, \quad n \geqslant n^*; \end{aligned} \tag{16}$$

so that,

$$\begin{aligned} V_n &= \frac{A}{n(n+1)}, \quad n < n^*, \\ &= Bn^{-5}, \quad n \geqslant n^*, \end{aligned} \tag{17}$$

and

$$\begin{aligned} E_n &= n(n+1)A, \quad n < n^*, \\ &= B(n+1)^2 n^{-3}, \quad n \geqslant n^*. \end{aligned} \tag{18}$$

Therefore, for large n we have (asymptotically) a −5 law for the variance of the stream function, a −3 law for kinetic energy and a −1 law for enstrophy.

Using the definition of the form of (7) we may now formulate the l spectra. We have,

$$\begin{aligned} V_l &= \sum_{n=l}^{\infty} V_n^l = \sum_{n=l}^{\infty} G(n), \\ &= \sum_{n=l}^{n^*-1} \frac{A}{n(n+1)^2} + \sum_{n=n^*}^{\infty} \frac{Bn^{-5}}{n+1}, \quad l < n^*, \\ &= \sum_{n=l}^{\infty} \frac{Bn^{-5}}{n+1}, \quad l \geqslant n^*. \end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned} K_l &= \sum_{n=l}^{n^*-1} \frac{A}{n+1} + \sum_{n=n^*}^{\infty} Bn^{-4}, \quad l < n^*, \\ &= \sum_{n=l}^{\infty} Bn^{-4}, \quad l \geqslant n^*, \end{aligned} \tag{20}$$

and

$$\begin{aligned} E_l &= \sum_{n=l}^{n^*-1} nA + \sum_{n=n^*}^{\infty} (n+1)\, Bn^{-3}, \quad l < n^*, \\ &= \sum_{n=l}^{\infty} (n+1)\, Bn^{-3}, \quad l \geqslant n^*, \end{aligned} \tag{21}$$

Interestingly, the infinite series can be evaluated in terms of polygamma functions. The polygamma function of order m is defined as (Abramowitz and Stegun, 1965, p. 260)

$$\Gamma^m(x) \equiv (-1)^{m+1}\, m! \sum_{k=0}^{\infty} (x+k)^{-m-1} \tag{22}$$

Therefore we can write that

$$\begin{aligned} K_l &= \sum_{n=l}^{n^*-1} \frac{A}{n+1} + \frac{B}{3!}\Gamma^3(n^*), \quad l < n^*, \\ &= \frac{B}{3!}\Gamma^3(l), \quad l > n^*. \end{aligned} \tag{23}$$

For large arguments $\Gamma^3(x) \sim 2x^{-3} + 3x^{-4}$, so that for large l, $K_l \sim \frac{1}{3}Bl^{-3}$; that is the l spectrum also follows a −3 law.

In a similar way it can be shown that the j spectrum of kinetic energy is given by

$$\begin{aligned} K_j &= \sum_{l=0}^{n^*-j-1} \frac{A}{l+j+1} + \frac{B}{3!}\Gamma^3(n^*), \quad j < n^*, \\ &= \frac{B}{3!}\Gamma^3(j), \quad j \geqslant n^*, \end{aligned} \tag{24}$$

so that the j-spectrum of kinetic energy follows a -3 law.

The l spectrum of variance in the power law range is given as

$$V_l = \sum_{n=l}^{\infty} \frac{Bn^{-5}}{n+1}.$$

If we expand the above summation into partial fractions we find that

$$V_l = \sum_{n=l}^{\infty}\left[\frac{1}{n^5} - \frac{1}{n^4} + \frac{1}{n^3} - \frac{1}{n^2}\right] + \sum_{n=l}^{\infty}\frac{1}{n(n+1)}, \tag{25}$$

and noting that the last summation has a value l^{-1}, we have

$$V_l = B\left[\frac{1}{l} - \Gamma^1(l) - \frac{\Gamma^2(l)}{2!} - \frac{\Gamma^3(l)}{3!} - \frac{\Gamma^4(l)}{4!}\right]. \tag{26}$$

The above expression is asymptotic to $(B/5)l^{-5}$ as we might expect. The j spectrum has exactly the same form as the l spectrum so that both V_l and V_j are asymptotic to -5 power laws.

