The Beauty of Beatty sequences

Sela Fried*

friedsela@gmail.com

Abstract

We visualize sequences corresponding to the parity of Beatty sequences of square roots of non-square natural numbers, that exhibit particularly beautiful symmetries.

1 Introduction

A Beatty sequence is the sequence of integers obtained by taking the floor of the positive multiples of a positive irrational number, i.e., if $r > 0$ is irrational, then the corresponding Beatty sequence B_r is given by

$$B_r = (\lfloor nr \rfloor)_{n \in \mathbb{N}}.$$

Beatty sequences are named after Samuel Beatty who brought them to the attention of the mathematical community by posing a problem in [1], in which the readers were asked to prove that, if $r > 1$ is irrational and $s = r/(r-1)$, then B_r and B_s are complementary sequences, i.e., every natural number belongs to exactly one of the two sequences.

During our work on a certain problem, we became aware of sequence A083035 in the On-Line Encyclopedia of Integer Sequences (OEIS) [3], that is concerned with the sequence of parities of $B_{\sqrt{2}}$. Benoit Cloitre, the author of A083035, provided a visualization of the sequence, that relies on a simple but ingenious method for visualizing any binary sequence $(a_n)_{n \in \mathbb{N}}$: Start at $(0,0)$, facing in the direction of the vector $(1,0)$. Then, for every $n \in \mathbb{N}$, turn right if $a_n = 0$ and left otherwise and proceed forward one step of unit length. The result of this walk is a path in the plane that we shall refer to as the *Cloitre walk of* $(a_n)_{n \in \mathbb{N}}$. Figure 1 below shows the Cloitre walk of the sequence $(1, 0, 0, 1)$.

---

*The author is a teaching fellow in the Department of Computer Science at the Israel Academic College in Ramat Gan.
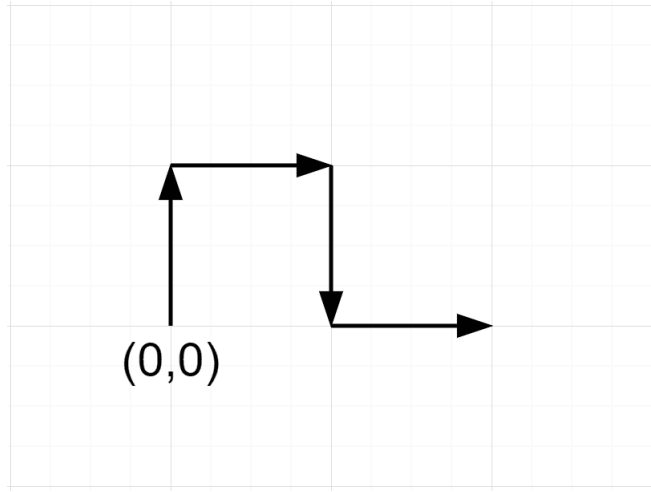
Figure 1: The Cloitre walk of the sequence $(1, 0, 0, 1)$.

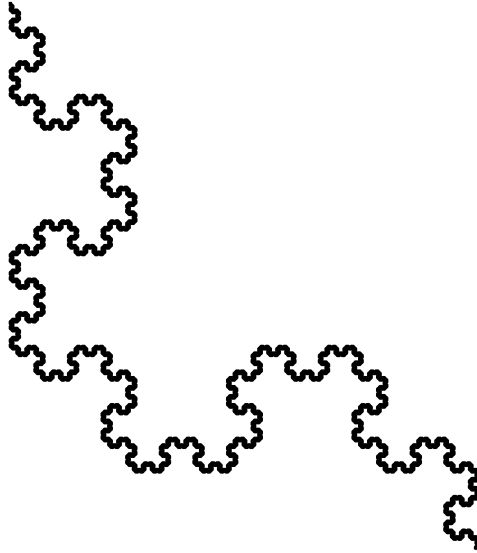The Cloitre walk of [A083035](#) is given in Figure 2 below.

Figure 2: The Cloitre walk of [A083035](#) that corresponds to the sequence of parities of $B_{\sqrt{2}}$.

2

We were curious about how the Cloitre walk would look like if we replaced $r = \sqrt{2}$ with $r = \sqrt{n}$, where n is a non-square natural number. We were amazed by what we have discovered. In the following we present 100 of our favorite Cloitre walks, where we considered values of n only up to 10^4. The code that generates these walks may be found at the end of this work.
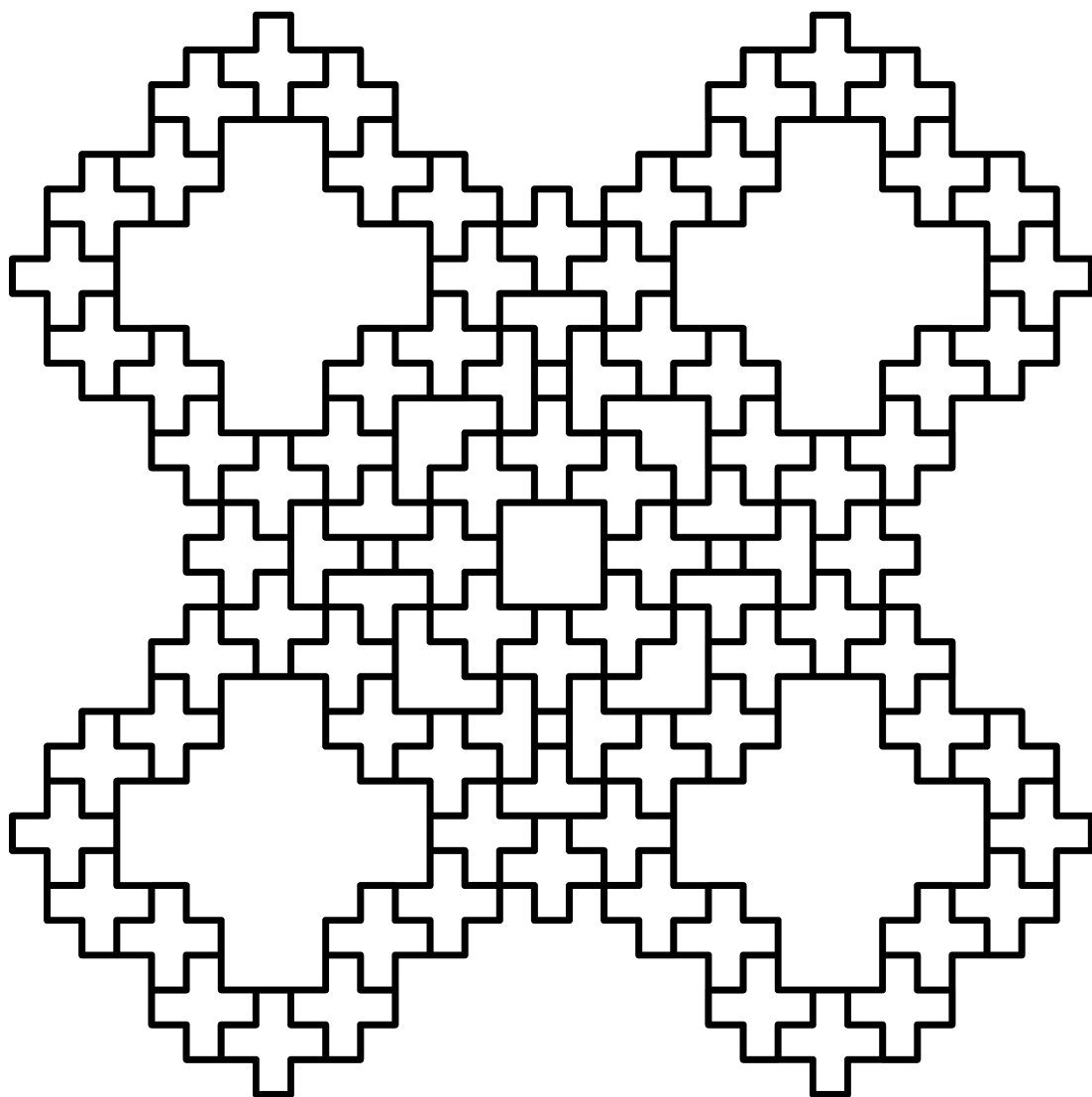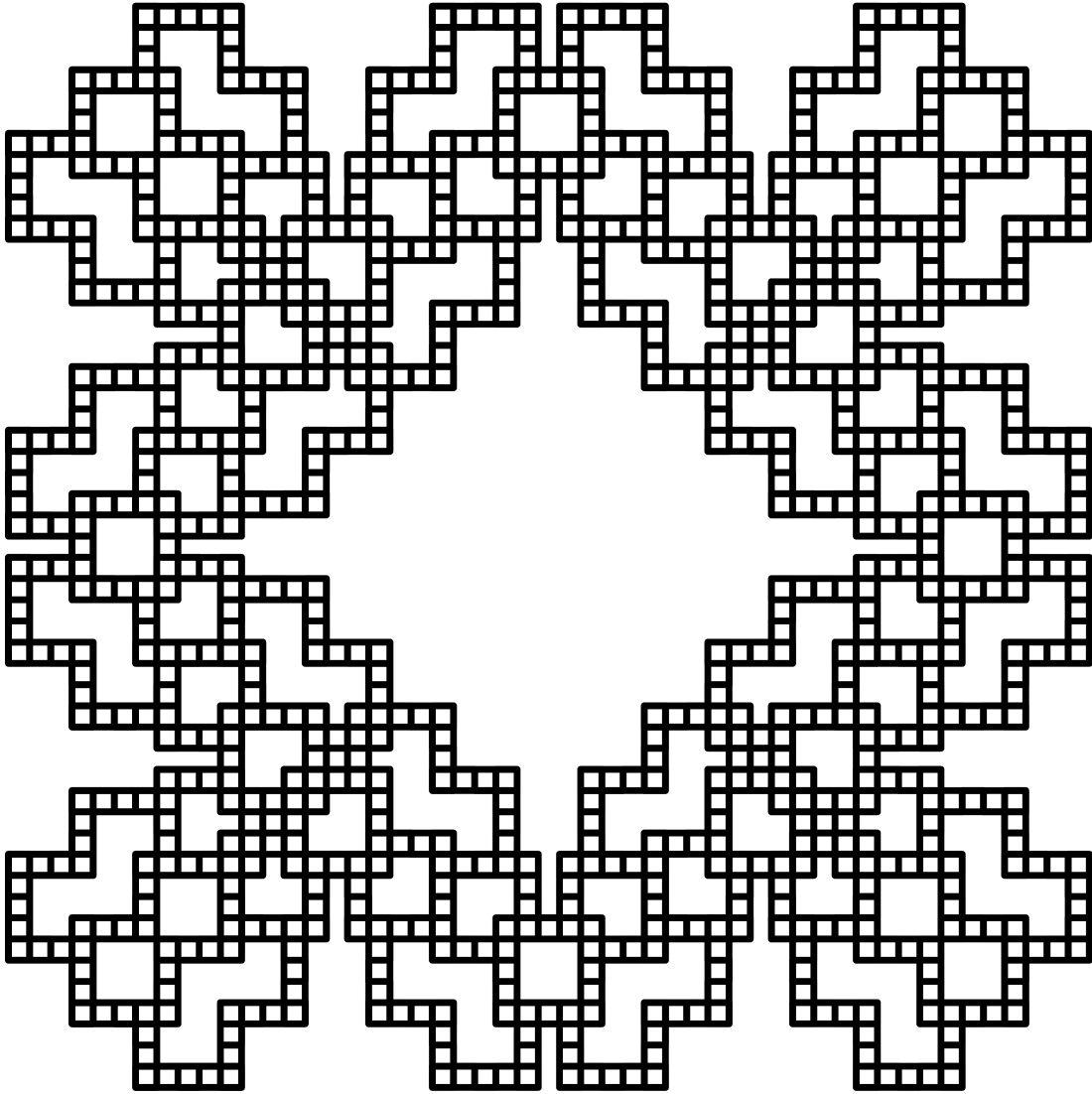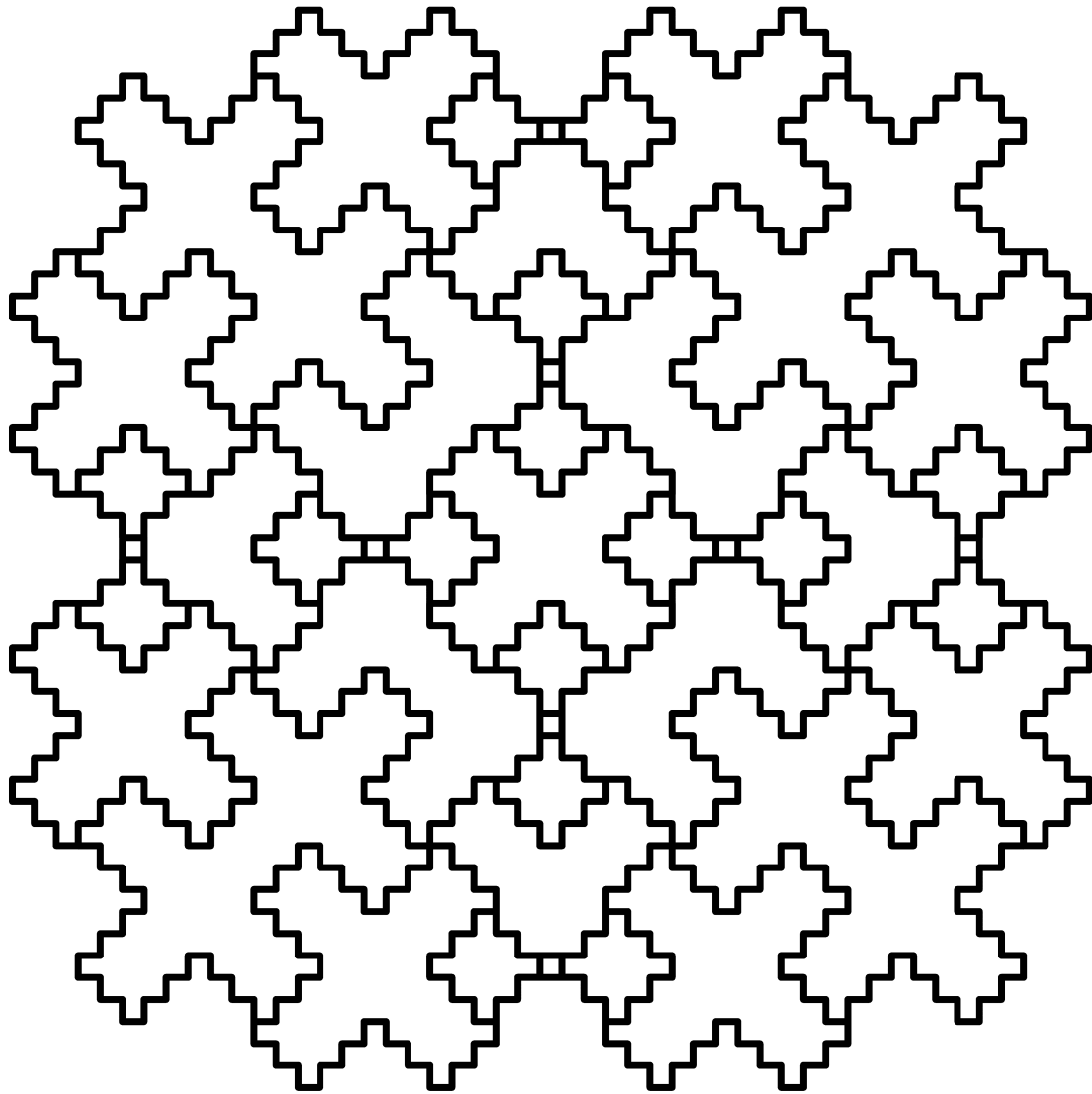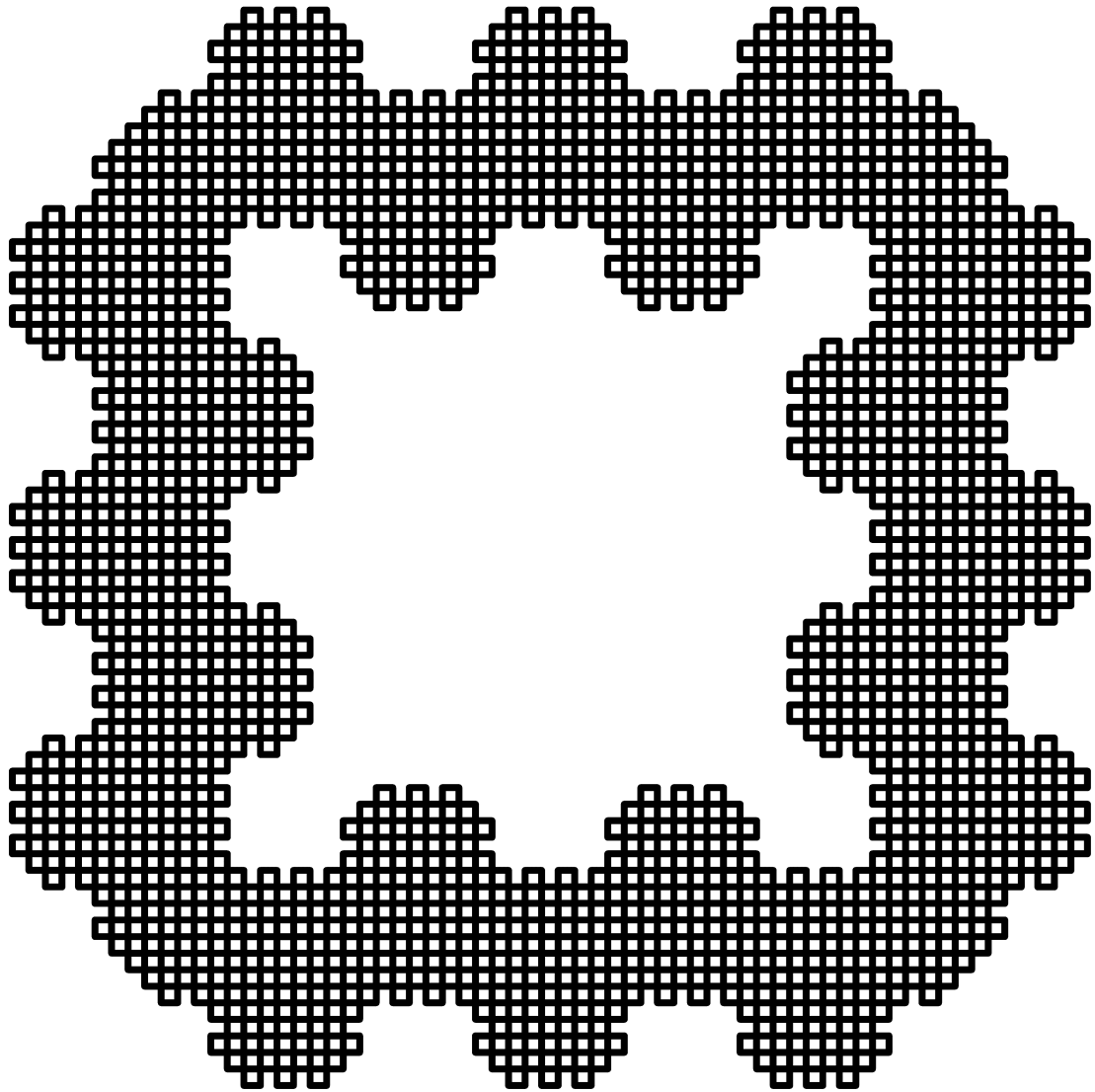
2 The Gallery
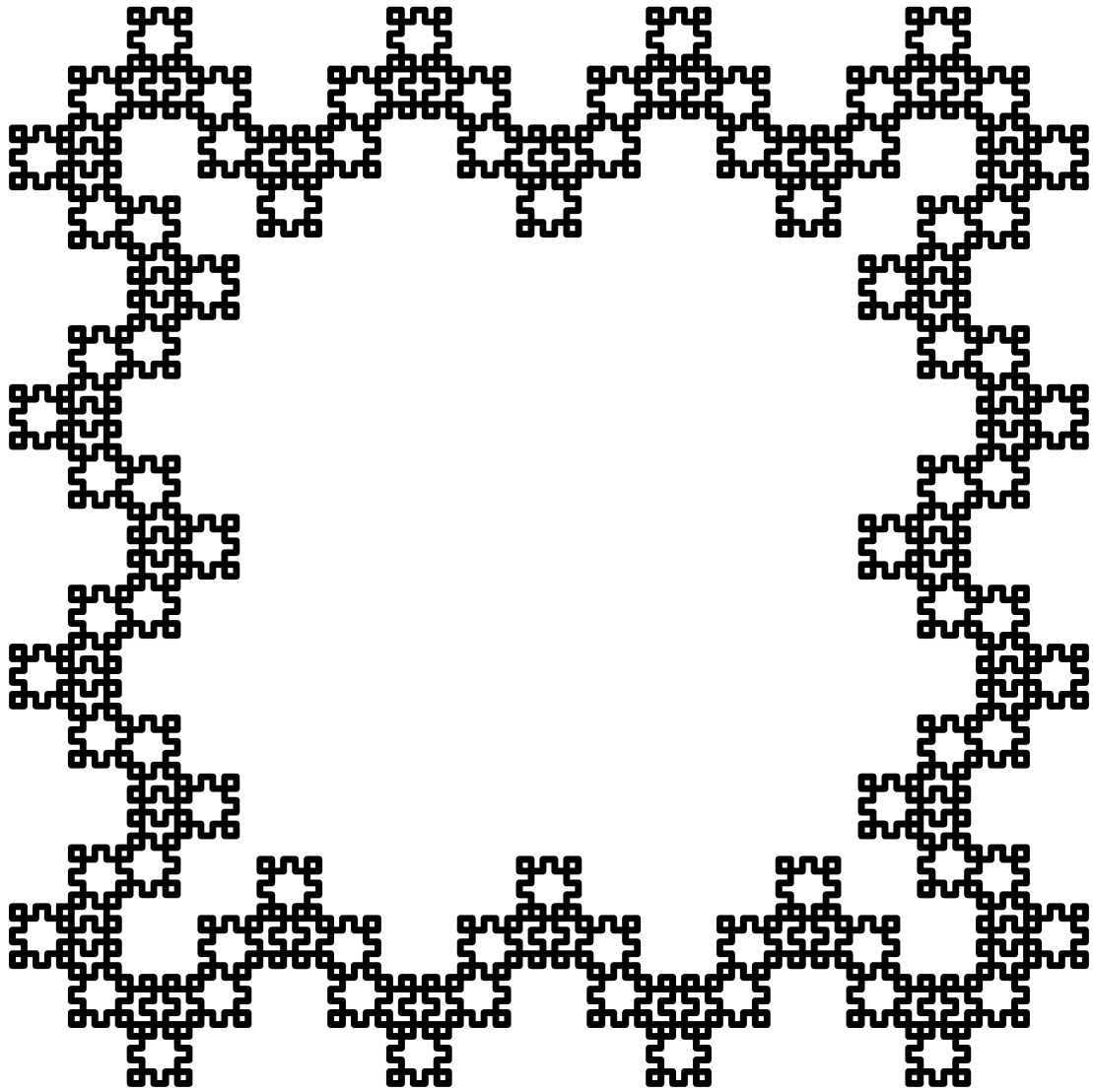
Figure 3: The Cloitre walk of the sequence of parities of $B_{\sqrt{109}}$.
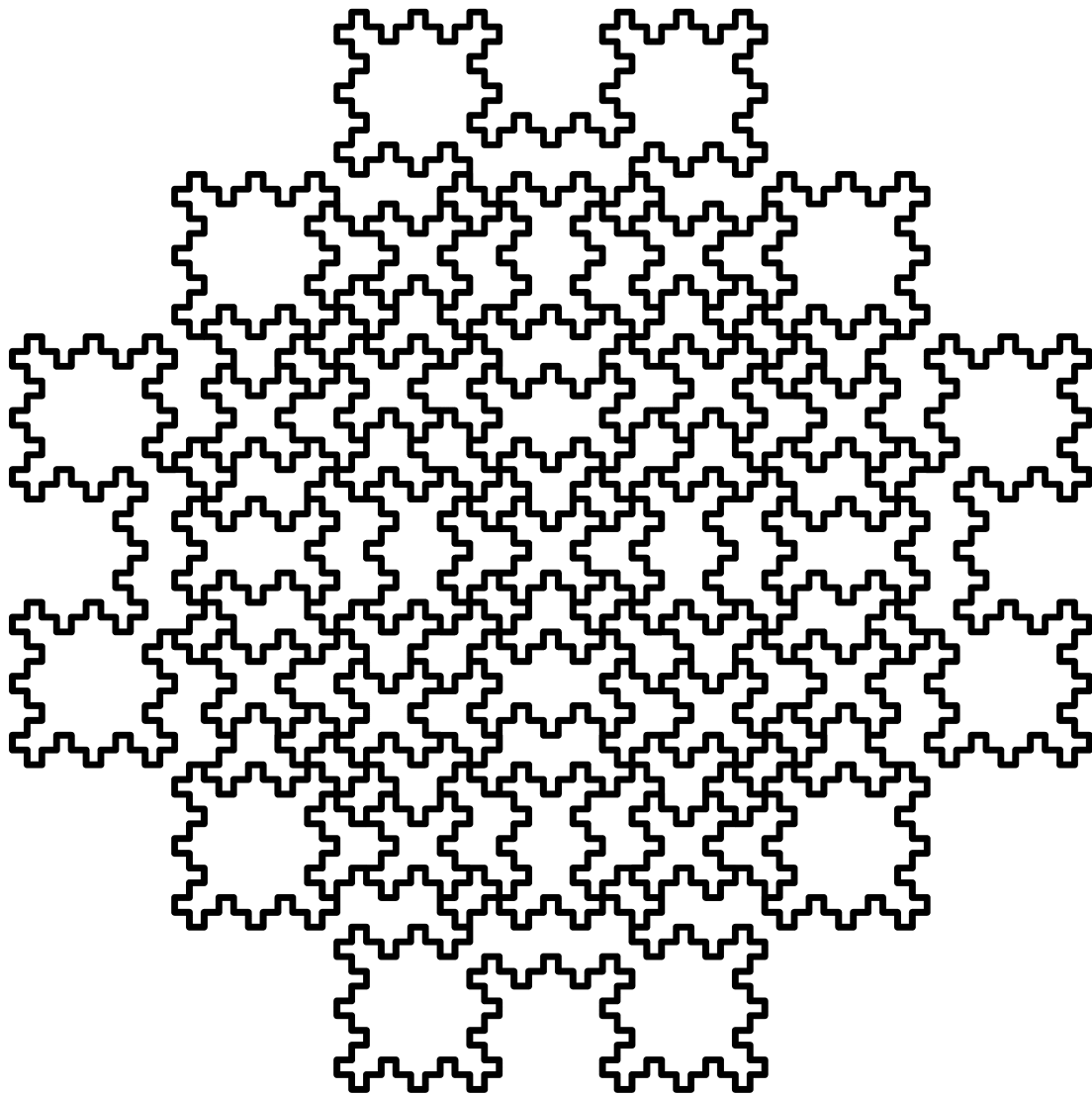
Figure 4: The Cloitre walk of the sequence of parities of $B_{\sqrt{233}}$.
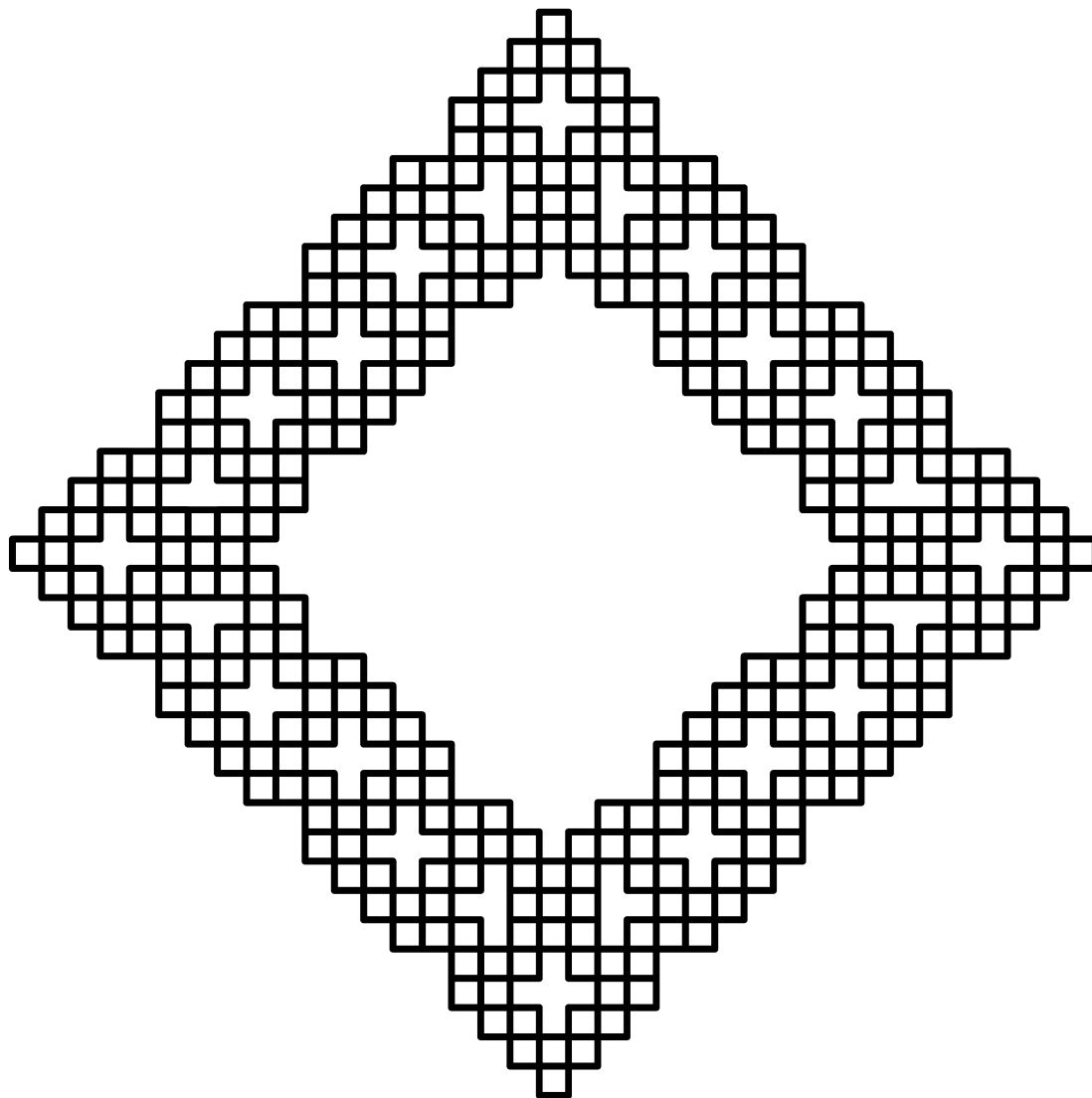
Figure 5: The Cloitre walk of the sequence of parities of $B_{\sqrt{498}}$.
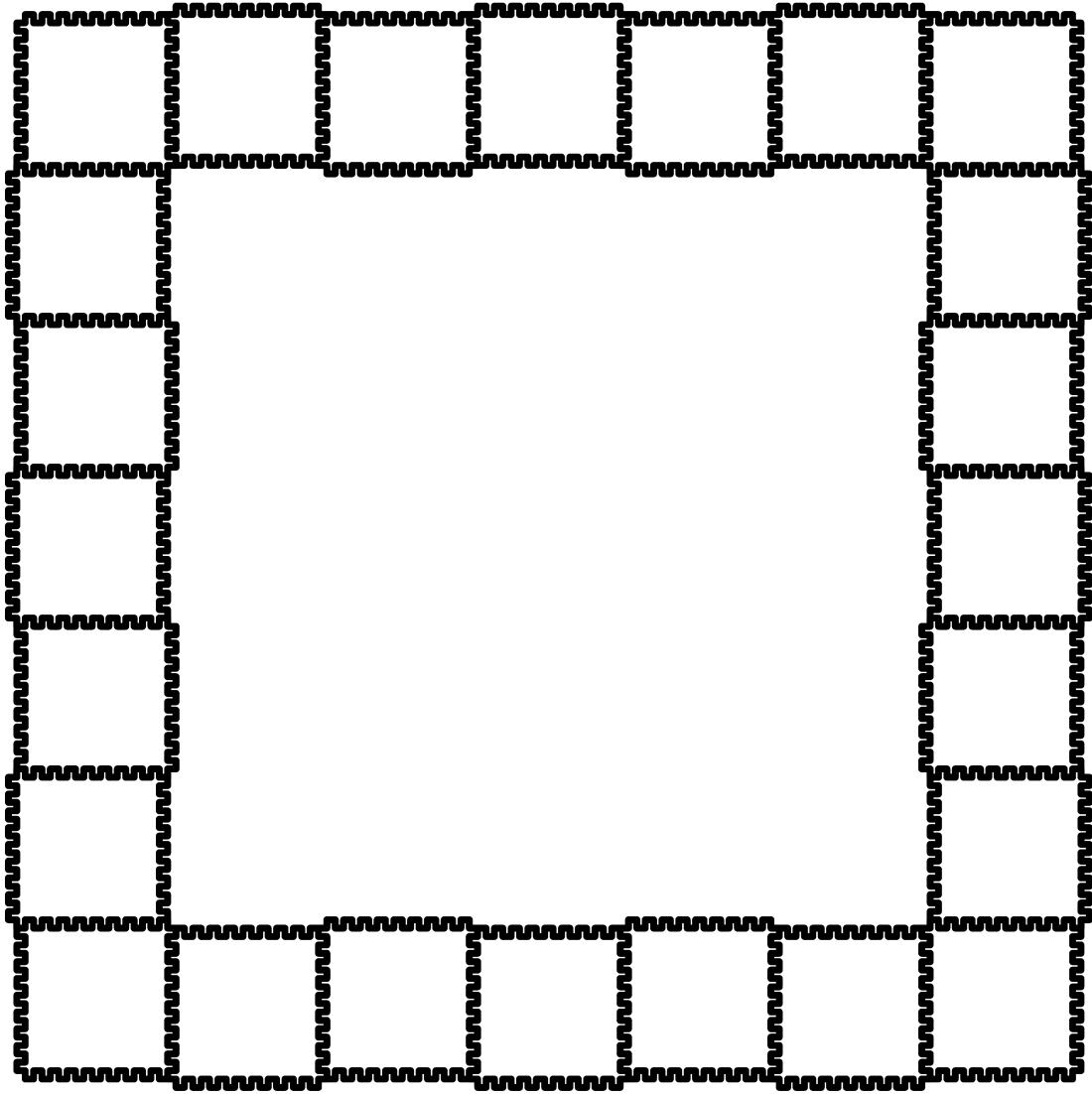
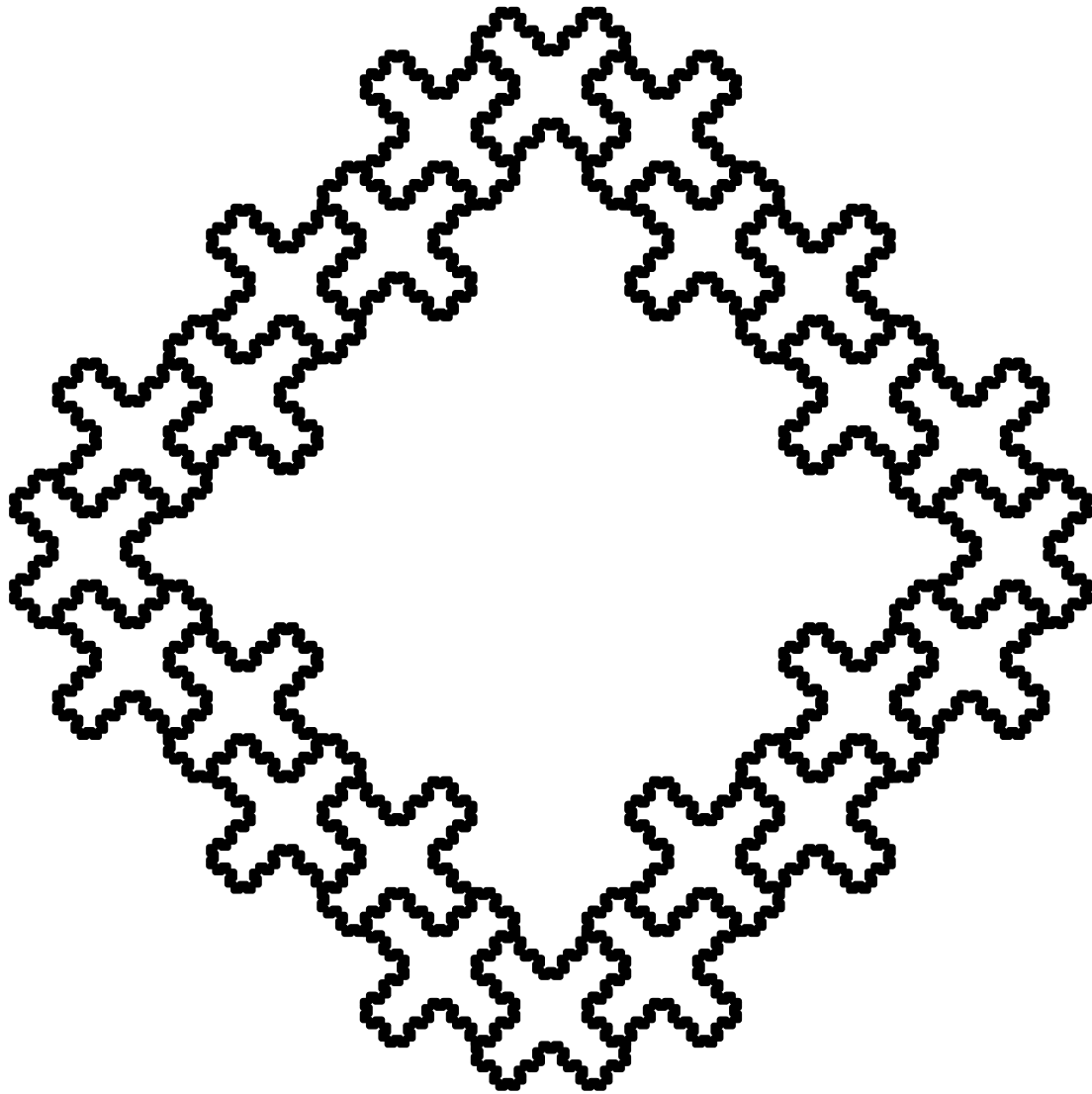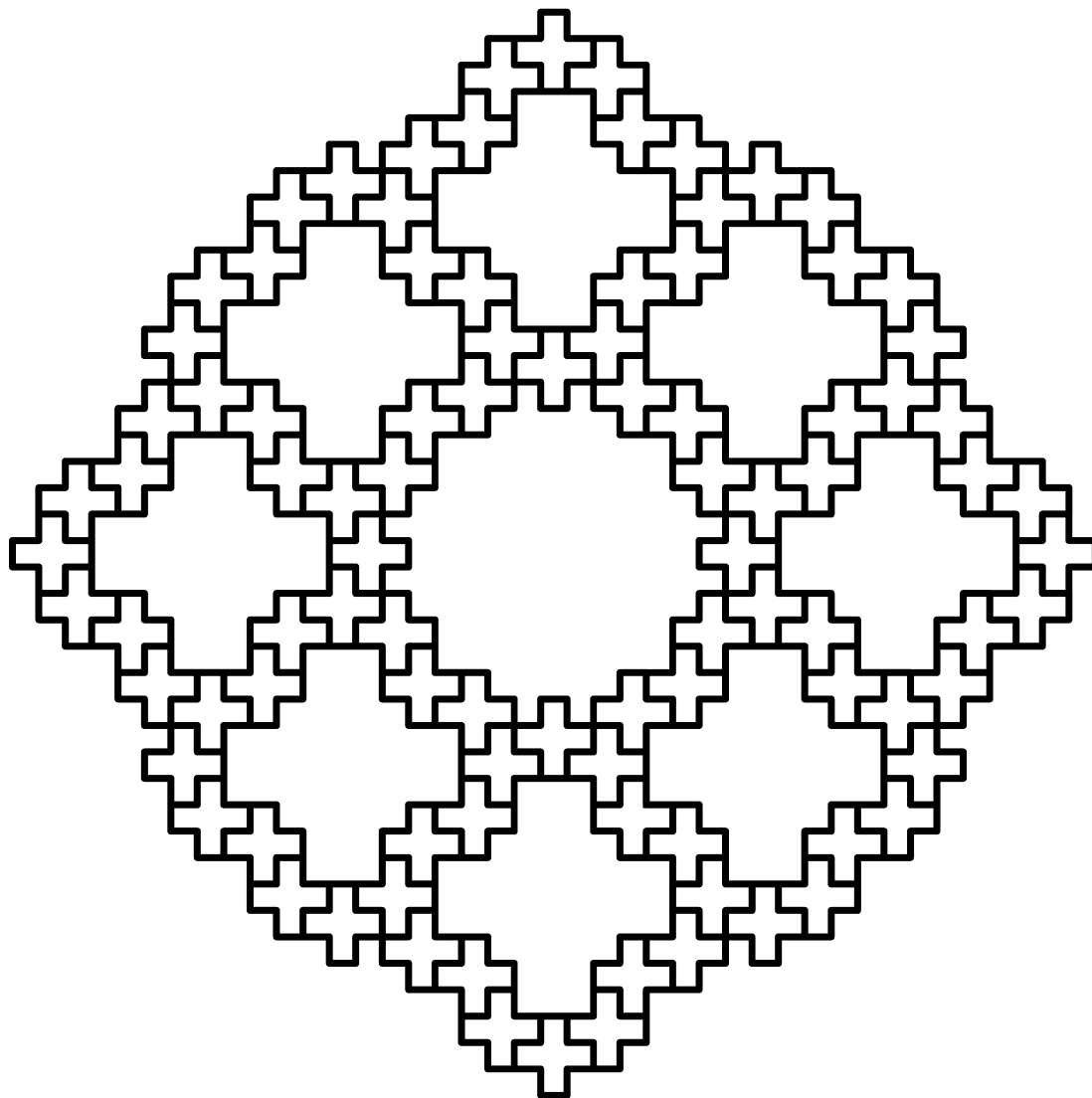Figure 6: The Cloitre walk of the sequence of parities of $B_{\sqrt{703}}$.
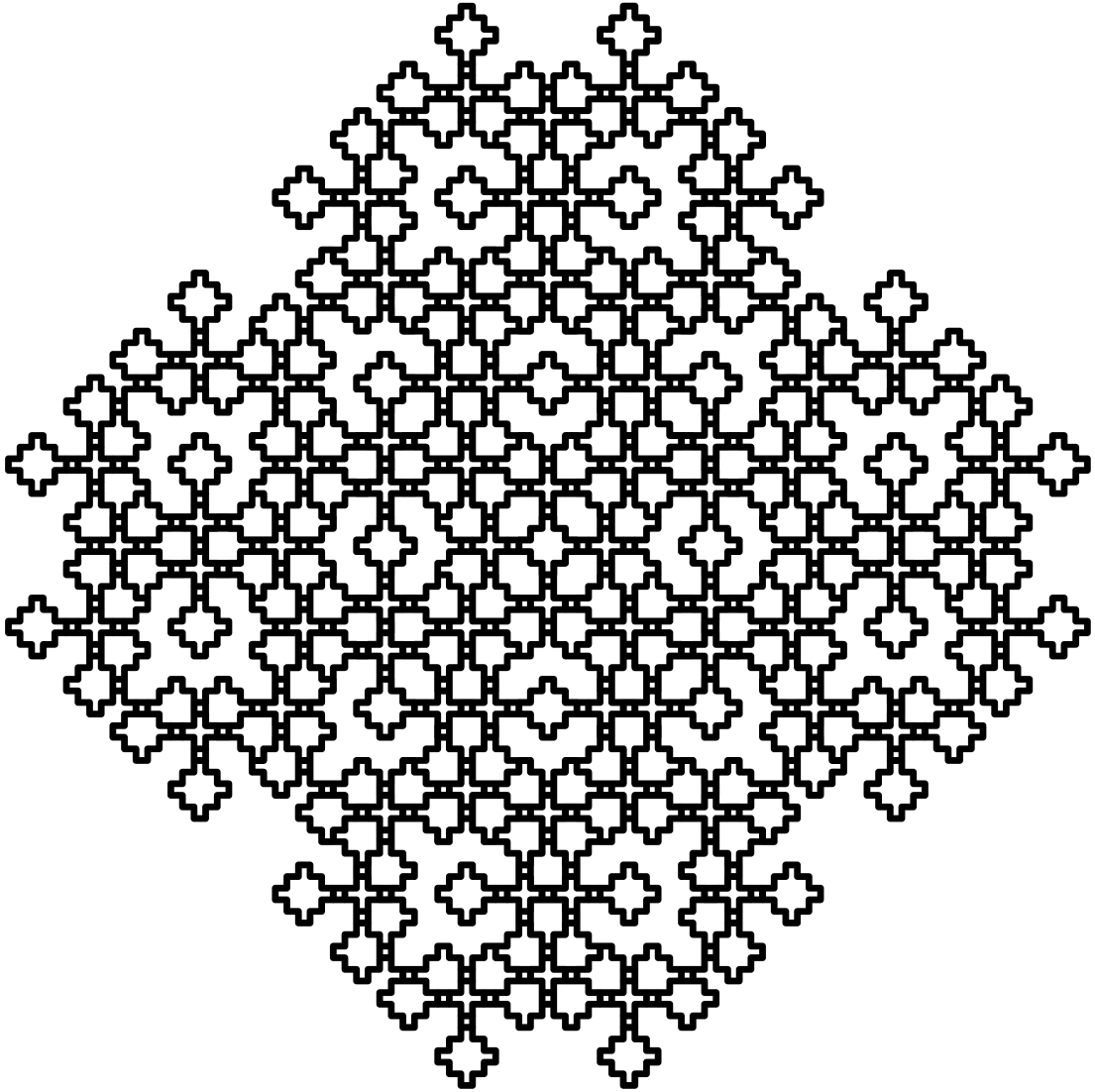
Figure 7: The Cloitre walk of the sequence of parities of $B_{\sqrt{719}}$.
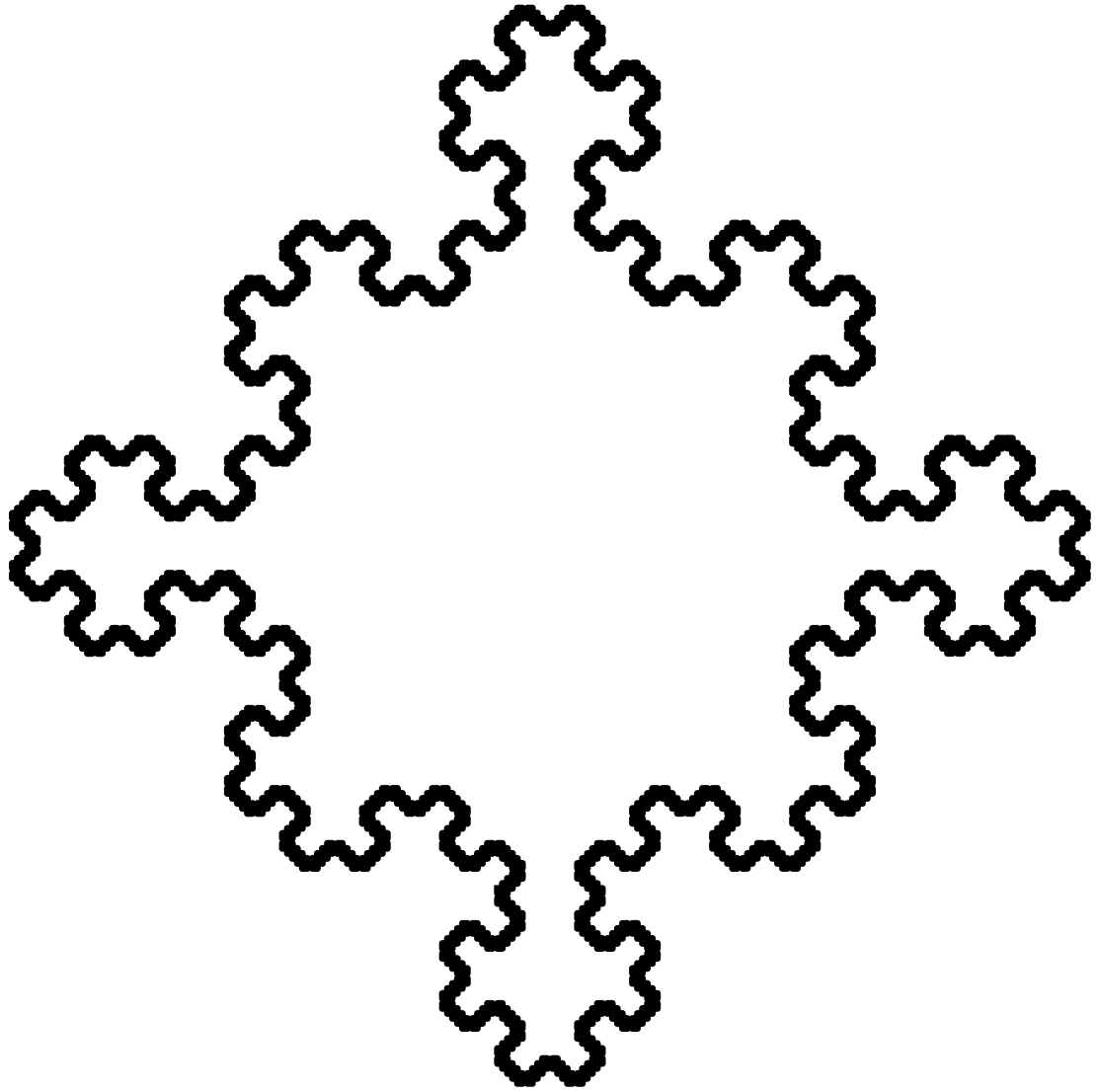
Figure 8: The Cloitre walk of the sequence of parities of $B_{\sqrt{1321}}$.
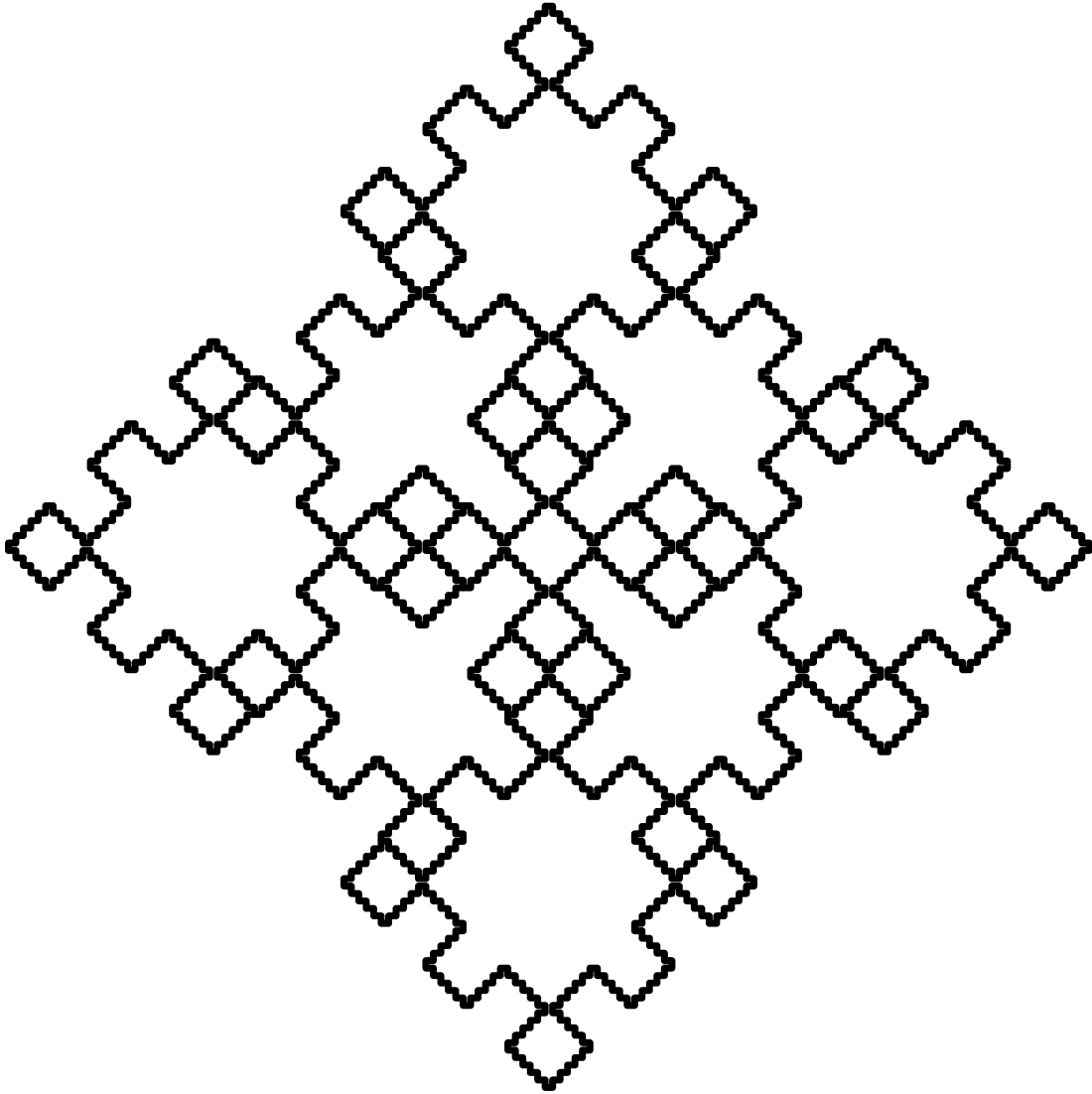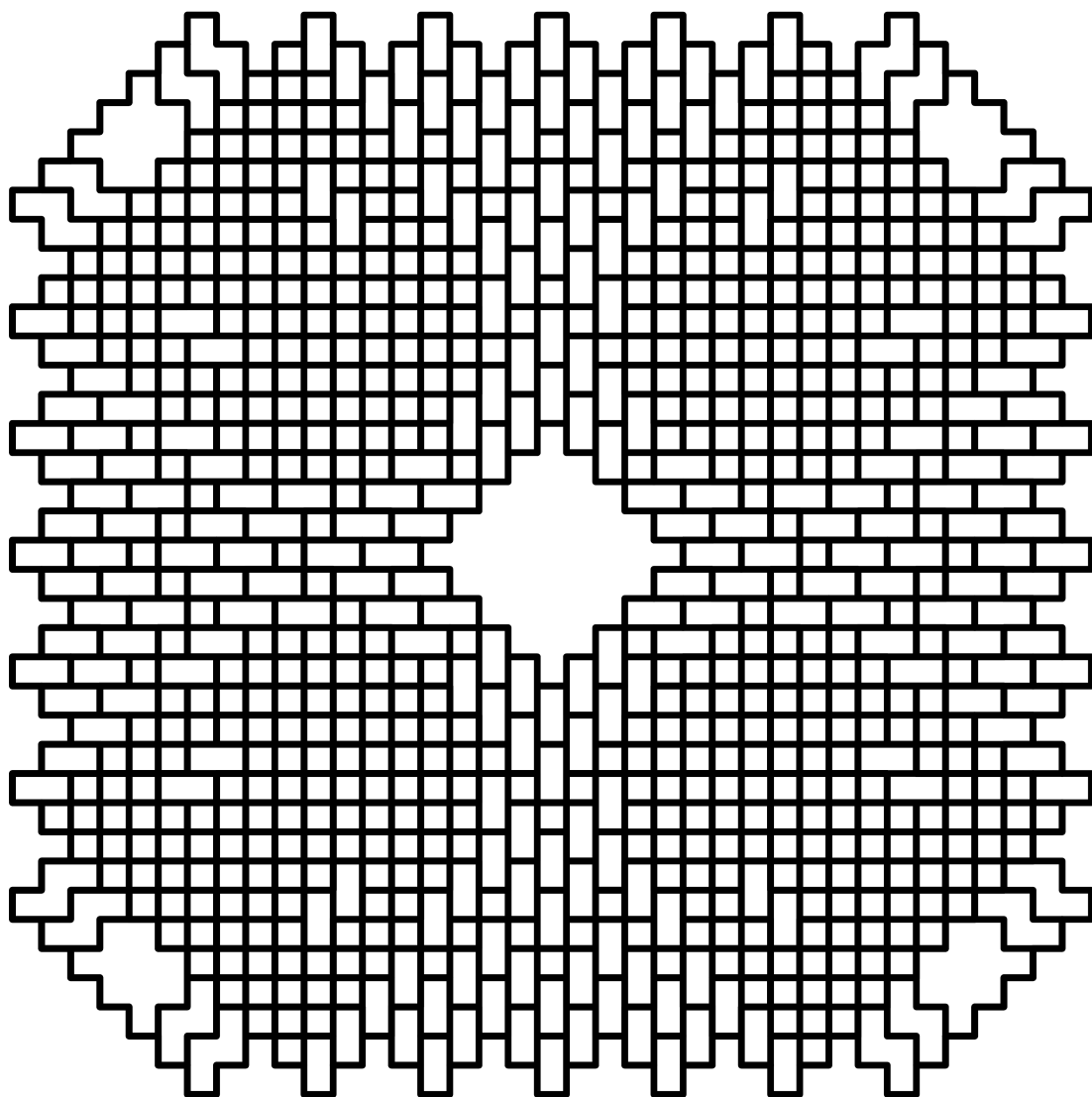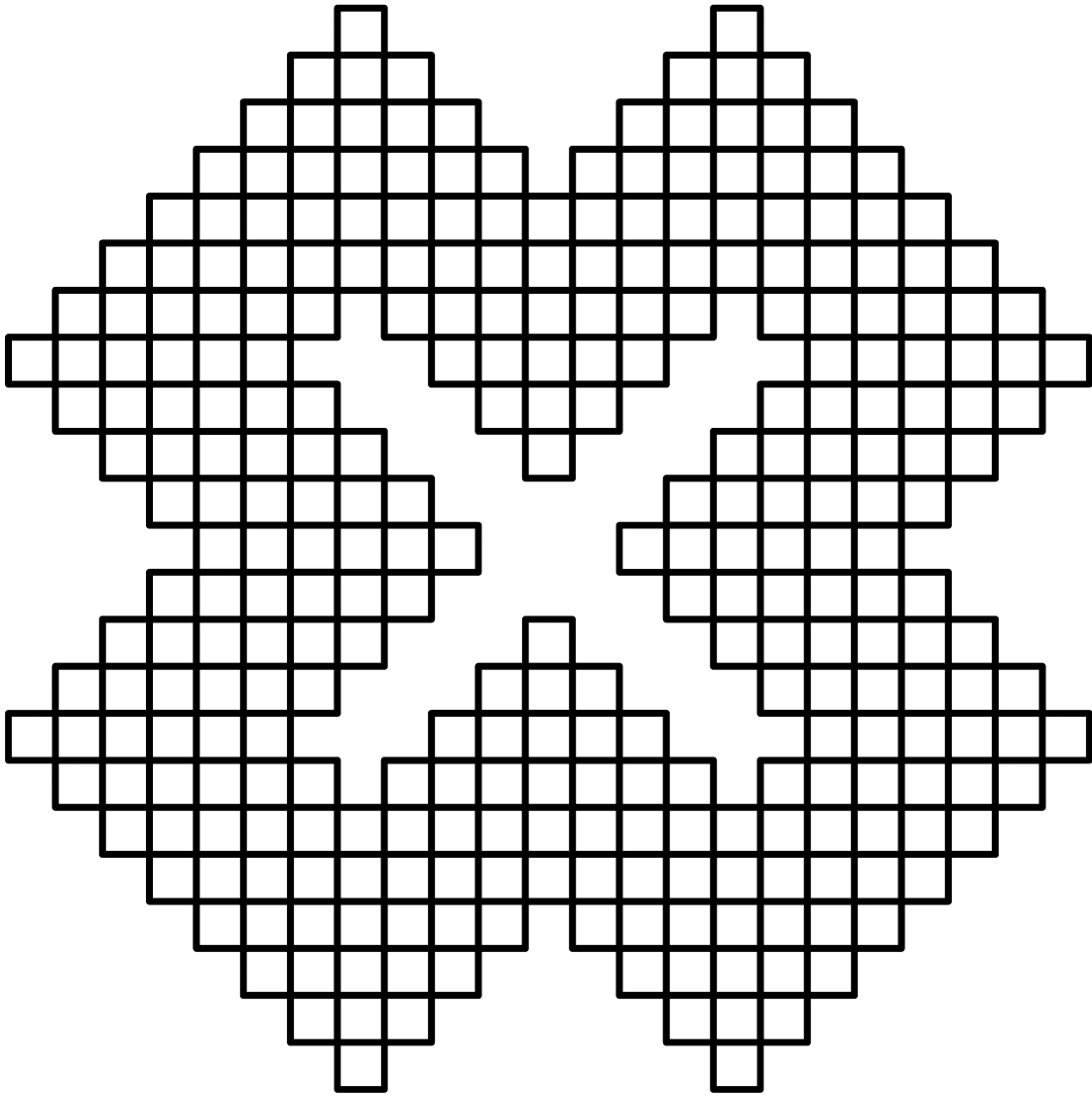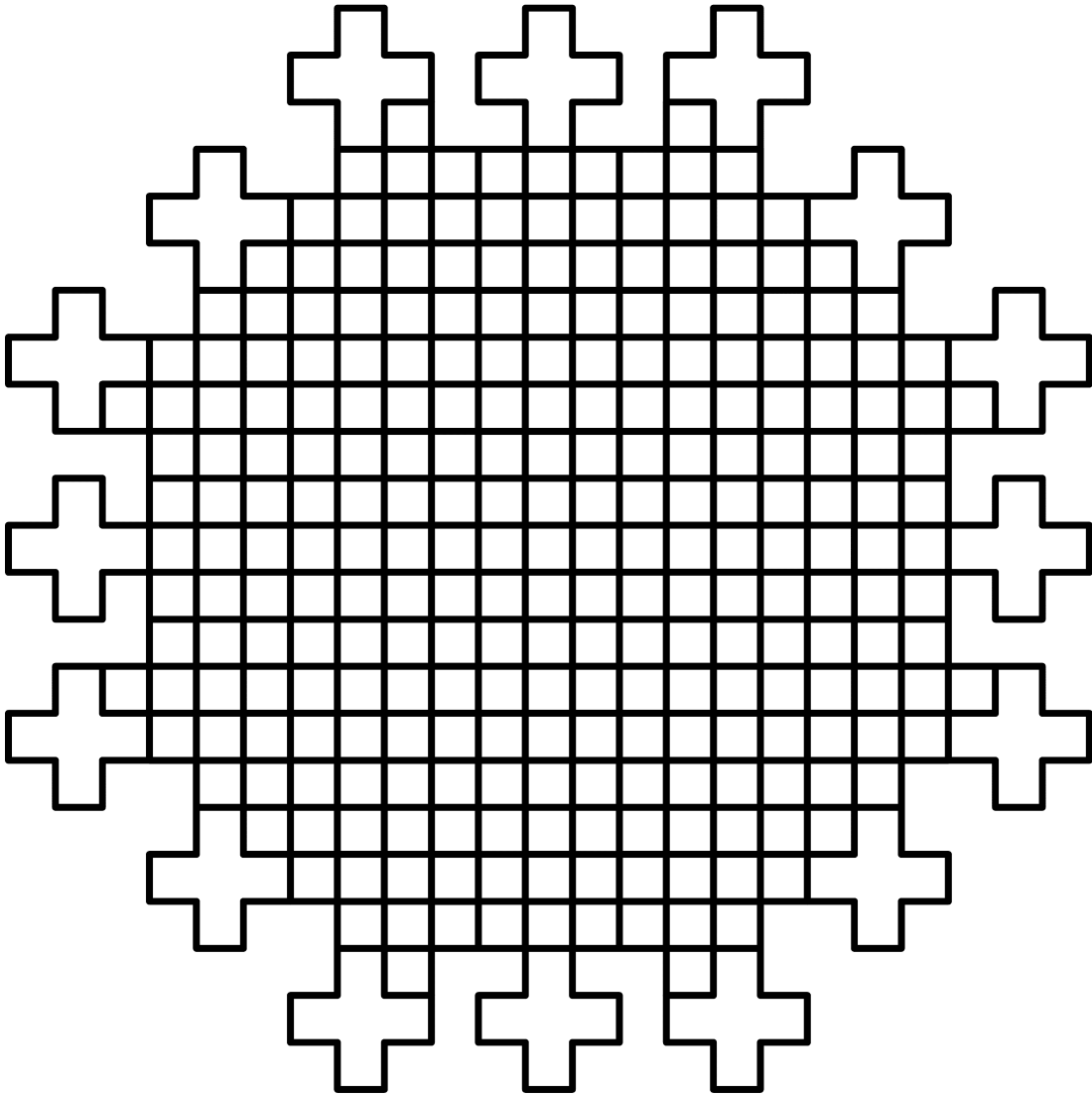
Figure 9: The Cloitre walk of the sequence of parities of $B_{\sqrt{1362}}$.
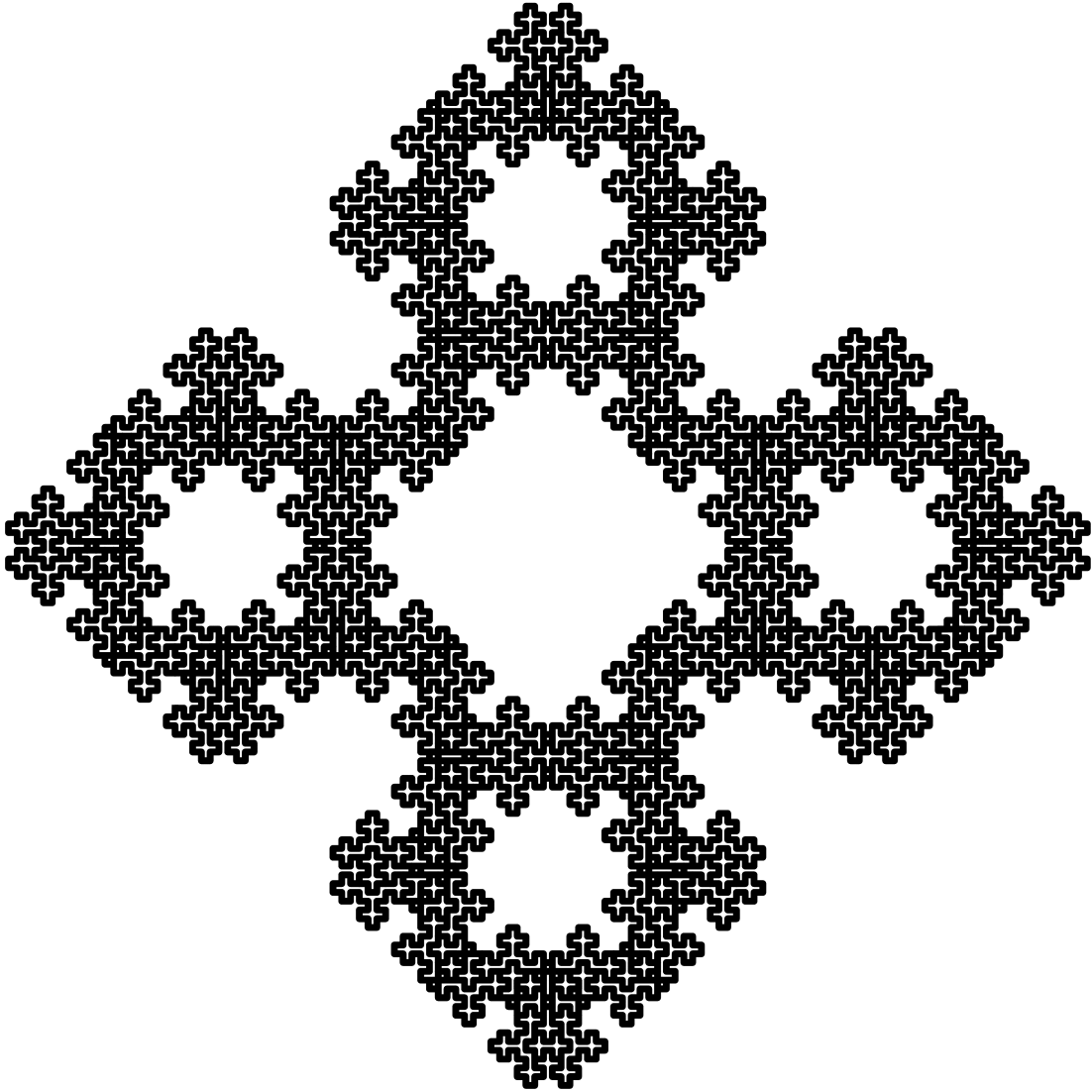
Figure 10: The Cloitre walk of the sequence of parities of $B_{\sqrt{1491}}$.
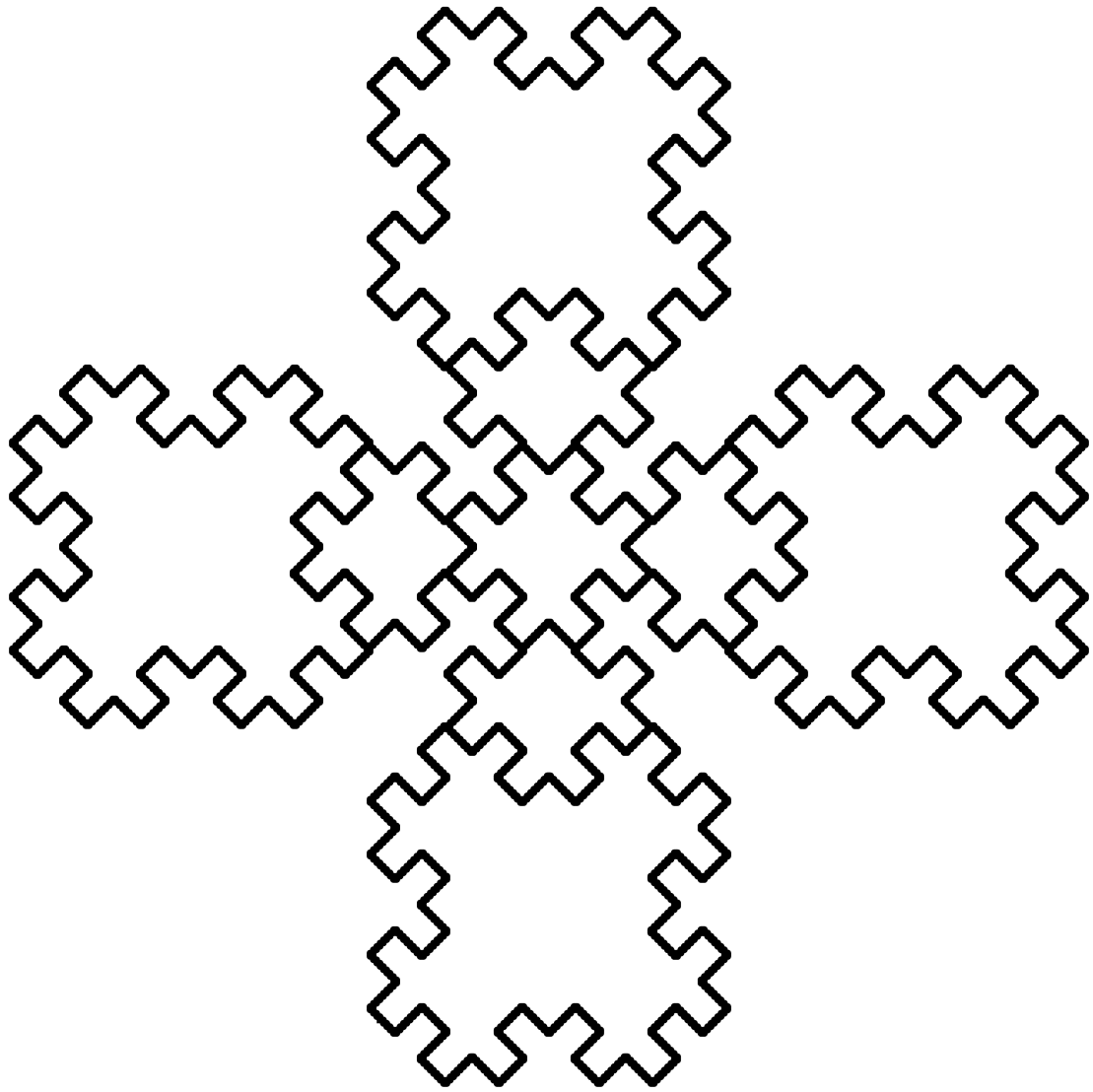
Figure 11: The Cloitre walk of the sequence of parities of $B_{\sqrt{1516}}$.
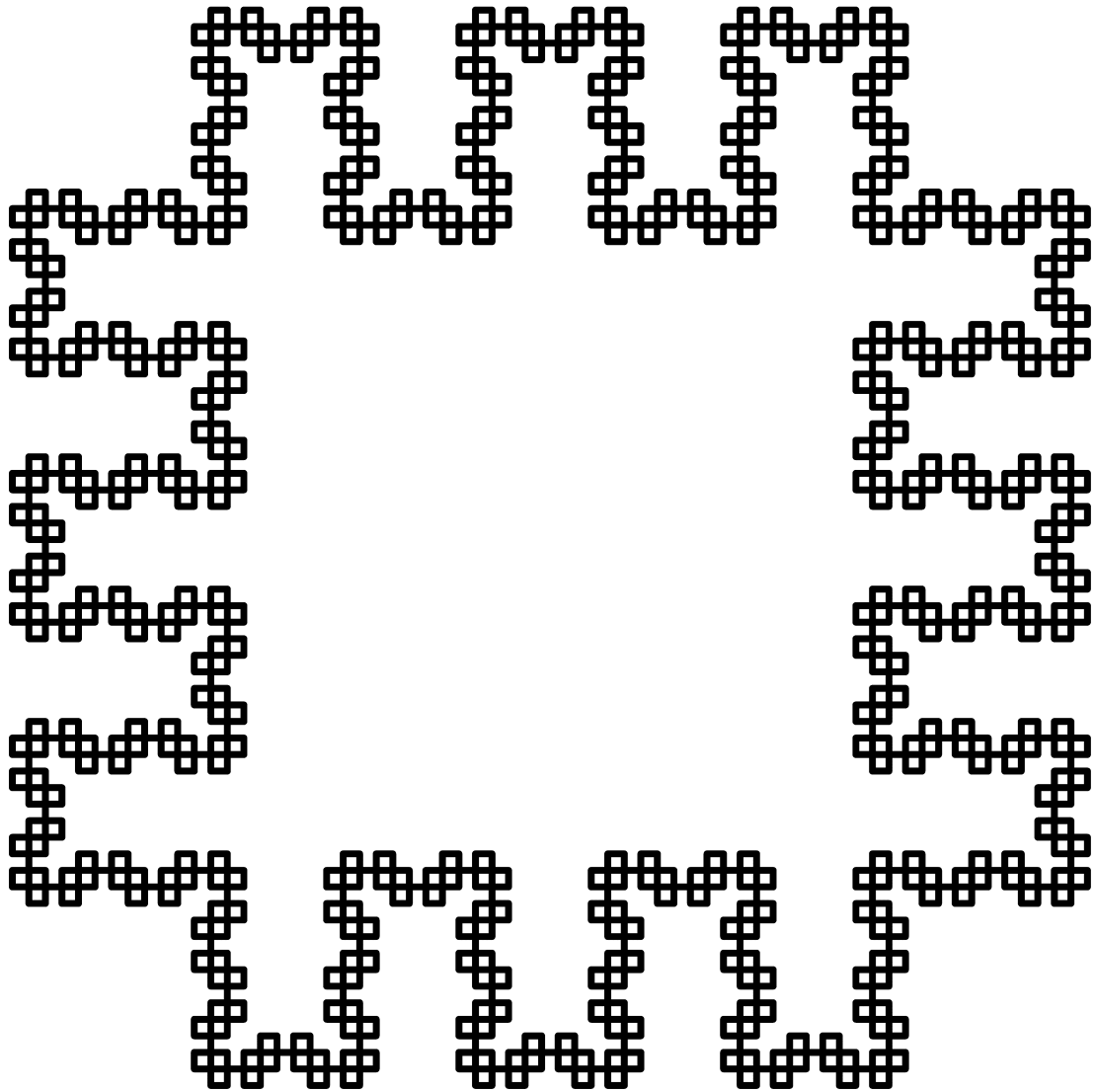
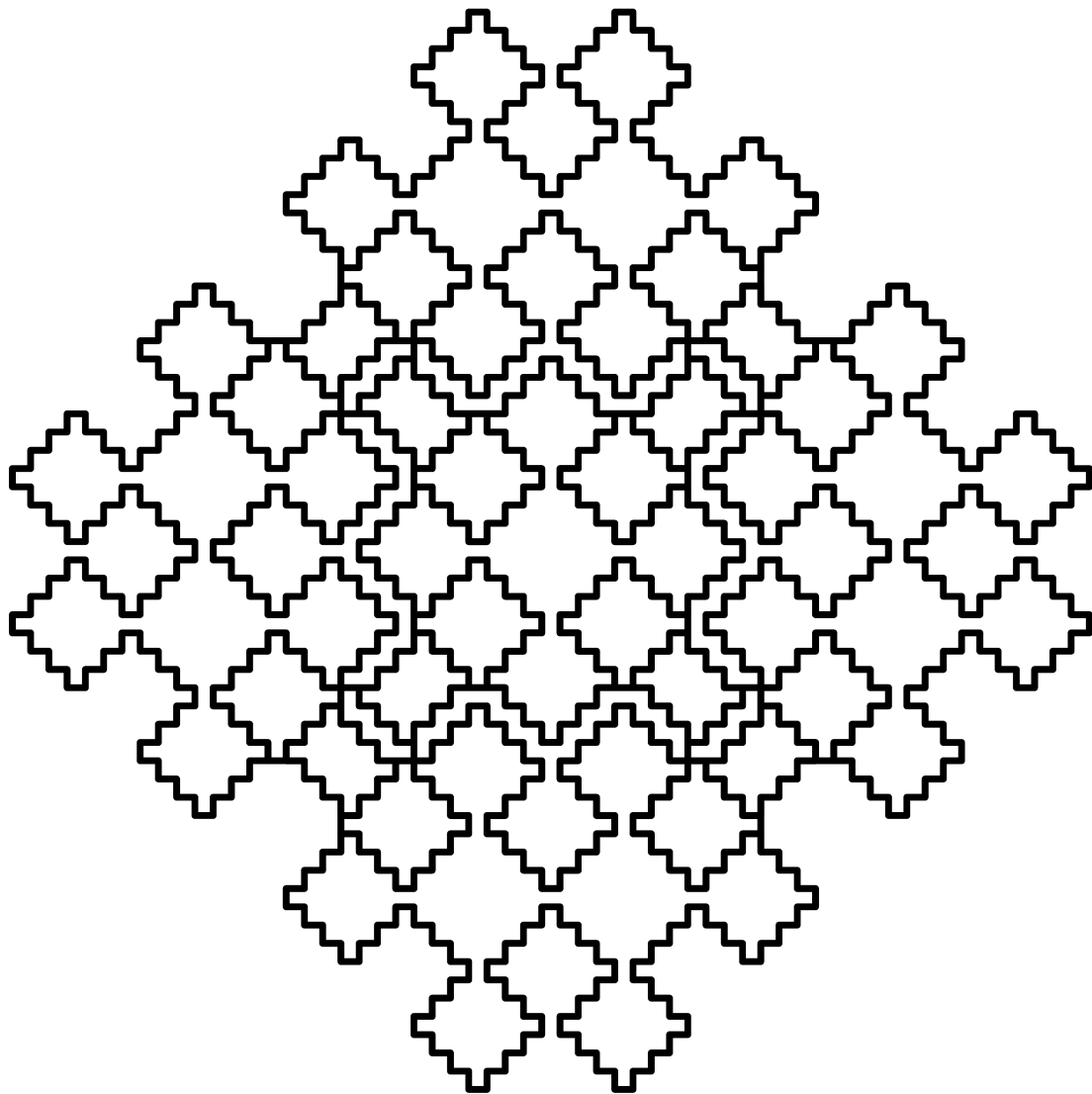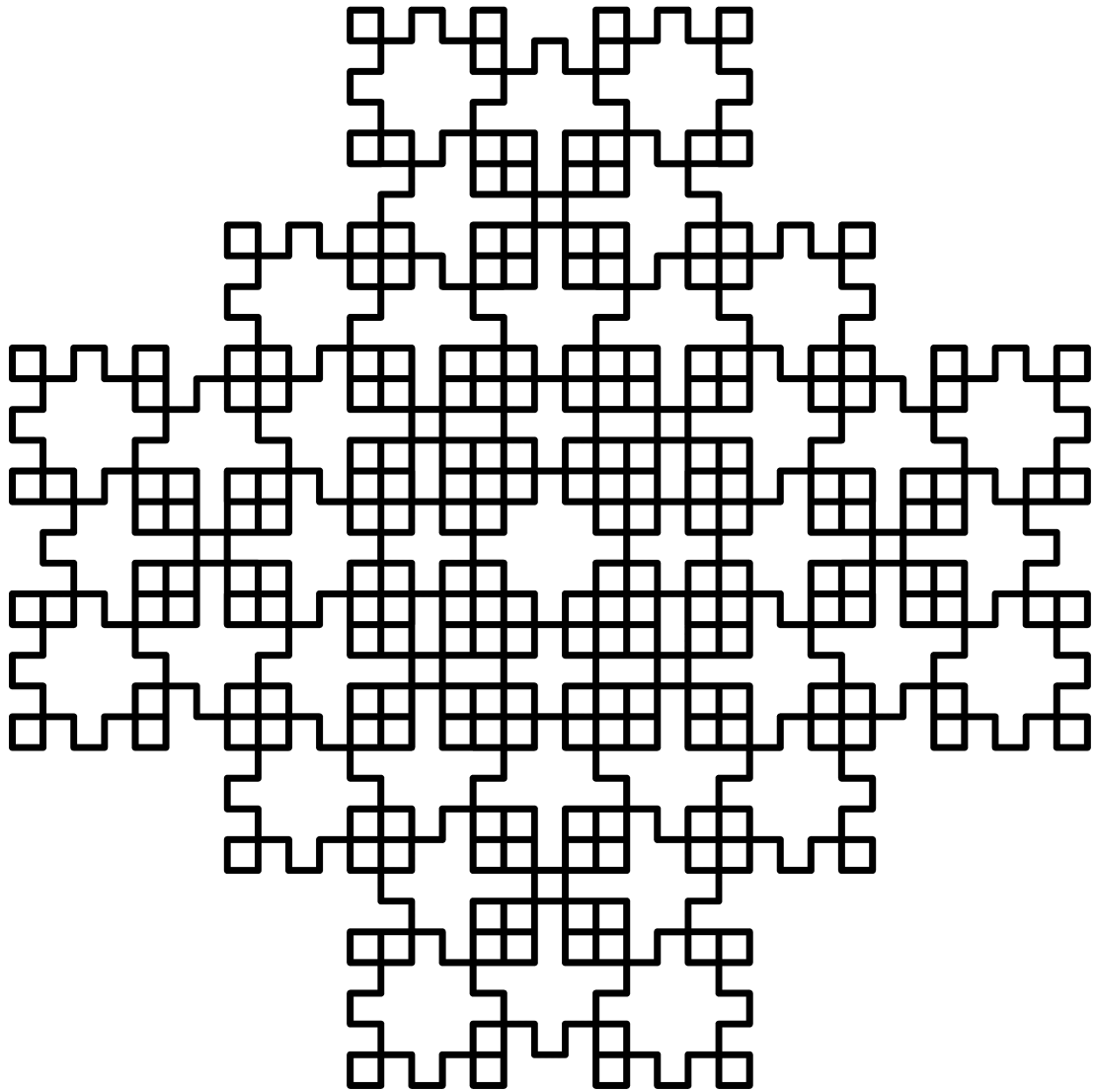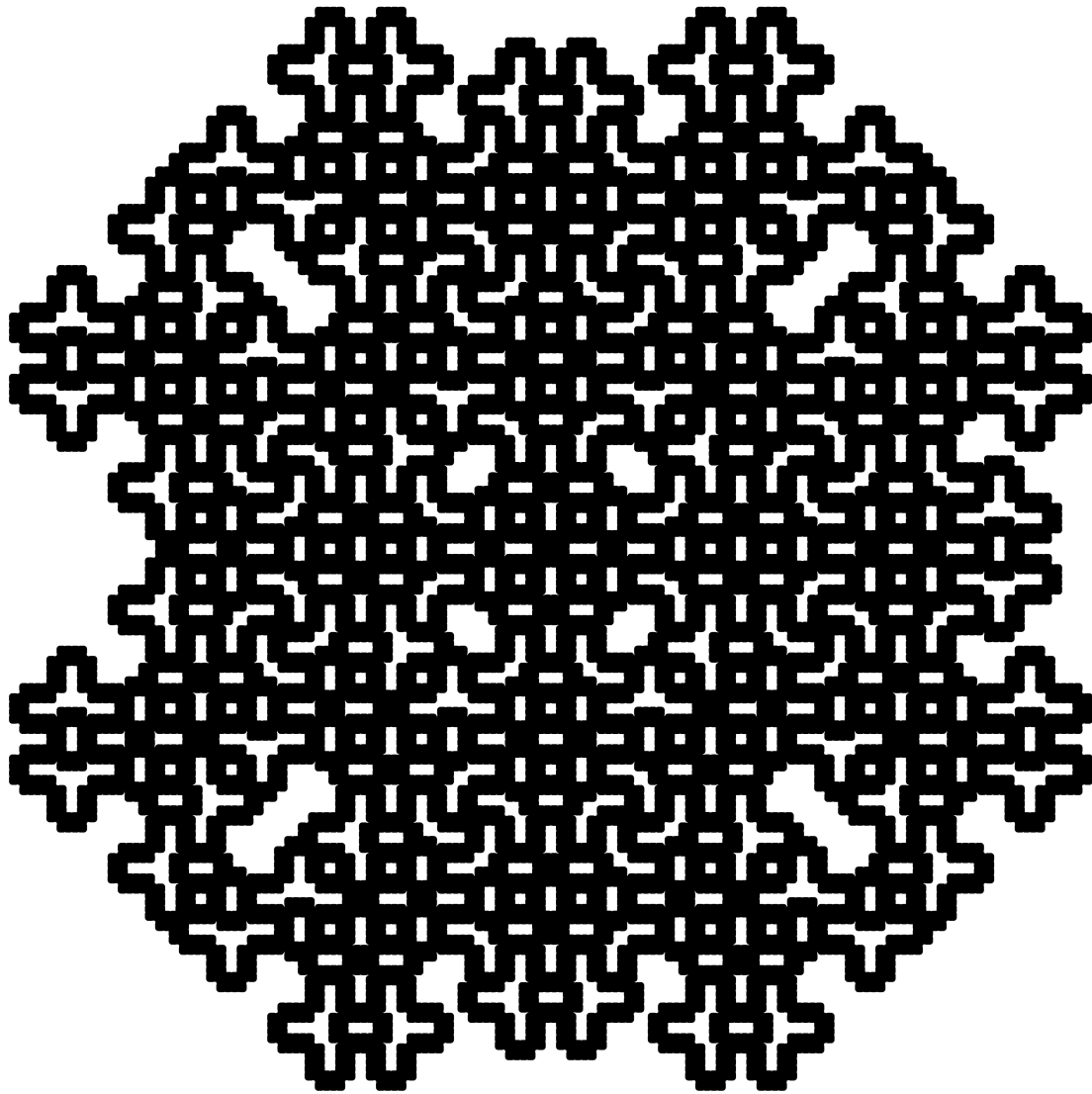Figure 12: The Cloitre walk of the sequence of parities of $B_{\sqrt{1734}}$.

Figure 13: The Cloitre walk of the sequence of parities of $B_{\sqrt{1822}}$.

Figure 14: The Cloitre walk of the sequence of parities of $B_{\sqrt{1950}}$.

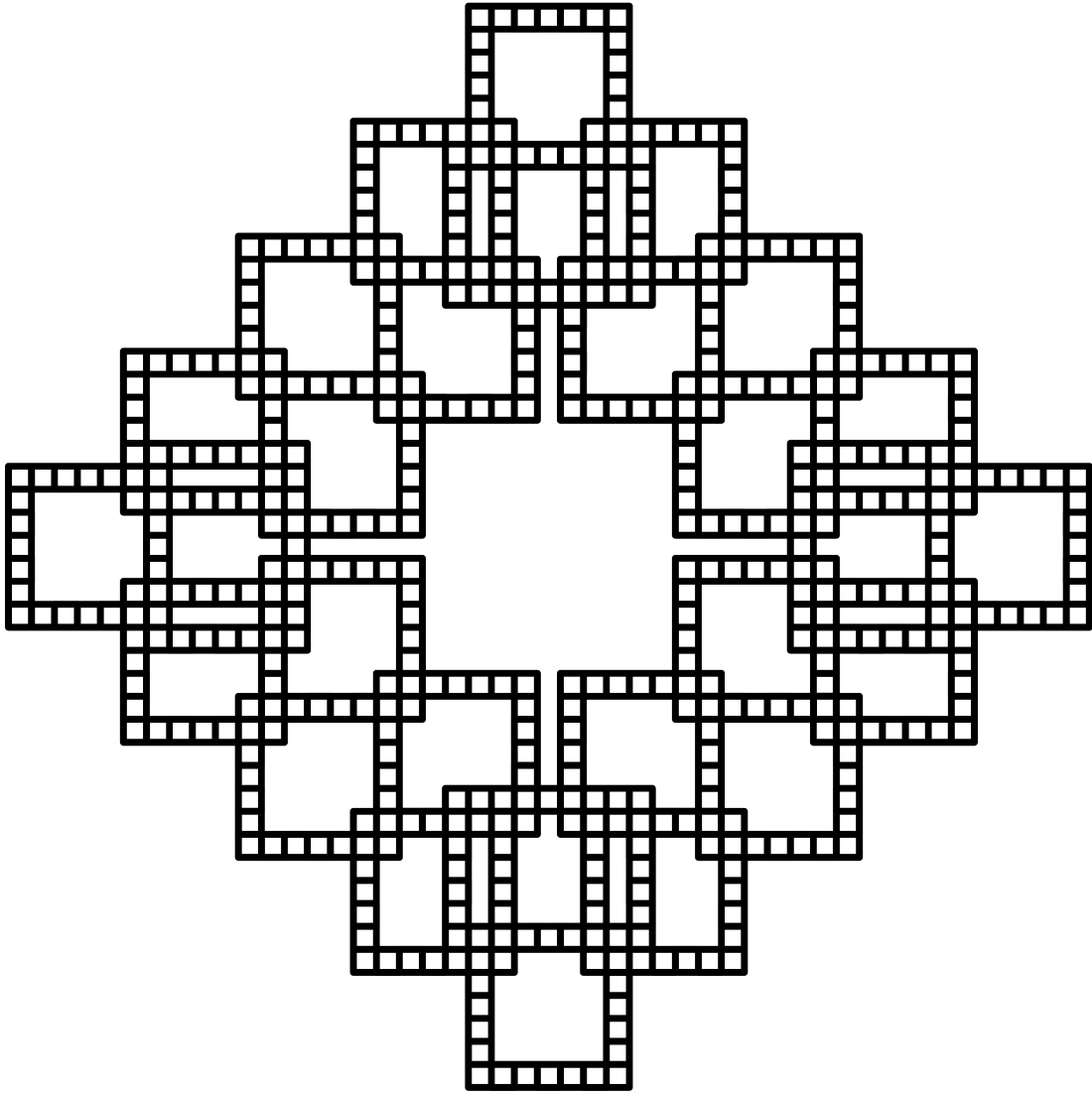Figure 15: The Cloitre walk of the sequence of parities of $B_{\sqrt{2009}}$.

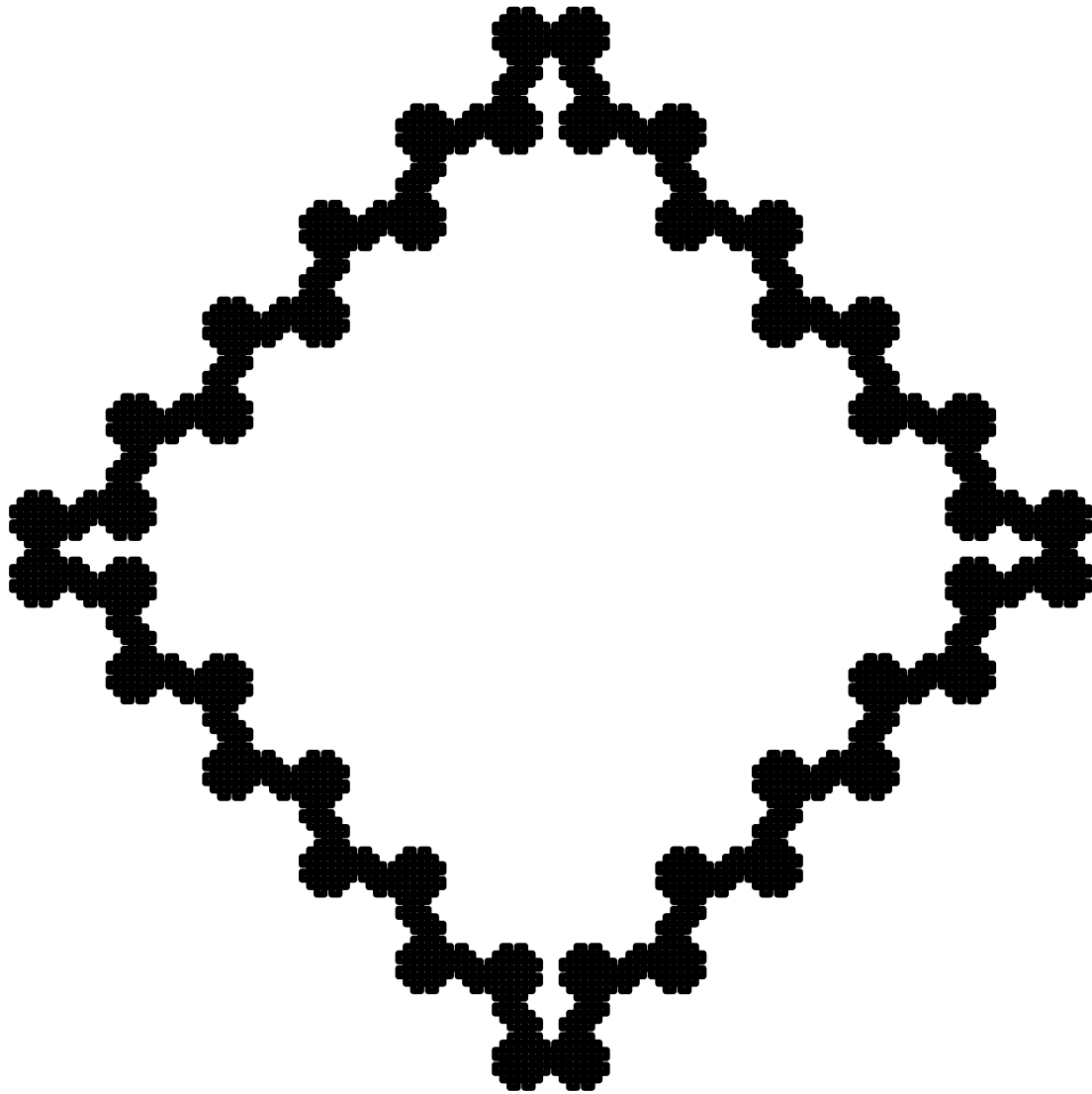Figure 16: The Cloitre walk of the sequence of parities of $B_{\sqrt{2327}}$.