The l spectrum of enstrophy in the power law range can also be expressed in terms of polygamma functions, namely,

$$E_l = B\left[\Gamma^1(l) - \frac{\Gamma^2(l)}{2!}\right], \tag{27}$$

with the same form for the j spectrum. Thus the l and j spectra of enstrophy follow the -1 law in the limit of large wavenumbers.

Thus we have shown that both the l and j spectra of variance, kinetic energy and enstrophy exhibit the same power laws obtained for the n spectra, in agreement with a similar property in plane geometry. In the next section we shall evaluate the form of these spectra when the representation is truncated in some way.

4. Effects on truncation on isotropic spectra

There are generally two types of truncation of spectral representation which are used for data analysis and model integrations, the triangular and rhomboidal. The terms arise because of the geometric shape outlined by the permitted degrees of freedom when they are mapped in the coordinates (l, n). The triangular truncation is defined by a number N such that no consideration is given to components ψ_n^l when $n > N$. The rhomboidal truncation permits each zonal wavenumber l within the truncation $l \leqslant L$ to have the same number of degrees of freedom for its latitudinal structure, i.e. $n - l = j \leqslant L$. We now consider the effects of these truncations on the various spectra.

4.1. *Triangular truncation*

For triangular truncation there will be no effect on the n spectrum, but the l and j spectra will be modified by the terms which are ignored. In the power law regime the truncated spectra of kinetic energy have the form,

$$K_l^T = \frac{B}{3!}\left[\Gamma^3(l) - \Gamma^3(N+1)\right],$$

$$K_j^T = \frac{B}{3!}\left[\Gamma^3(j) - \Gamma^3(N+1)\right], \tag{28}$$

so that both the l and j spectra will exhibit a fall off with wavenumber which is faster than -3. In Fig. 1 we give examples of the spectra described by (28) for a few values of N. In the case of the spectra of variance the effect of truncation is less because of the more rapid decrease with wavenumber as shown in Fig. 2, while the effect is much more for the case of enstrophy spectra (Fig. 3).

4.2. *Rhomboidal truncation*

Under rhomboidal truncation the n spectra are affected by the truncation. In the case of kinetic energy we find that

$$K_n^T = B(n+1)\,n^{-4}, \quad n^* < n \leqslant L, \tag{29}$$

$$= B(2L + 1 - n)\,n^{-4}, \quad L < n \leqslant 2L.$$

The shape of this spectrum is quite different from the -3 law as shown in Fig. 4.

The l and j spectra of kinetic energy in the power law range $(l, j > n^*)$ are also affected

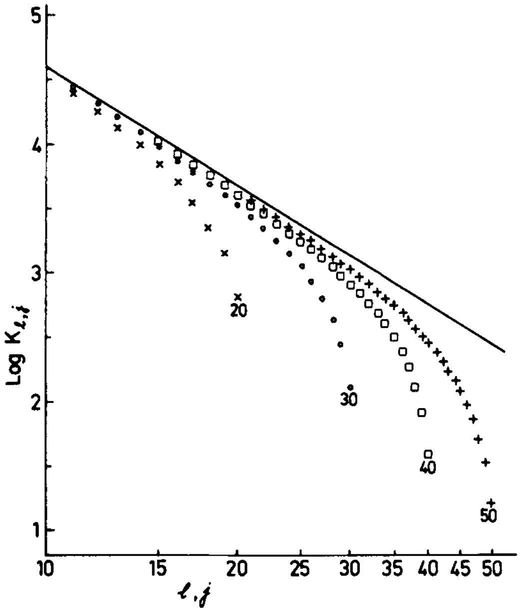

Fig. 1. The l and j spectra of kinetic energy under triangular truncations $N = 20, 30, 40, 50$; in the power law range. The solid curve corresponds to infinite resolution. The kinetic energy is in units of joules m^{-2}.