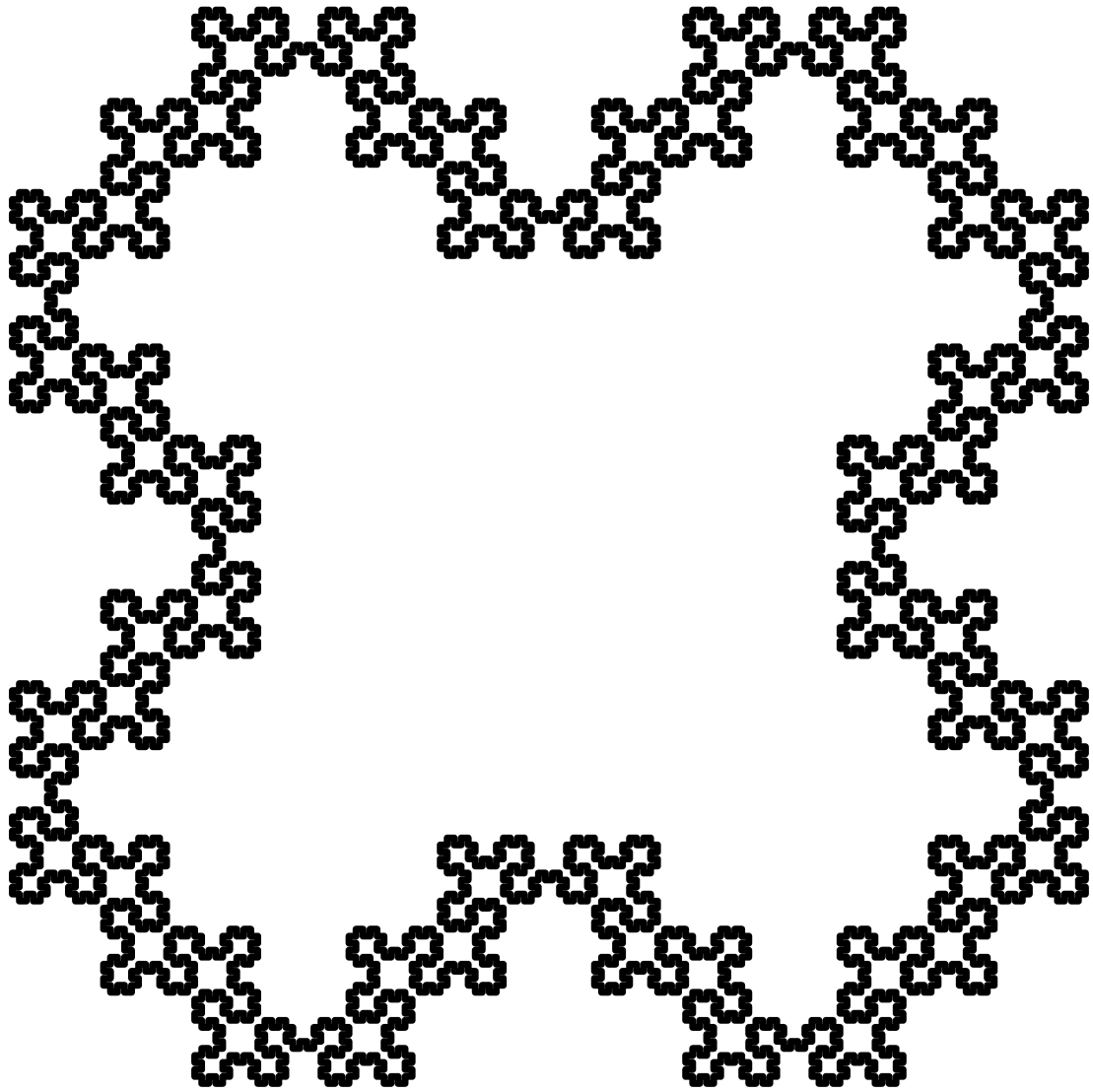Figure 17: The Cloitre walk of the sequence of parities of $B_{\sqrt{2361}}$.

Figure 18: The Cloitre walk of the sequence of parities of $B_{\sqrt{2433}}$.

Figure 19: The Cloitre walk of the sequence of parities of $B_{\sqrt{2590}}$.

Figure 20: The Cloitre walk of the sequence of parities of $B_{\sqrt{2638}}$.

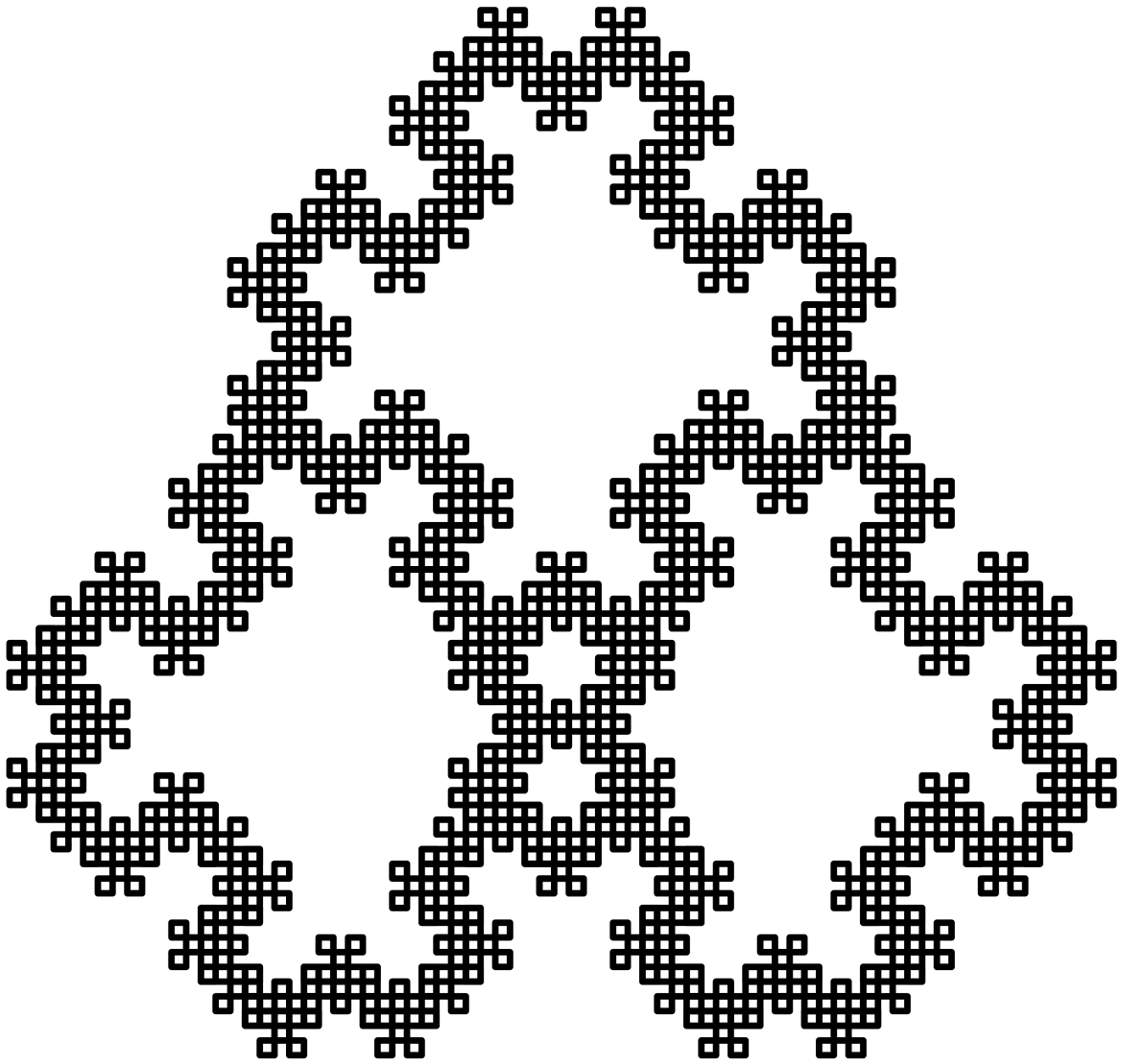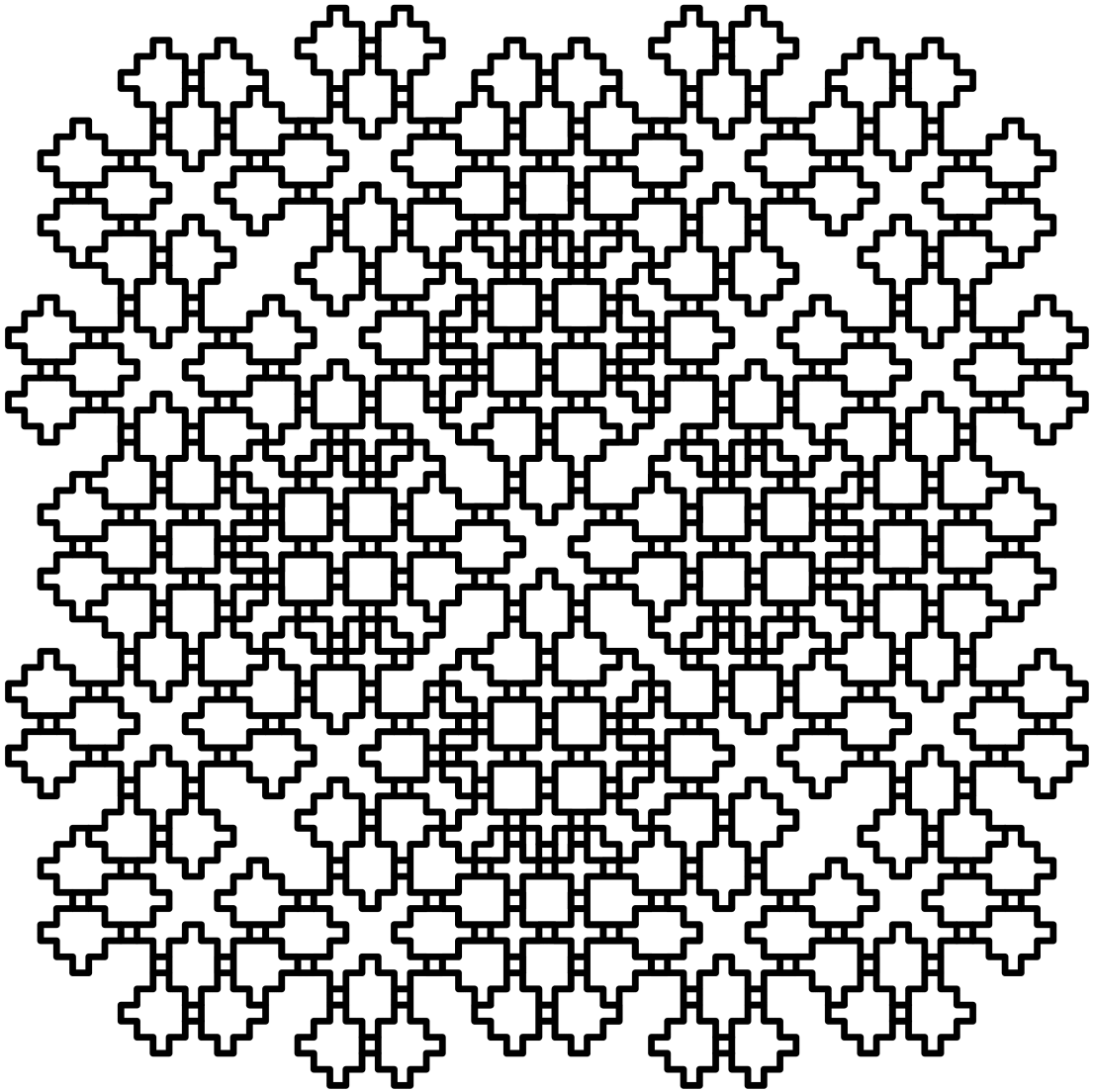Figure 21: The Cloitre walk of the sequence of parities of $B_{\sqrt{2778}}$.

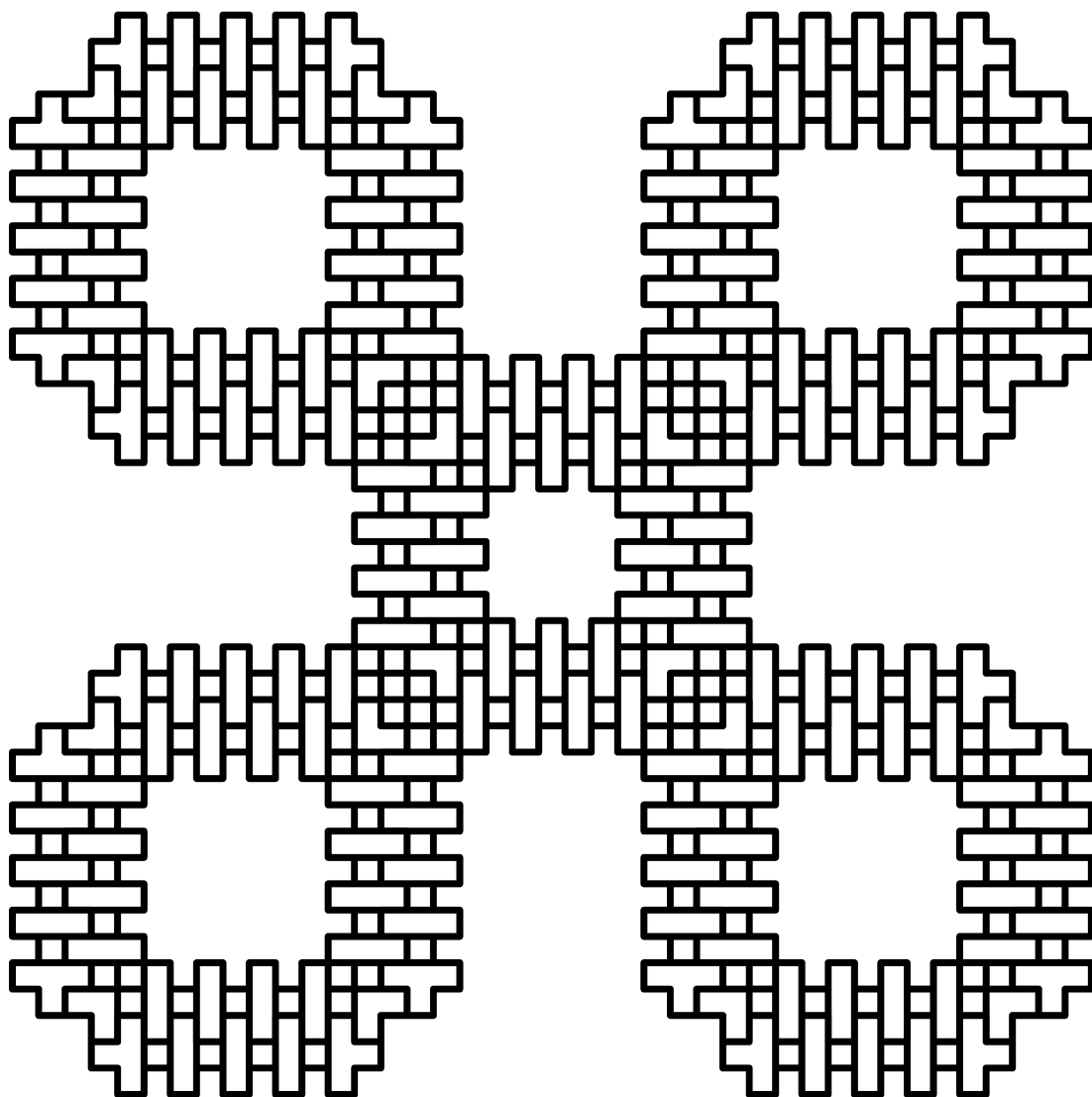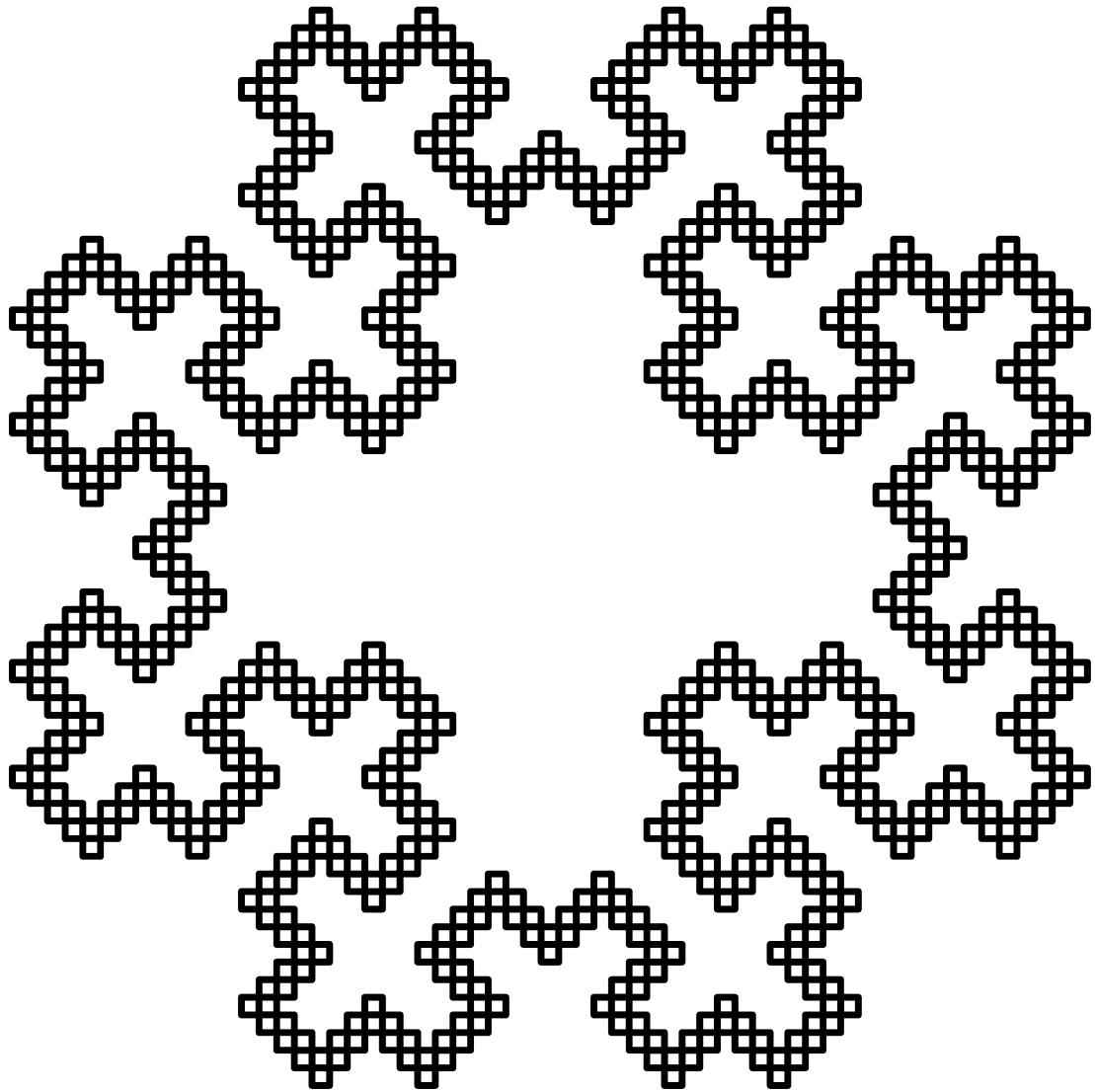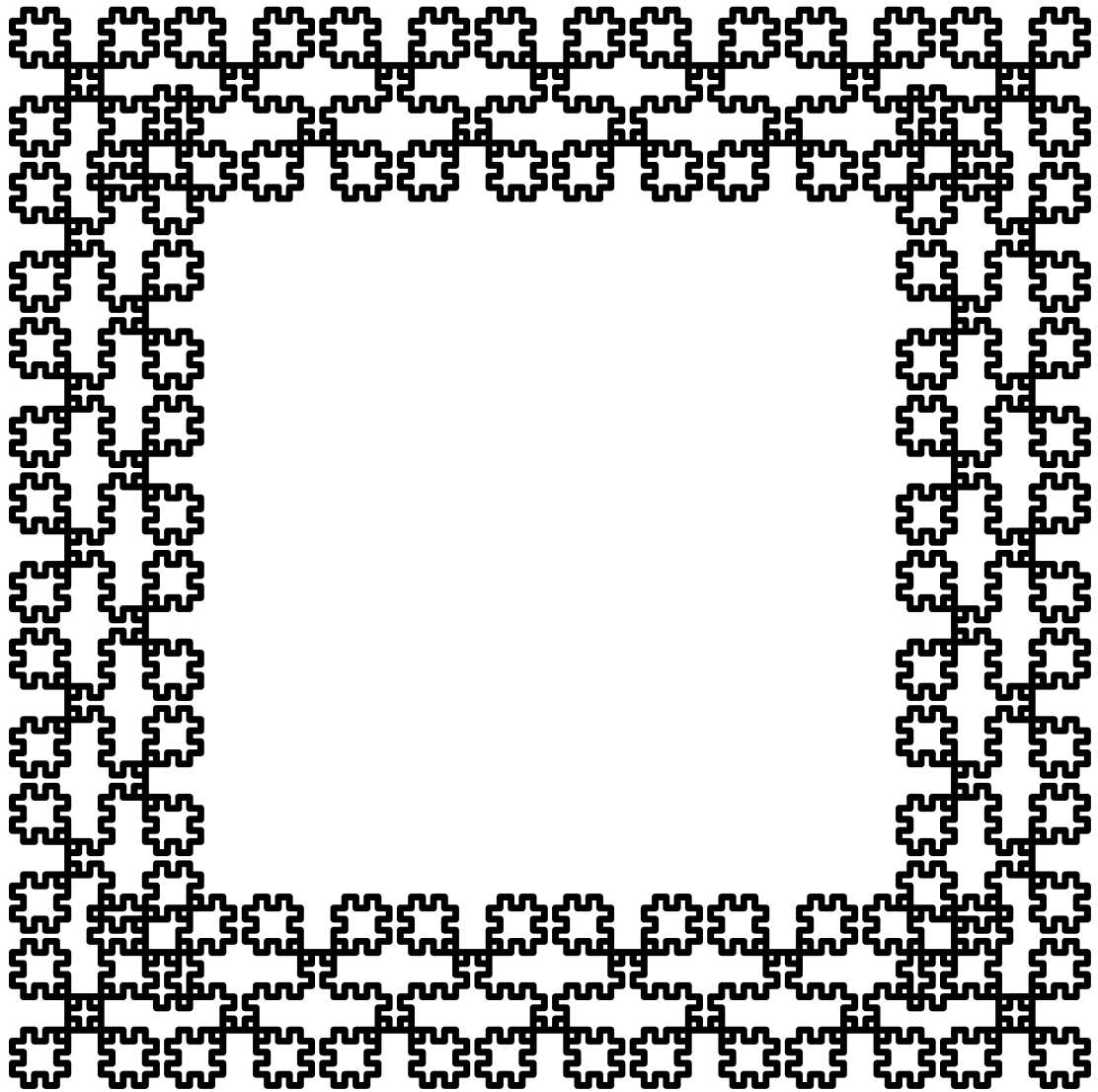Figure 22: The Cloitre walk of the sequence of parities of $B_{\sqrt{2825}}$.

Figure 23: The Cloitre walk of the sequence of parities of $B_{\sqrt{2826}}$.
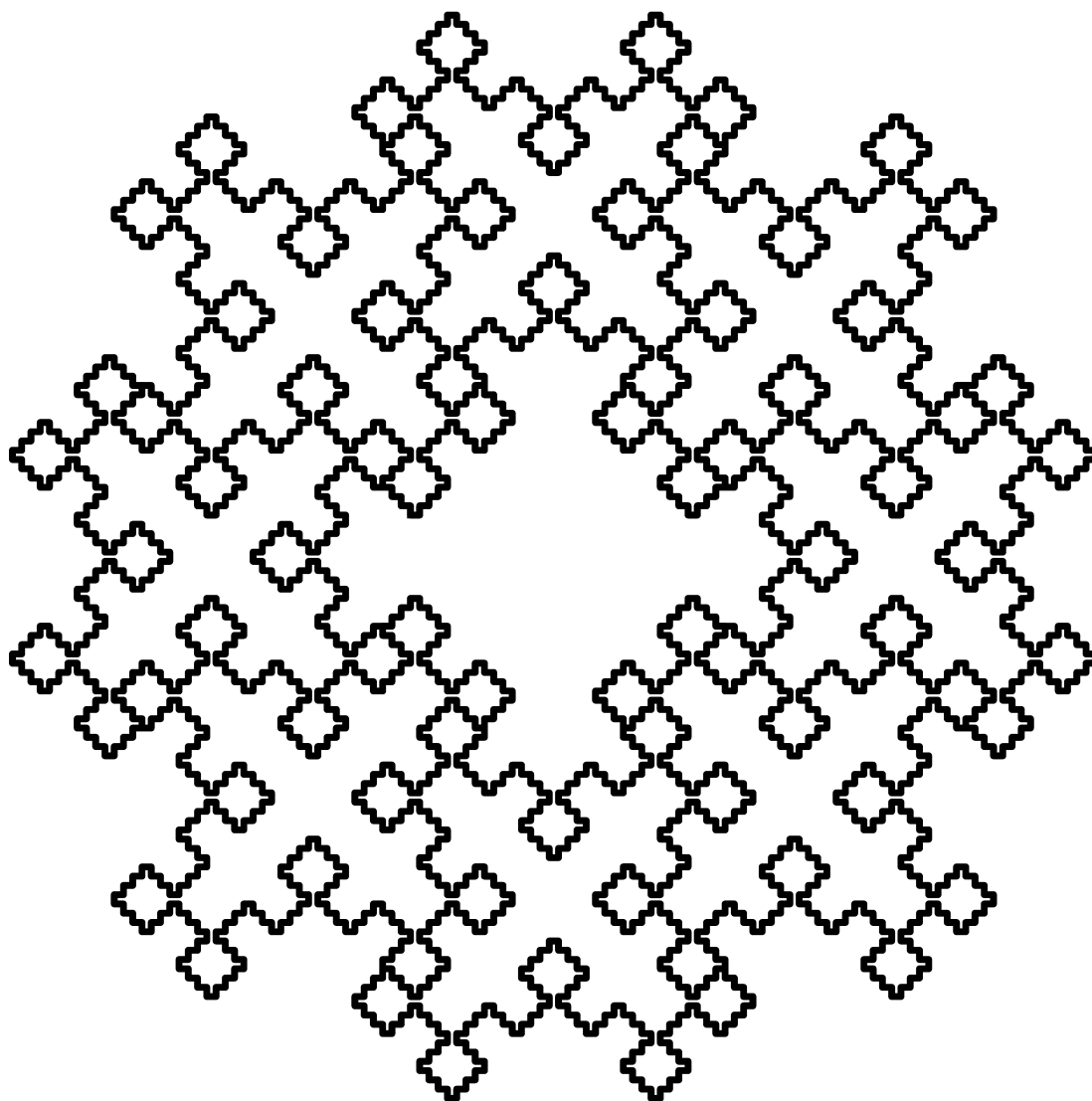
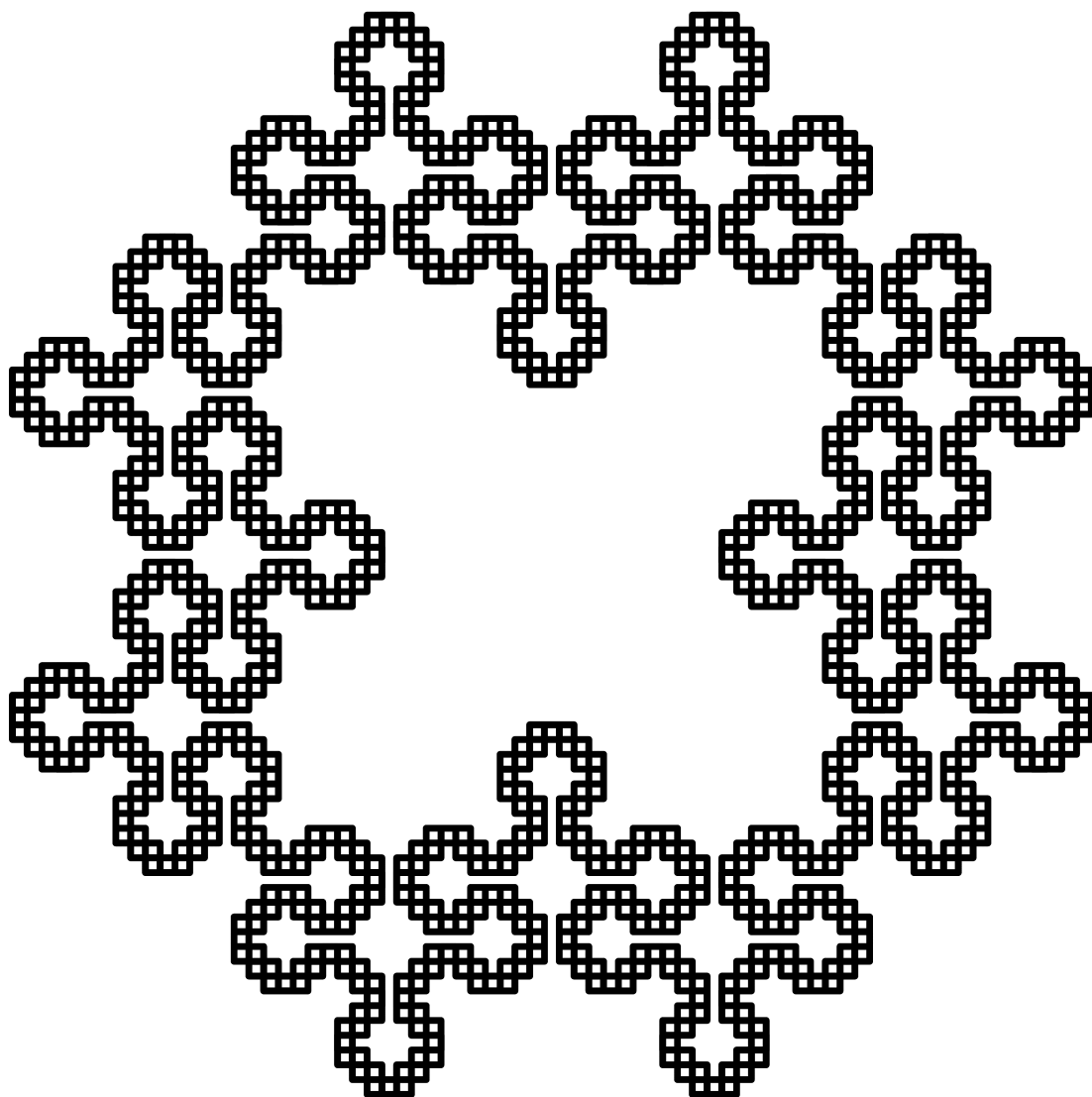Figure 24: The Cloitre walk of the sequence of parities of $B_{\sqrt{2868}}$.

Figure 25: The Cloitre walk of the sequence of parities of $B_{\sqrt{2871}}$.

Figure 26: The Cloitre walk of the sequence of parities of $B_{\sqrt{2927}}$.

Figure 27: The Cloitre walk of the sequence of parities of $B_{\sqrt{2939}}$.

Figure 28: The Cloitre walk of the sequence of parities of $B_{\sqrt{2979}}$.

Figure 29: The Cloitre walk of the sequence of parities of $B_{\sqrt{4410}}$.

Figure 30: The Cloitre walk of the sequence of parities of $B_{\sqrt{4514}}$.

Figure 31: The Cloitre walk of the sequence of parities of $B_{\sqrt{4533}}$.
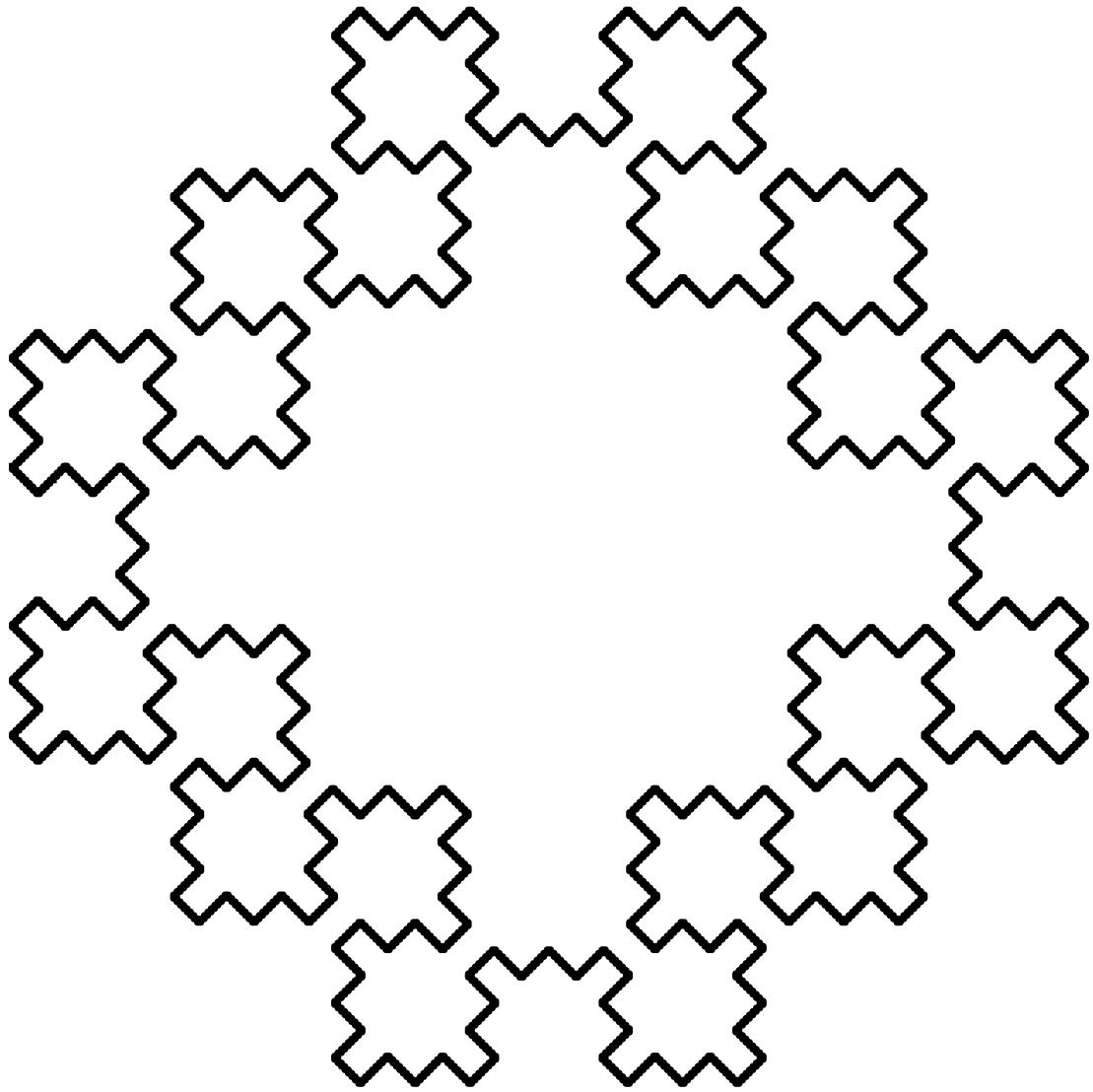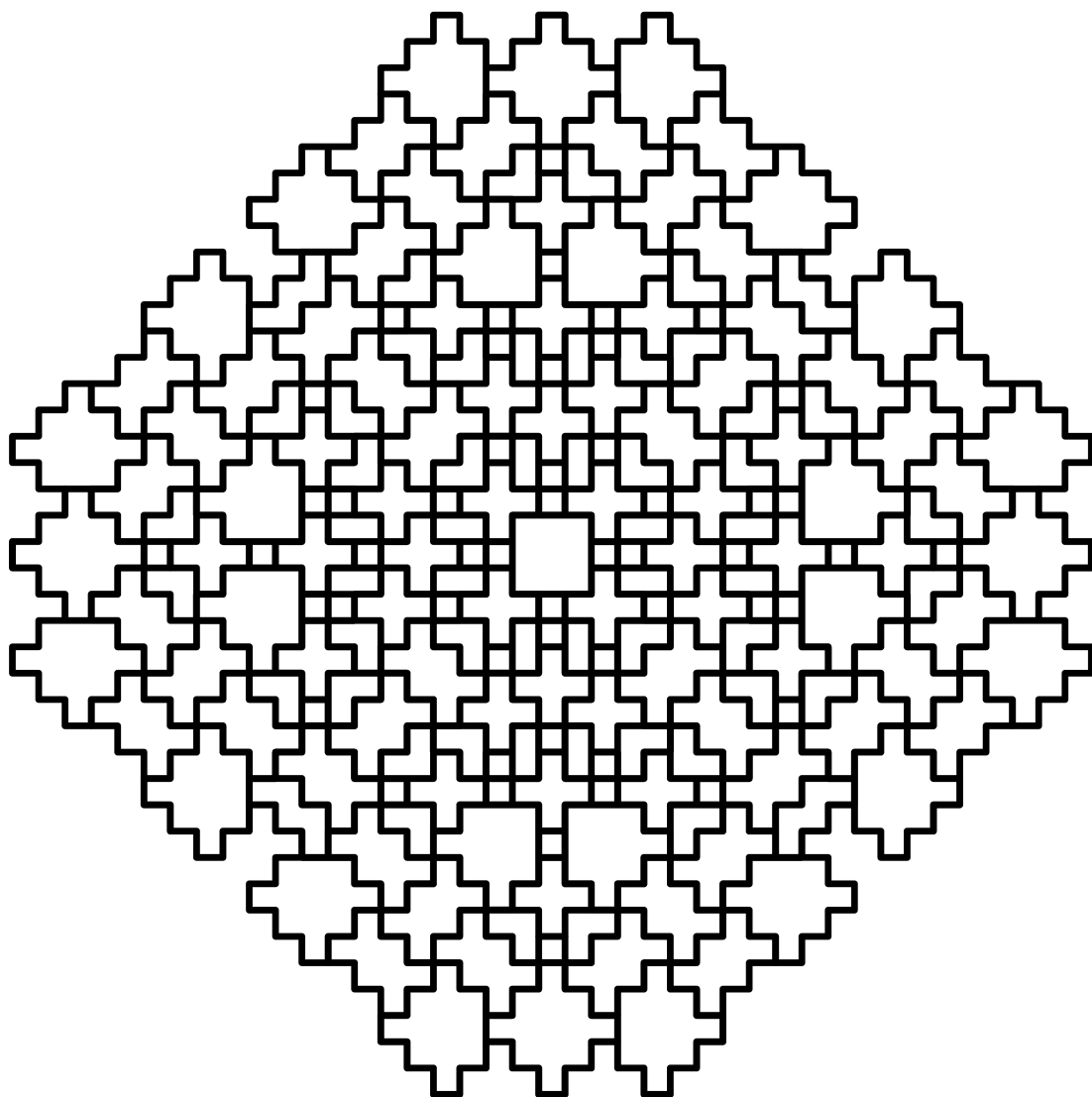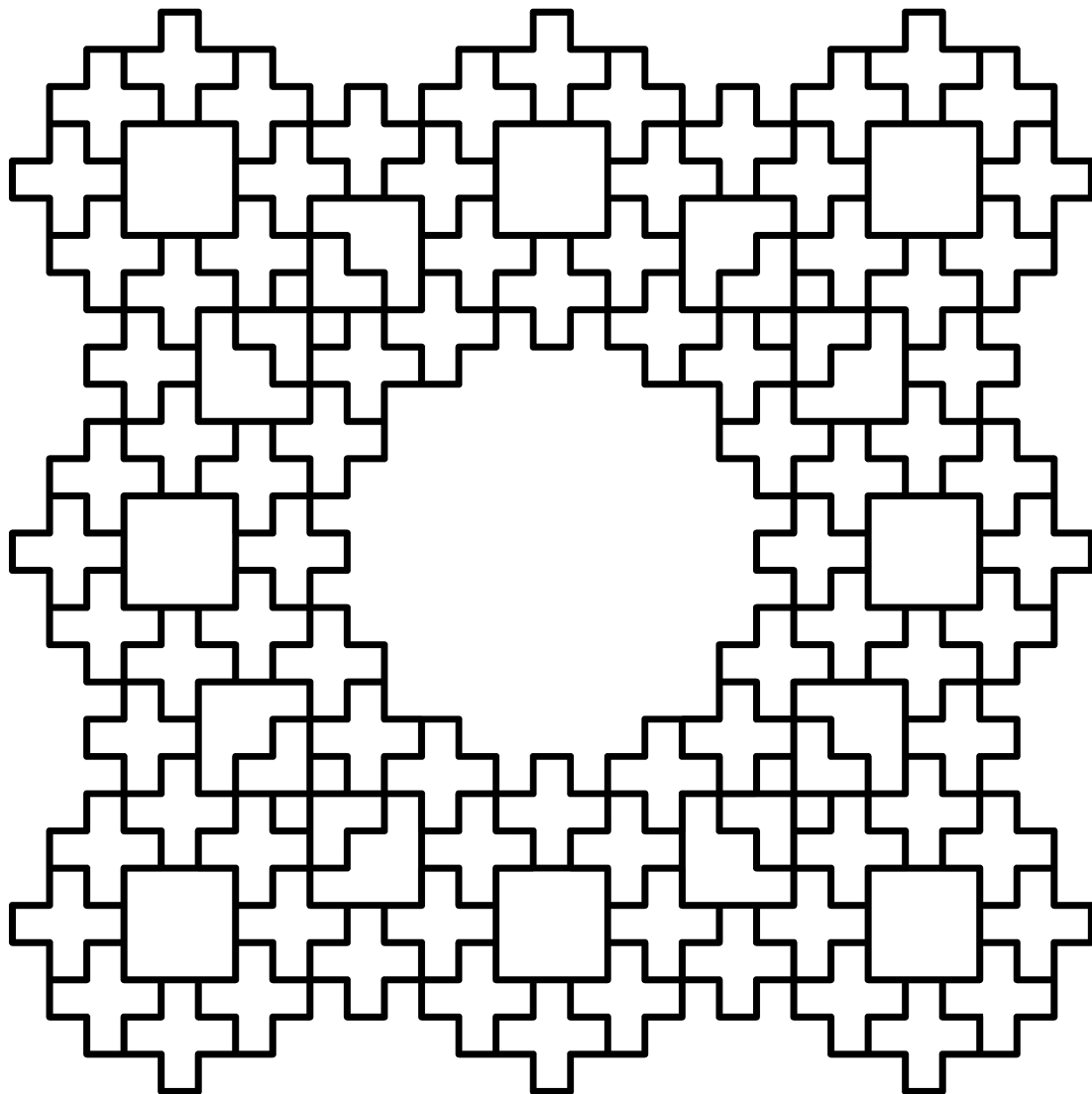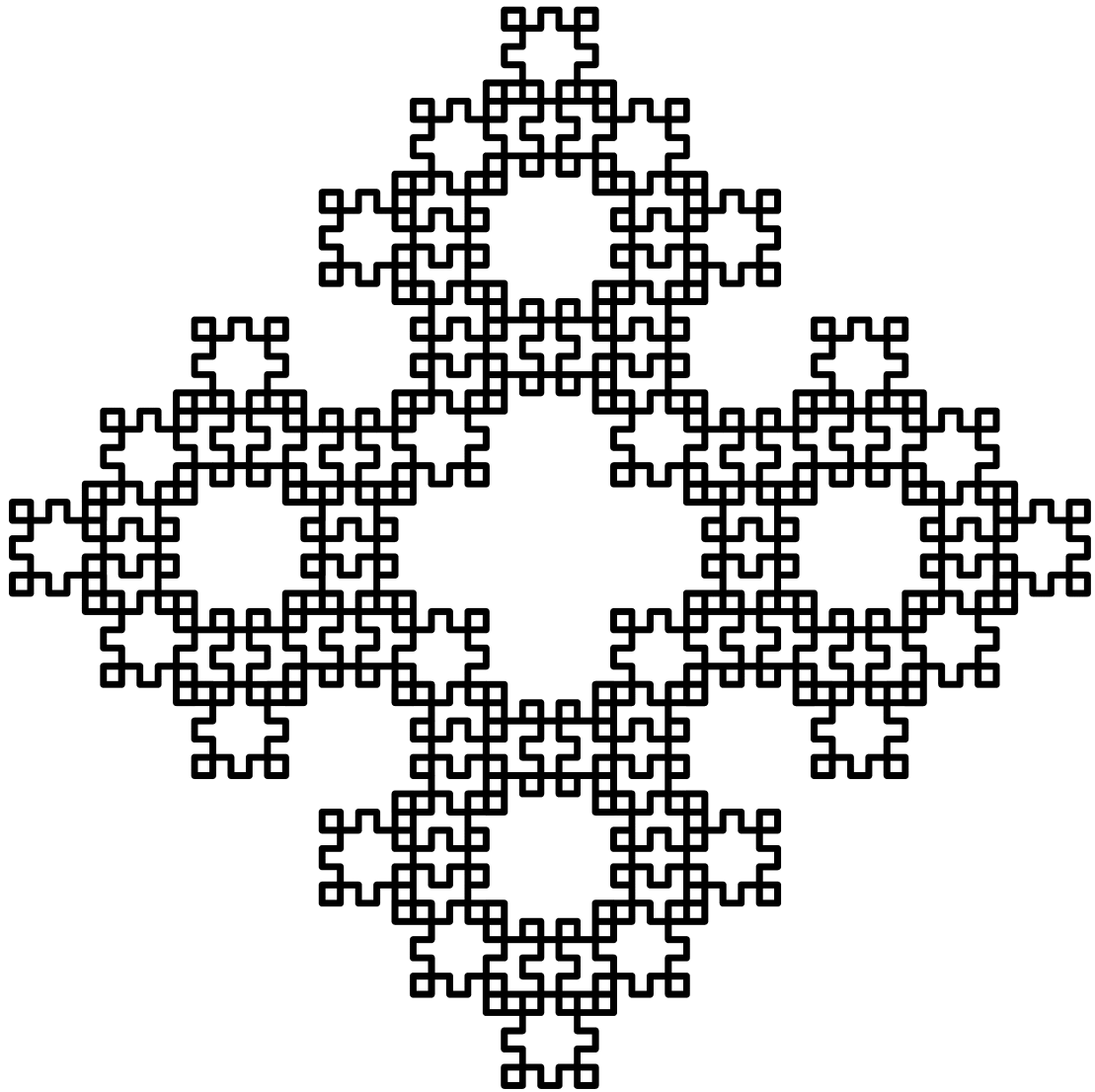
Figure 32: The Cloitre walk of the sequence of parities of $B_{\sqrt{4652}}$.
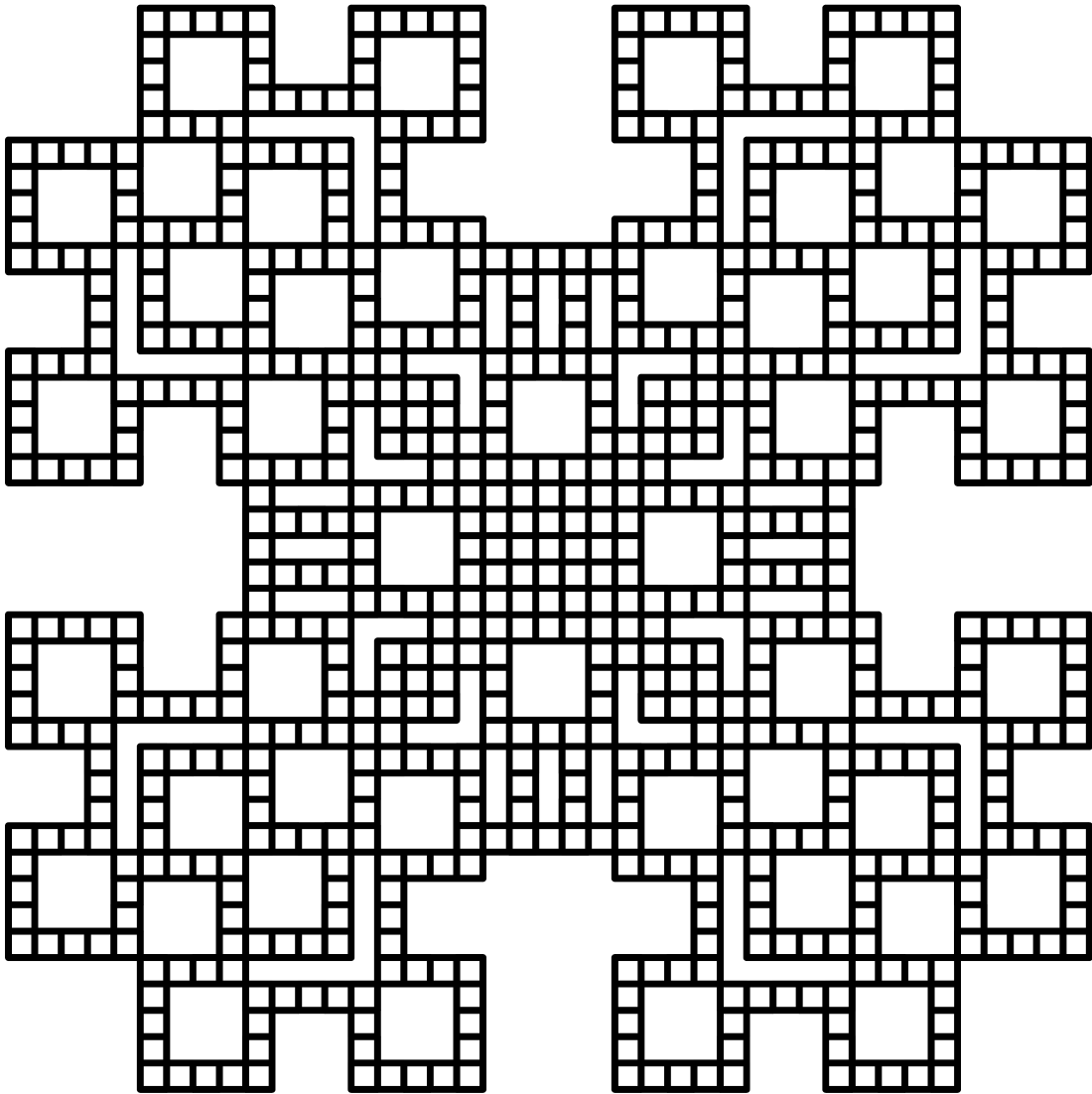
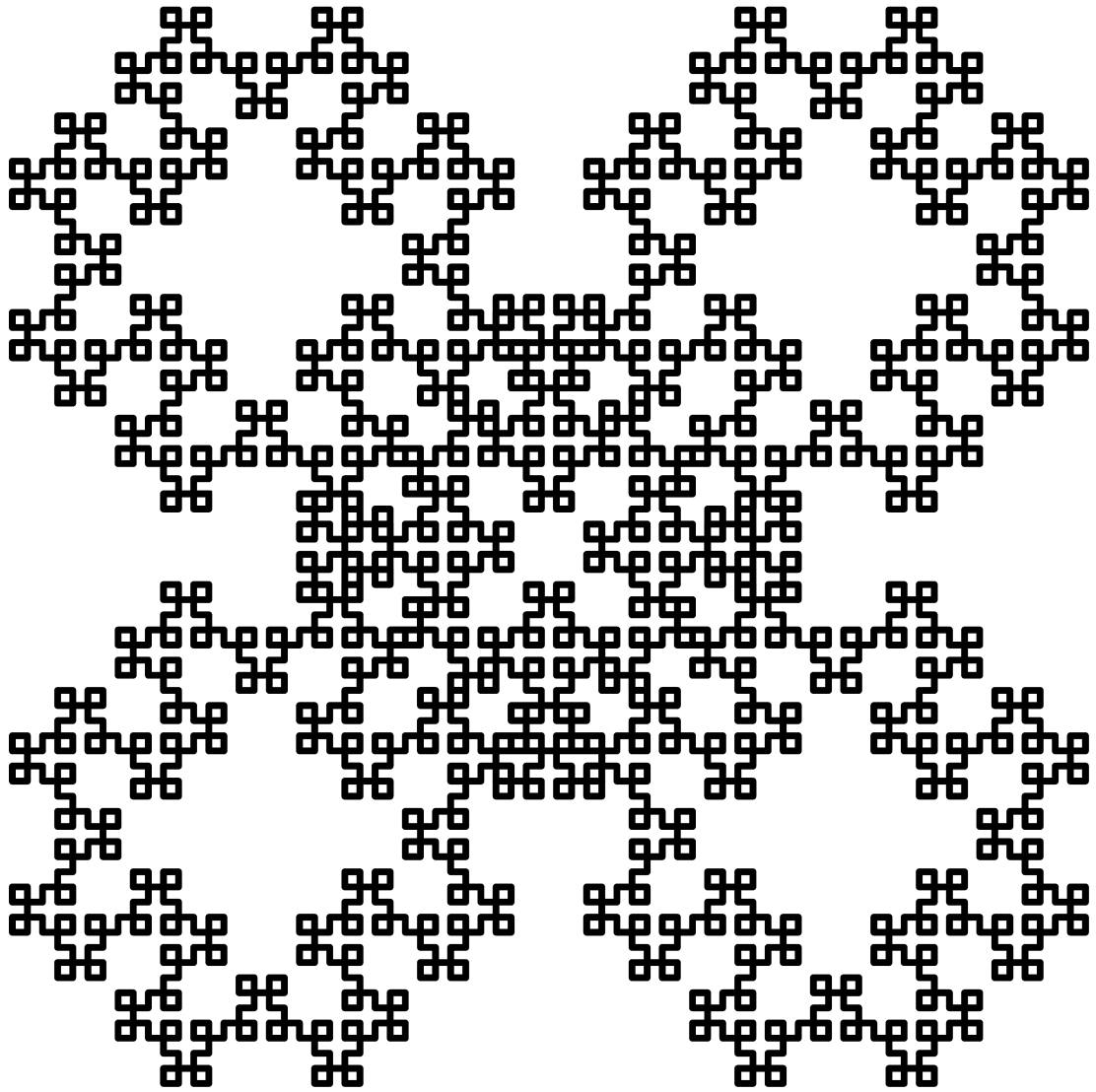Figure 33: The Cloitre walk of the sequence of parities of $B_{\sqrt{4701}}$.
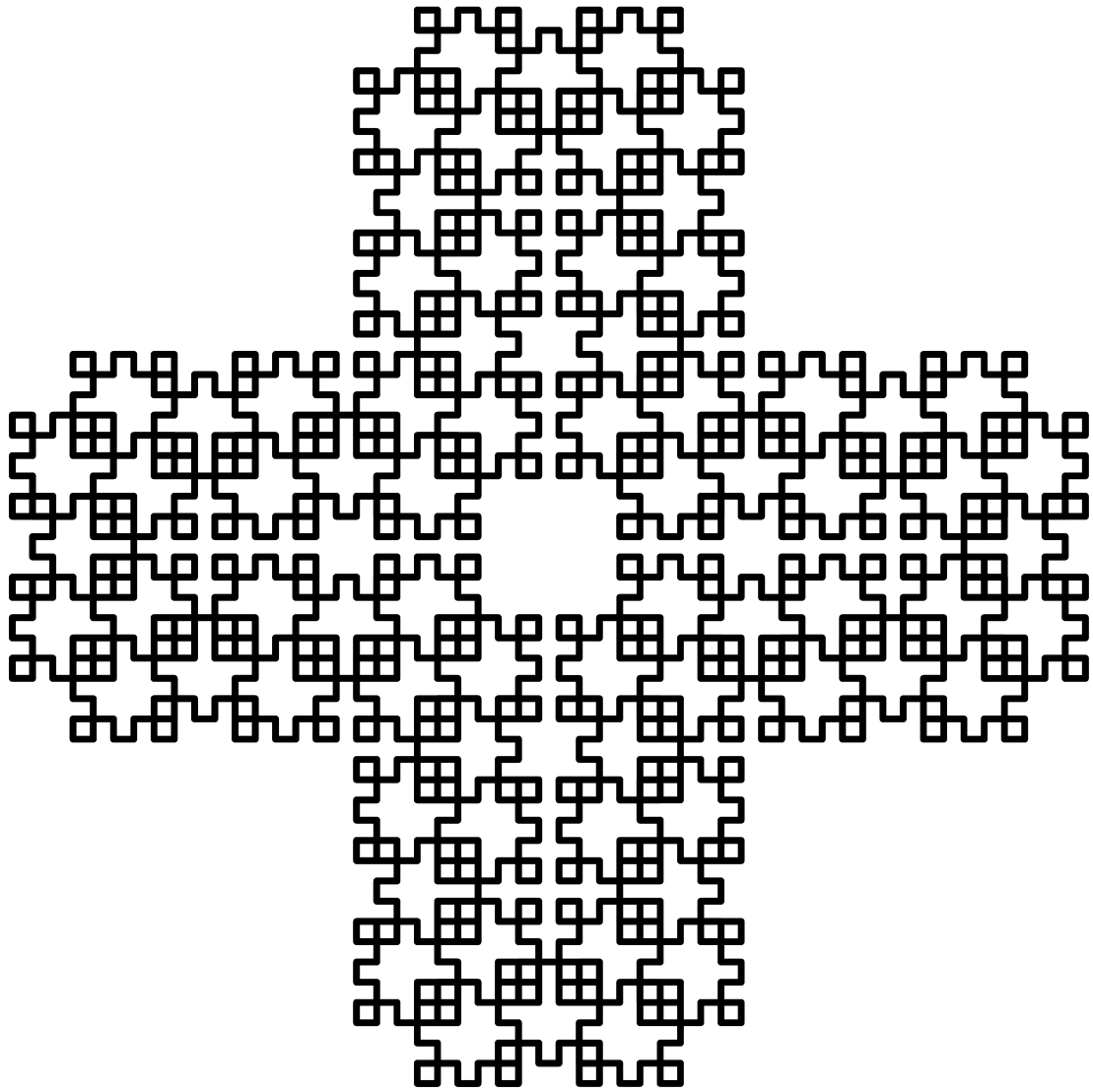
Figure 34: The Cloitre walk of the sequence of parities of $B_{\sqrt{4782}}$.

Figure 35: The Cloitre walk of the sequence of parities of $B_{\sqrt{4873}}$.
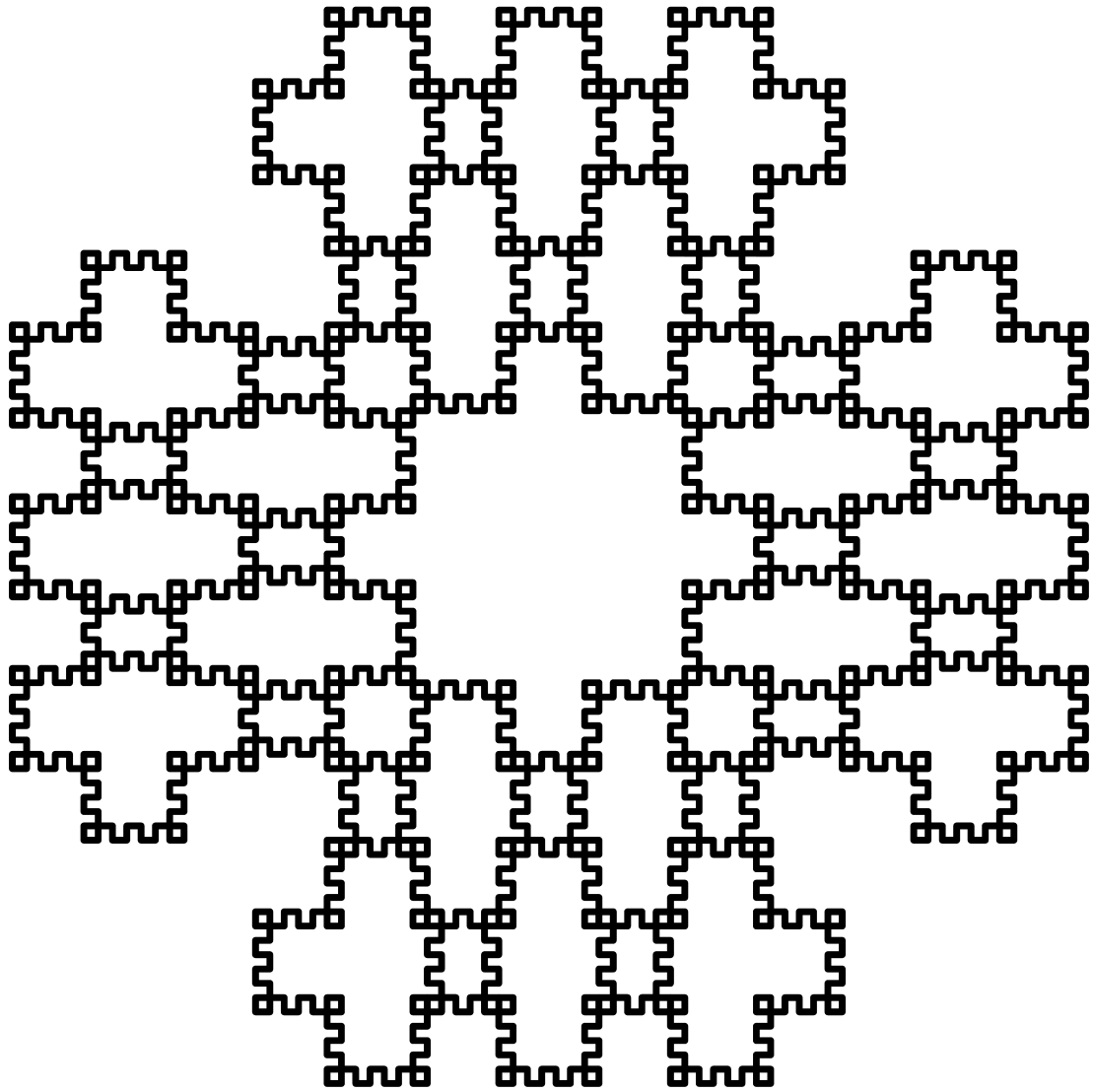
Figure 36: The Cloitre walk of the sequence of parities of $B_{\sqrt{5050}}$.
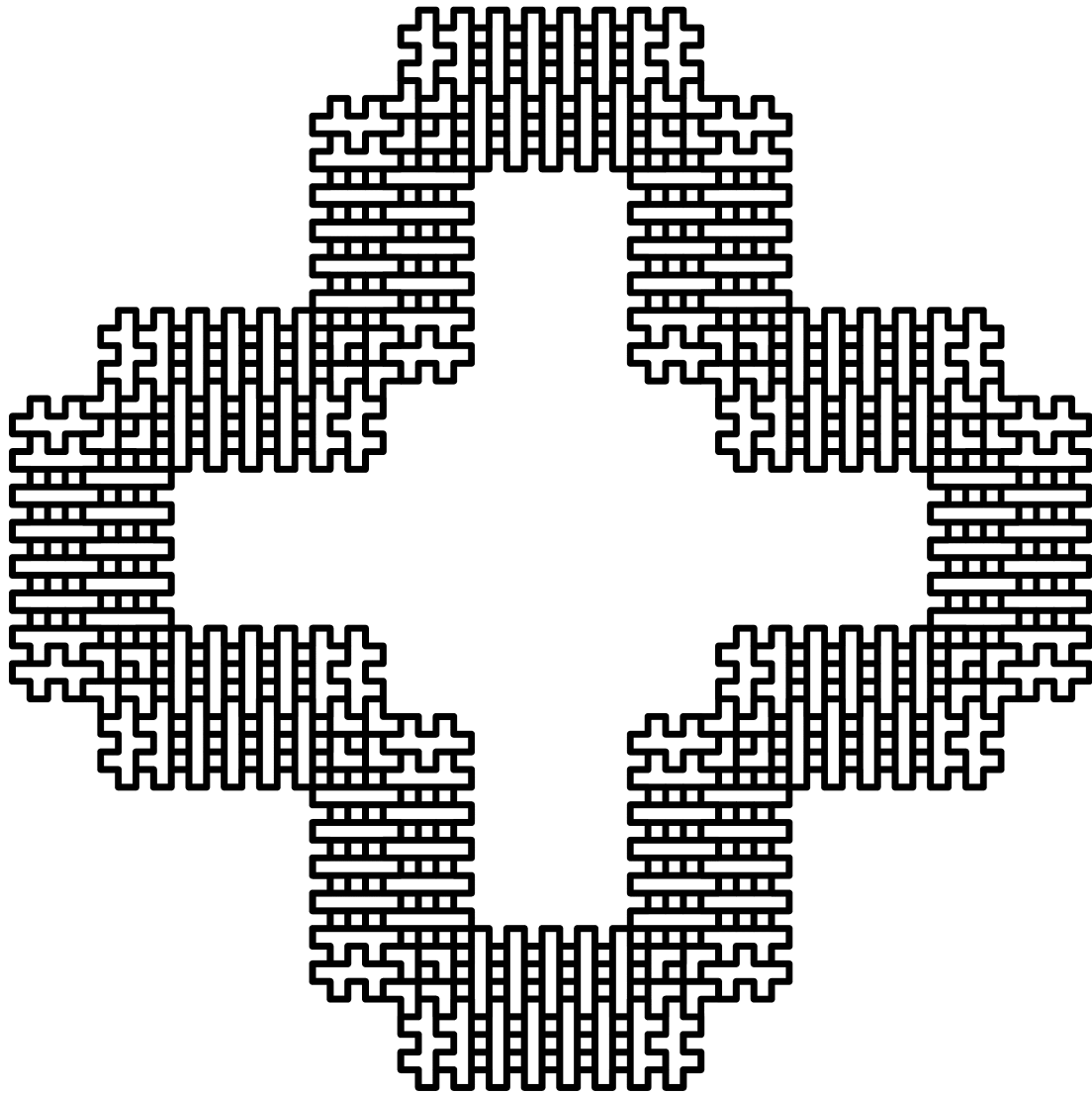
Figure 37: The Cloitre walk of the sequence of parities of $B_{\sqrt{5074}}$.

Figure 38: The Cloitre walk of the sequence of parities of $B_{\sqrt{5091}}$.

Figure 39: The Cloitre walk of the sequence of parities of $B_{\sqrt{5120}}$.

40

Figure 40: The Cloitre walk of the sequence of parities of $B_{\sqrt{5170}}$.

Figure 41: The Cloitre walk of the sequence of parities of $B_{\sqrt{5245}}$.
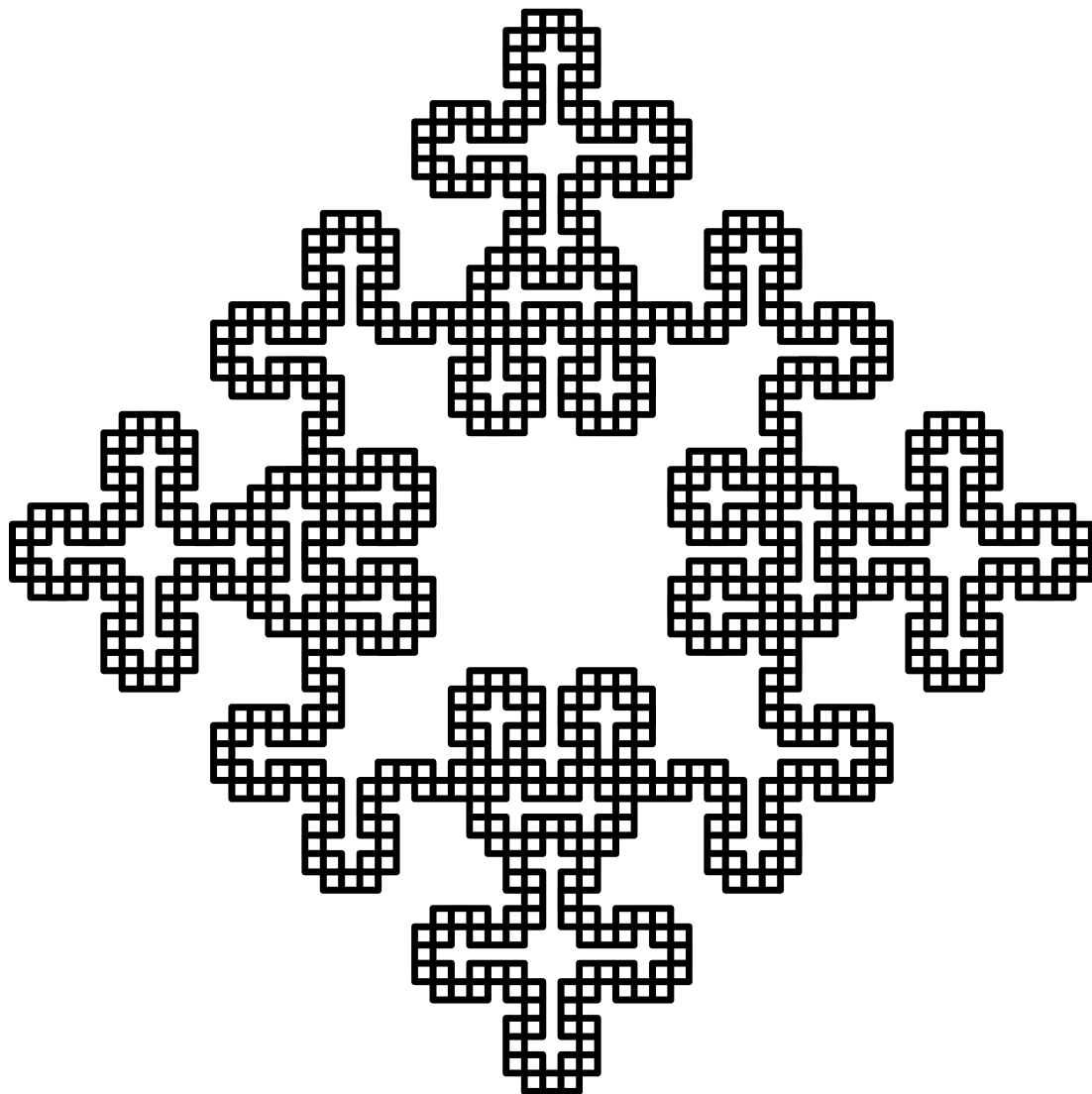
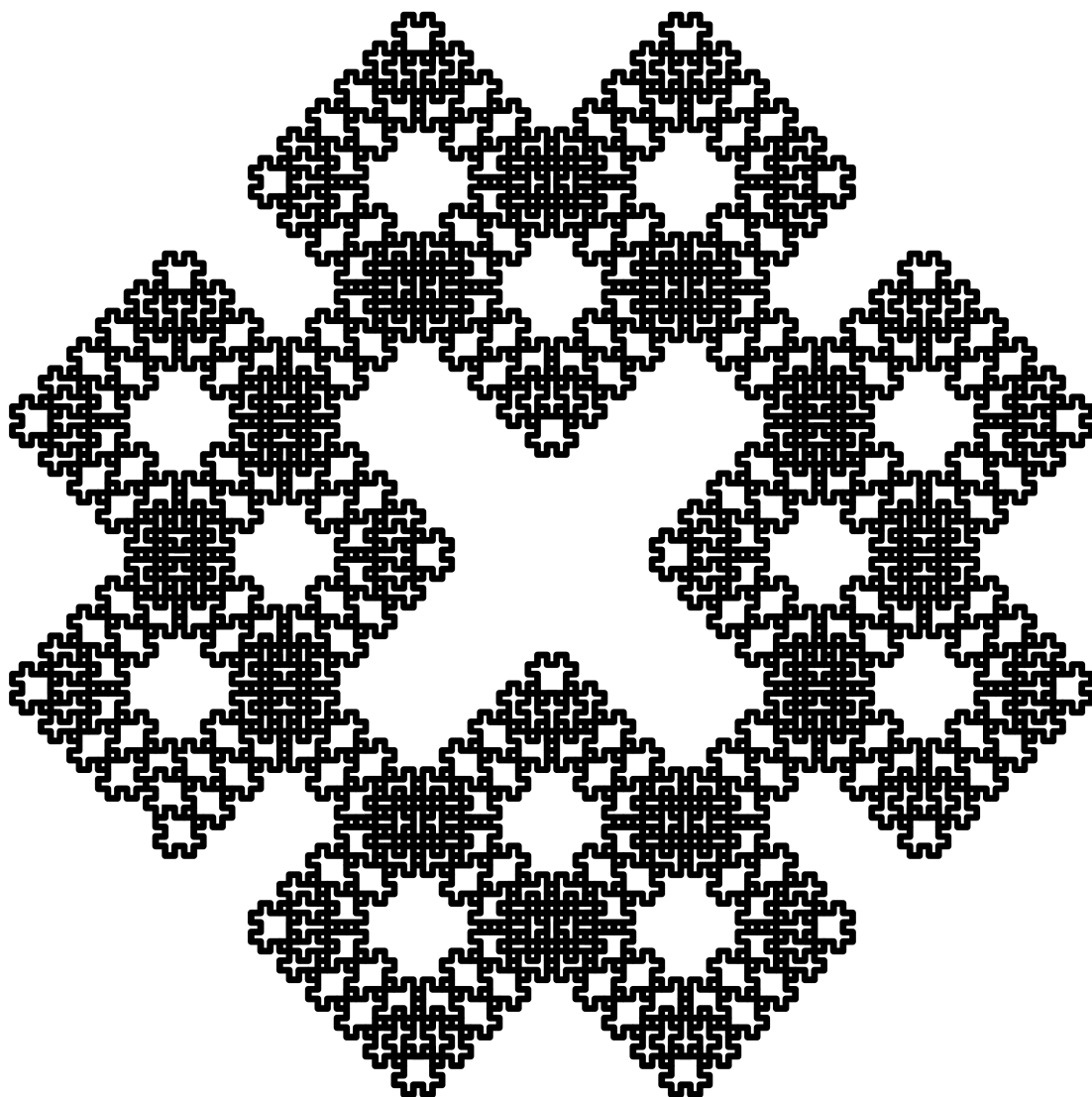Figure 42: The Cloitre walk of the sequence of parities of $B_{\sqrt{5248}}$.

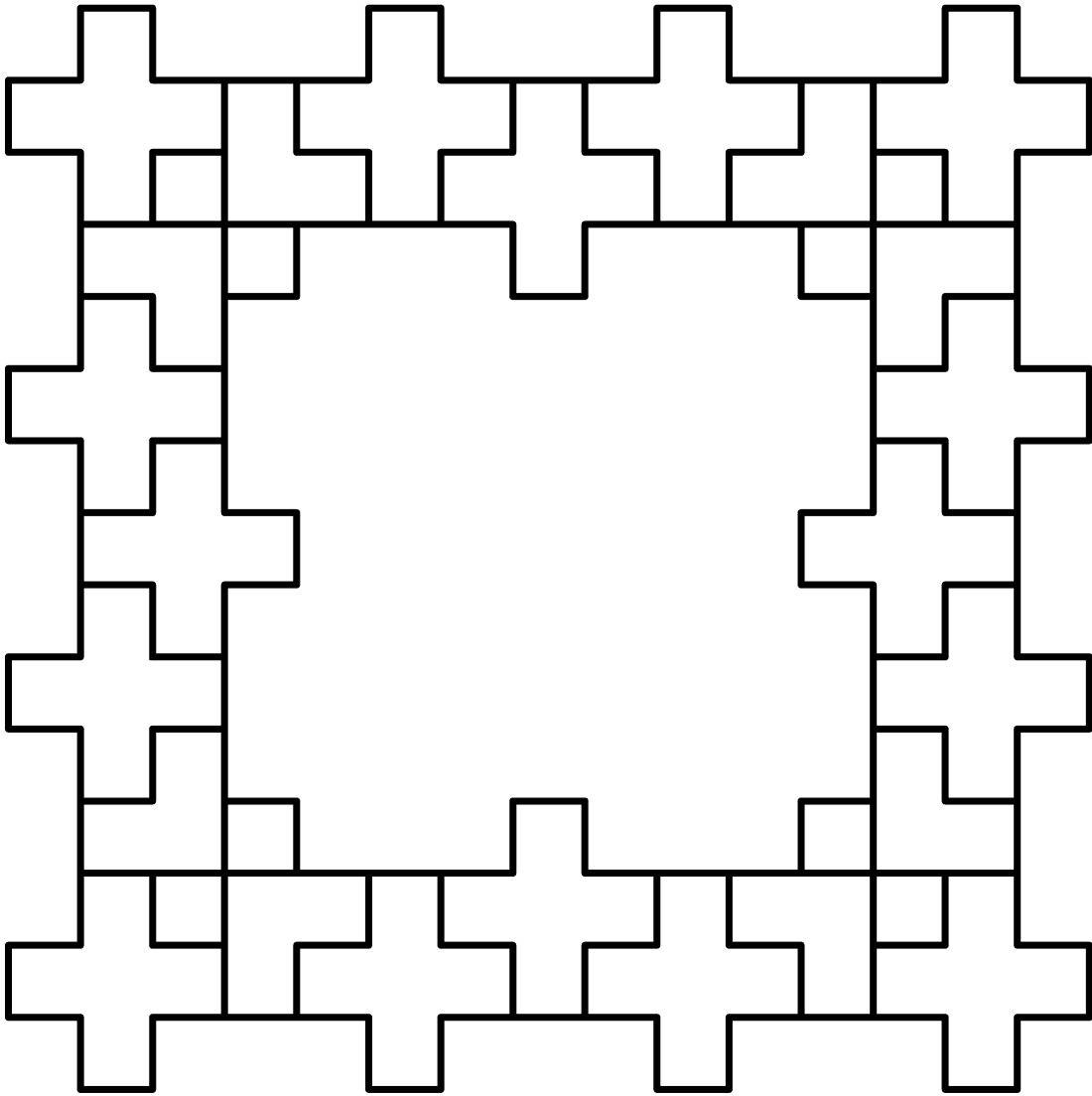Figure 43: The Cloitre walk of the sequence of parities of $B_{\sqrt{5250}}$.

Figure 44: The Cloitre walk of the sequence of parities of $B_{\sqrt{5267}}$.
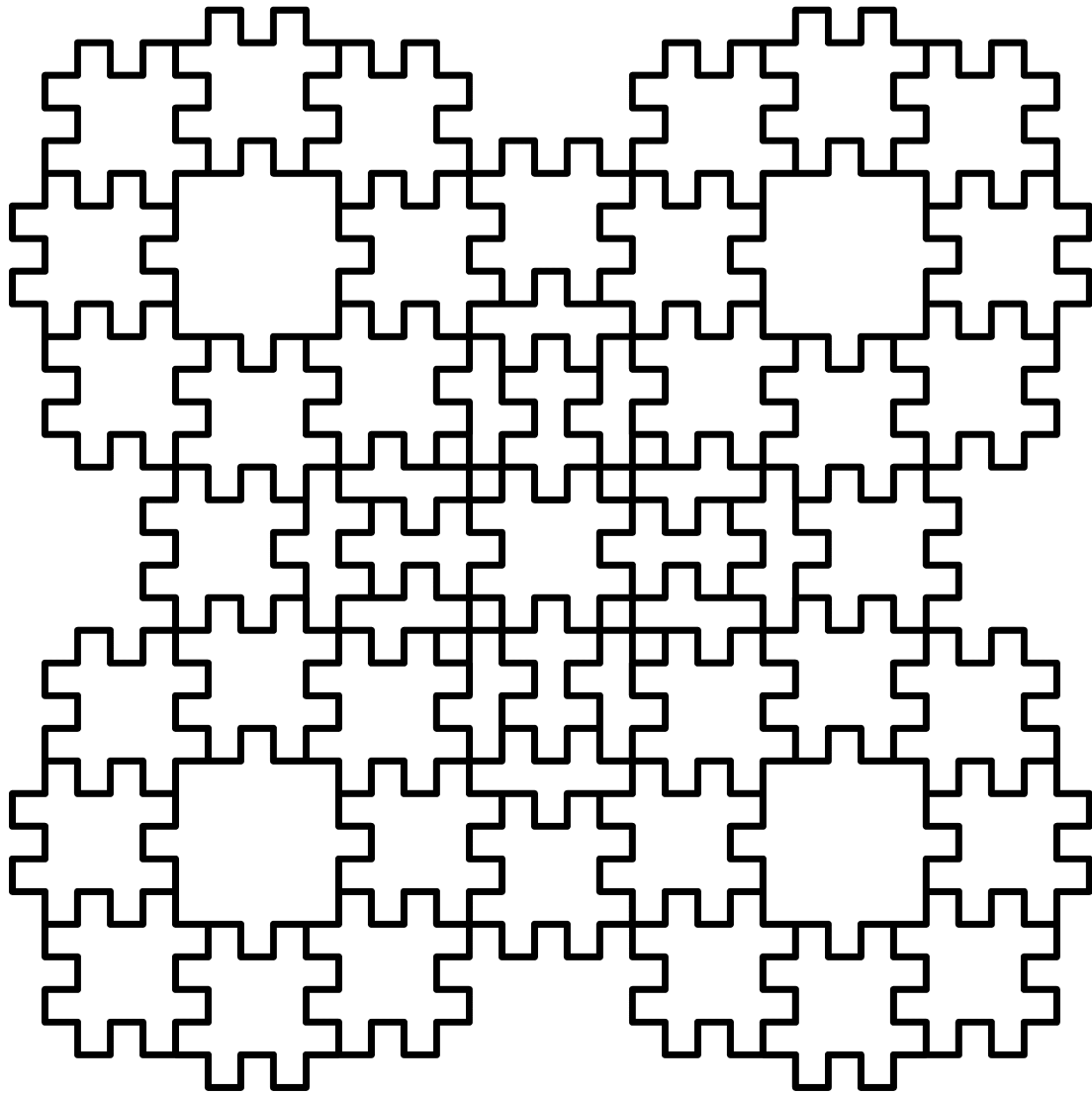
45

Figure 45: The Cloitre walk of the sequence of parities of $B_{\sqrt{5448}}$.

Figure 46: The Cloitre walk of the sequence of parities of $B_{\sqrt{5560}}$.

Figure 47: The Cloitre walk of the sequence of parities of $B_{\sqrt{5676}}$.

Figure 48: The Cloitre walk of the sequence of parities of $B_{\sqrt{5690}}$.
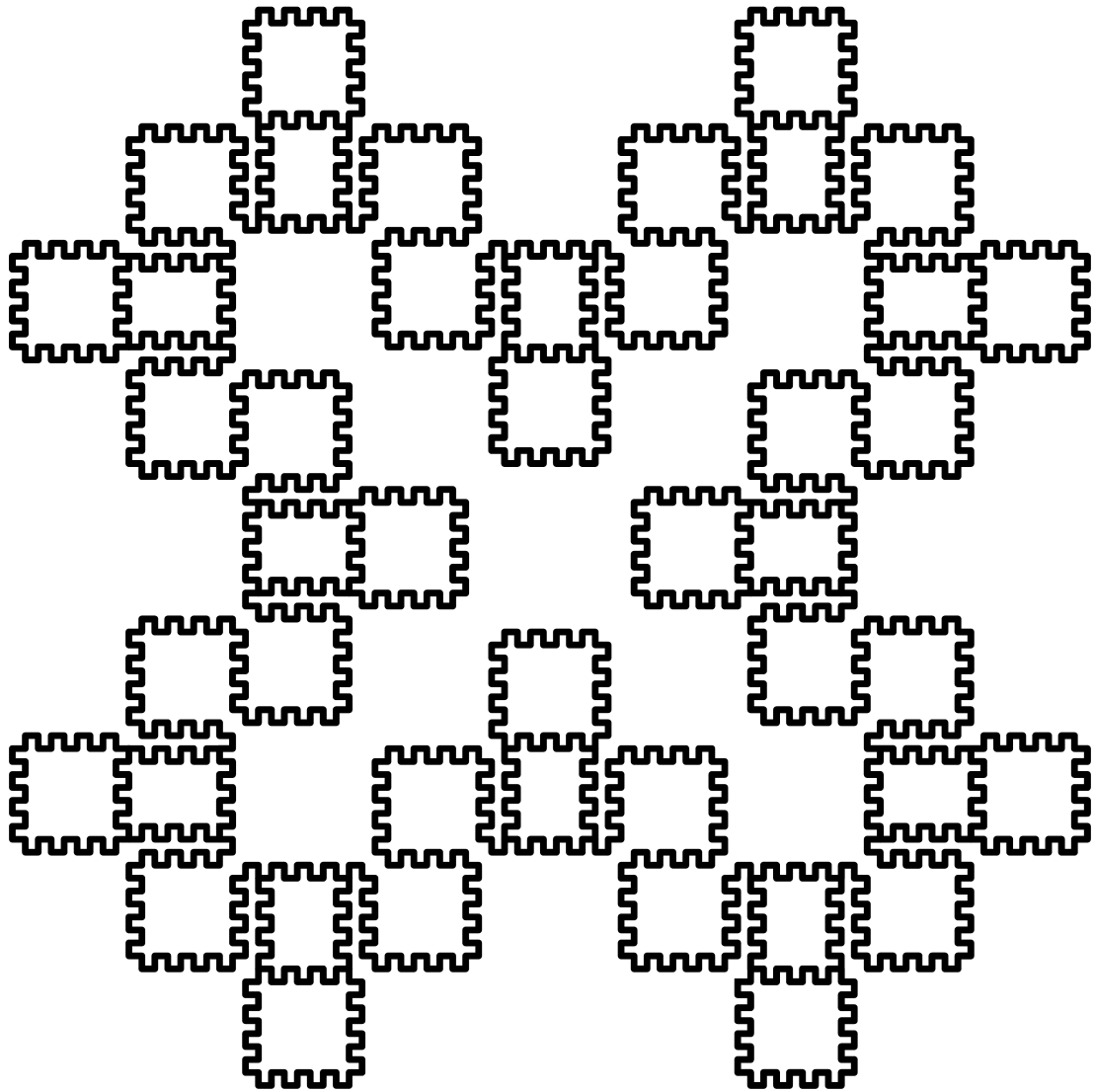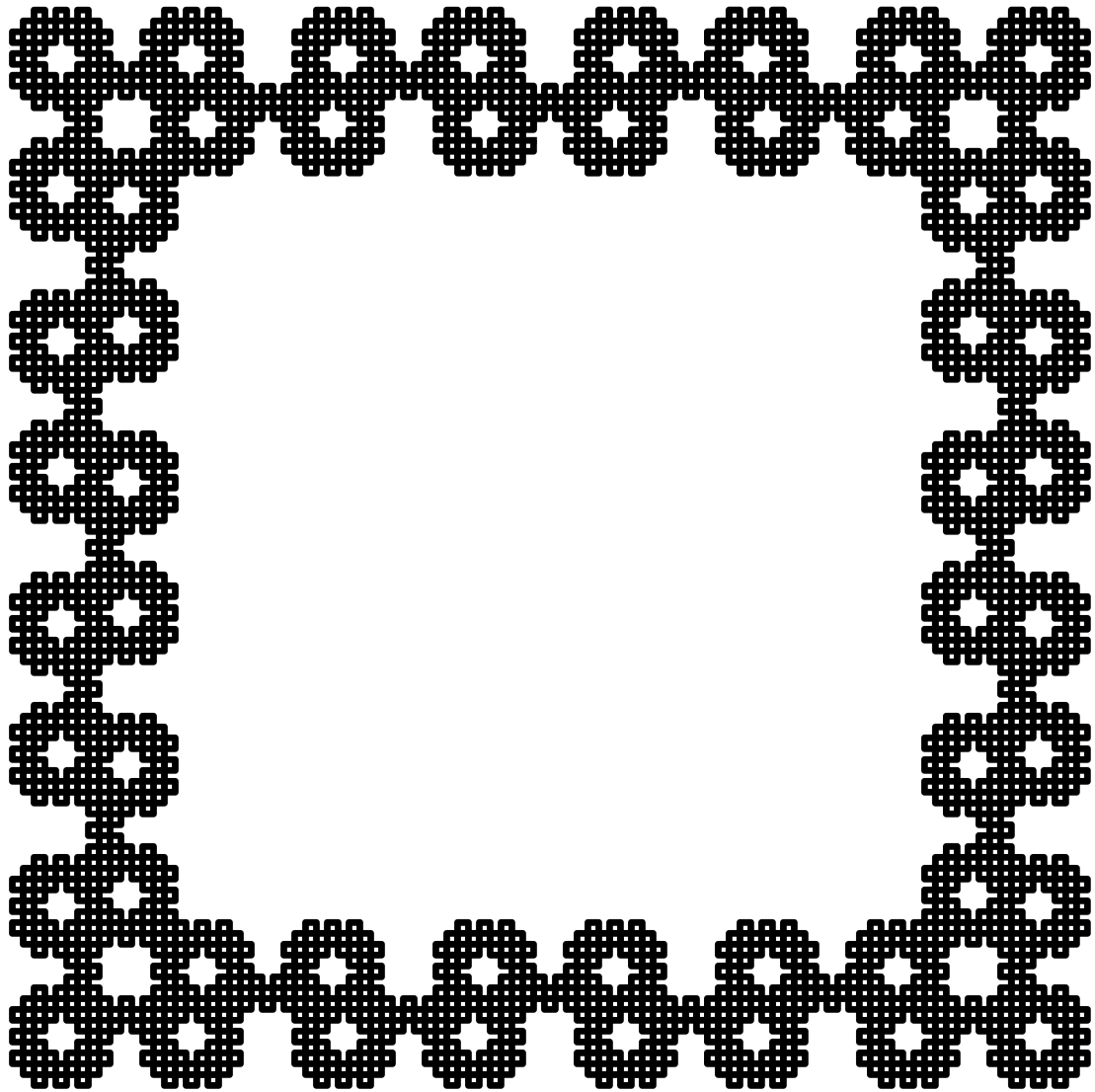
Figure 49: The Cloitre walk of the sequence of parities of $B_{\sqrt{5695}}$.

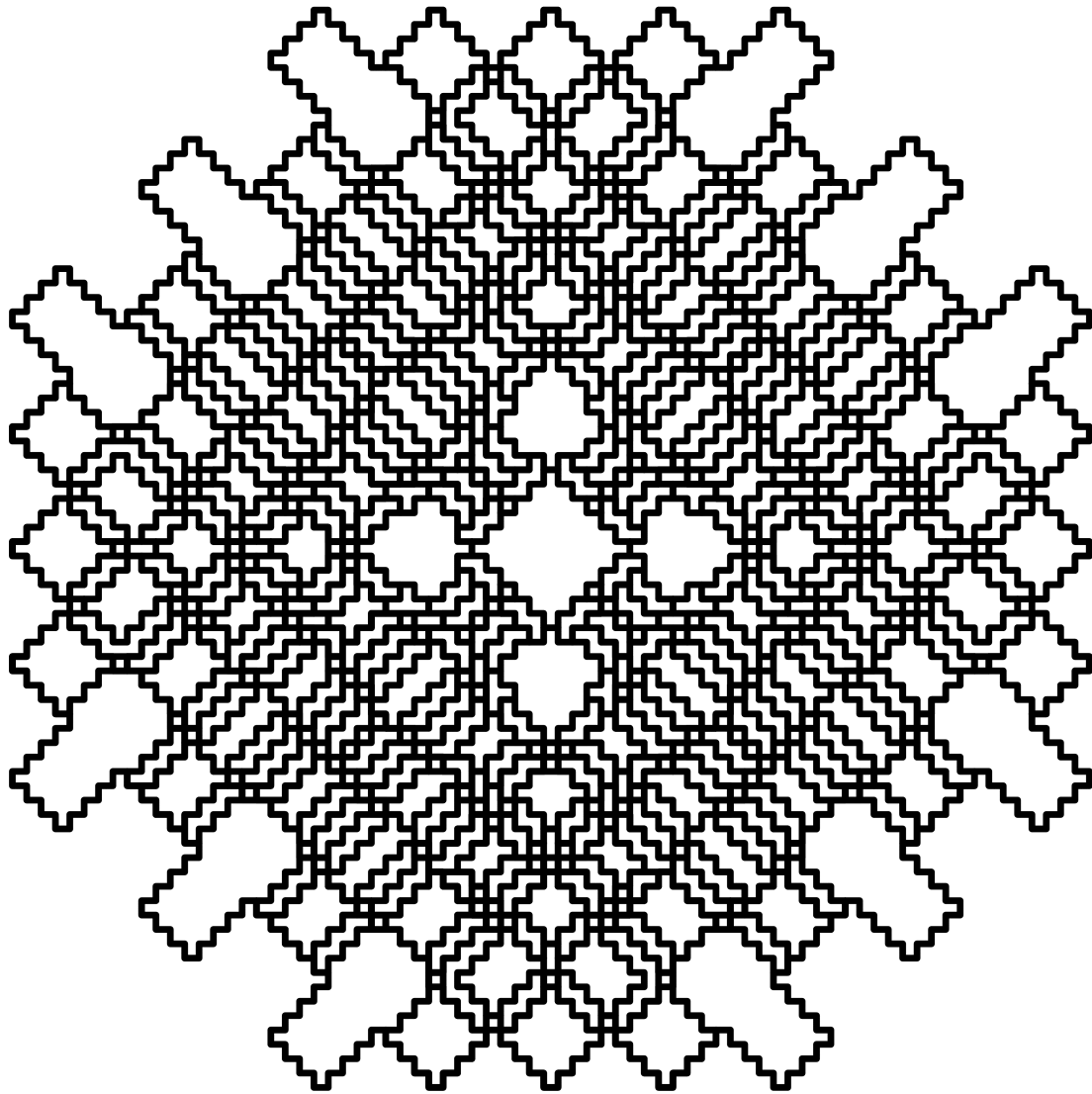Figure 50: The Cloitre walk of the sequence of parities of $B_{\sqrt{5822}}$.

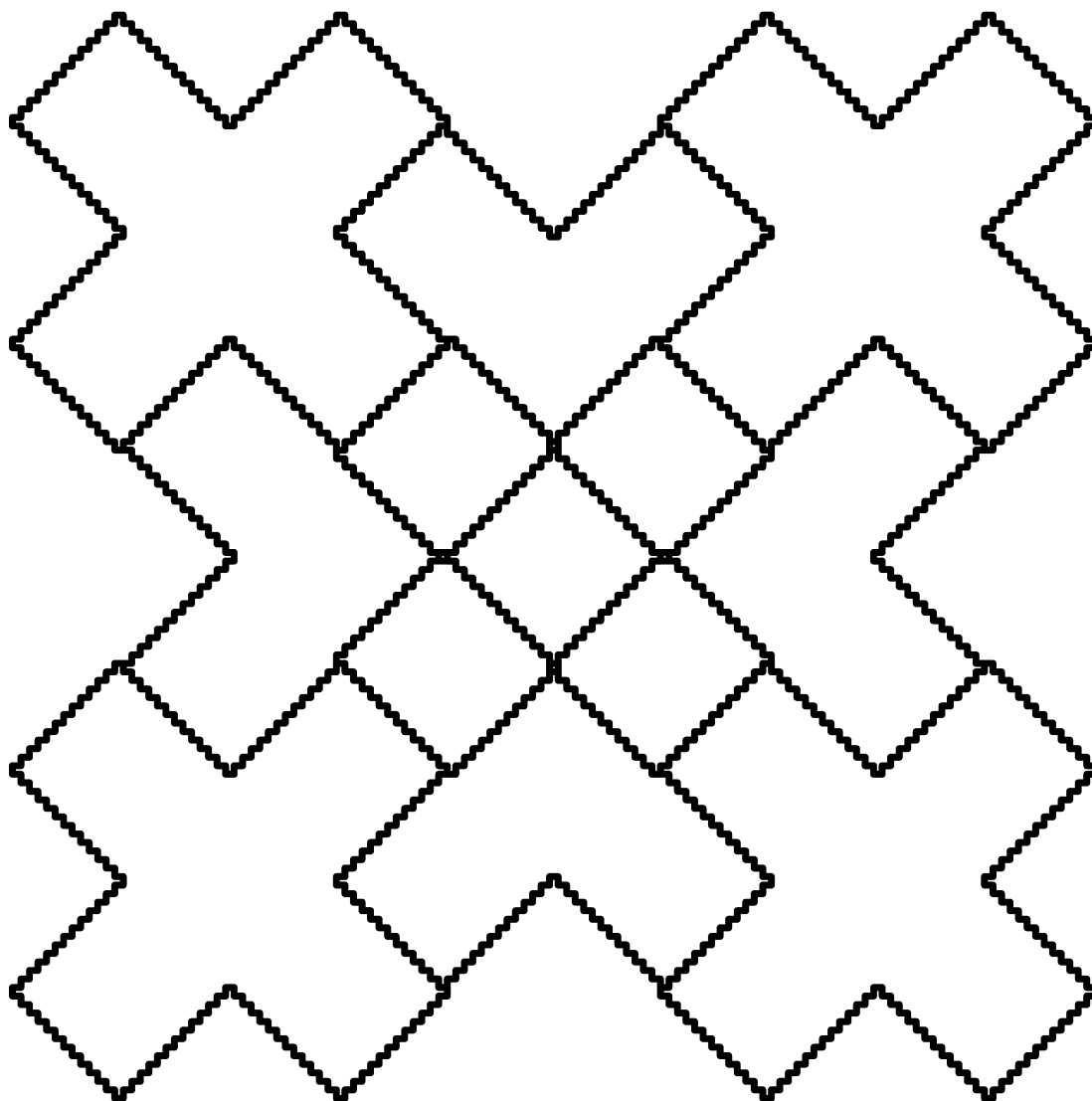Figure 51: The Cloitre walk of the sequence of parities of $B_{\sqrt{5909}}$.