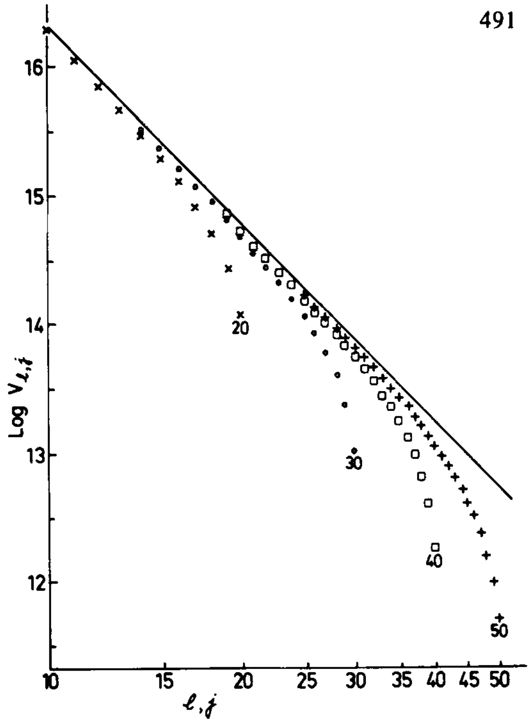

Fig. 2. The l and j spectra of variance under triangular truncations $N = 20, 30, 40, 50$; in the power law range. The solid curve corresponds to infinite resolution. The variance is expressed in joules.

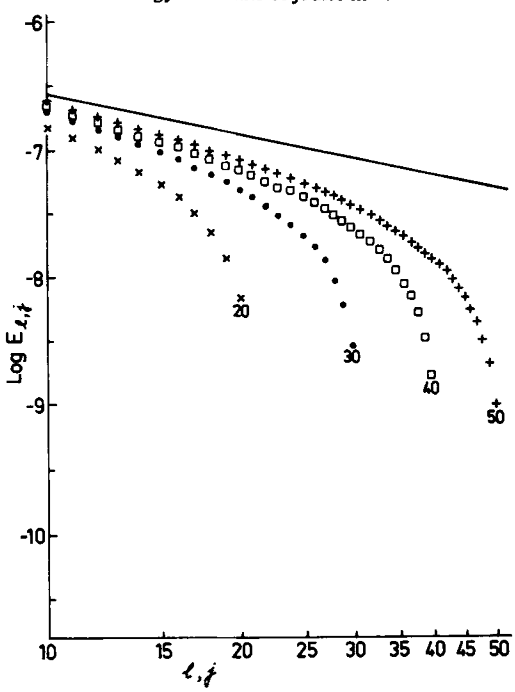

Fig. 3. The l and j spectra of enstrophy under triangular truncations $N = 20, 30, 40, 50$; in the power law range. The solid curve corresponds to infinite resolution. The enstrophy is expressed in joules m^{-4}.

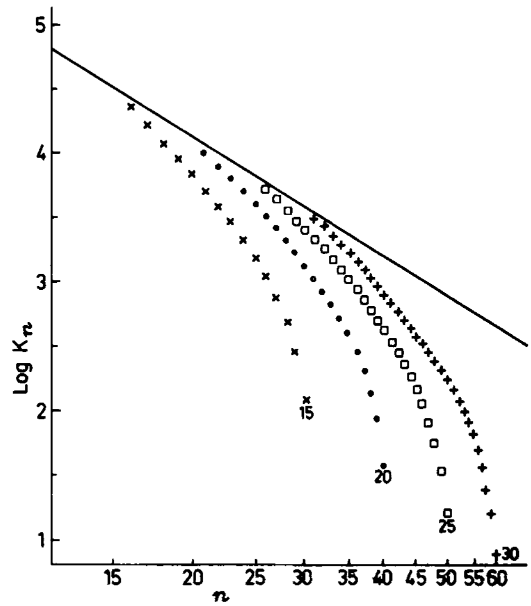

Fig. 4. The n spectra of kinetic energy under rhomboidal truncations $L = 15, 20, 25, 30$; in the power law range. The solid curve corresponds to infinite resolution. The kinetic energy is in units of joules m^{-2}.