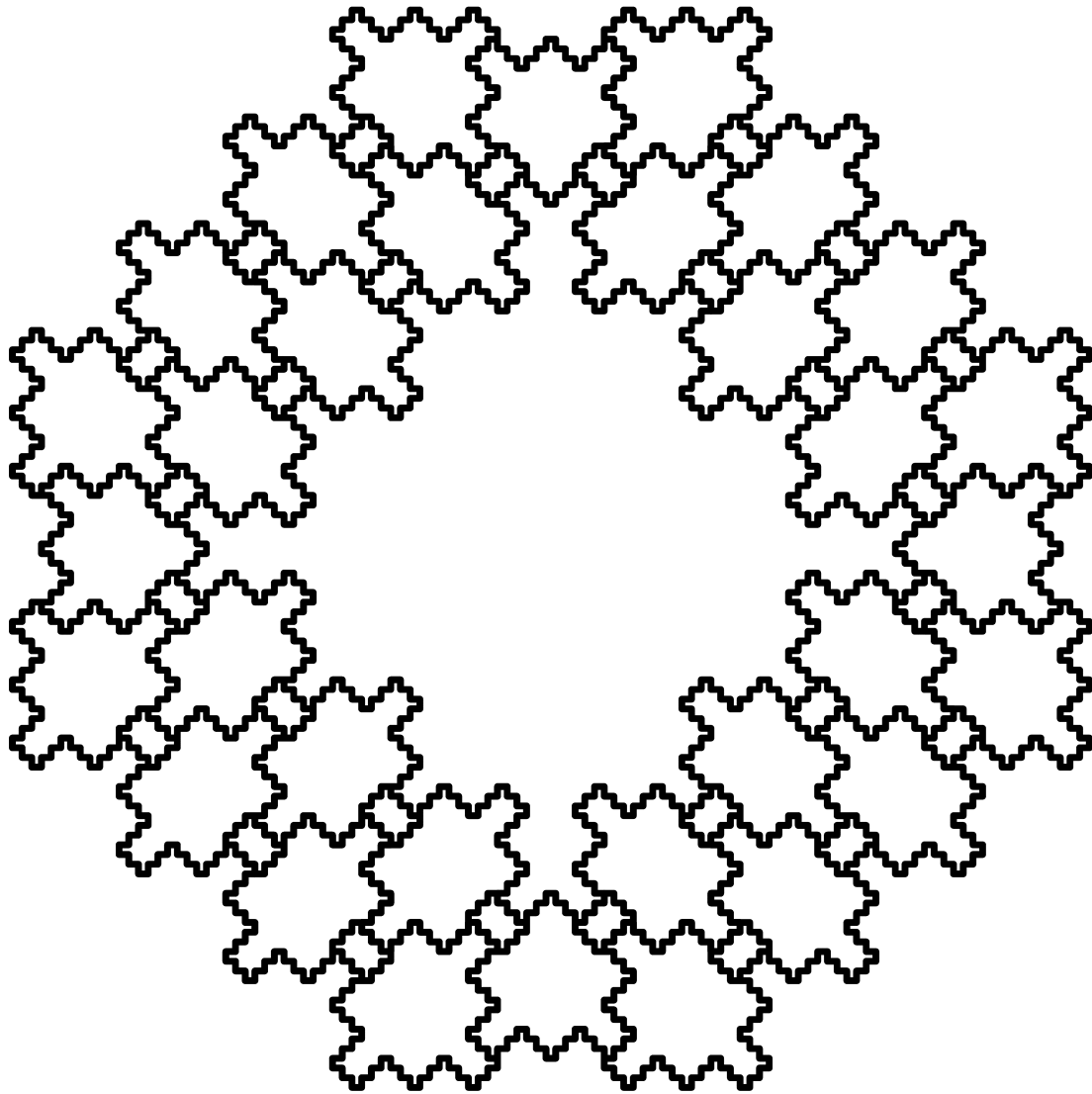Figure 52: The Cloitre walk of the sequence of parities of $B_{\sqrt{5923}}$.

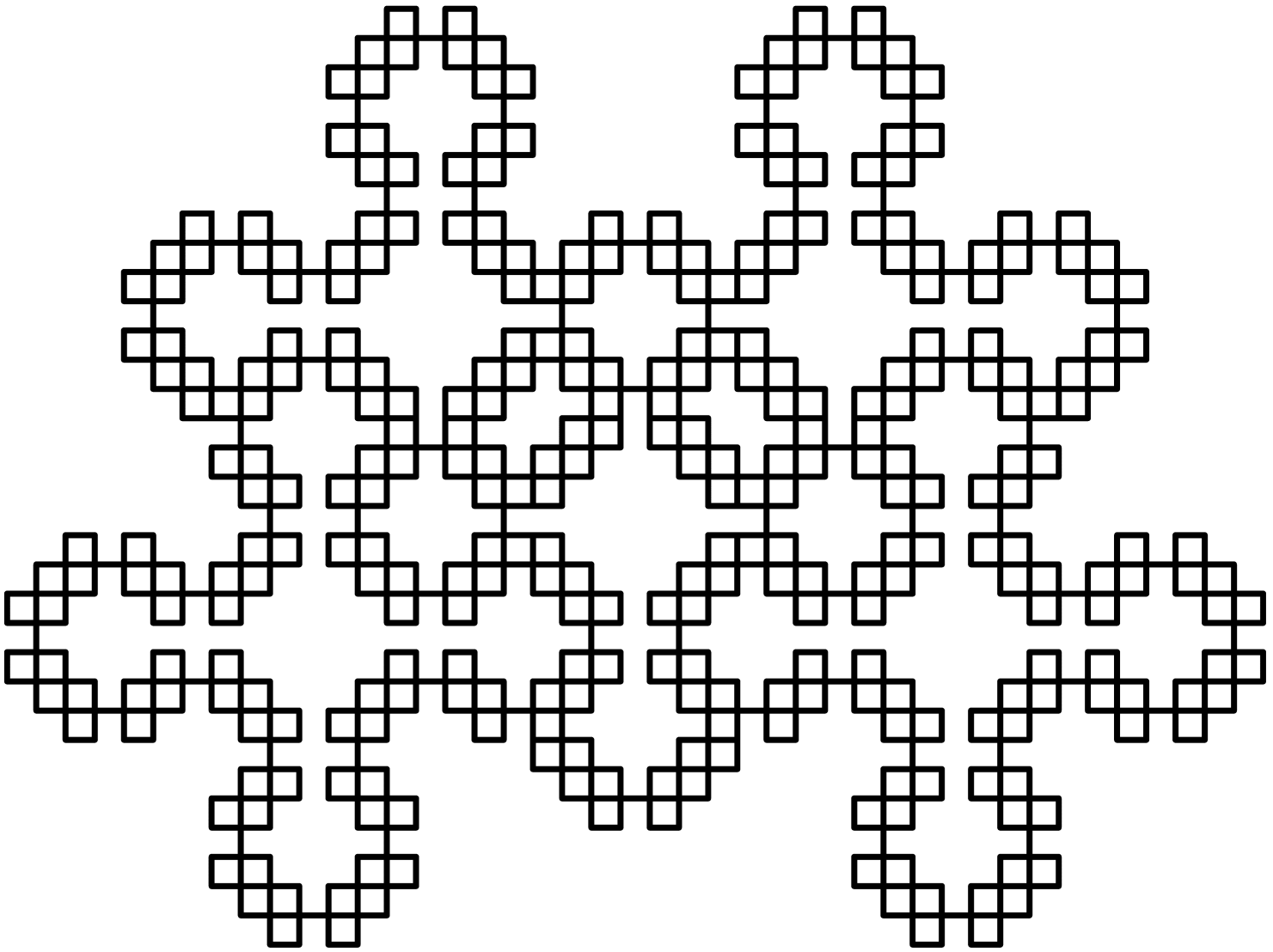Figure 53: The Cloitre walk of the sequence of parities of $B_{\sqrt{6589}}$.

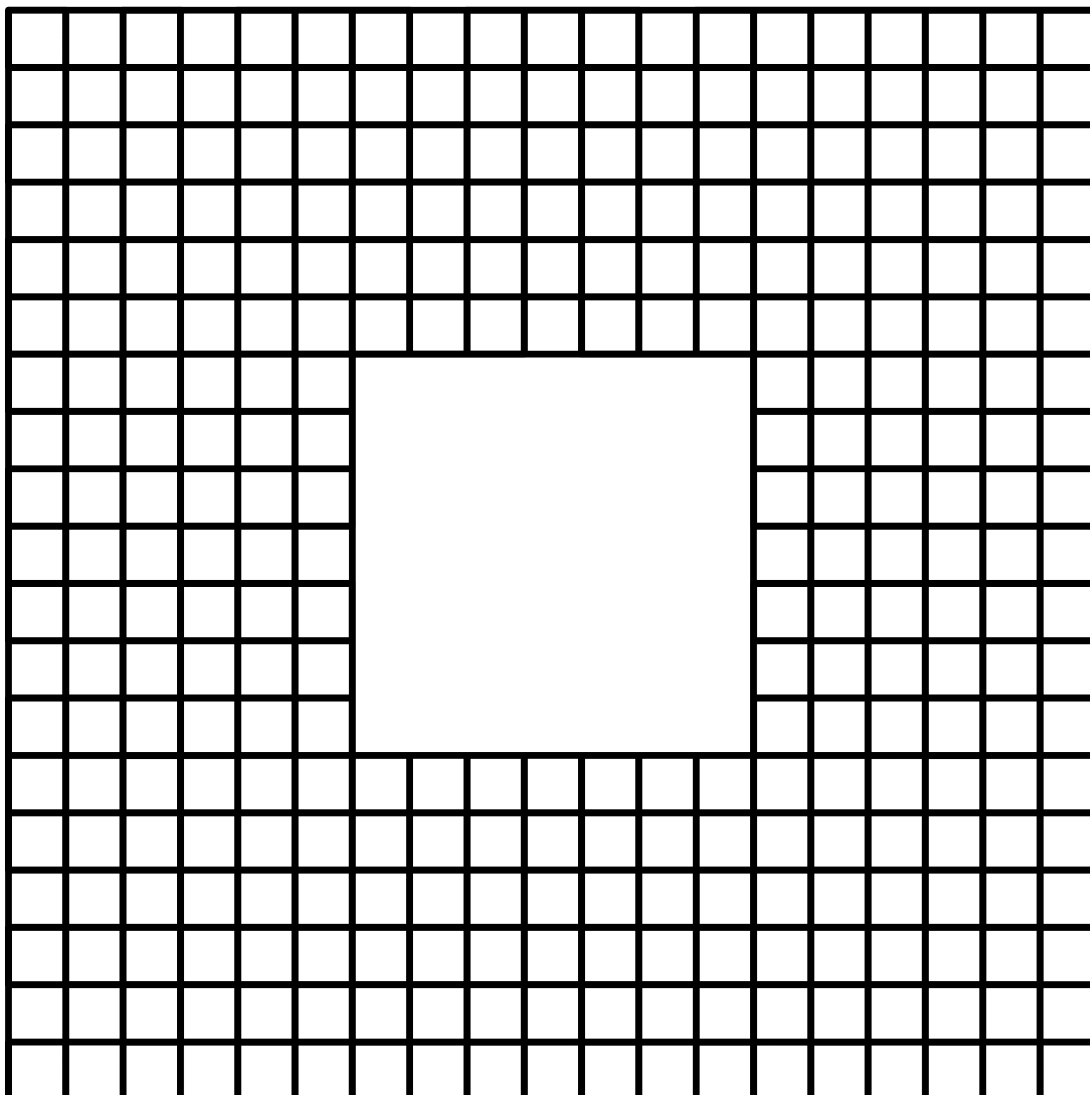Figure 54: The Cloitre walk of the sequence of parities of $B_{\sqrt{6666}}$.

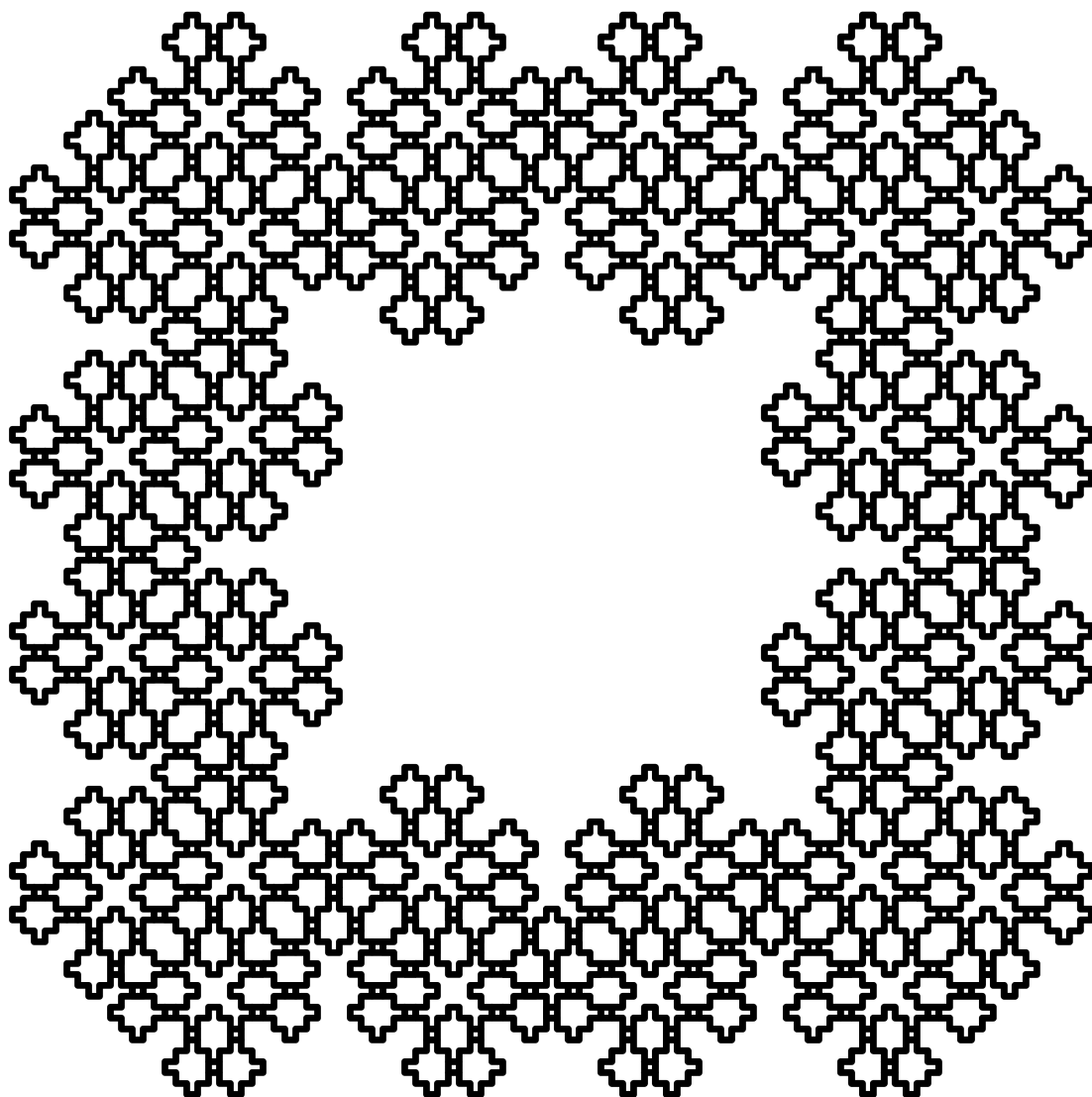Figure 55: The Cloitre walk of the sequence of parities of $B_{\sqrt{6749}}$.

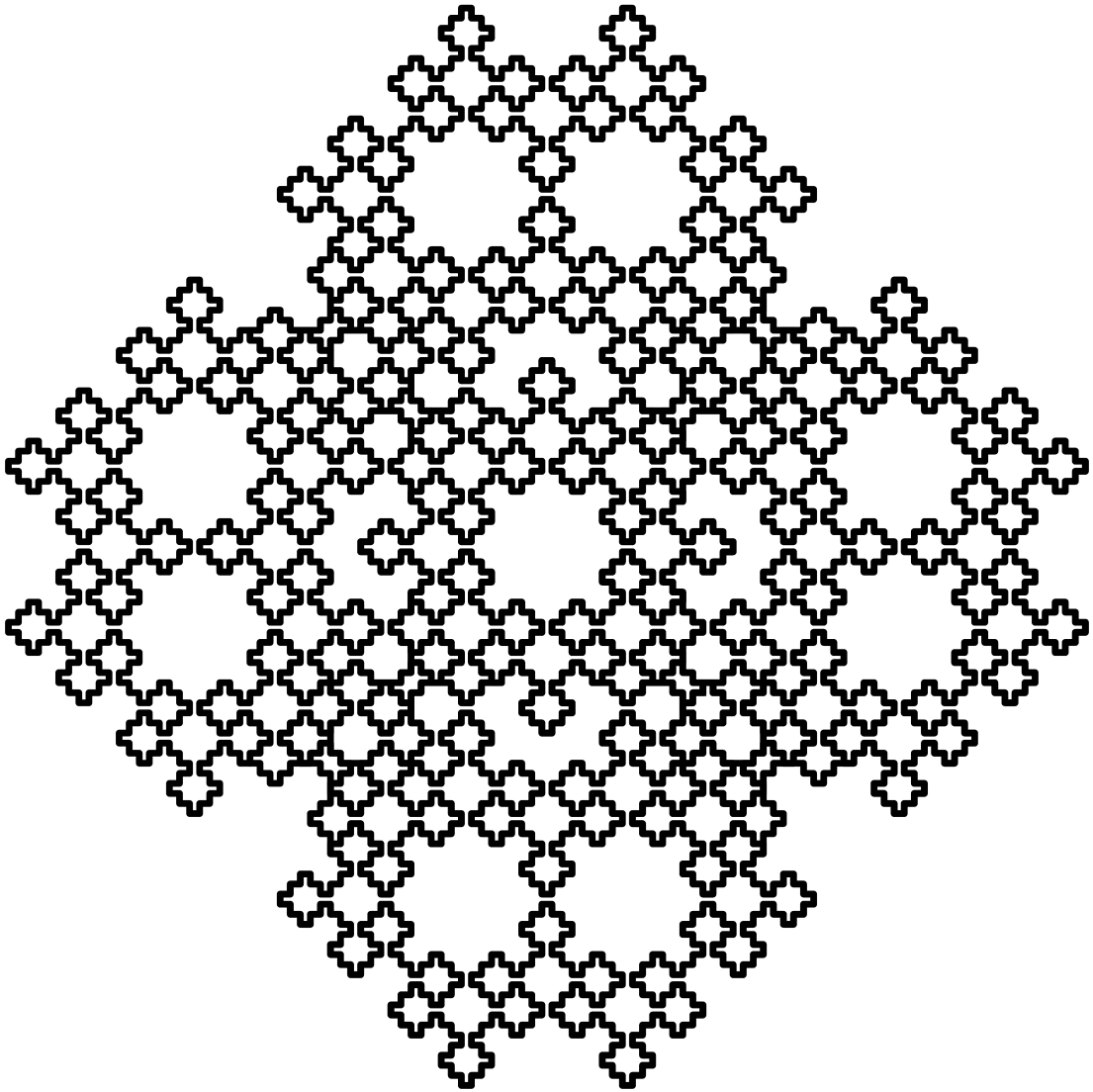Figure 56: The Cloitre walk of the sequence of parities of $B_{\sqrt{6920}}$.

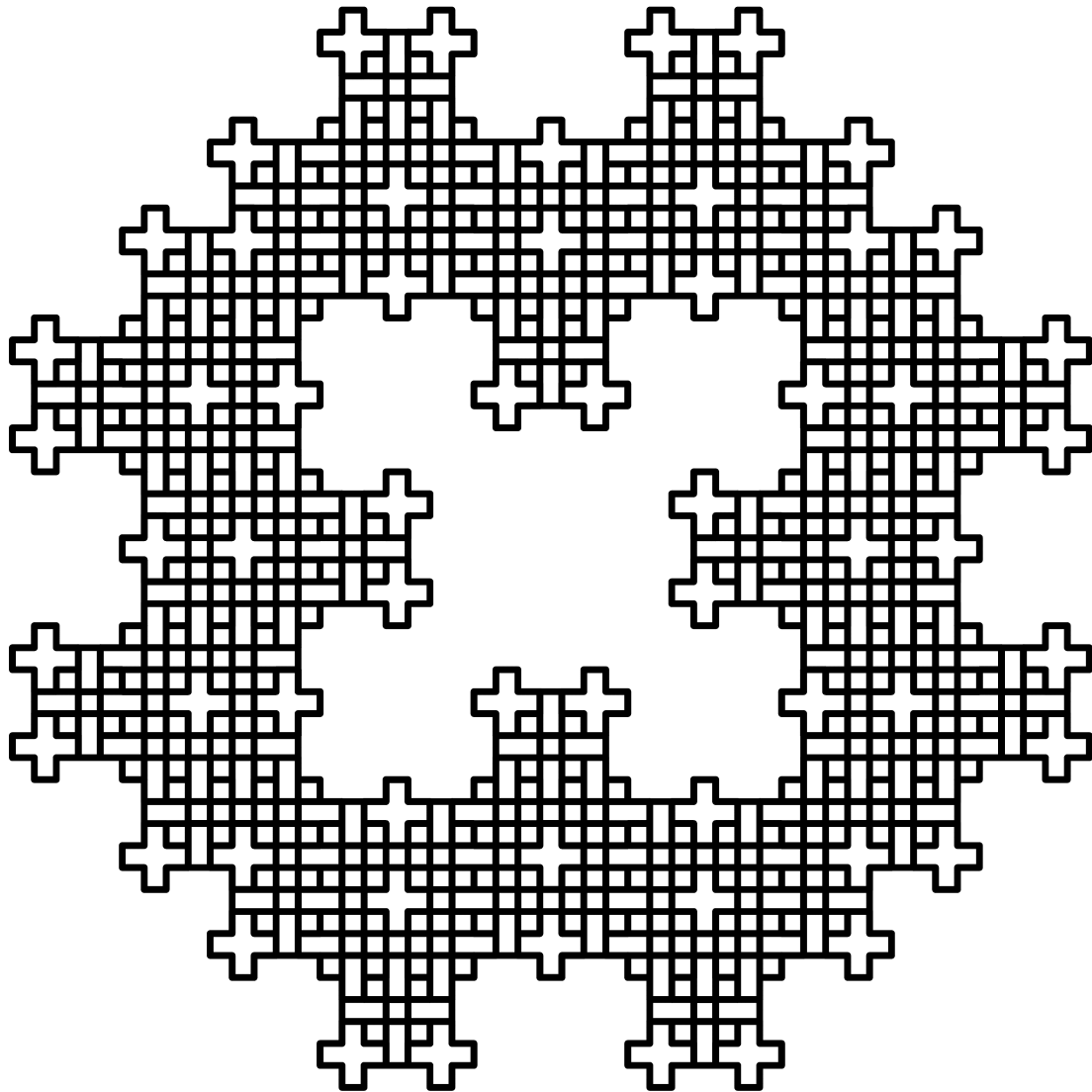Figure 57: The Cloitre walk of the sequence of parities of $B_{\sqrt{6925}}$.

Figure 58: The Cloitre walk of the sequence of parities of $B_{\sqrt{6947}}$.
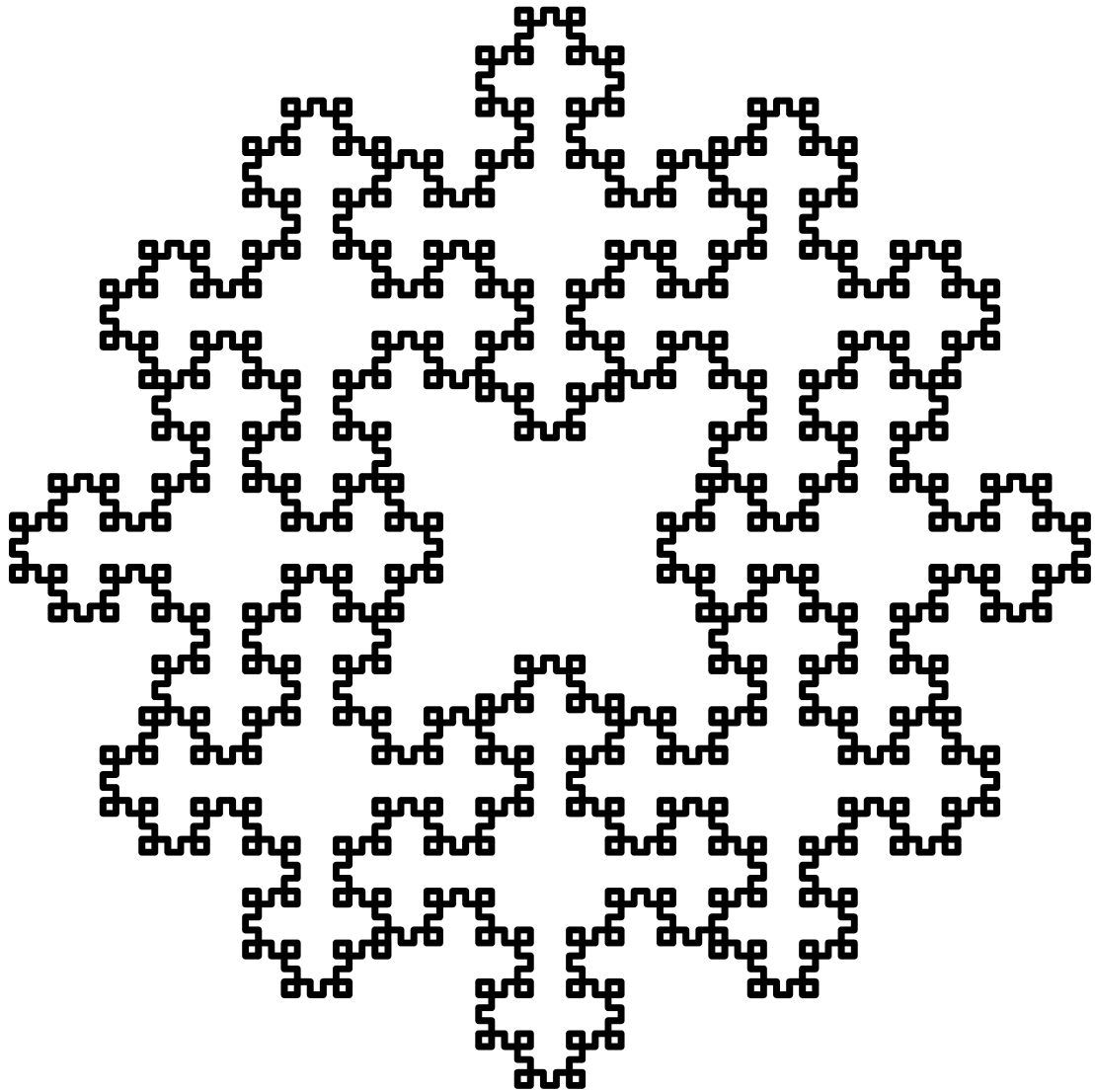
Figure 59: The Cloitre walk of the sequence of parities of $B_{\sqrt{6983}}$.
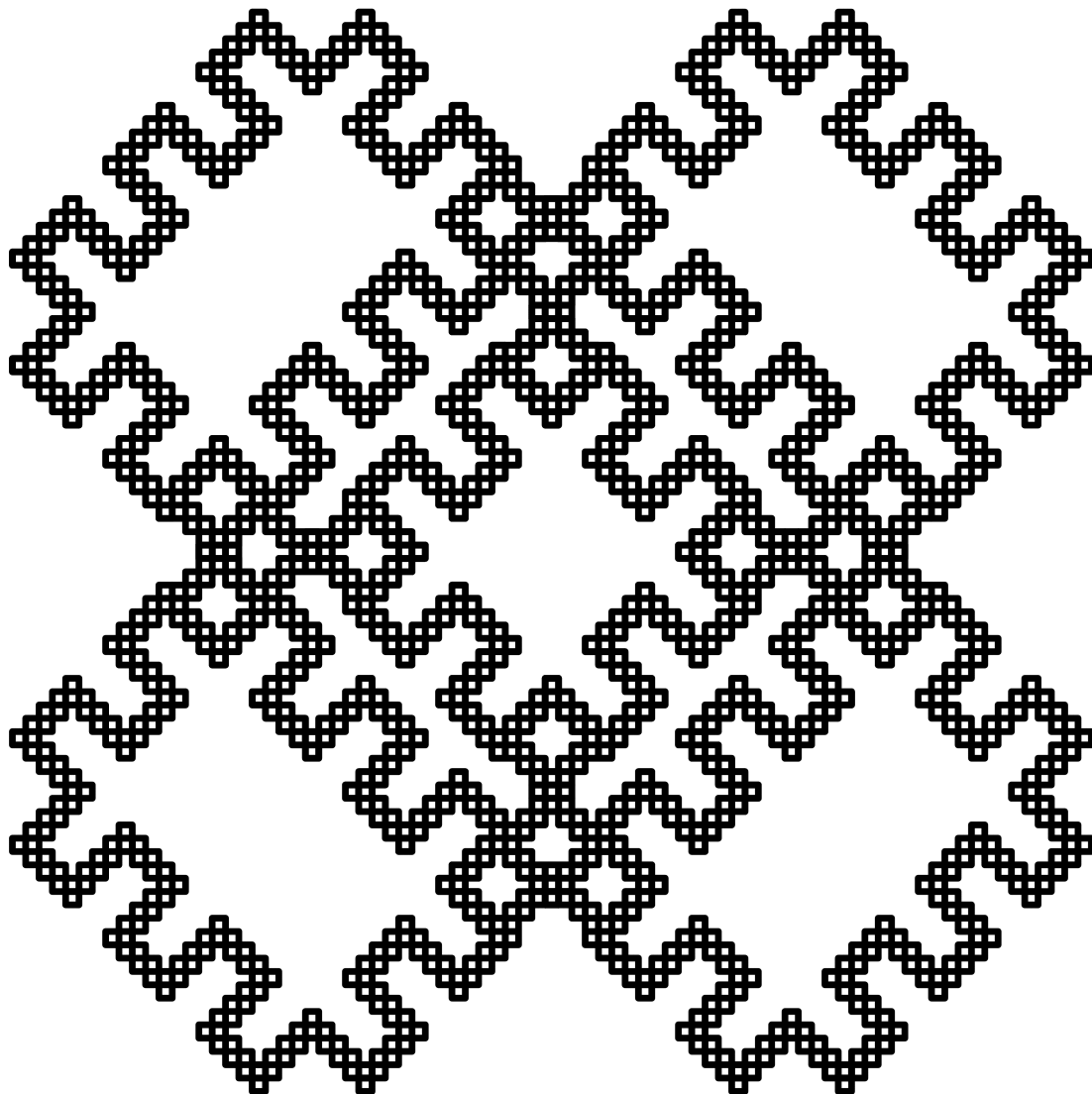
Figure 60: The Cloitre walk of the sequence of parities of $B_{\sqrt{7110}}$.
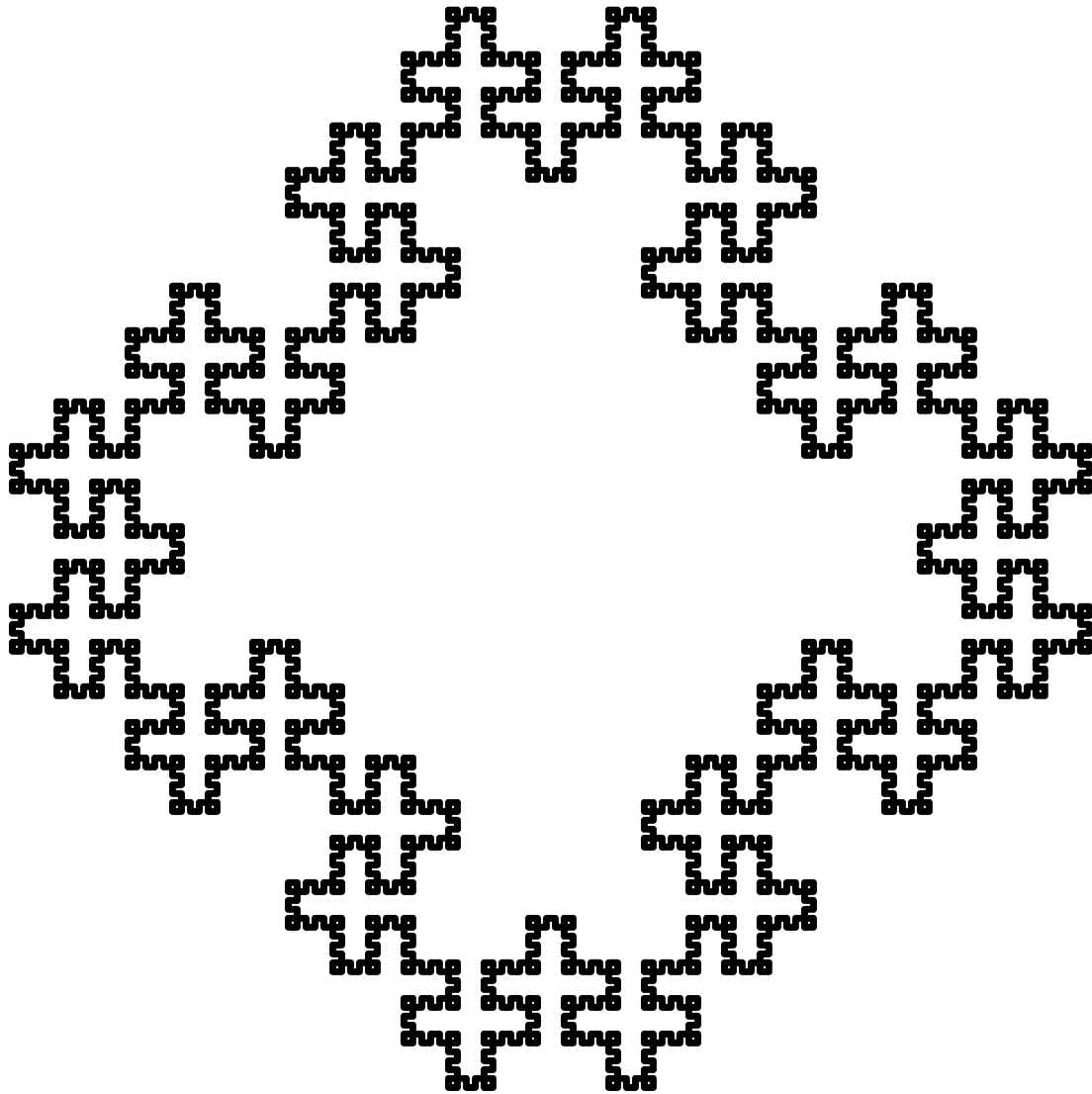
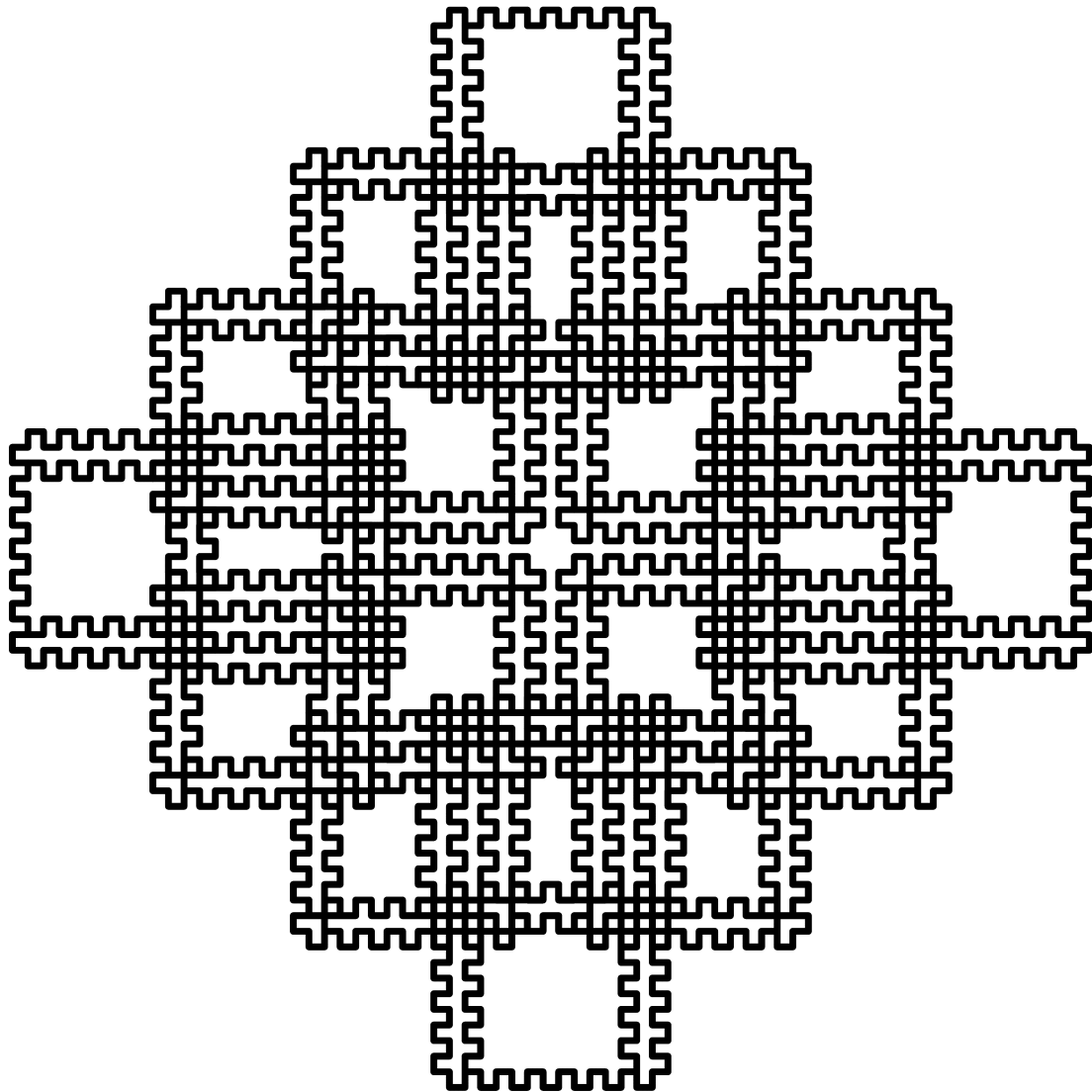Figure 61: The Cloitre walk of the sequence of parities of $B_{\sqrt{7132}}$.

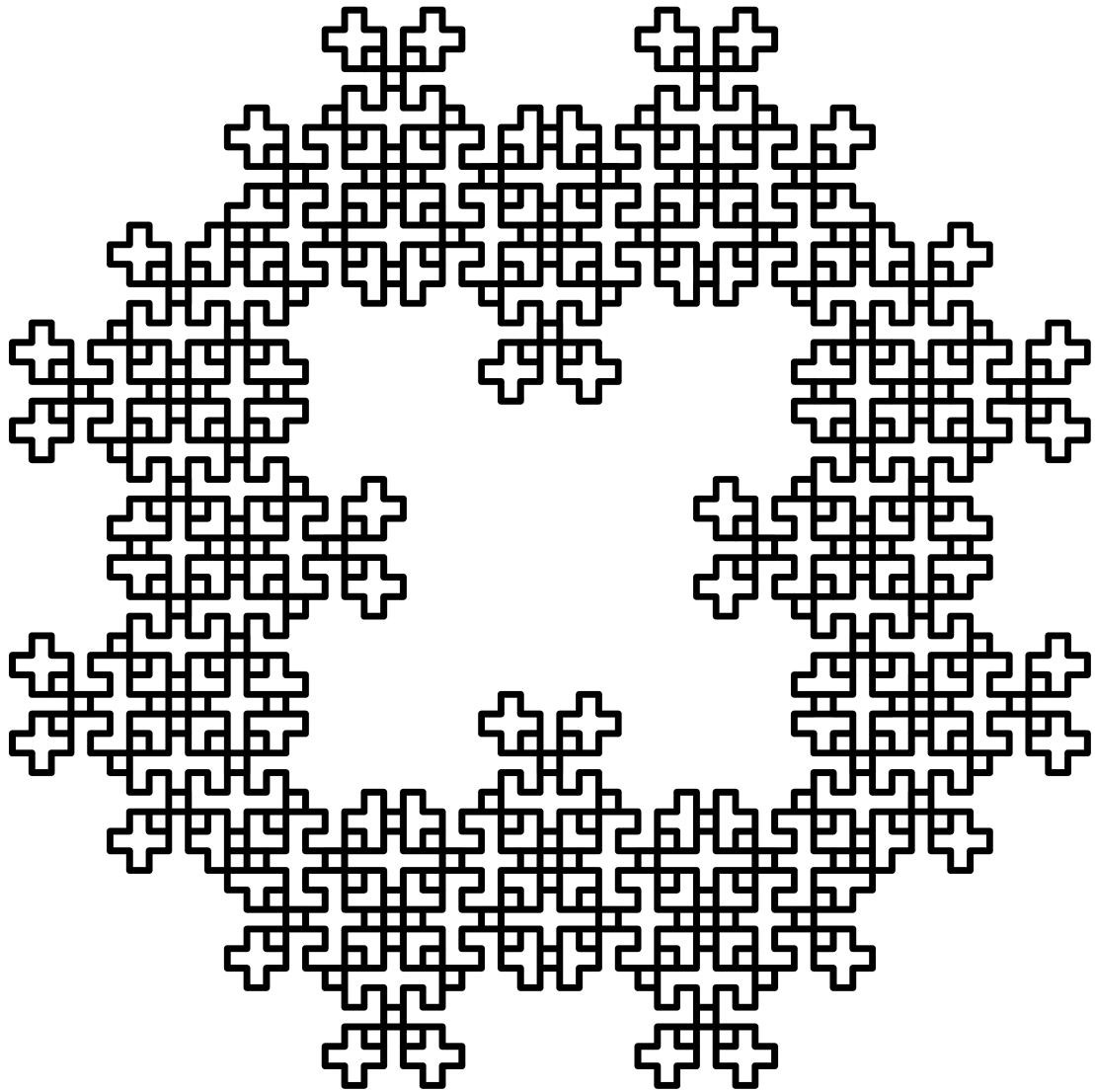Figure 62: The Cloitre walk of the sequence of parities of $B_{\sqrt{7144}}$.

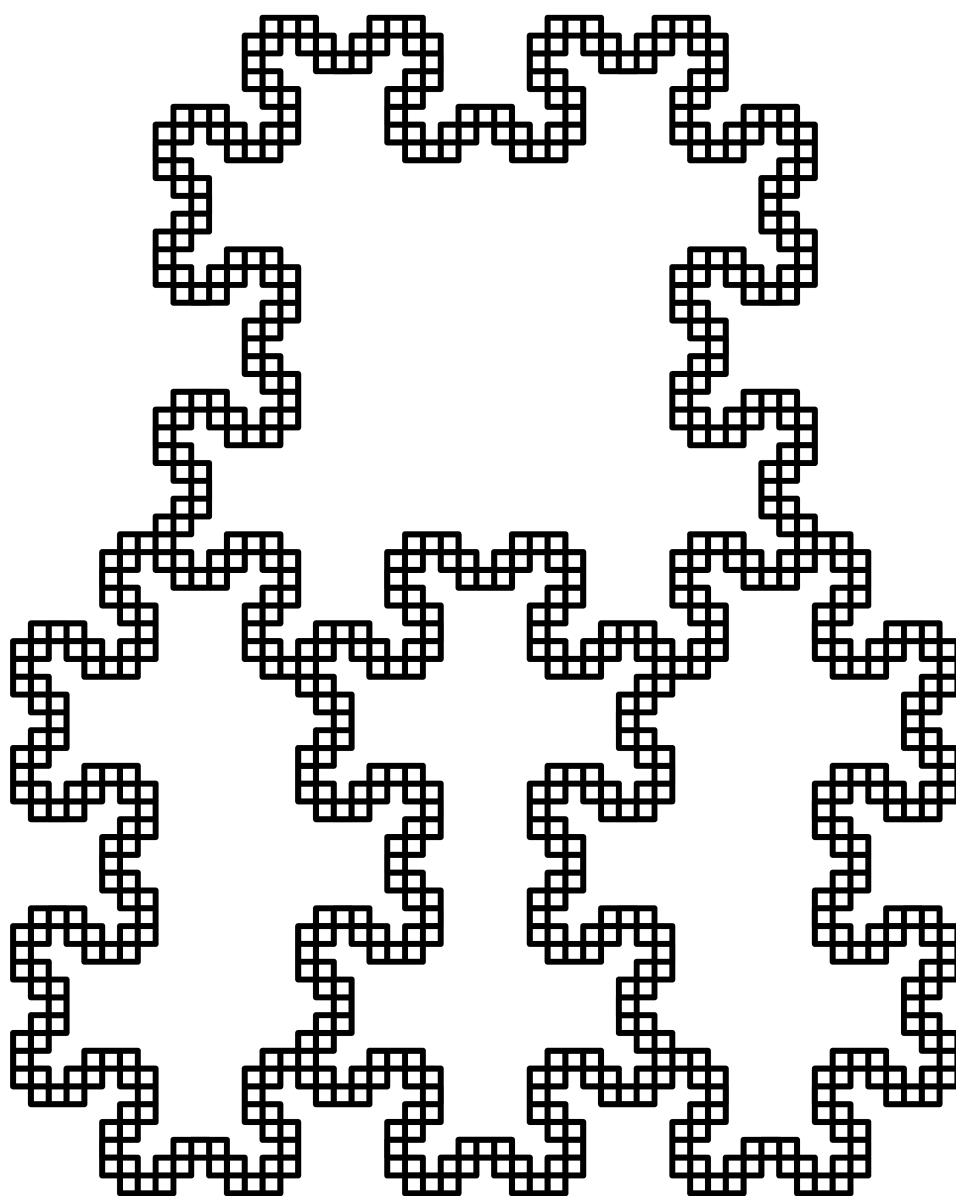Figure 63: The Cloitre walk of the sequence of parities of $B_{\sqrt{7288}}$.

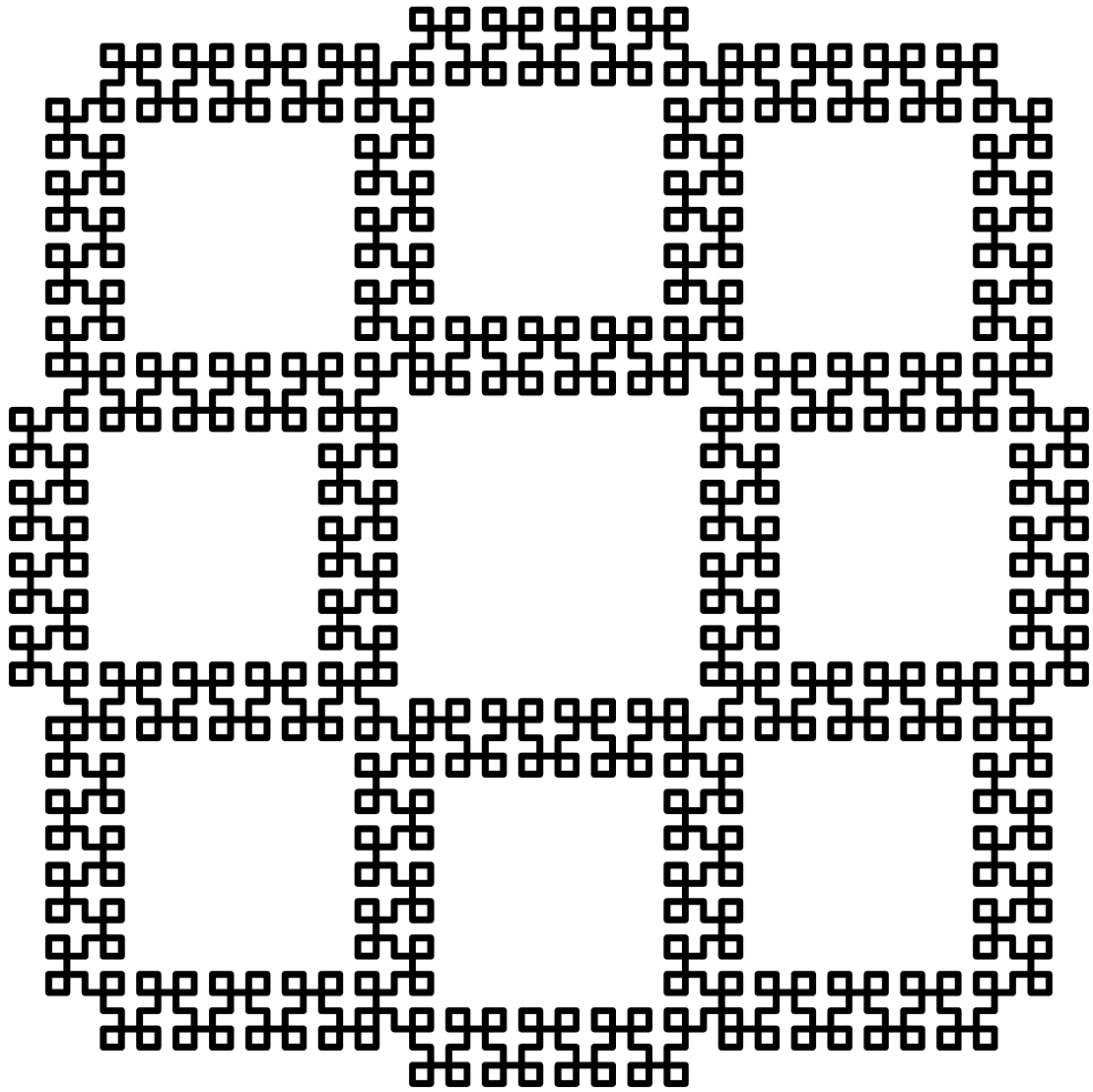Figure 64: The Cloitre walk of the sequence of parities of $B_{\sqrt{7429}}$.

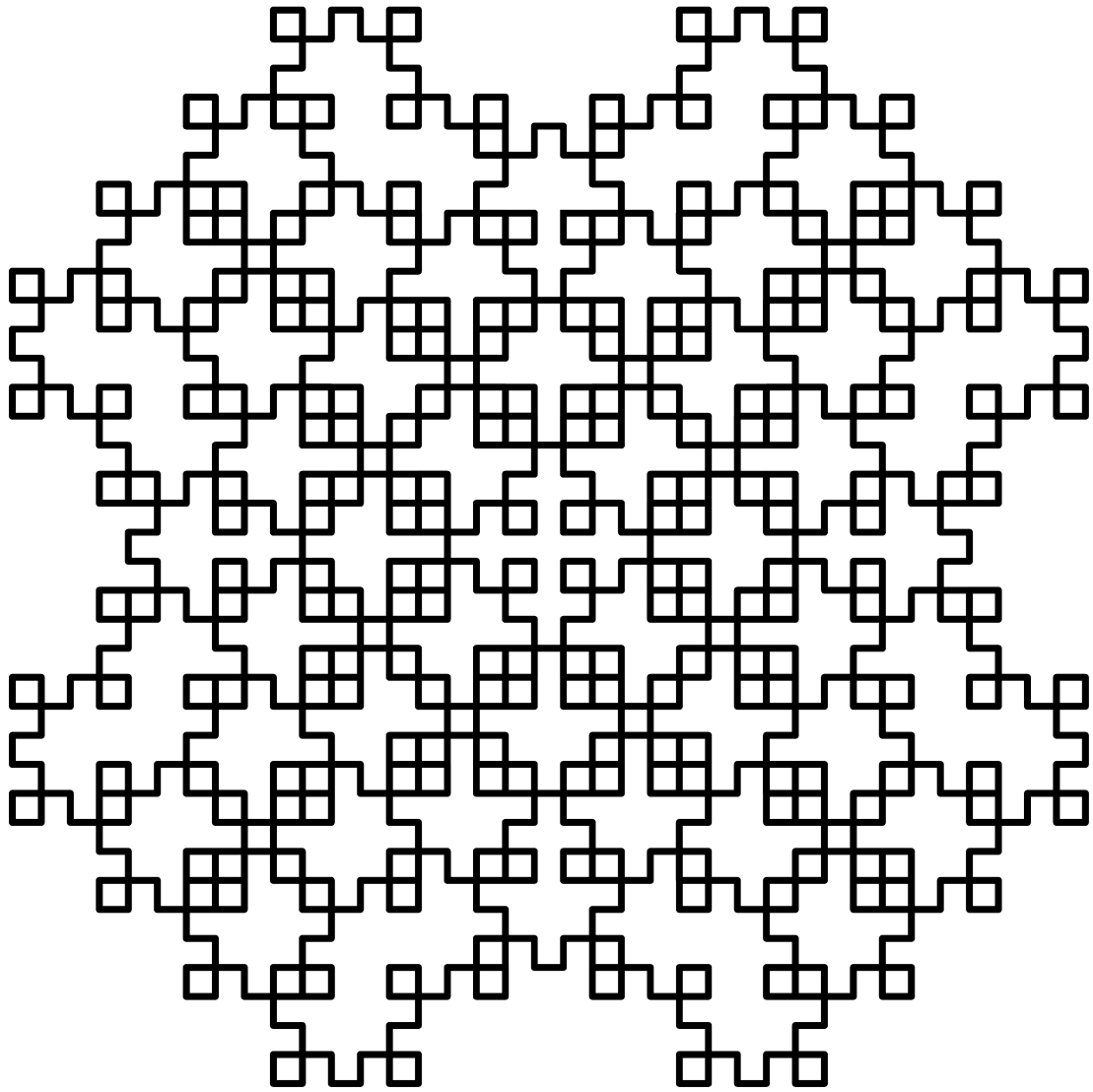Figure 65: The Cloitre walk of the sequence of parities of $B_{\sqrt{7468}}$.

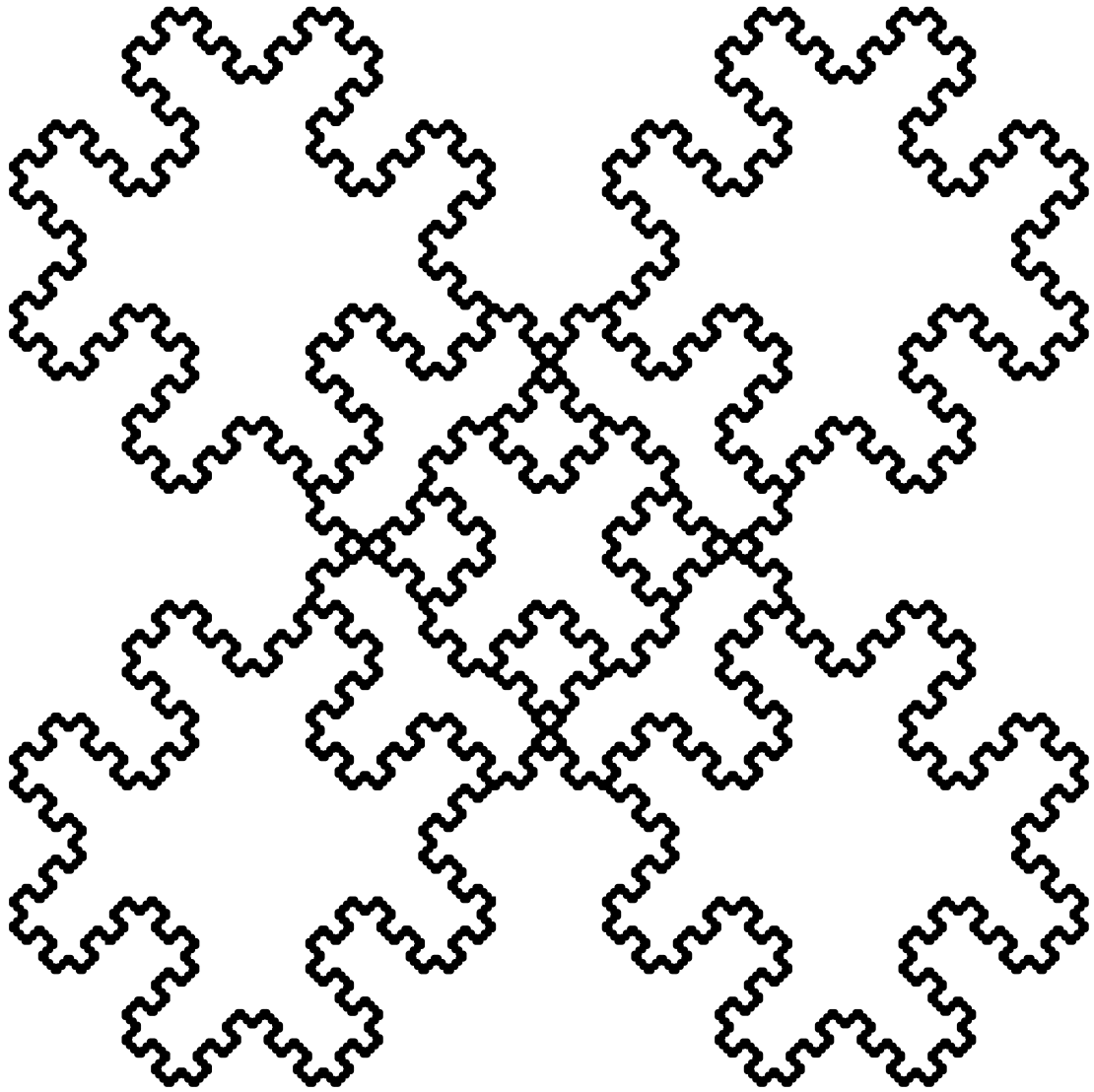Figure 66: The Cloitre walk of the sequence of parities of $B_{\sqrt{7471}}$.

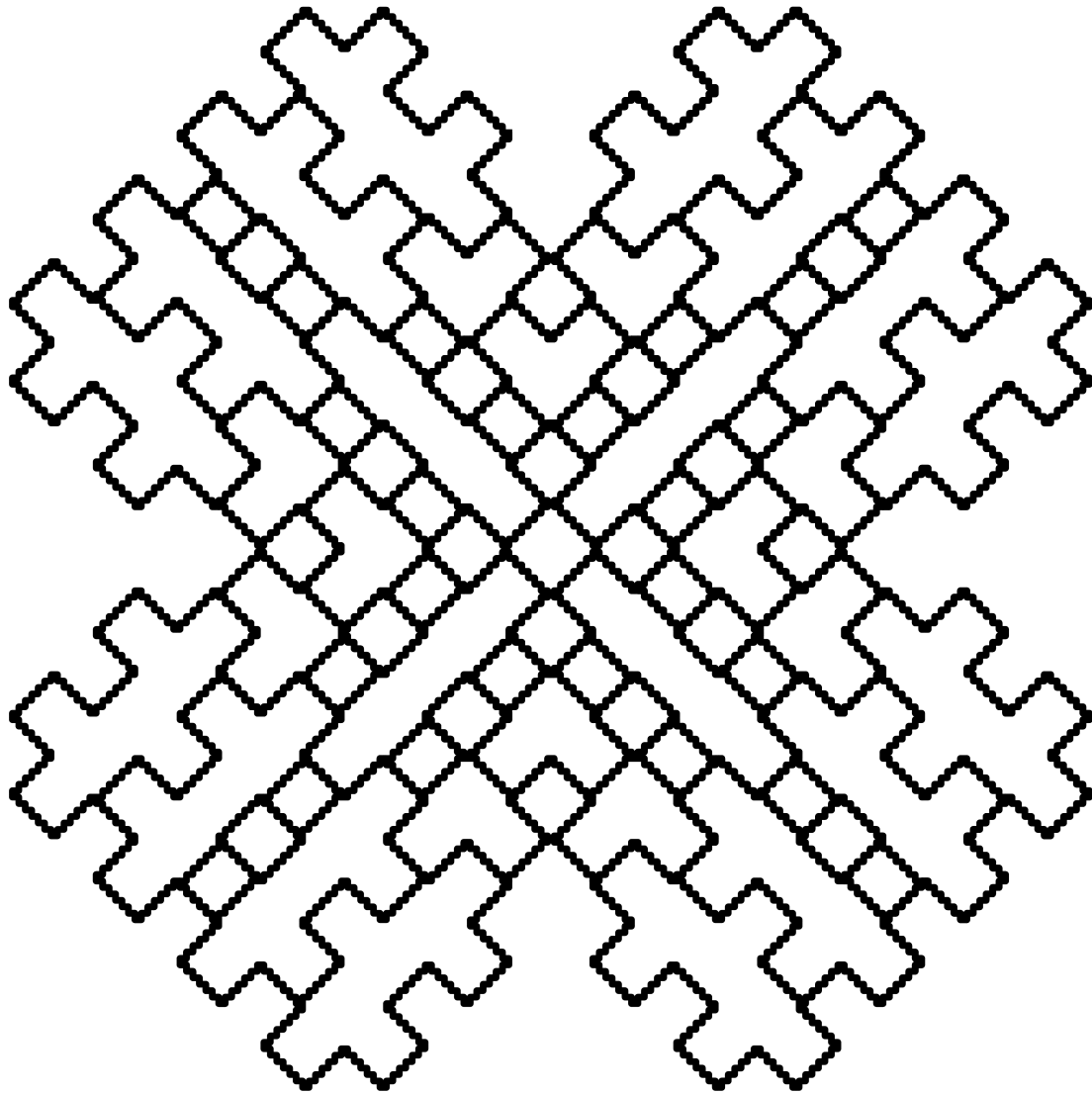Figure 67: The Cloitre walk of the sequence of parities of $B_{\sqrt{7542}}$.

Figure 68: The Cloitre walk of the sequence of parities of $B_{\sqrt{7555}}$.
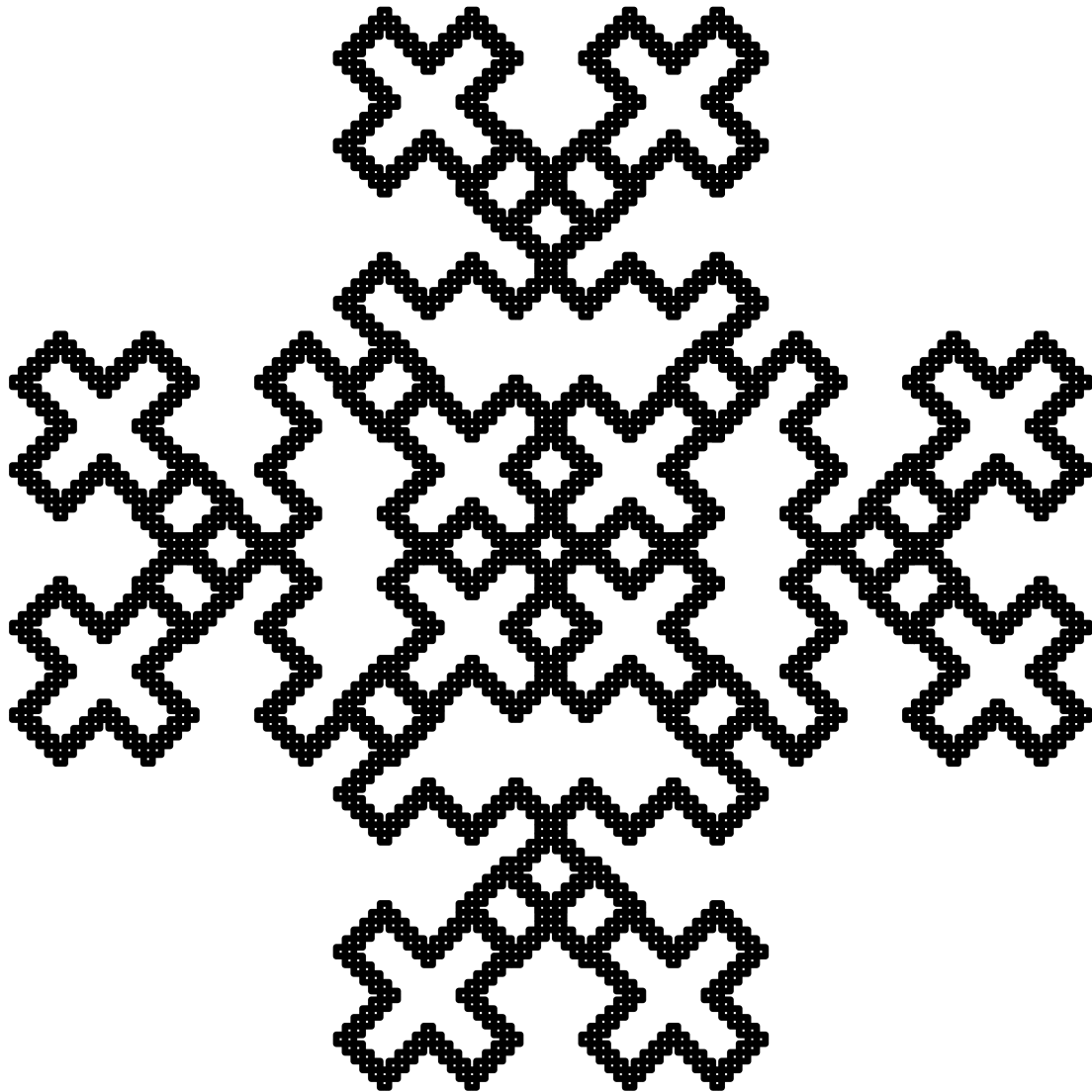
Figure 69: The Cloitre walk of the sequence of parities of $B_{\sqrt{7801}}$.
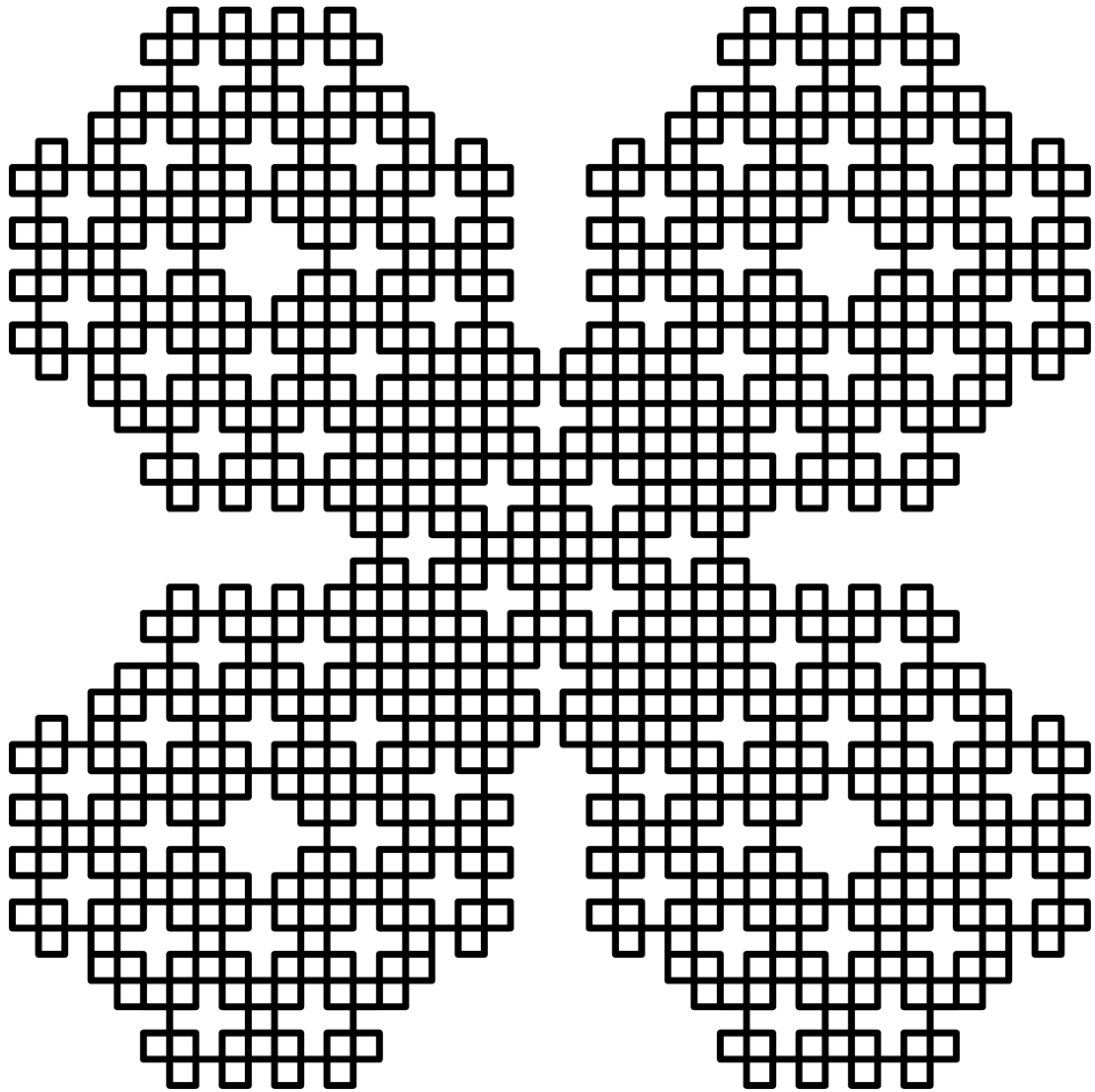
Figure 70: The Cloitre walk of the sequence of parities of $B_{\sqrt{7808}}$.
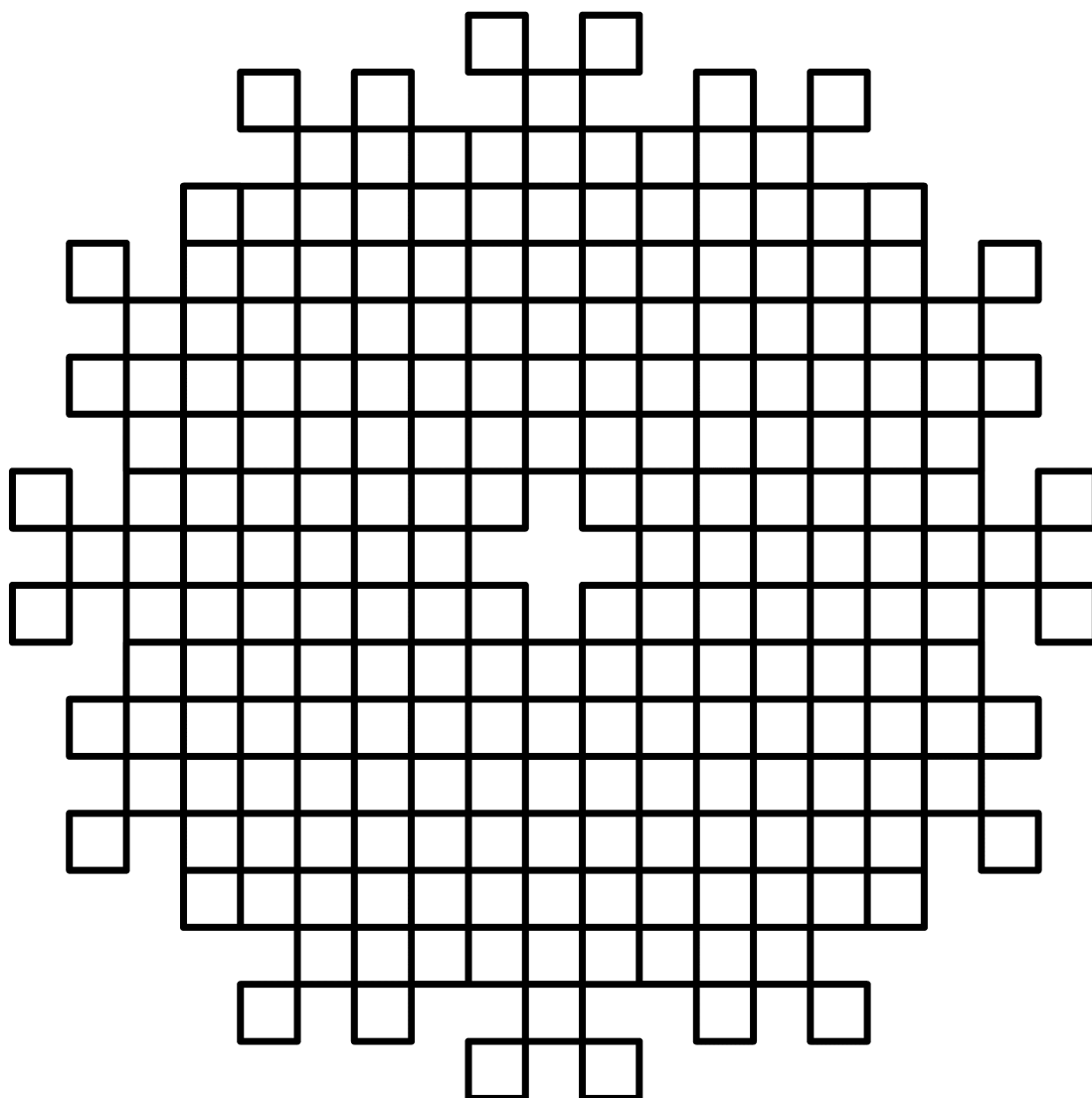
71

Figure 71: The Cloitre walk of the sequence of parities of $B_{\sqrt{7816}}$.
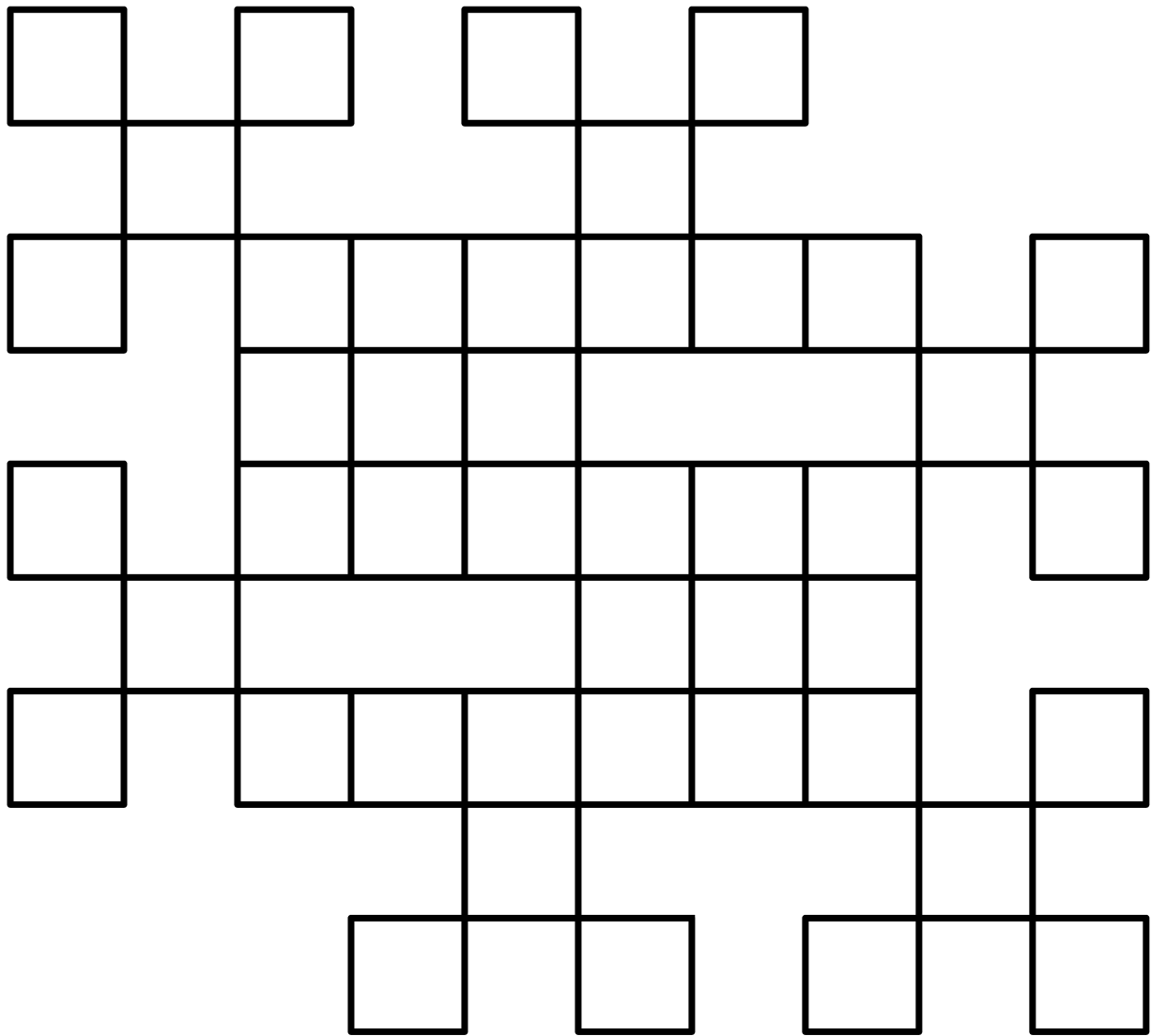
Figure 72: The Cloitre walk of the sequence of parities of $B_{\sqrt{7817}}$.

Figure 73: The Cloitre walk of the sequence of parities of $B_{\sqrt{7824}}$.

Figure 74: The Cloitre walk of the sequence of parities of $B_{\sqrt{8033}}$.
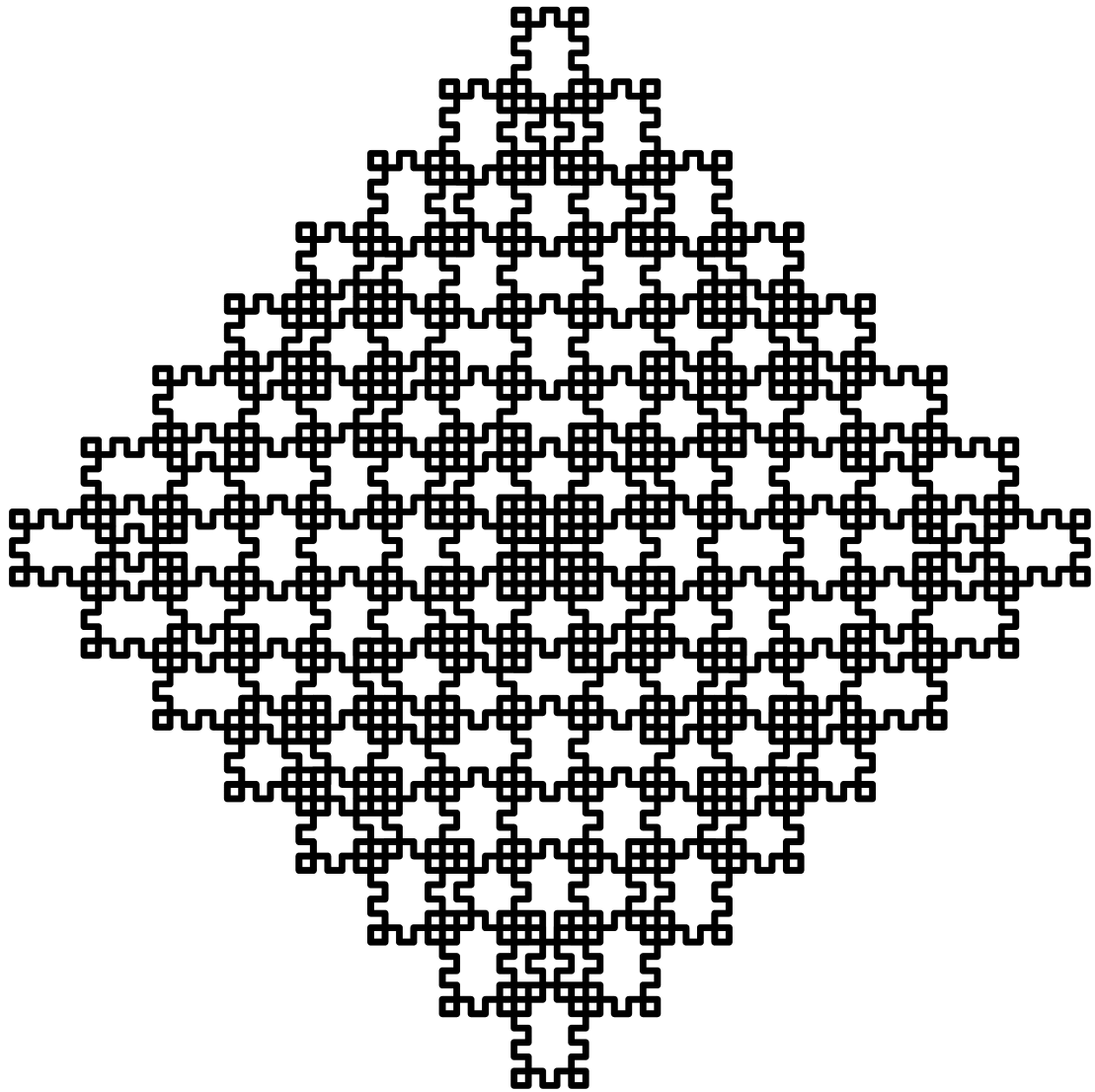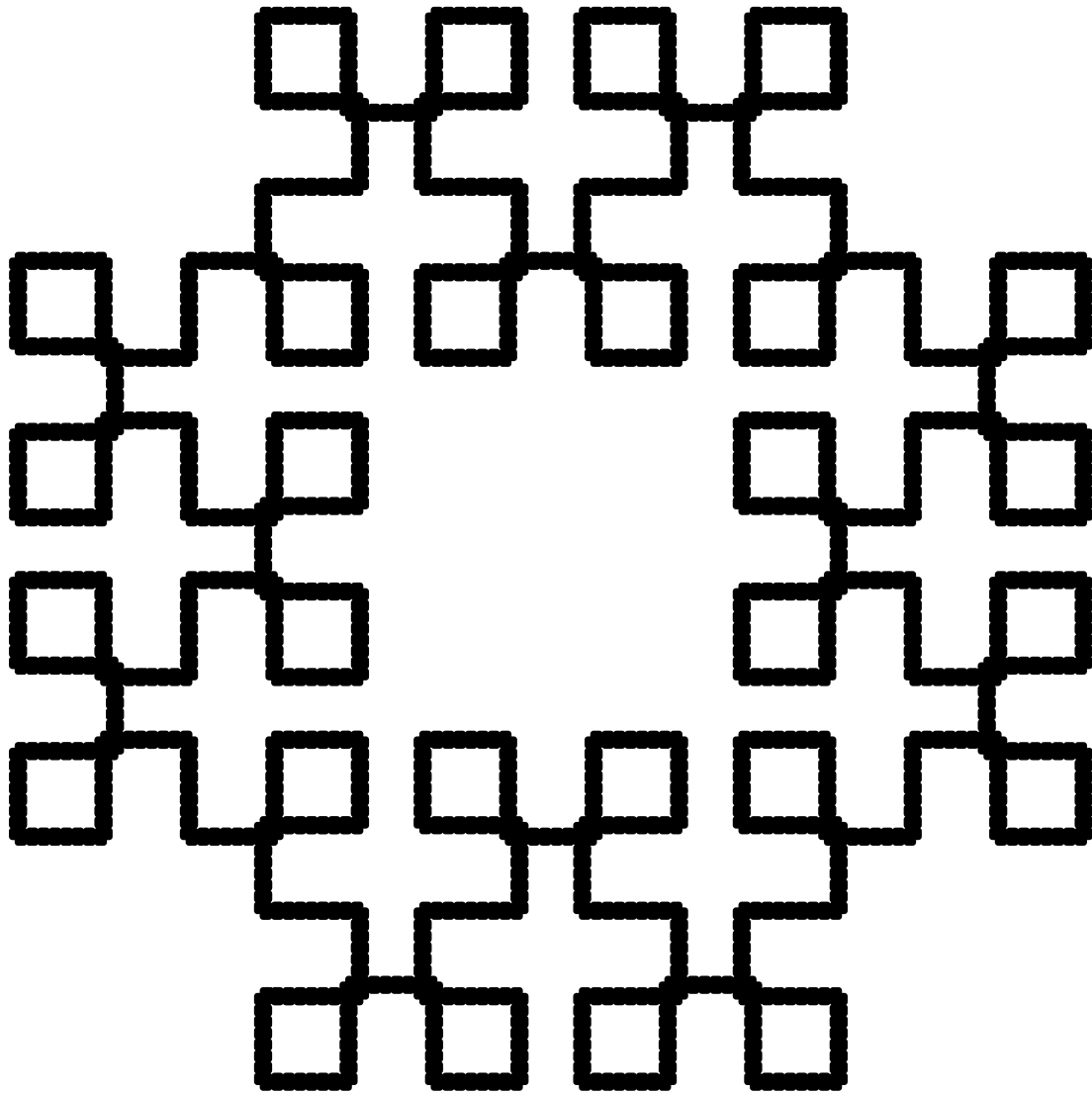
Figure 75: The Cloitre walk of the sequence of parities of $B_{\sqrt{8042}}$.

Figure 76: The Cloitre walk of the sequence of parities of $B_{\sqrt{8043}}$.
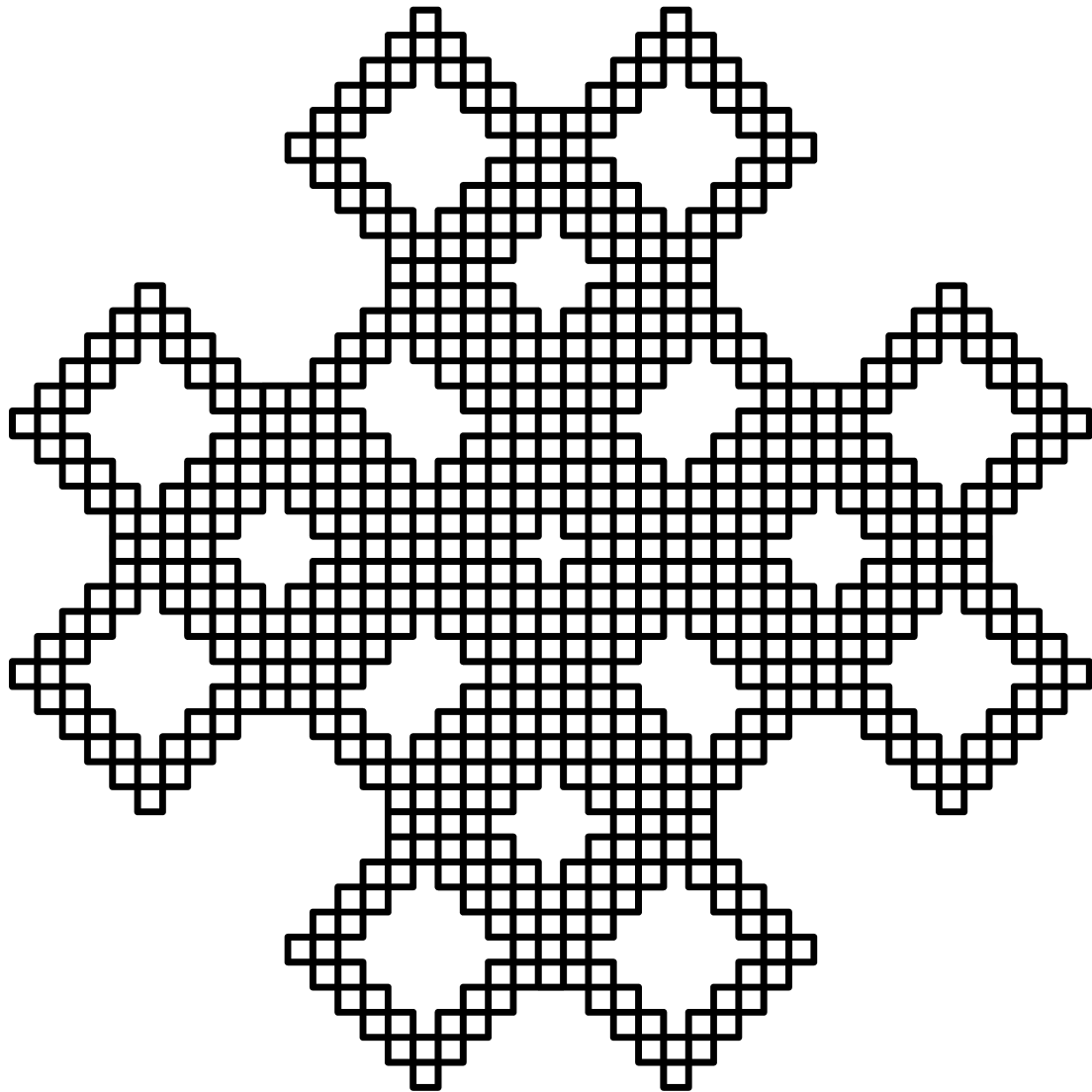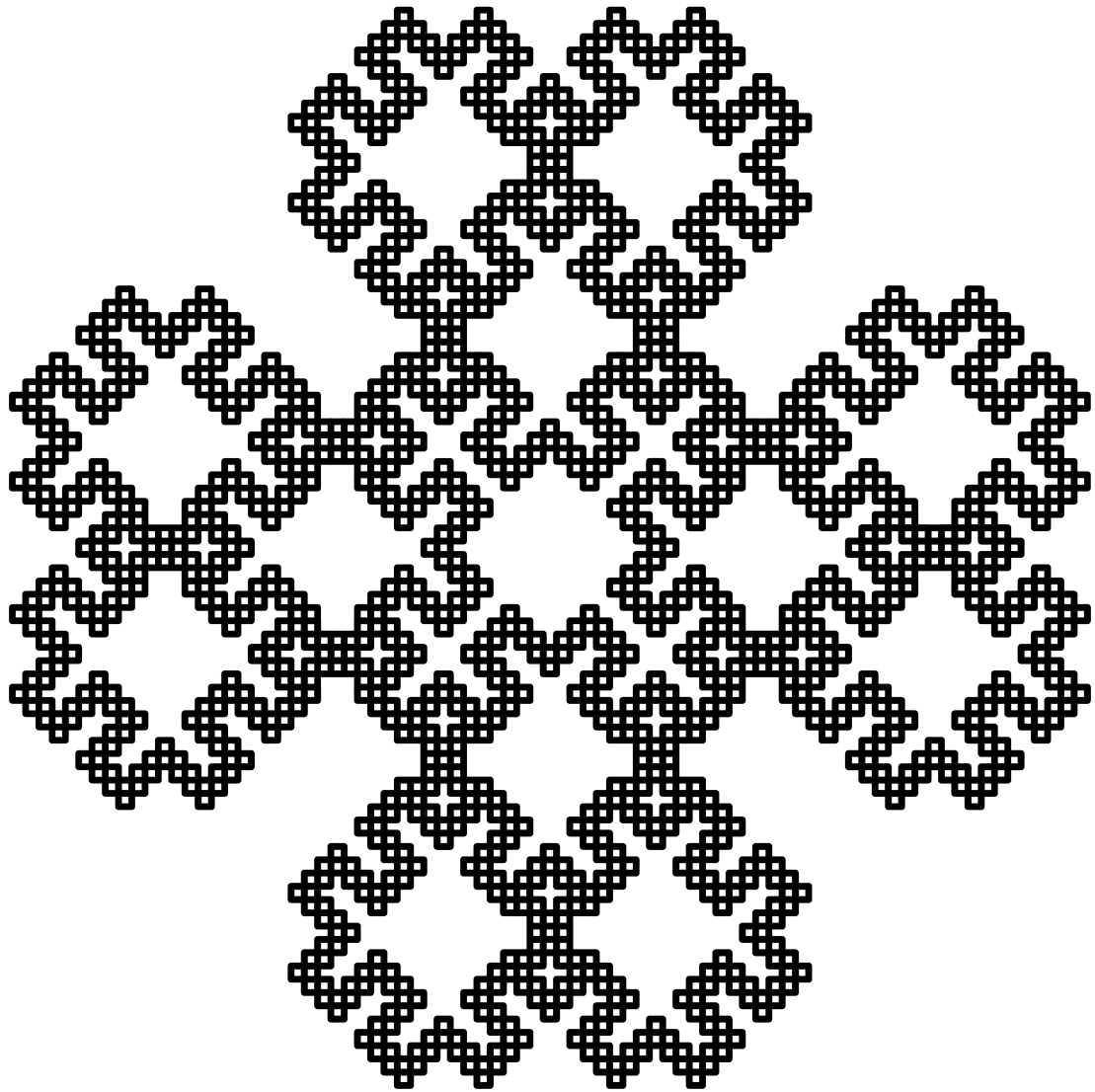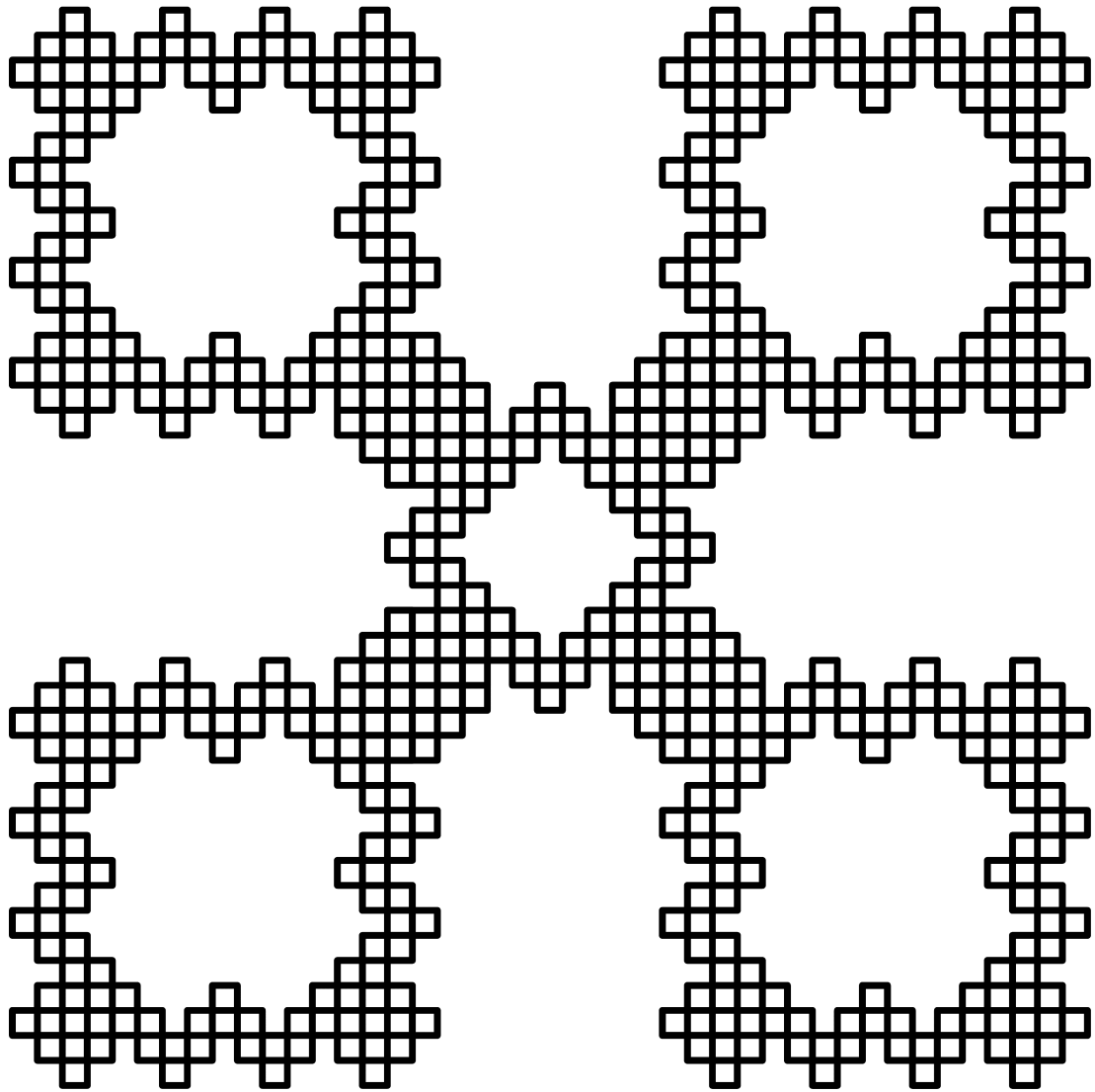
Figure 77: The Cloitre walk of the sequence of parities of $B_{\sqrt{8125}}$.
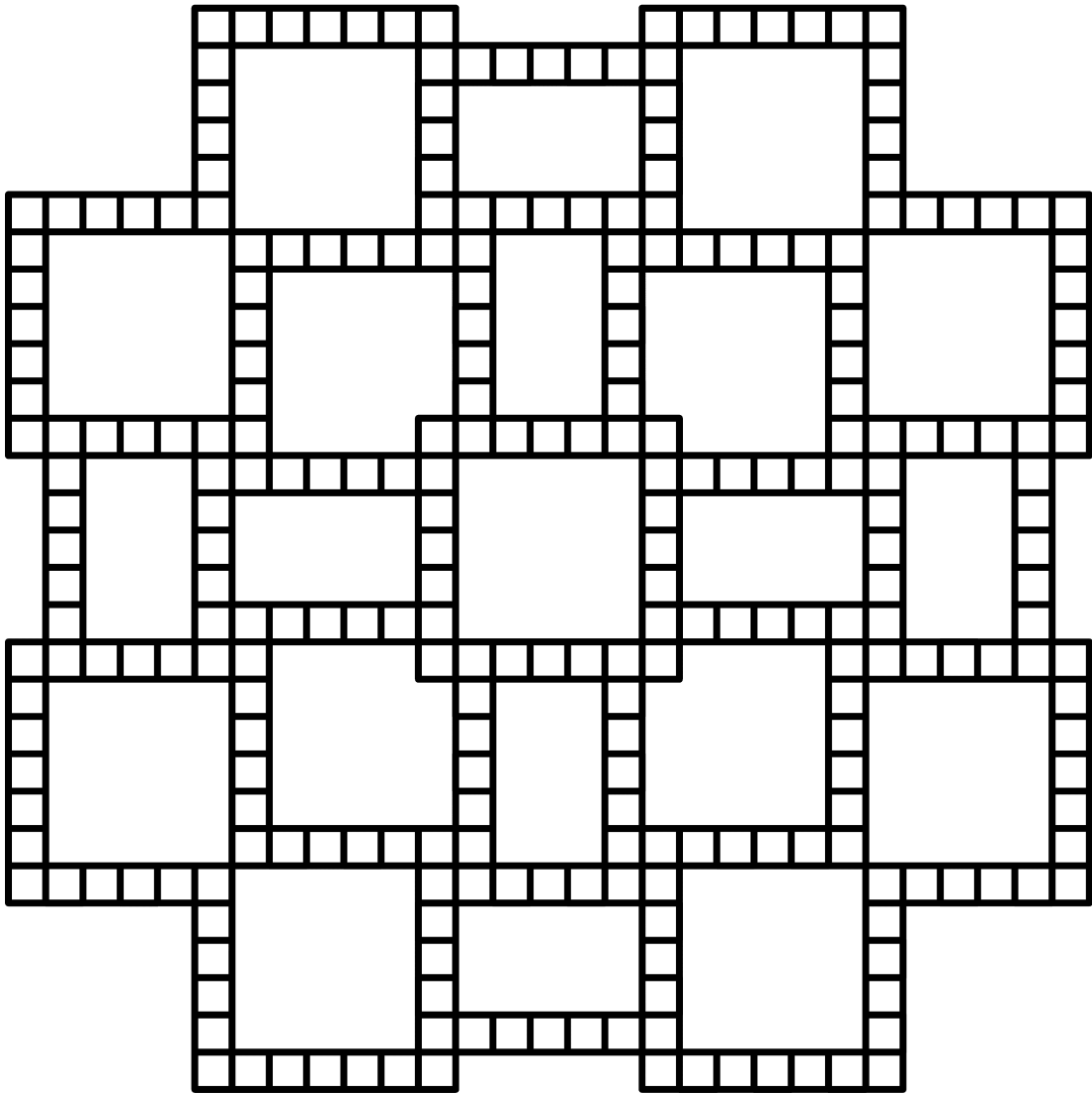
Figure 78: The Cloitre walk of the sequence of parities of $B_{\sqrt{8131}}$.