by the truncation but to a negligible degree. We find that

$$K_l^T = \frac{B}{3!}\,[\Gamma^3(l) - \Gamma^3(l + L + 1)], \quad l > n^*,$$
$$K_j^T = \frac{B}{3!}\,[\Gamma^3(j) - \Gamma^3(j + L + 1)], \quad j > n^*, \tag{30}$$

so that for any reasonable value of $L \gtrsim 20$, the second terms in the parentheses in (30) are negligible. As one would expect there is even less distortion in the l and j spectra of the variance and slightly more in the case of enstrophy spectra.

5. Discussion

The above results show that if the variance, kinetic energy and enstrophy spectra are described by power laws in terms of the two-dimensional index n, then similar power laws in terms of the zonal index l and north-south index j are obtained. To the extent that a representation is truncated, the spectra obtained will fall off more rapidly than these power laws. The l and j spectra are affected by triangular truncation whereas the n spectra are affected by rhomboidal truncation. Indeed the results of Baer (1972) in which he calculated l spectra from a truncated representation are quite similar to the Fig. 1. It is noteworthy that the l and j spectra are not strongly affected by rhomboidal truncation. This implies that l spectra which are computed by carrying out Fourier analyses at latitude circles and then averaging over latitude circles should exhibit a -3 power law since such a procedure is equivalent to computing the l spectrum using rhomboidal truncation in a spherical harmonic representation. As such they are direct evidence of a -3 law for kinetic energy in terms of the index n.

It is clear that spectra obtained from spectral models should be interpreted in the light of their particular truncation procedure. Indeed it would appear that the important property to simulate and to use for verification is the extent that the averages of *individual kinetic energy components* K_n^l are described by a -4 law of the two dimensional index n.

REFERENCES

Abramowitz, M. and Stegun, I. A. 1965. Handbook of mathematical functions, Dover, New York.

Baer, F. 1972. An alternate scale representation of atmospheric energy spectra. *J. Atmos. Sci. 29*, 649–664.

Baer, F. 1974. Hemispherical spectral statistics of available potential energy. *J. Atmos. Sci. 31*, 932–941.

Chen, T.-C. and Wiin-Nielsen, A. 1978. On nonlinear cascades of atmospheric energy and enstrophy in a two-dimensional spectral index. *Tellus 30*, 311–322.

Leith, C. E. 1971. Atmospheric predictability and two-dimensional turbulence. *J. Atmos. Sci. 28*, 145–161.

Wellck, R. E., Kasahara, A., Washington, W. M. and De Santo, G. 1971. The effect of horizontal resolution in a finite difference model of the general circulation. *Mon. Wea. Rev. 99*, 673–683.

Wiin-Nielsen, A. 1967. On the annual variation and spectral distribution of atmospheric energy. *Tellus 19*, 540–559.

Wiin-Nielsen, A. 1972. A study of power laws in the atmospheric kinetic energy spectrum using spherical harmonic functions. *Meteor Ann. 6*, 107–124.

О КИНЕМАТИЧЕСКИХ СВОЙСТВАХ "ЗАКОНА – 3" ДЛЯ КИНЕТИЧЕСКОЙ ЭНЕРГИИ

Показано, что если кинетическая энергия подчиняется закону степени – 3 для двумерного индекса n, то она должна следовать тому же закону для зонального волнового числа l и для меридионального волнового числа j. Для облегчения интерпретации анализа реальных данных и модельных результатов демонстрируется влияние обрезания на эти степенные законы.