Figure 79: The Cloitre walk of the sequence of parities of $B_{\sqrt{8365}}$.

Figure 80: The Cloitre walk of the sequence of parities of $B_{\sqrt{8401}}$.

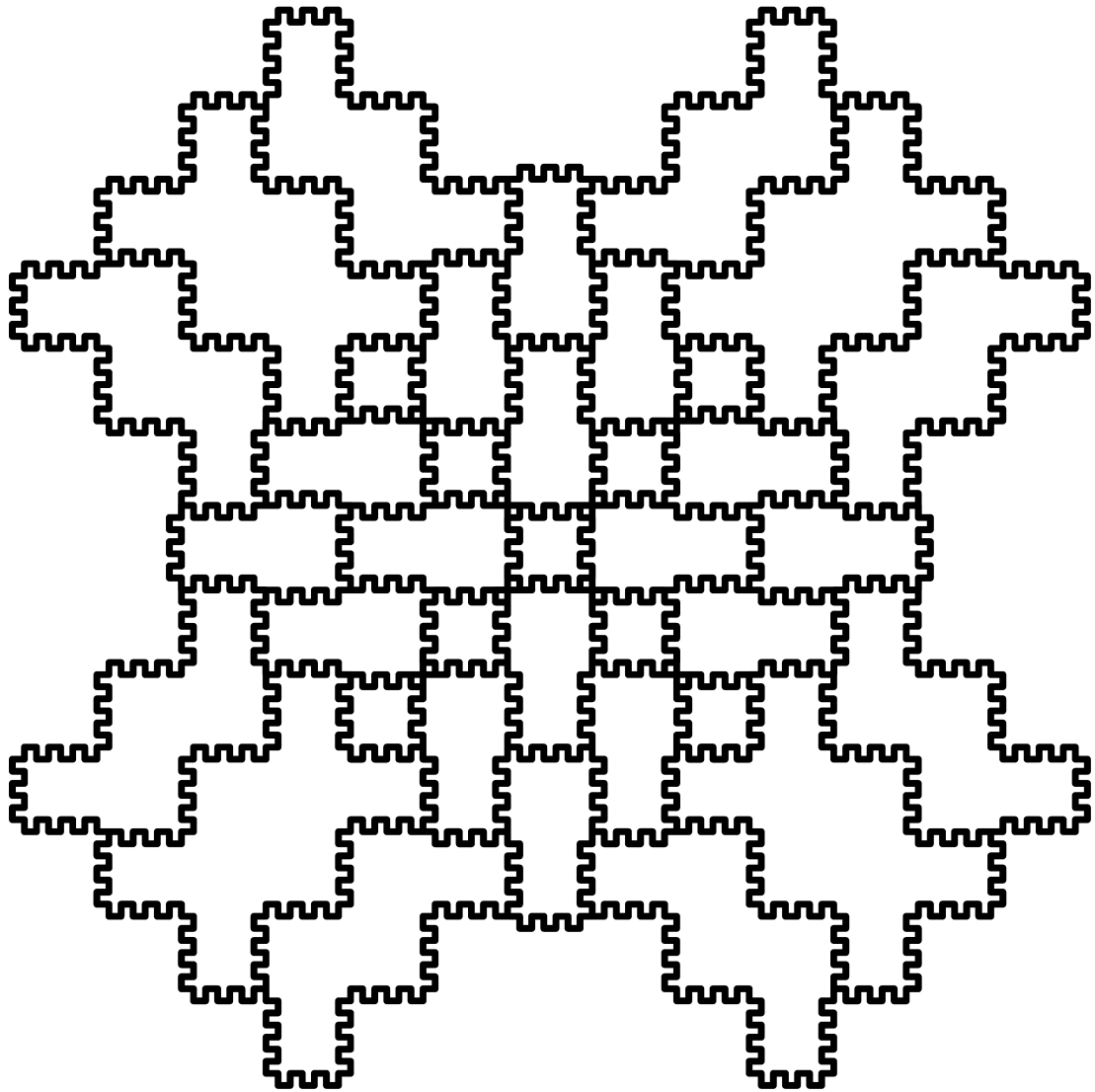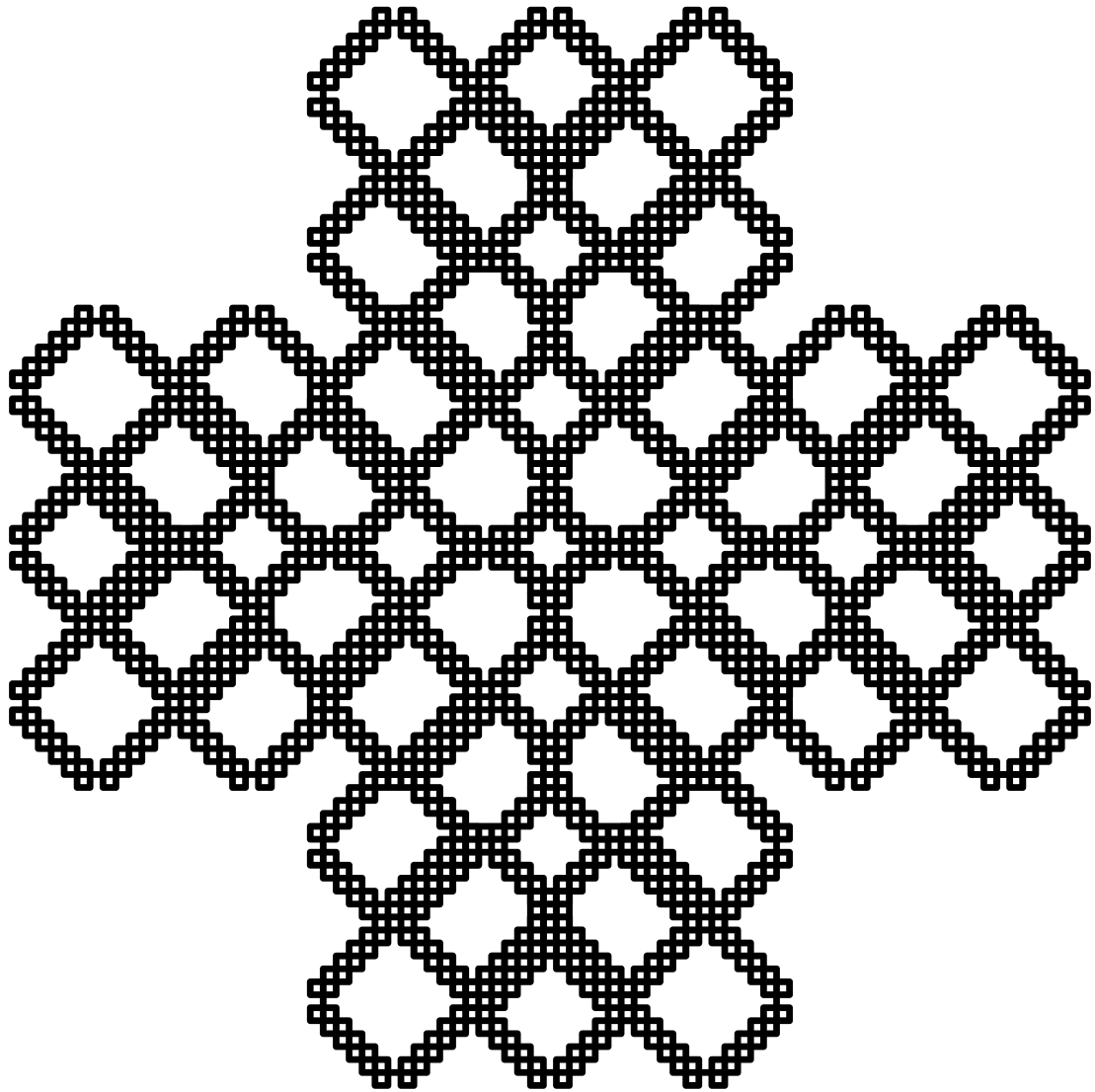Figure 81: The Cloitre walk of the sequence of parities of $B_{\sqrt{8588}}$.

Figure 82: The Cloitre walk of the sequence of parities of $B_{\sqrt{8595}}$.

Figure 83: The Cloitre walk of the sequence of parities of $B_{\sqrt{8606}}$.

Figure 84: The Cloitre walk of the sequence of parities of $B_{\sqrt{8705}}$.

Figure 85: The Cloitre walk of the sequence of parities of $B_{\sqrt{8725}}$.

Figure 86: The Cloitre walk of the sequence of parities of $B_{\sqrt{8764}}$.
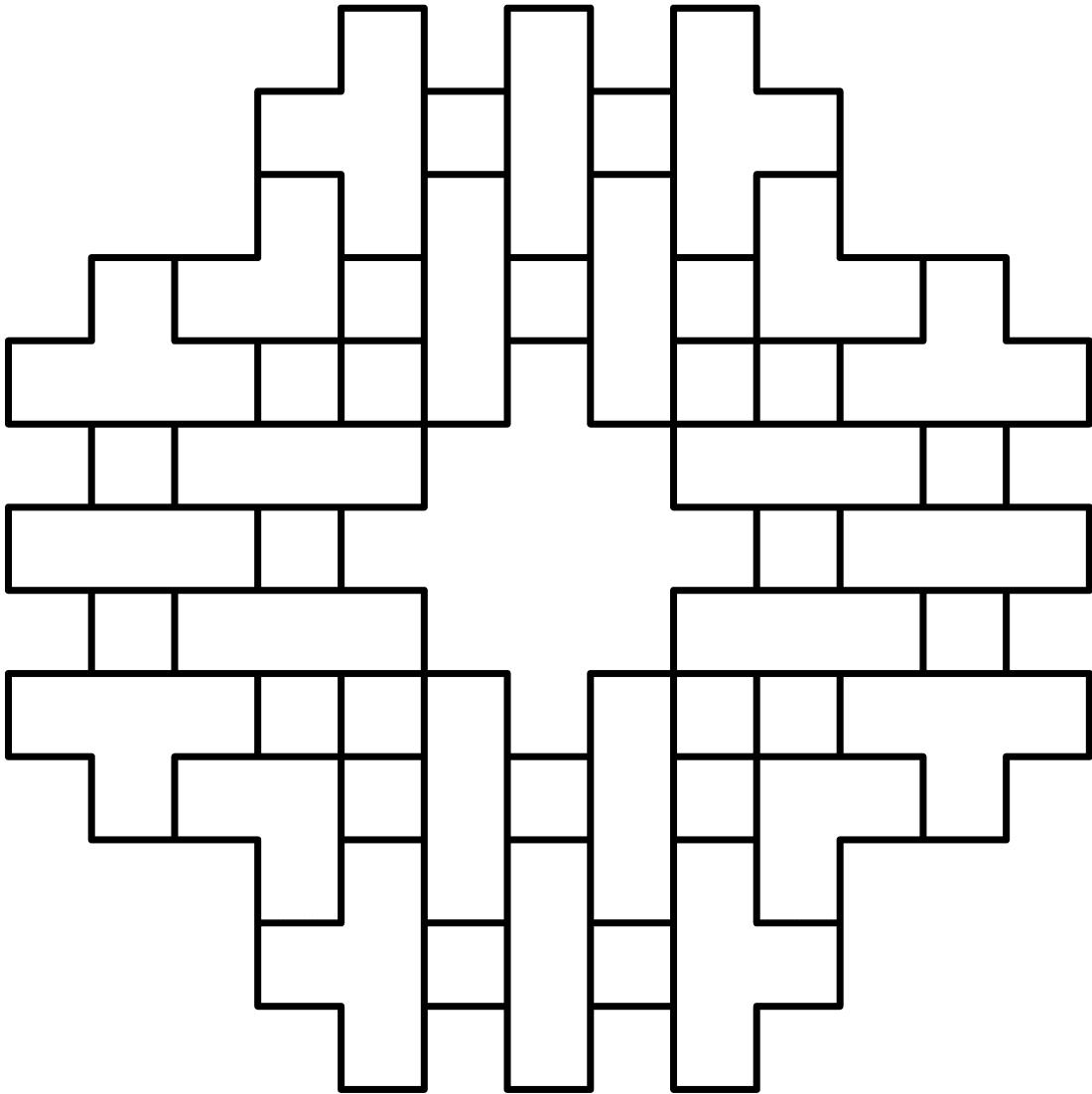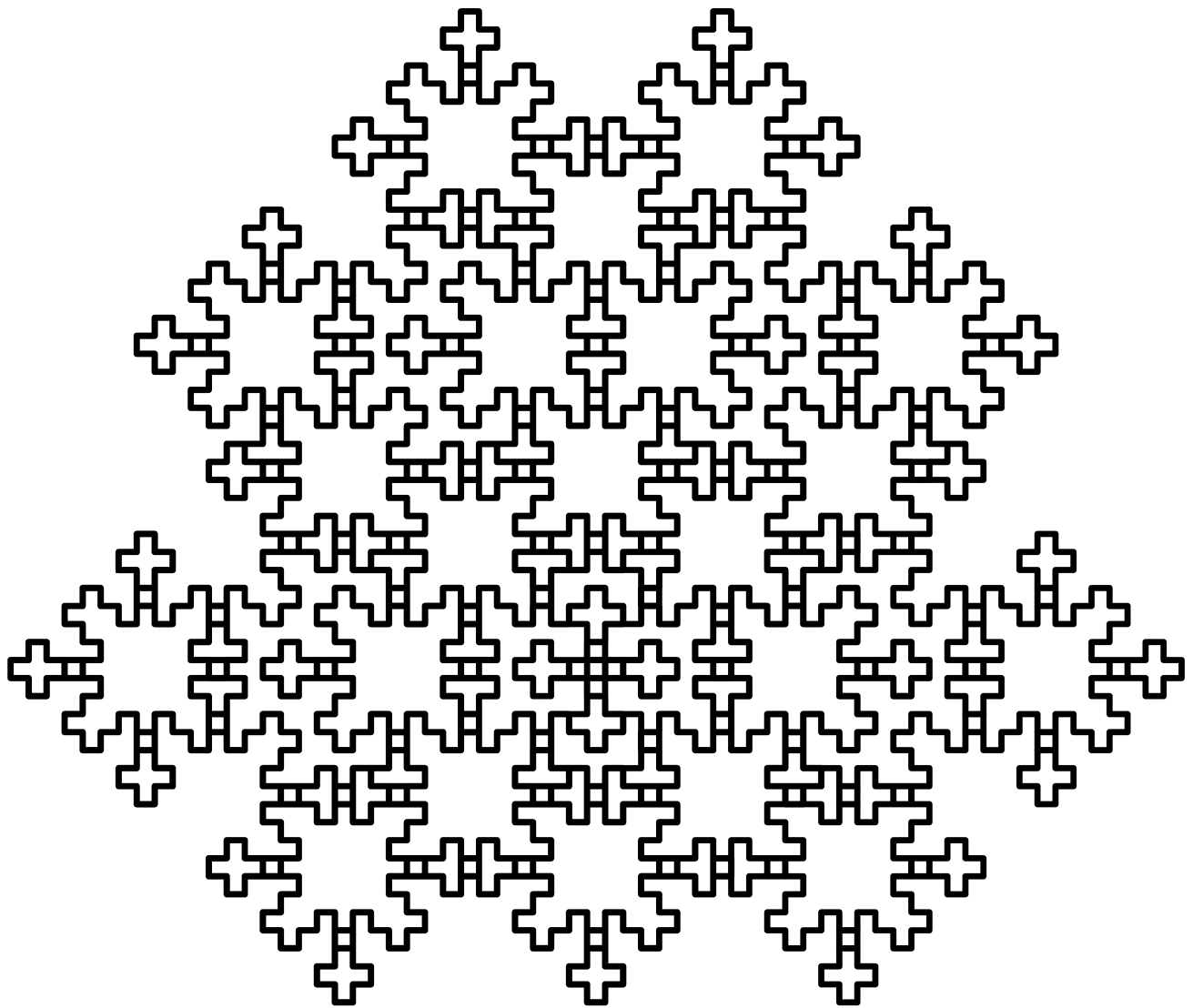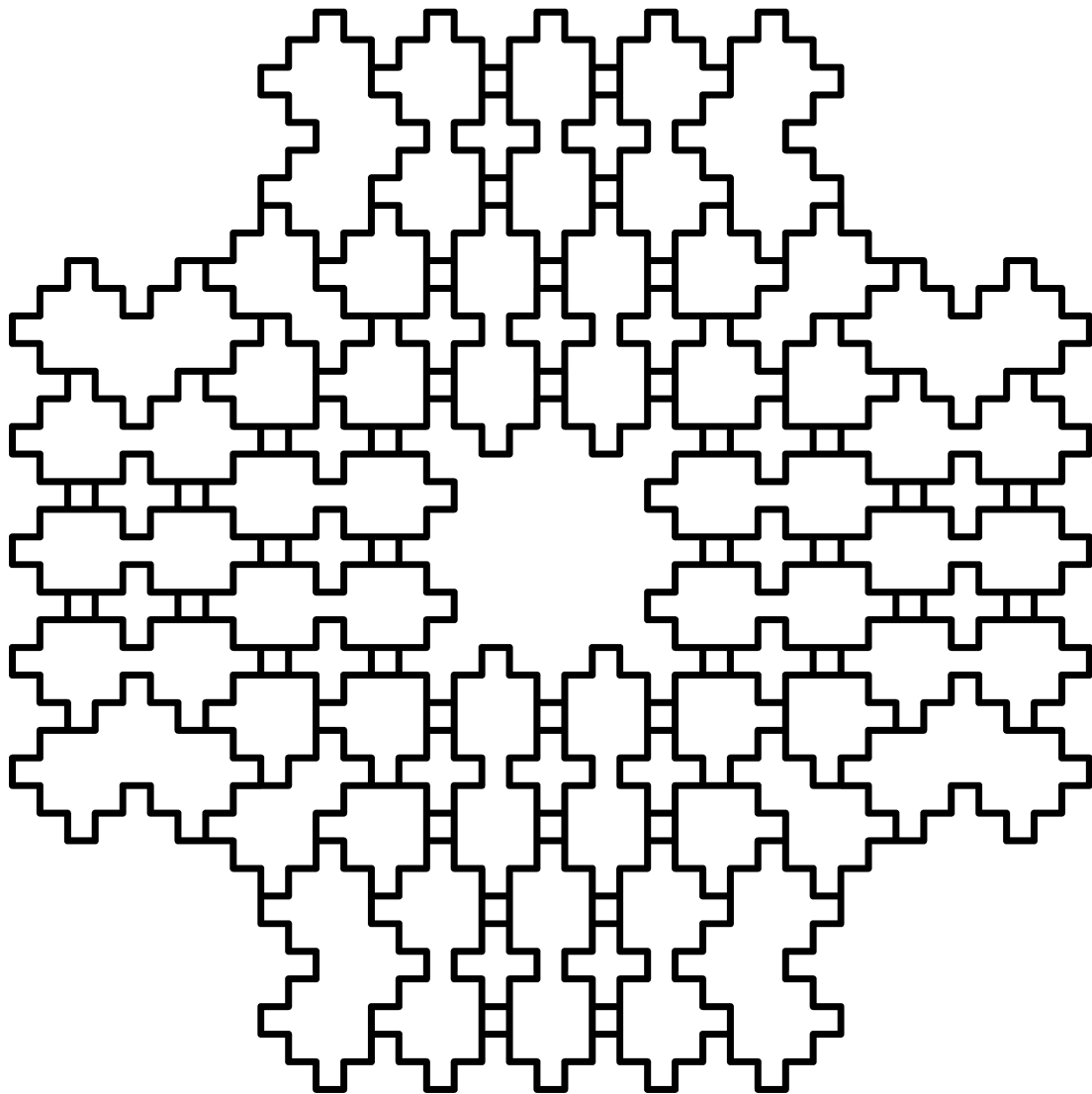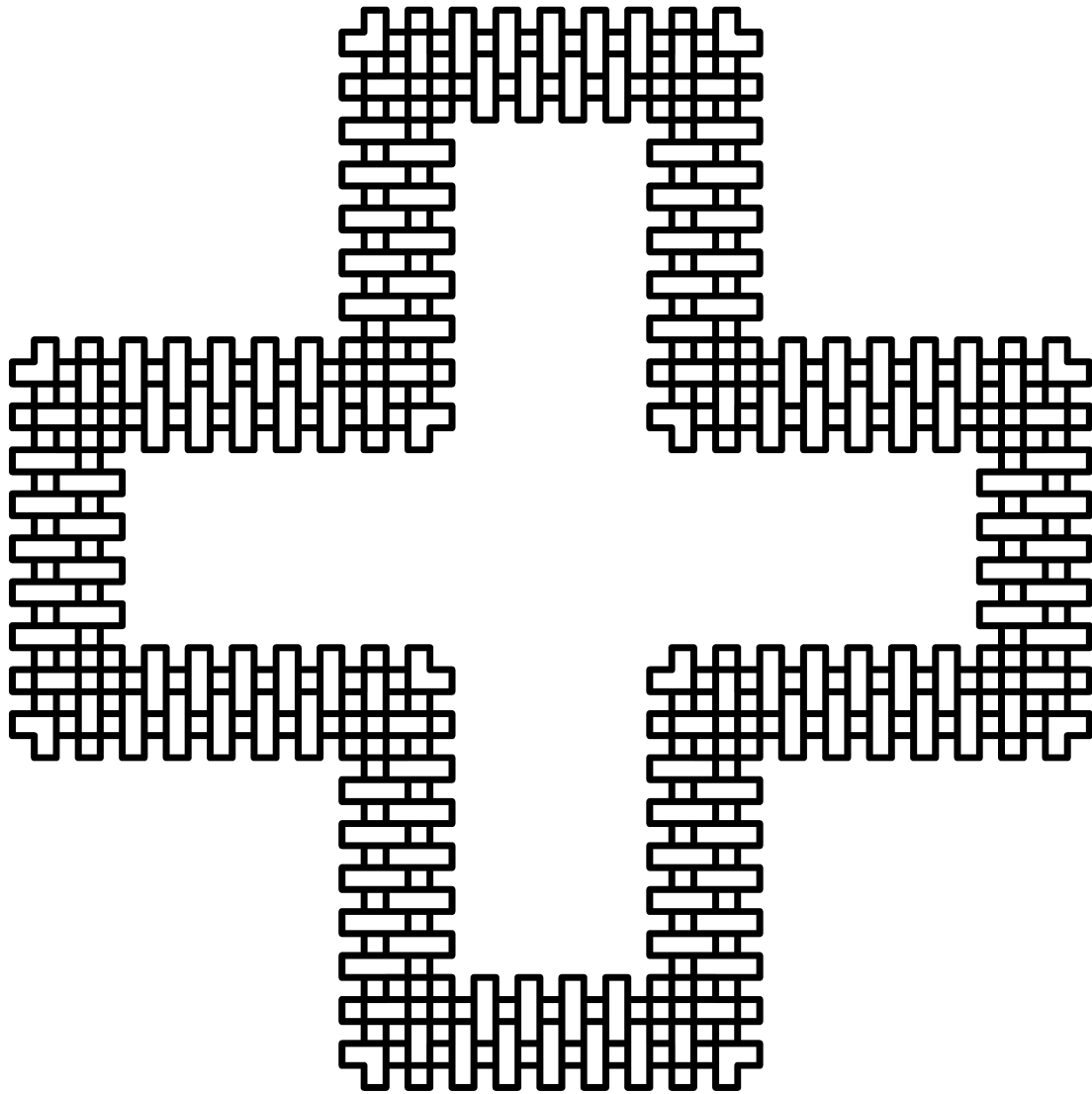
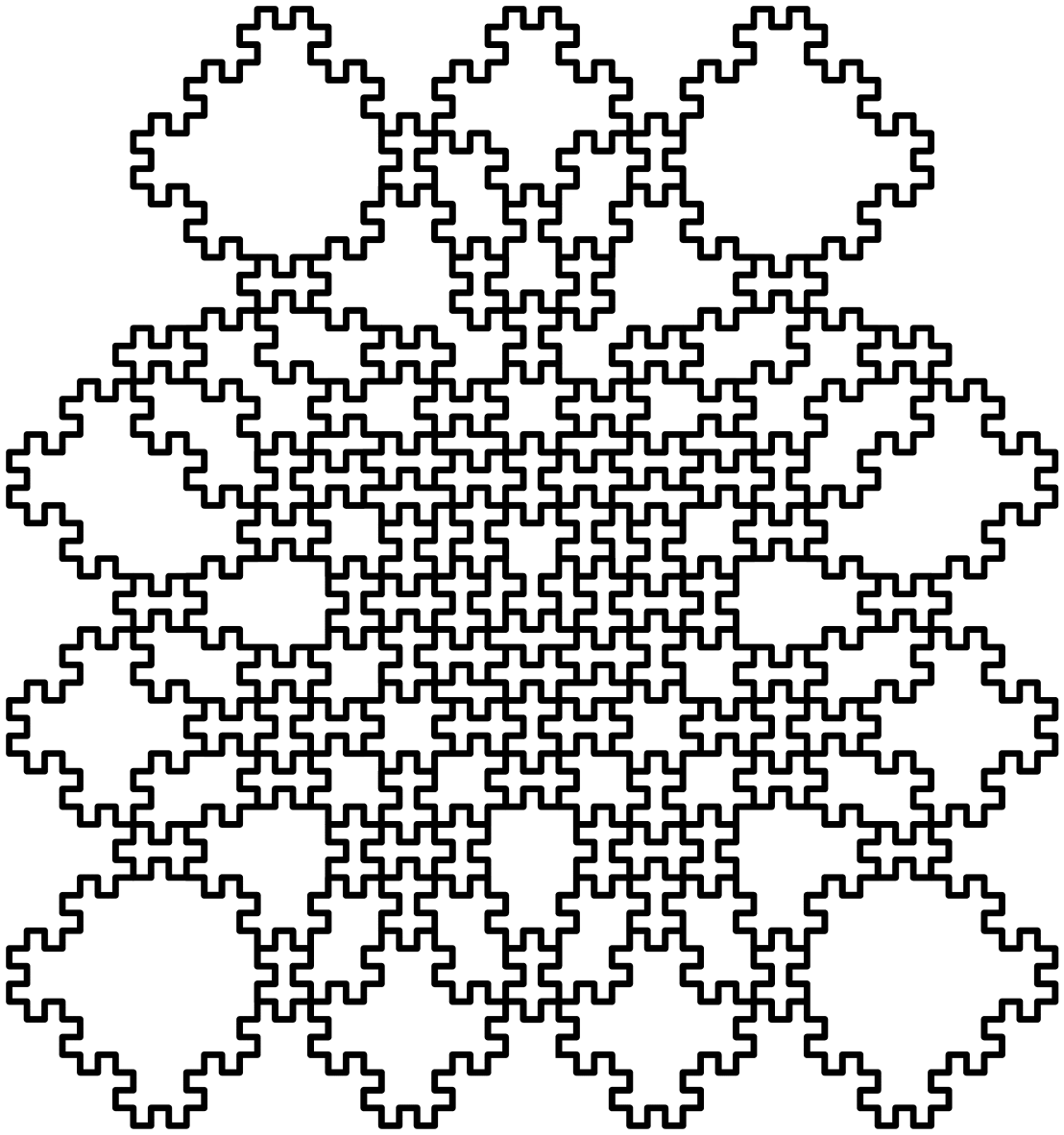Figure 87: The Cloitre walk of the sequence of parities of $B_{\sqrt{8810}}$.
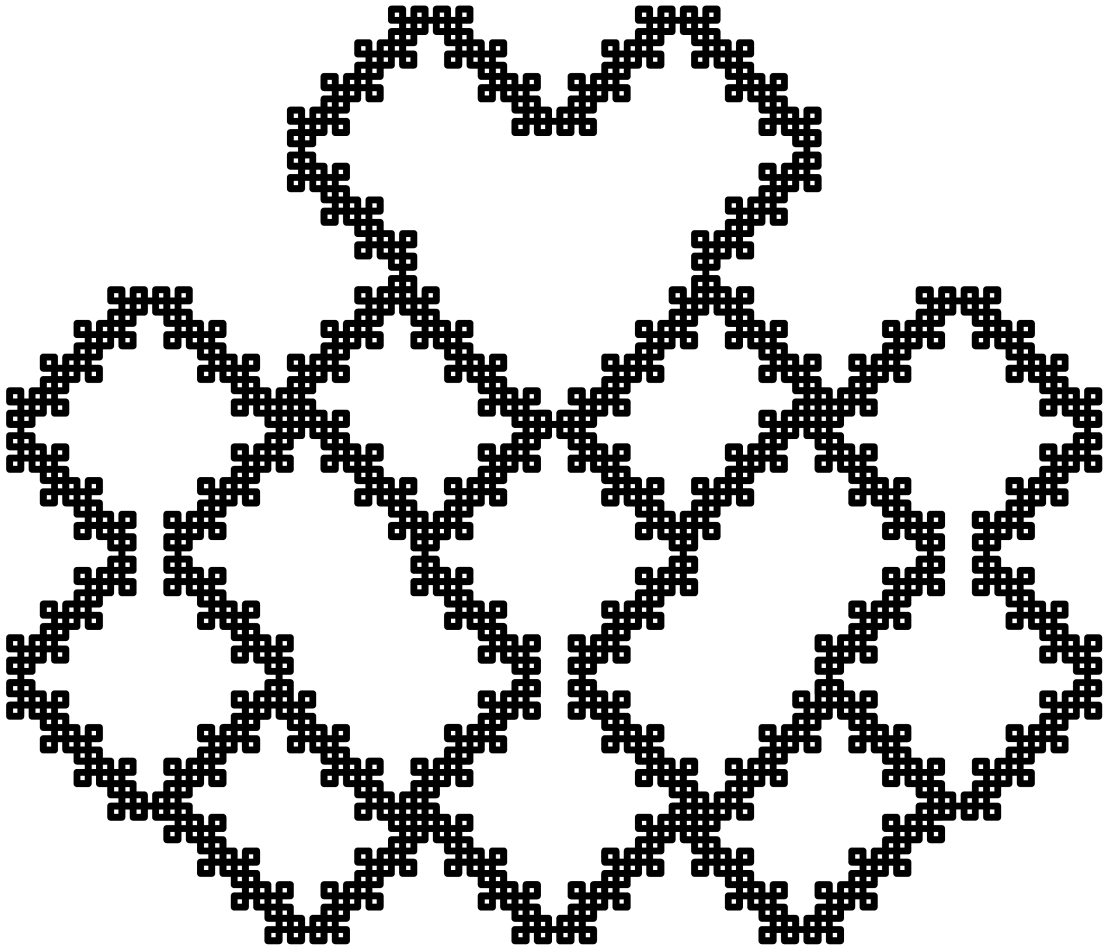
Figure 88: The Cloitre walk of the sequence of parities of $B_{\sqrt{8892}}$.
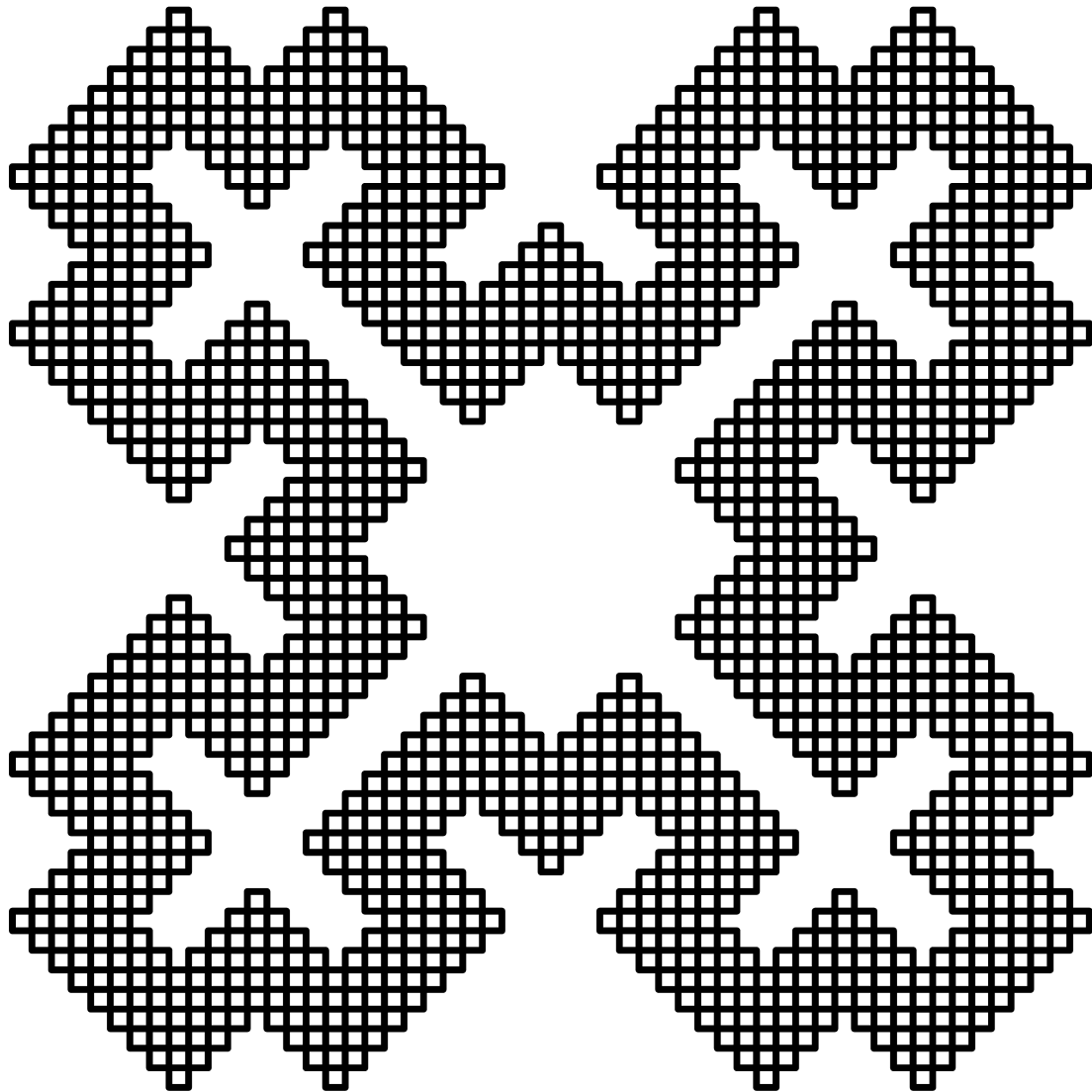
Figure 89: The Cloitre walk of the sequence of parities of $B_{\sqrt{8893}}$.

Figure 90: The Cloitre walk of the sequence of parities of $B_{\sqrt{8905}}$.

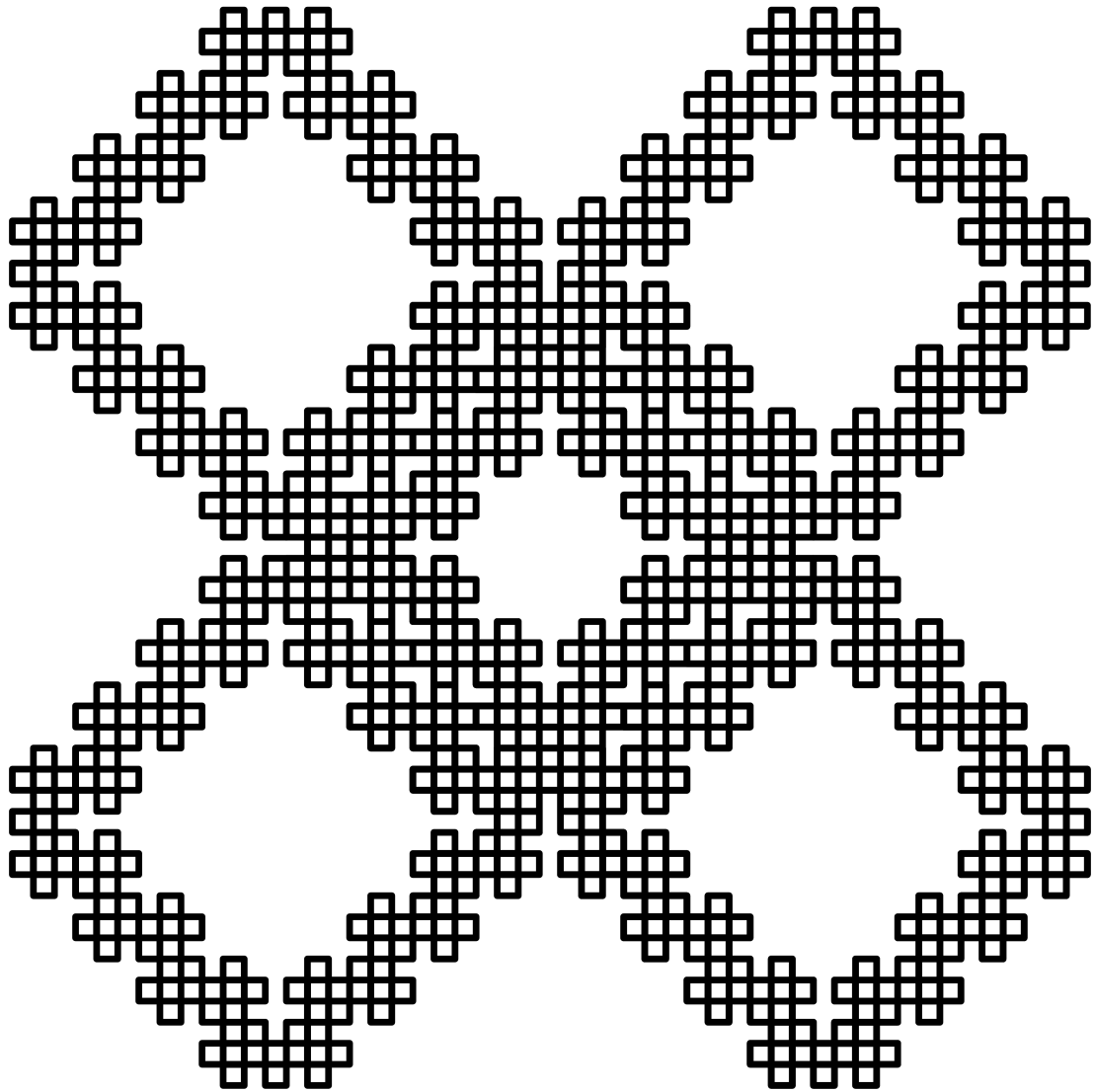Figure 91: The Cloitre walk of the sequence of parities of $B_{\sqrt{8945}}$.

Figure 92: The Cloitre walk of the sequence of parities of $B_{\sqrt{8972}}$.

Figure 93: The Cloitre walk of the sequence of parities of $B_{\sqrt{8973}}$.

Figure 94: The Cloitre walk of the sequence of parities of $B_{\sqrt{9032}}$.

Figure 95: The Cloitre walk of the sequence of parities of $B_{\sqrt{9065}}$.

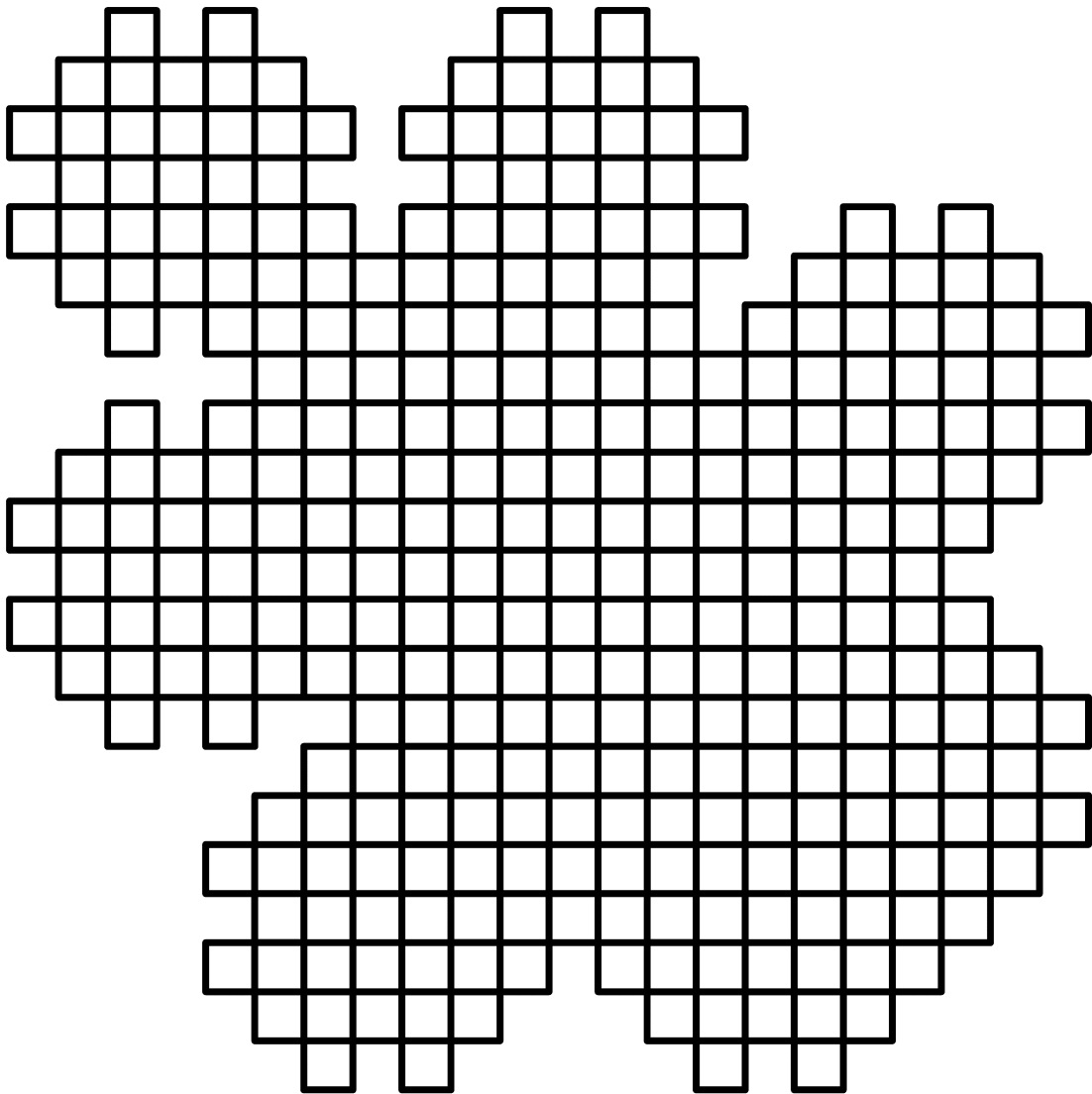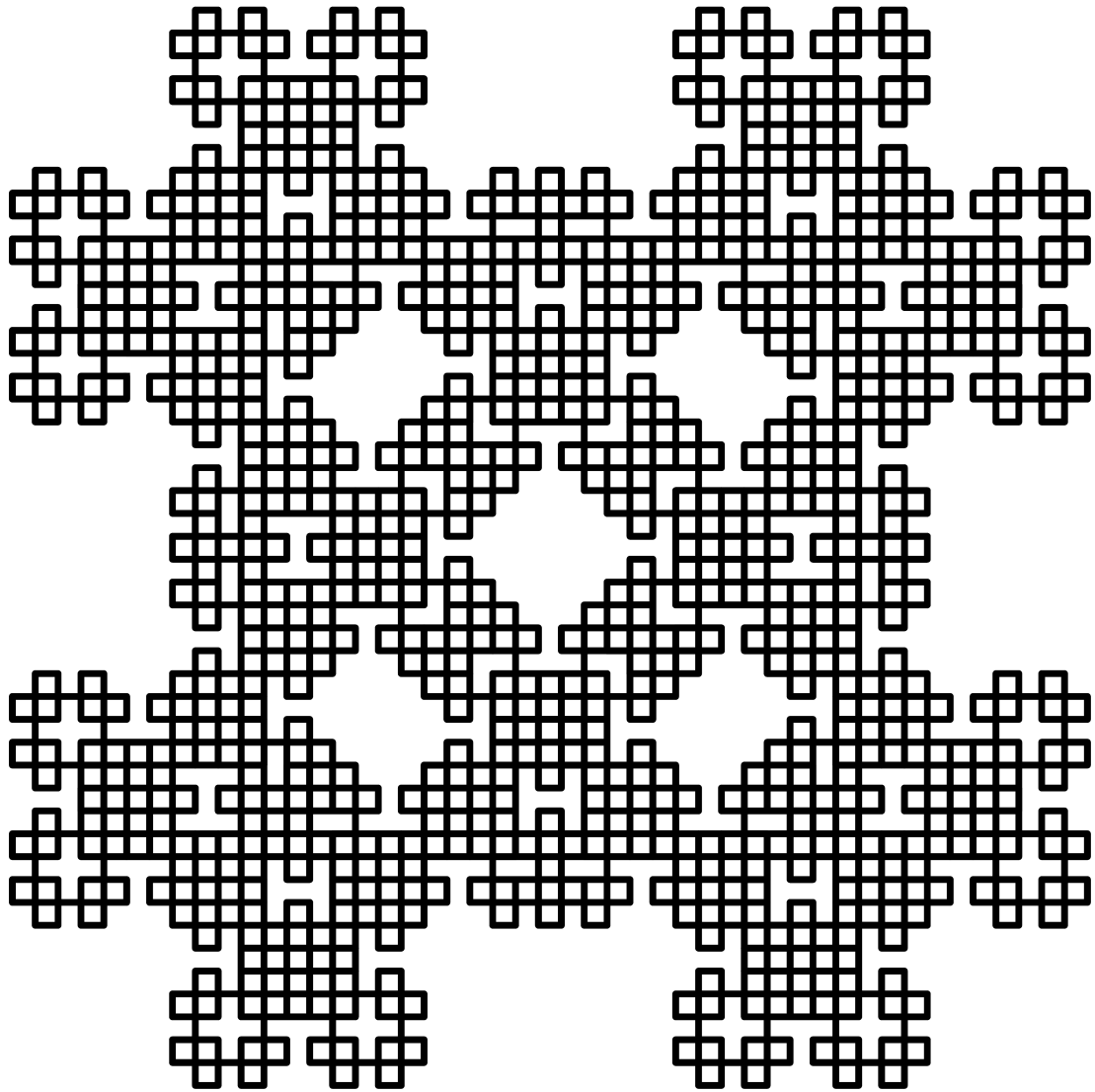Figure 96: The Cloitre walk of the sequence of parities of $B_{\sqrt{9182}}$.

Figure 97: The Cloitre walk of the sequence of parities of $B_{\sqrt{9329}}$.
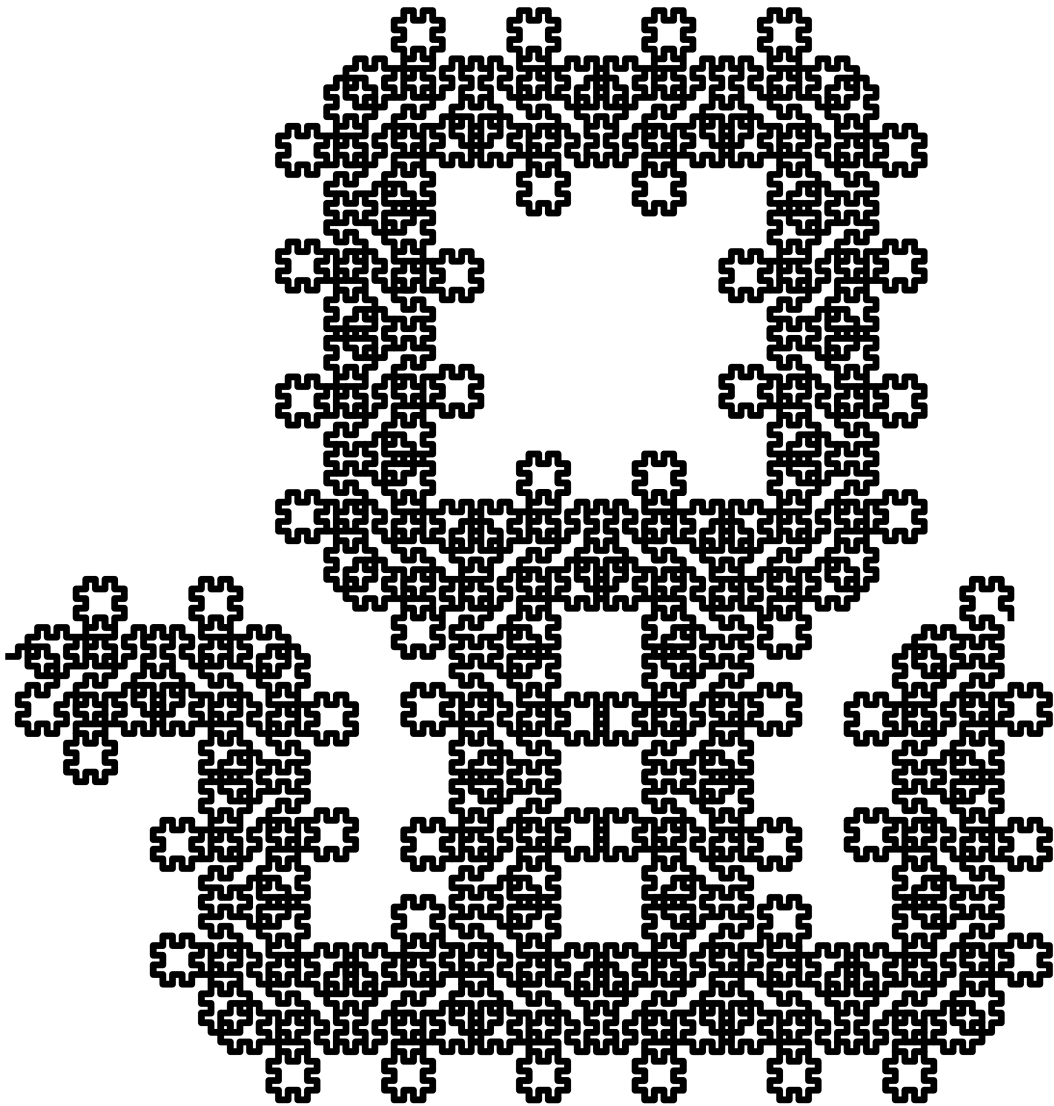
Figure 98: The Cloitre walk of the sequence of parities of $B_{\sqrt{9542}}$.

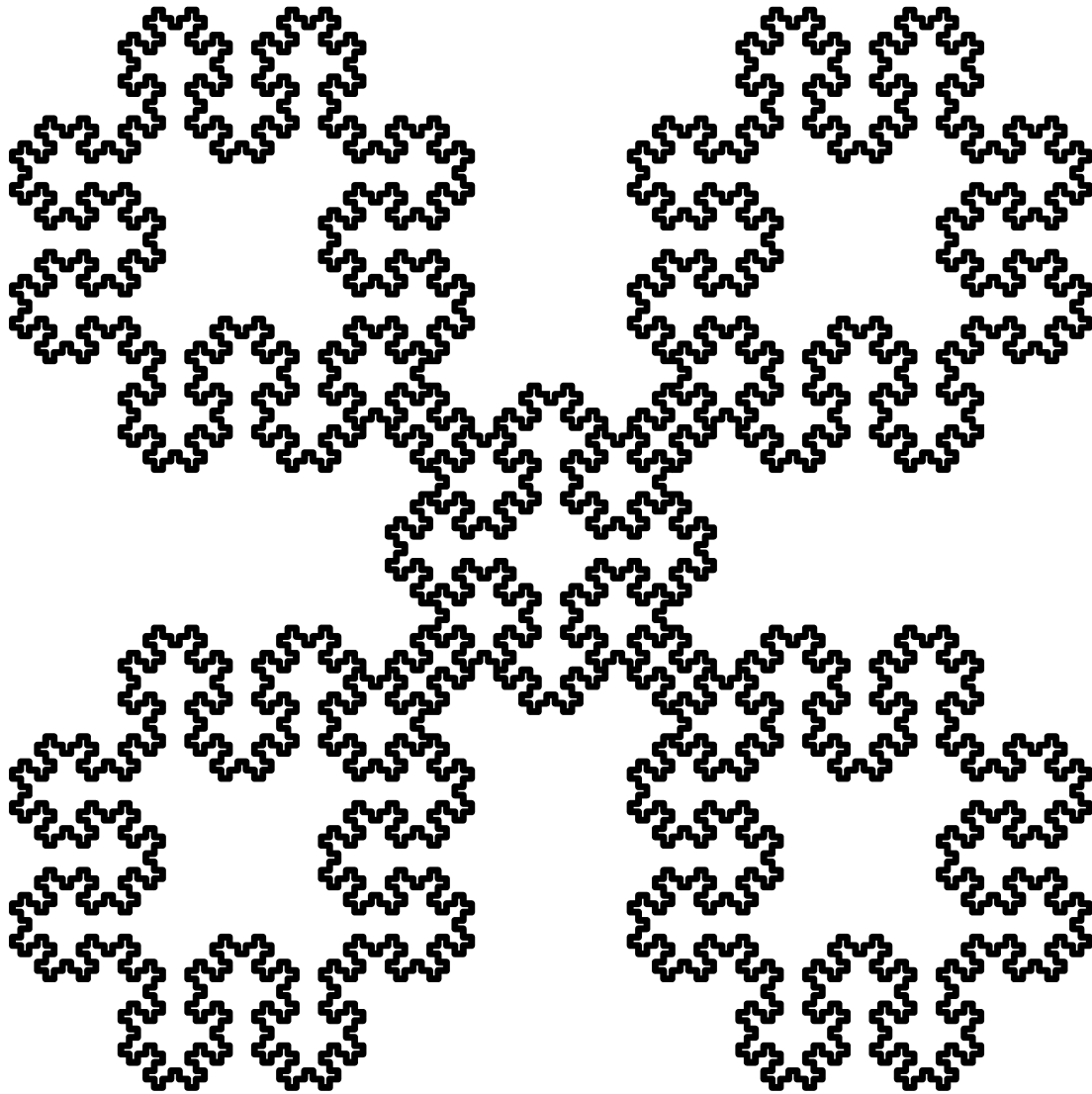Figure 99: The Cloitre walk of the sequence of parities of $B_{\sqrt{9614}}$.
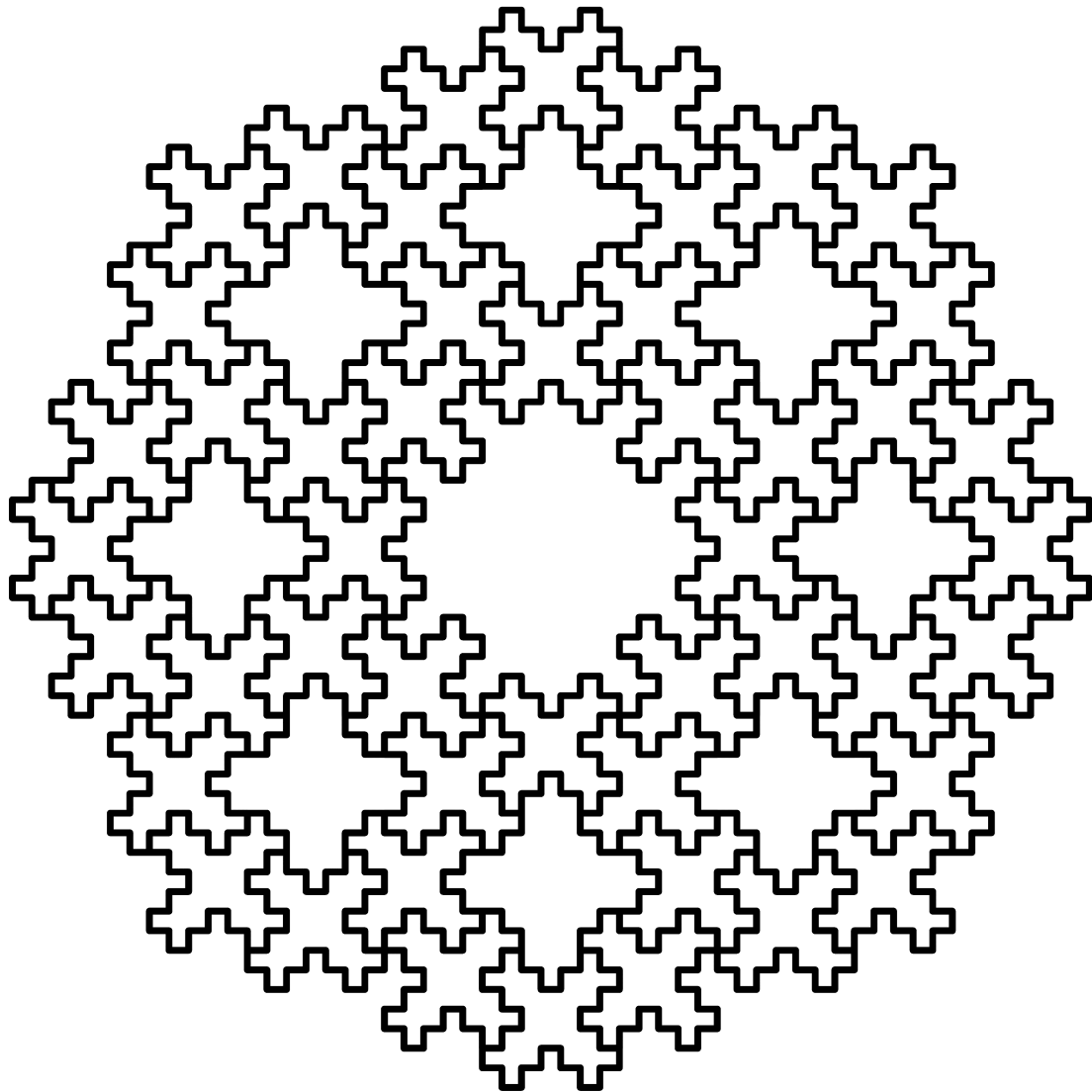
Figure 100: The Cloitre walk of the sequence of parities of $B_{\sqrt{9754}}$.

Figure 101: The Cloitre walk of the sequence of parities of $B_{\sqrt{9780}}$.

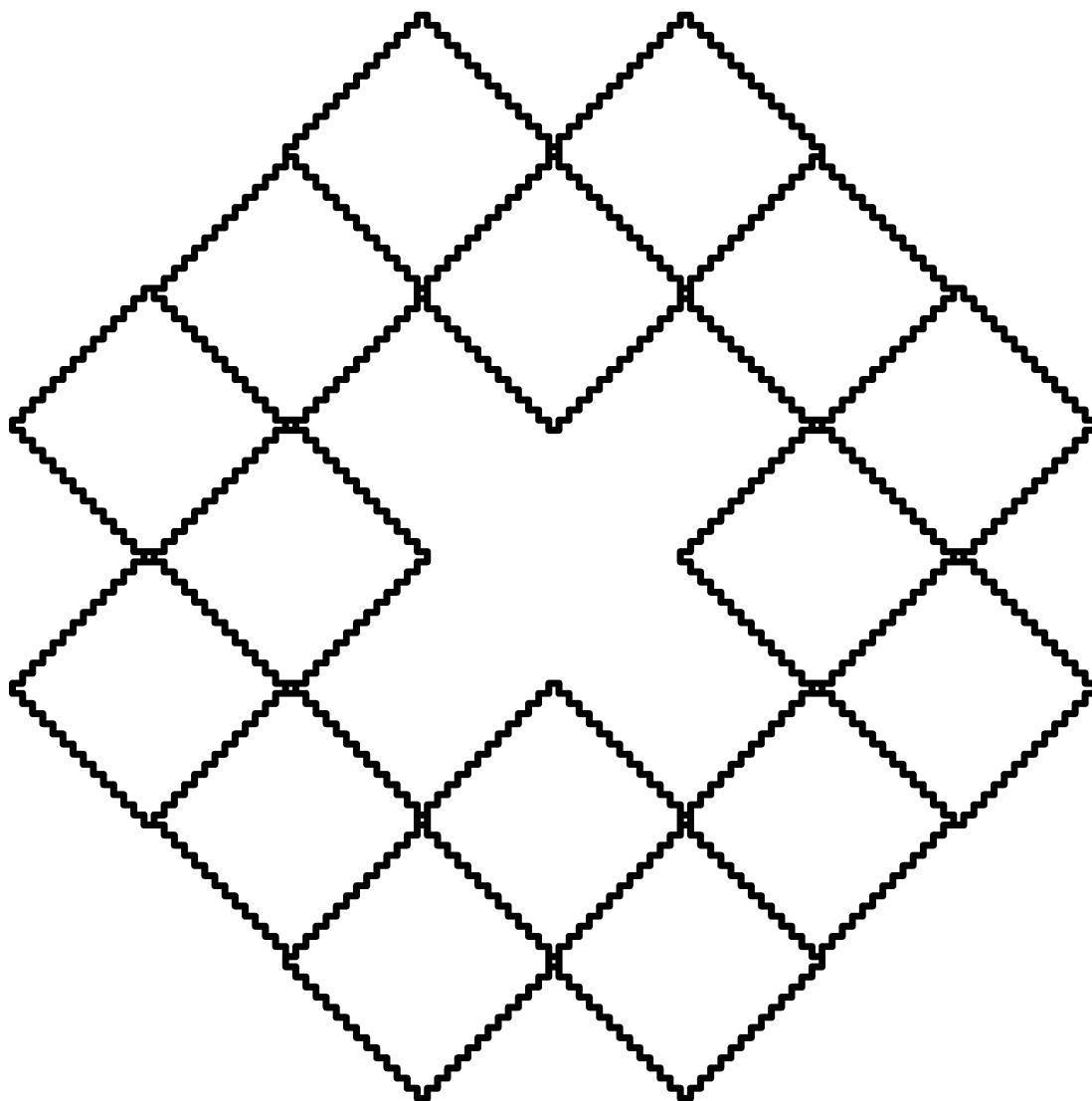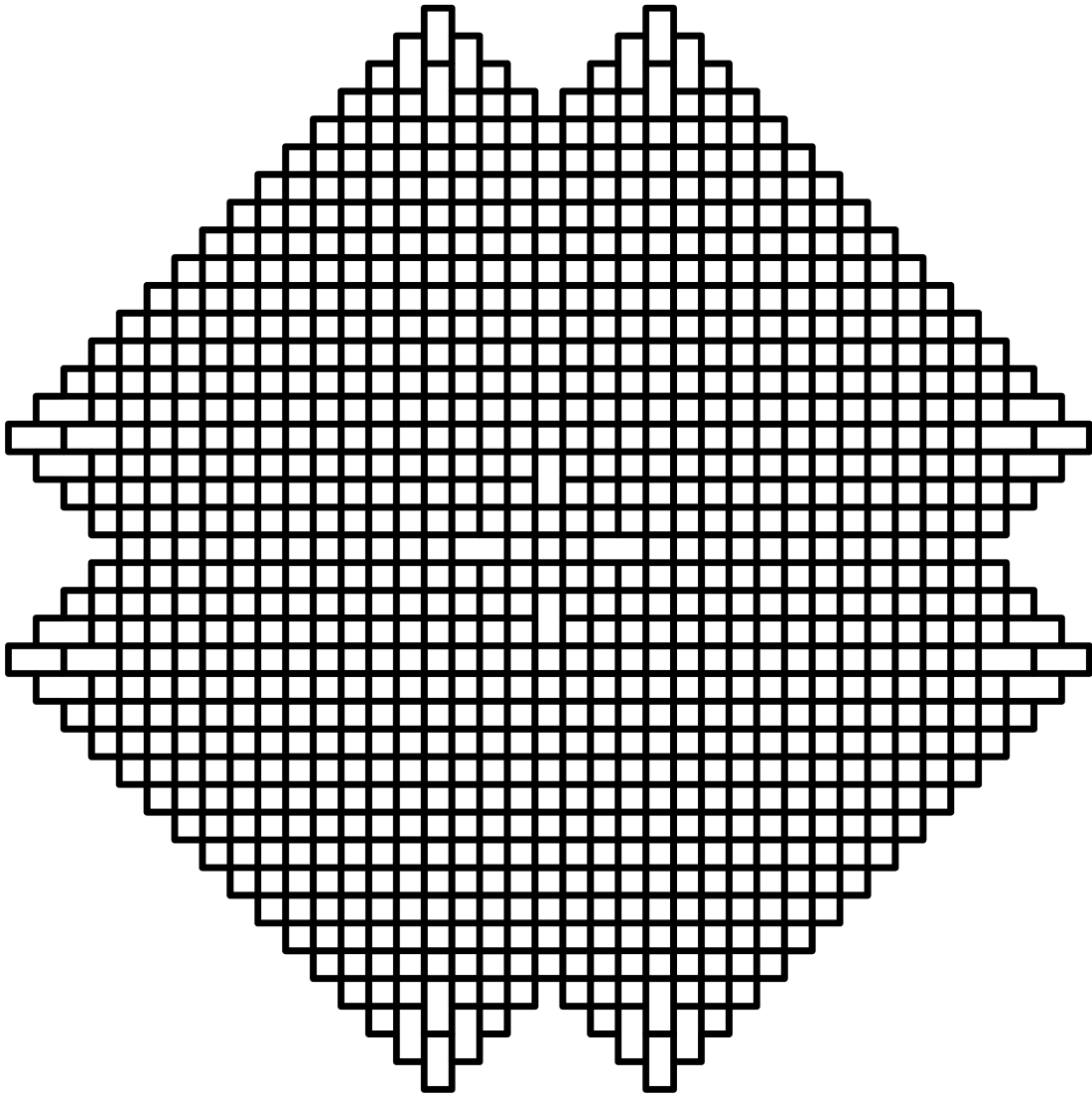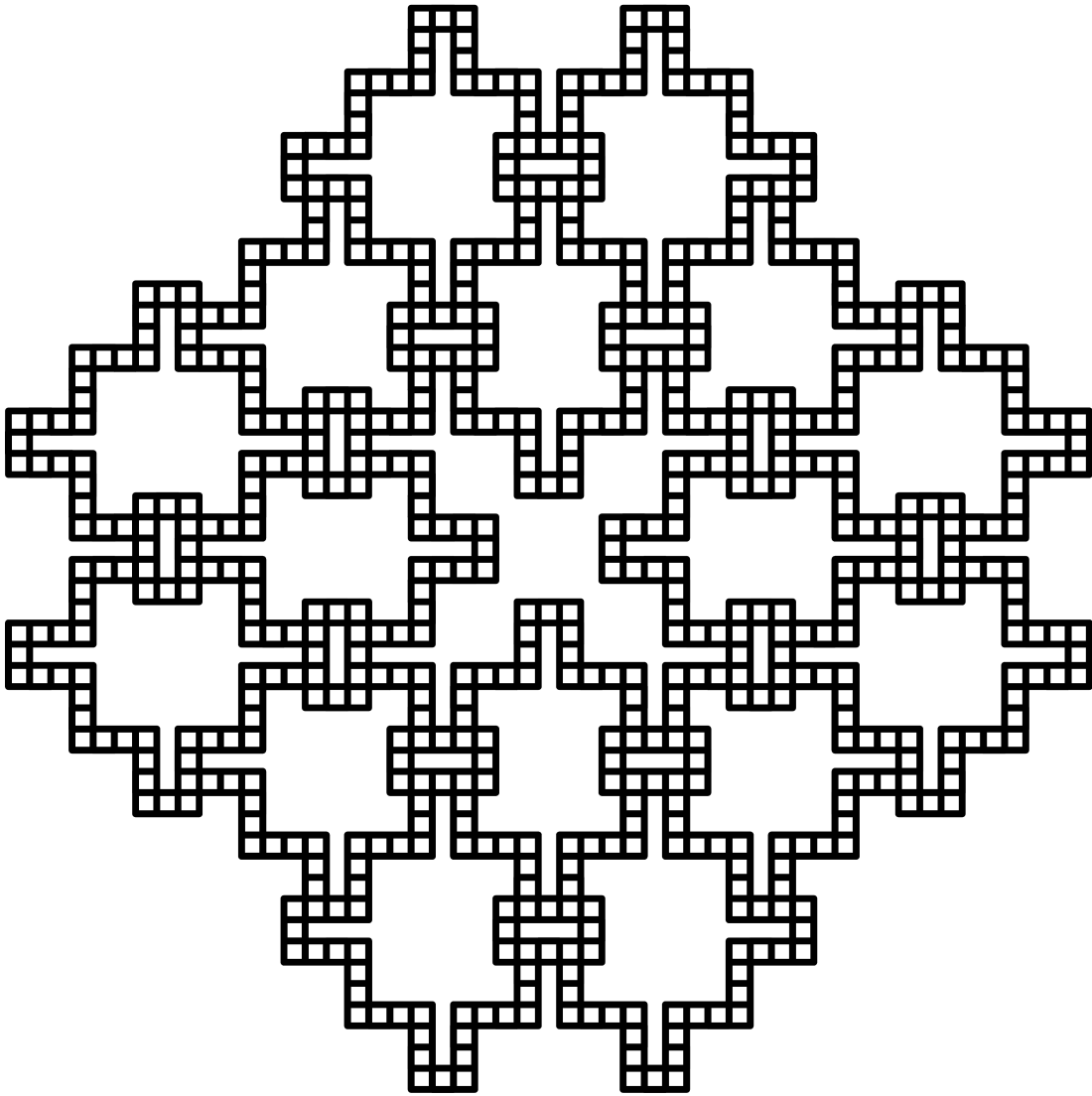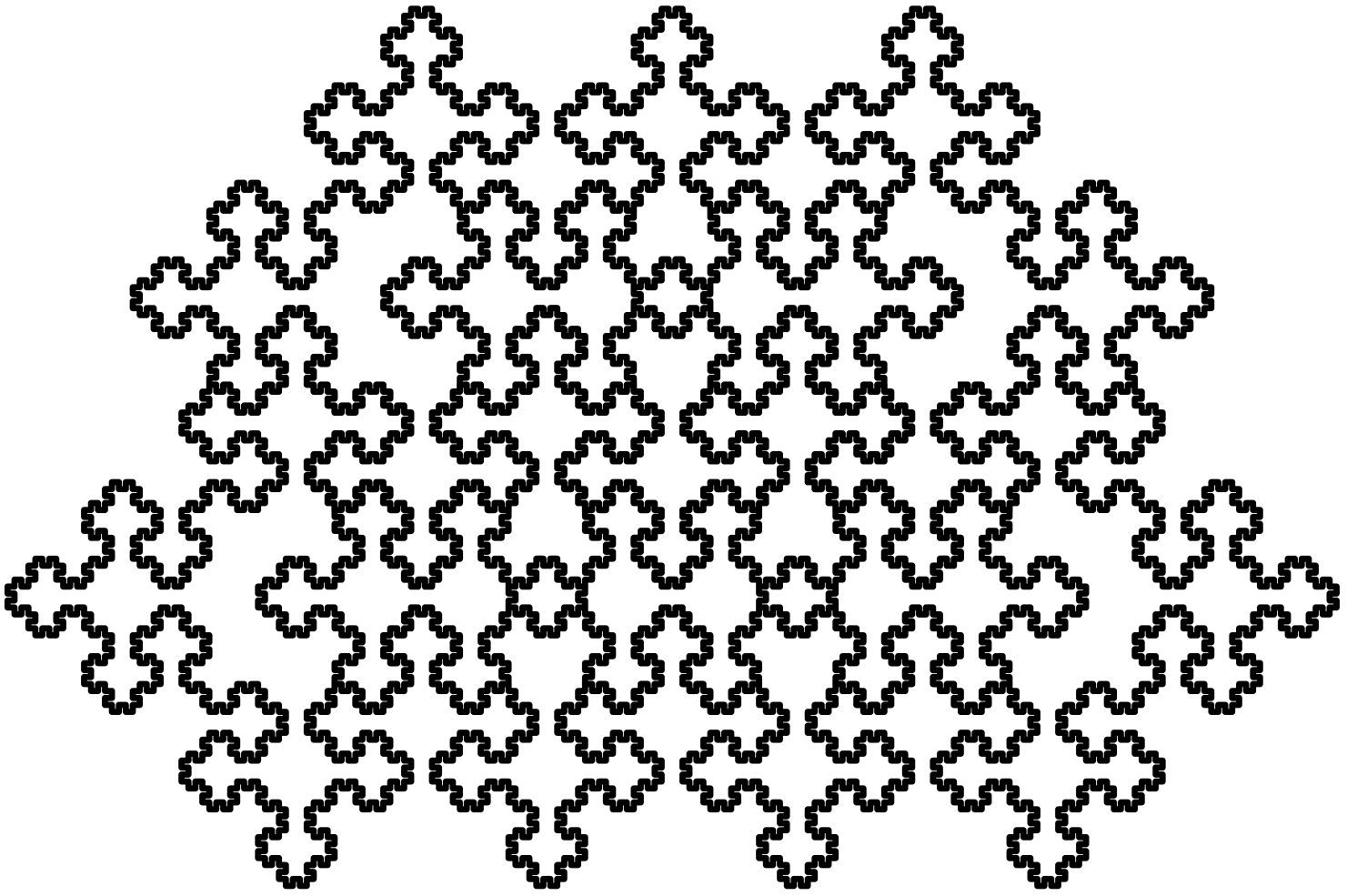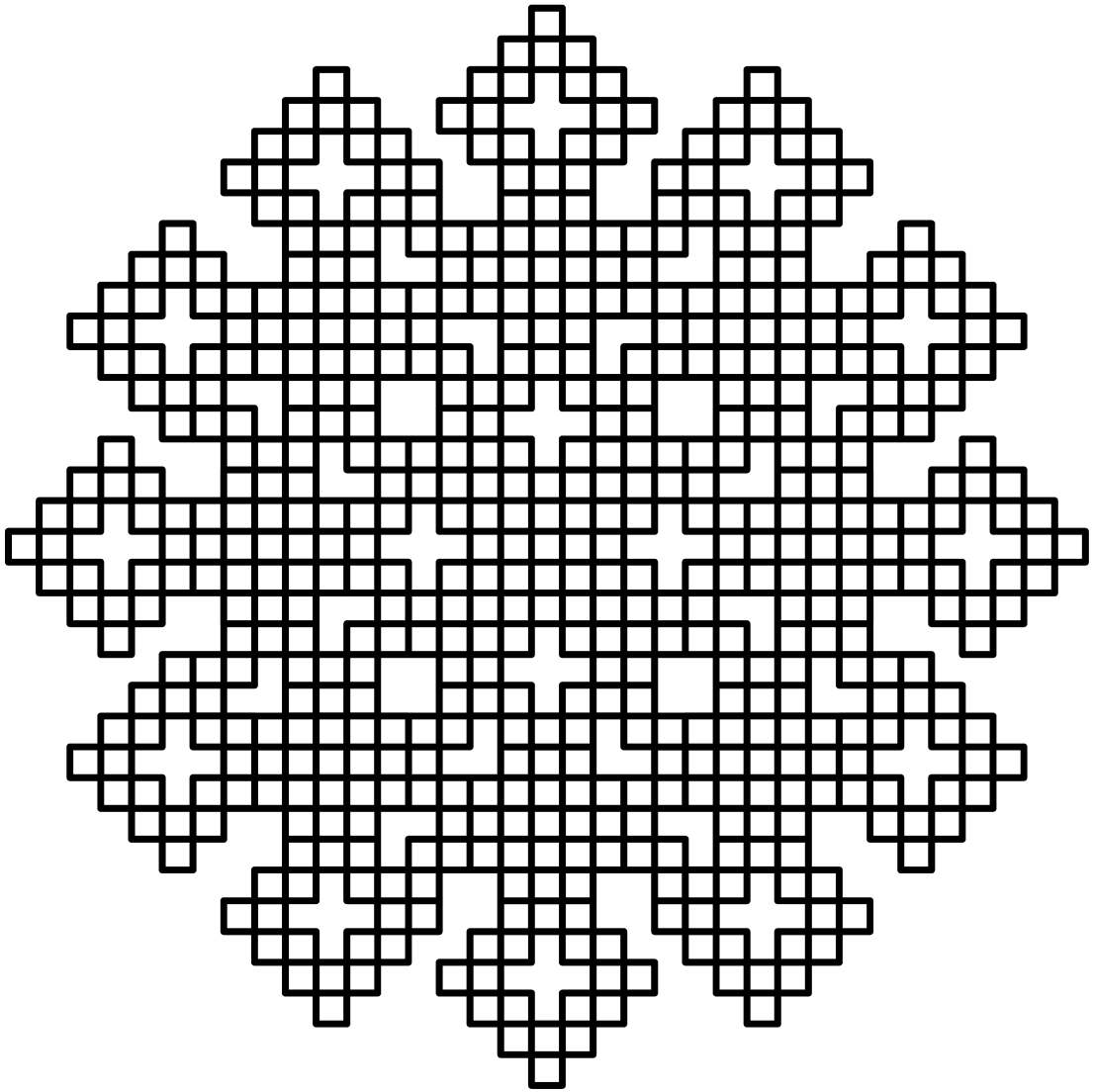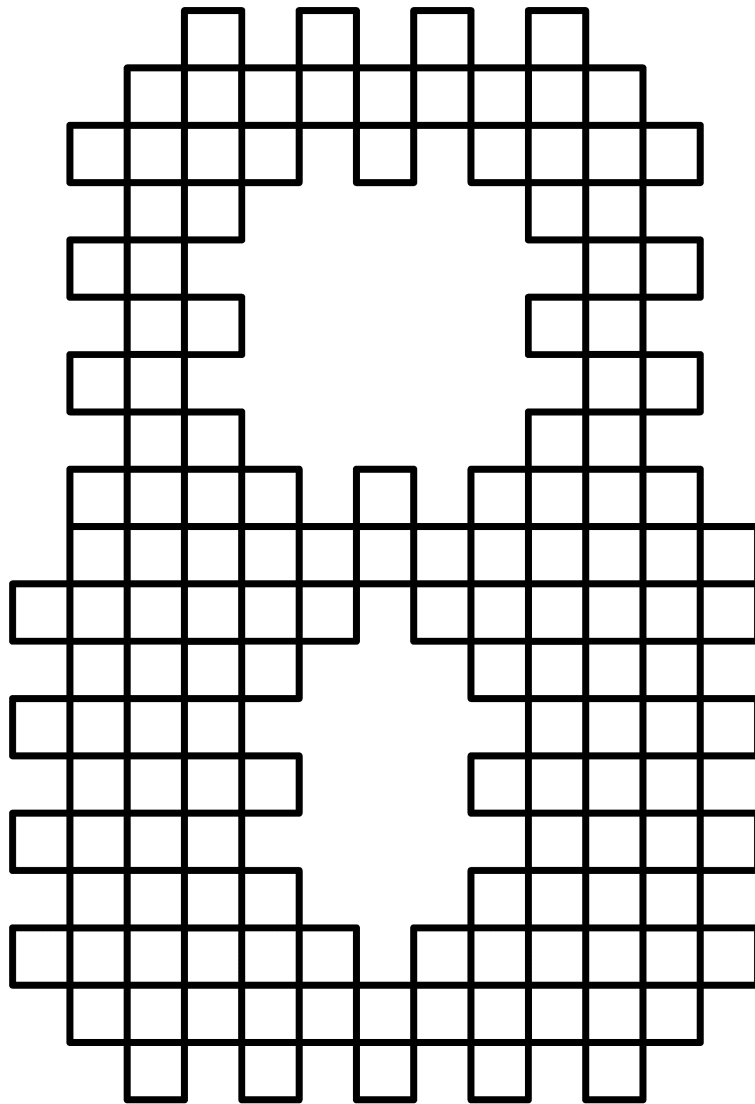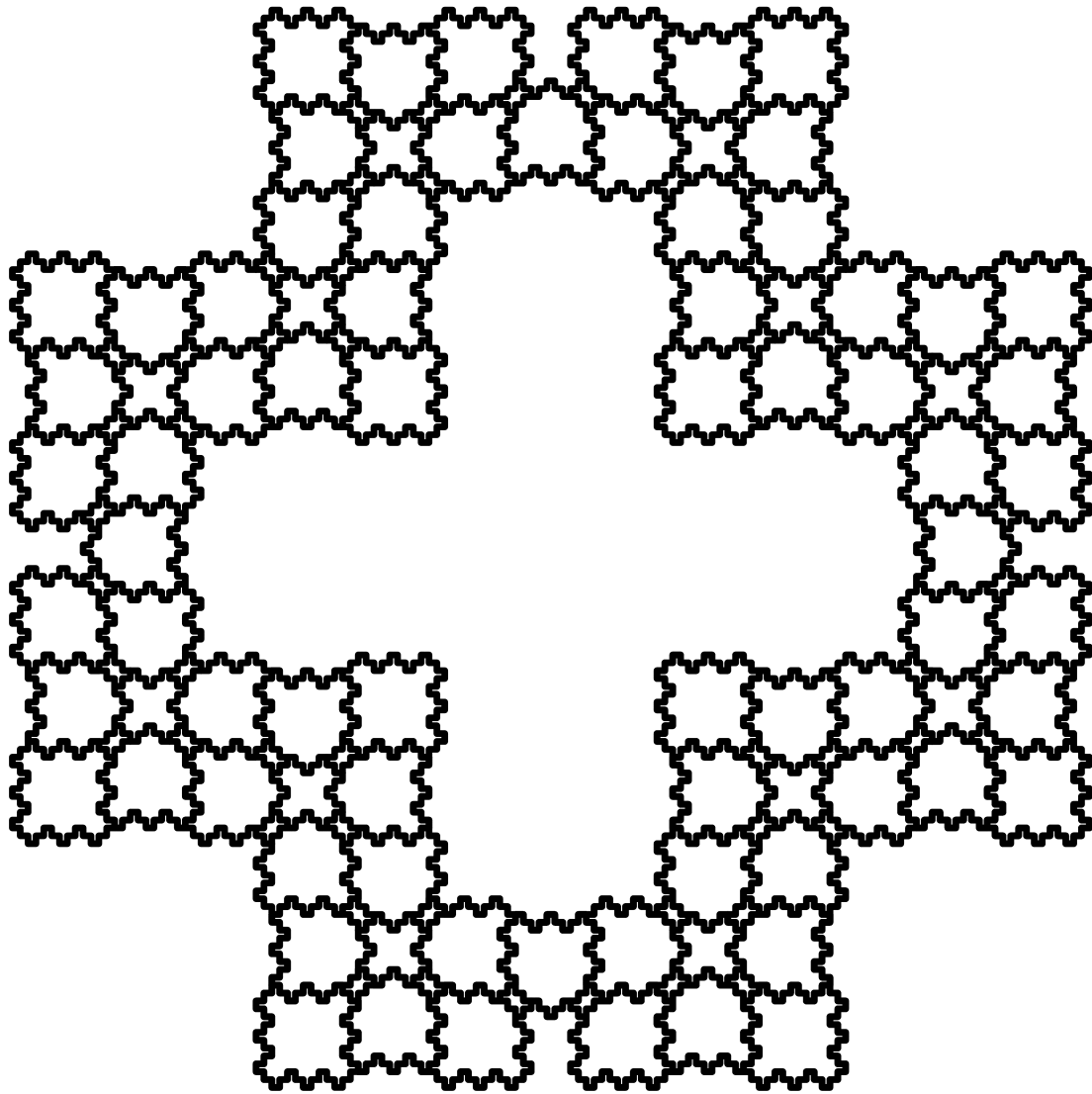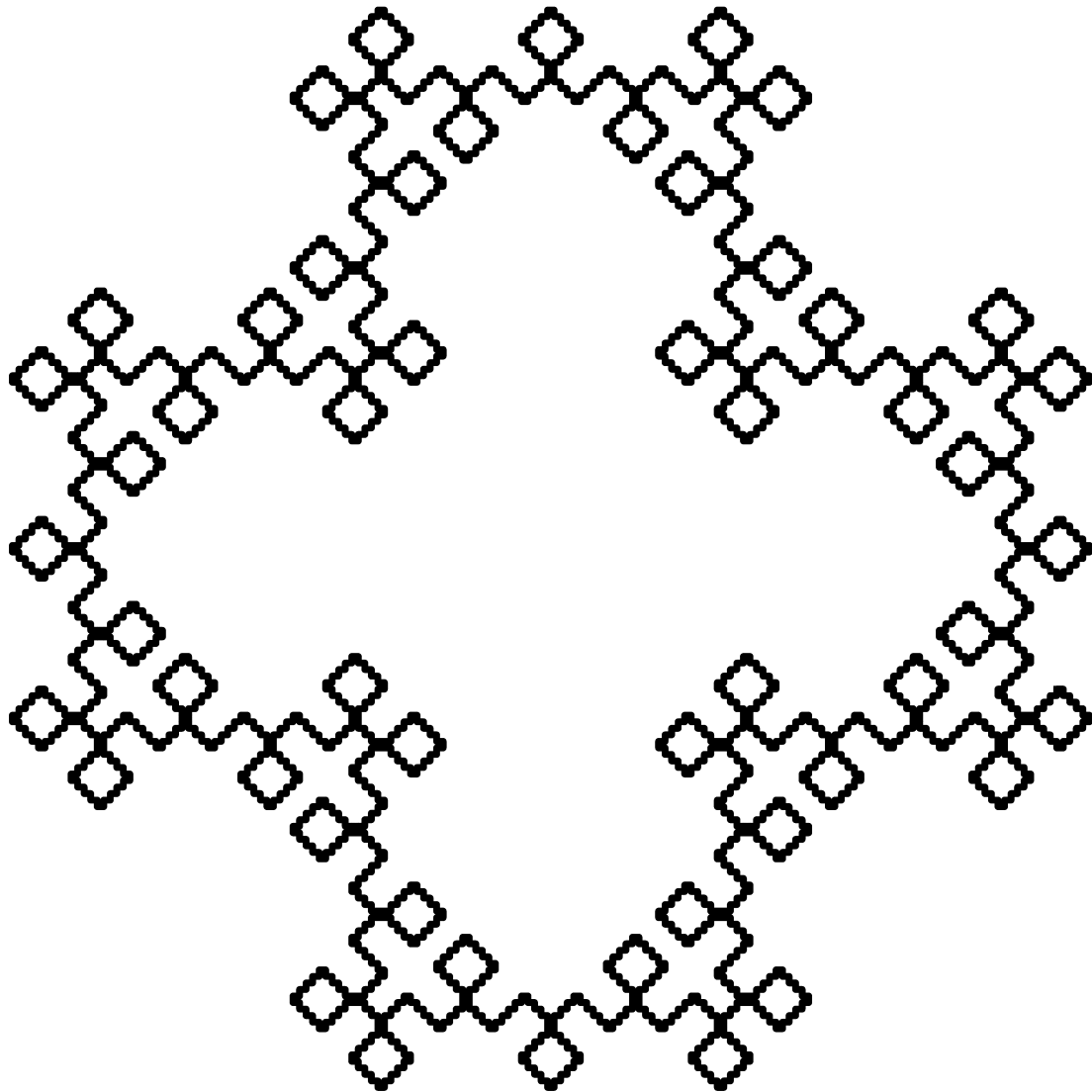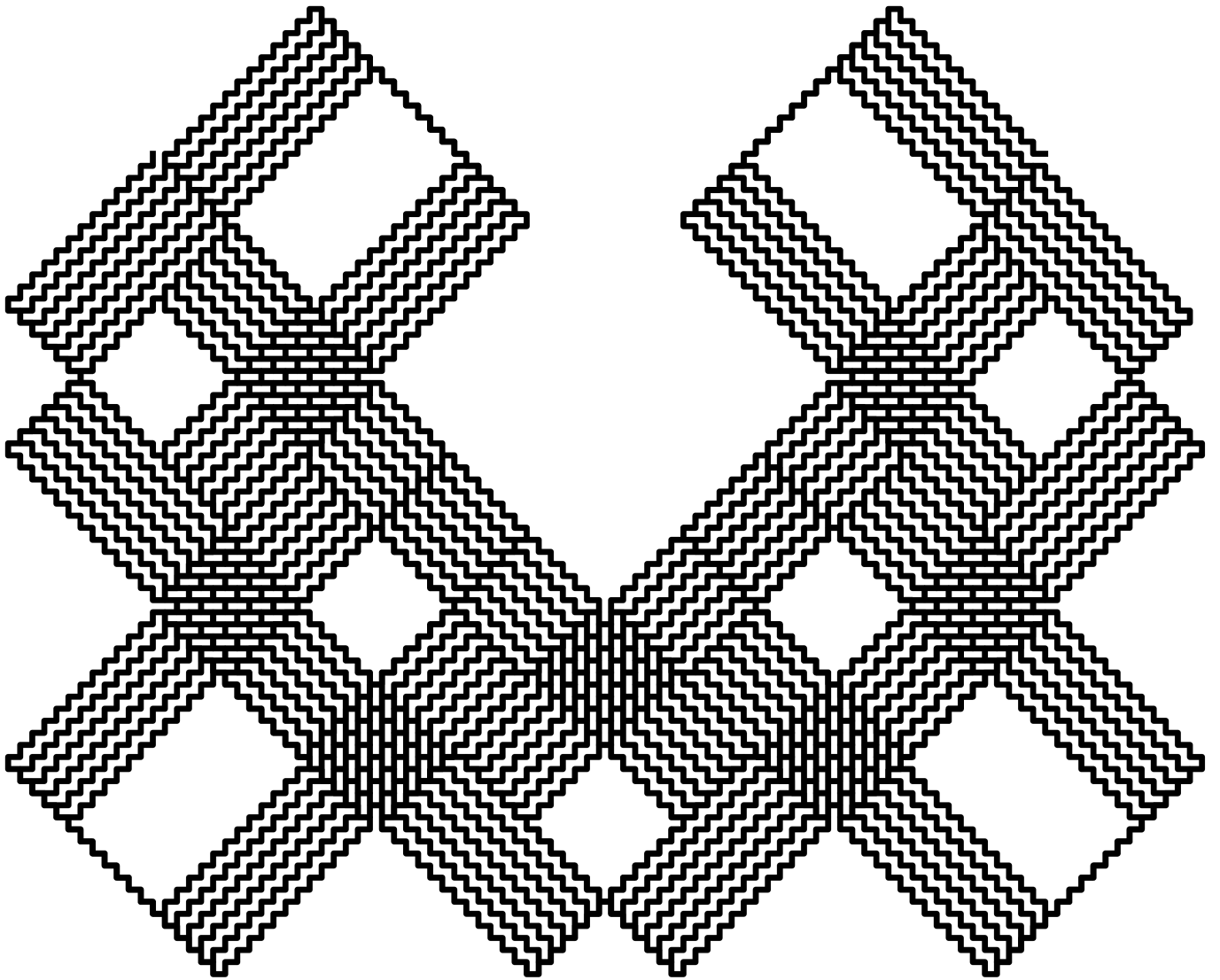Figure 102: The Cloitre walk of the sequence of parities of $B_{\sqrt{9793}}$.

Appendix

The following Python code generates num_steps steps of the Cloitre walk of the sequence of parities of the Beatty sequence $B_{\sqrt{n}}$.

```python
import matplotlib.pyplot as plt
import numpy as np
from copy import deepcopy

def draw_walk(n, num_steps = 10000):

    directions = [np.array([1, 0]),
                  np.array([0, 1]),
                  np.array([-1, 0]),
                  np.array([0, -1])]

    position = np.array([0, 0])
    r = np.sqrt(n)
    p = 0
    walk = [deepcopy(position)]

    for i in range(1, num_steps + 1):

        if np.floor(i*r).astype(int) % 2 == 1: p -= 1
        else: p += 1
        if p < 0: p = 3
        if p > 3: p = 0

        position += directions[p]
        walk.append(deepcopy(position))

    walk = np.array(walk)

    plt.plot(walk[:,0], walk[:,1], color='k')
    plt.axis('equal')
    plt.show()

draw_walk(1950)
```

References

[1] S. Beatty, Problem 3173, *Amer. Math. Monthly*, **3** (1926), 159.

[2] K. B. Stolarsky, Beatty sequences, continued fractions, and certian shift operators, *Canad. Math. Bull.* **19** (1976), 473–482.

[3] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., https://oeis.org.