Order Dimension, Strong Bruhat Order and Lattice Properties for Posets

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Abstract. We determine the order dimension of the strong Bruhat order on finite Coxeter groups of types A, B and H. The order dimension is determined using a generalization of a theorem of Dilworth: $\dim(P) = \text{width}(\text{Irr}(P))$, whenever P satisfies a simple order-theoretic condition called here the dissective property (or "clivage" in [16, 21]). The result for dissective posets follows from an upper bound and lower bound on the dimension of any finite poset. The dissective property is related, via MacNeille completion, to the distributive property of lattices. We show a similar connection between quotients of the strong Bruhat order with respect to parabolic subgroups and lattice quotients.

1. Introduction

We give here three short summaries of the main results of this paper, from three points of view. We conclude the introduction by outlining the organization of the paper.

Strong Bruhat Order. From the point of view of strong Bruhat order, the first main result of this paper is the following:

Theorem 1. *The order dimension of the Coxeter group A_n under the strong Bruhat order is:*

$$\dim(A_n) \quad = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$$

The upper bound $\dim(A_n) \leq \frac{(n+1)^2}{4}$ appeared as an exercise in [3], but the proof given here does not rely on the previous bound. In [27], the same methods are used to prove the following theorem. The result for type I (dihedral groups) is trivial.

Theorem 2. *The order dimensions of the finite Coxeter groups of types B, H and I under the strong Bruhat order are:*

$$\begin{aligned} \dim(B_n) &= \binom{n}{2} + 1 \\ \dim(H_3) &= 6 \\ \dim(H_4) &= 25 \\ \dim(I_2(m)) &= 2. \end{aligned}$$

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A finite poset is *dissective* if every join-irreducible element generates a principal order filter whose complement is a principal order ideal. Lascoux and Schützenberger [21] show that the strong Bruhat order on Coxeter groups of types A and B is dissective (or exhibits "clivage"). In types A and B, the dissective property of the strong order is closely related to the *tableau criterion* [3, 4]. Geck and Kim [16] show that strong Bruhat order on types D, E and F is not dissective. They also cite computer calculations to the effect that the exceptional type H is dissective. Type I is easily seen to be dissective. The following theorem applies:

Theorem 3. *If P is a dissective poset then* $\dim(P) = \text{width}(\text{Irr}(P))$.

Here, $\text{Irr}(P)$ is the subposet of join-irreducible elements.

The subposet $\text{Irr}(A_n)$ can be realized as a lattice tetrahedron in $\mathbb{R}^4$ [10] or can be characterized by considering certain "rectangular" words in the Coxeter group.

Theorem 4. $\text{Irr}(A_n)$ *has a symmetric chain decomposition.*

In particular, $\text{width}(\text{Irr}(A_n))$ is the number of chains in the symmetric chain decomposition. The distributive lattice $J(\text{Irr}(A_n))$ is the lattice of monotone triangles [21], which are in bijection with alternating sign matrices. The lattice of monotone triangles is the MacNeille completion of the strong order on A_n and has the same dimension as A_n. (The MacNeille completion of a finite poset P is the smallest lattice containing P as a subposet.) The poset $\text{Irr}(B_n)$ is less well-behaved, but its width can be determined by finding an antichain and a chain-decomposition of the same size. The results for H_3 and H_4 are obtained by computer calculations of the width.

The dissective property is inherited by quotients with respect to parabolic subgroups, so Theorem 3 can be used to determine the order dimensions of quotients in types A, B, H and I (see Theorems 40 and 41). Theorem 6, below, can in principle be used to compute bounds on the order dimensions of types D, E and F.

The order-dimension calculations reflect a deeper insight into the structure of strong Bruhat orders and quotients. For a poset, being dissective is, in a very strong sense, analogous to a lattice being distributive—for a precise statement, see Theorem 7 below. Bruhat orders and quotients of types A, B, H and I are, in some sense, "distributive non-lattices". The fact that Bruhat quotients inherit the dissective property reflects the intimate relationship of Bruhat quotients to lattice quotients. The equivalence relation on the strong order arising from cosets of a parabolic subgroup is an example of a poset congruence, which is in the same strong sense analogous to a lattice congruence (Theorem 8). Theorem 8 also shows that given any quotient of strong Bruhat order on type A, there is a unique corresponding lattice quotient on the lattice of monotone triangles.

The reader who is primarily interested in Theorem 1 may wish to skip Sections 3 through 6 on the first reading.

Order Dimension. From the point of view of order dimension, the main result of this paper is Theorem 3, which generalizes the following result of Dilworth:

Theorem 5. [9] *If L is a distributive lattice, then* $\dim(L) = \text{width}(\text{Irr}(L))$.

Theorem 3 is a generalization in the sense that a lattice is dissective if and only if it is distributive. The generalization is meaningful because there is an important class of dissective posets, namely the strong Bruhat orders on finite Coxeter groups of types A, B, H and I.

Theorem 3 follows from a more general result:

Theorem 6. *For a finite poset P,* $\text{width}(\text{Dis}(P)) \leq \dim(P) \leq \text{width}(\text{Irr}(P))$*.*

Here $\text{Dis}(P)$ is a subposet of $\text{Irr}(P)$ consisting of *dissectors* of P, those elements which generate a principal order filter whose complement is a principal order ideal. The upper bound in Theorem 6 also appears in [29]. The lower bound, in the case where P is a lattice, is implicit in [11]. A poset P is *dissective* if $\text{Dis}(P) = \text{Irr}(P)$. The dissective posets include, for example, distributive lattices and the "standard examples" of order dimension.

Both Theorem 6 and Theorem 3 can be expressed geometrically in terms of the *critical complex* $\mathcal{C}(P)$, a simplicial complex such that the dimension of P is the size of a smallest set of faces of $\mathcal{C}(P)$ covering the vertices of $\mathcal{C}(P)$. The critical complex is "dual" to the hypergraph H_P^c of critical pairs in [12], in that H_P^c is a hypergraph on the same vertex set whose edges are the minimal non-faces of $\mathcal{C}(P)$. The critical complex sheds light on the connection between dimension and width: The width of a poset is the size of a smallest covering set of its order complex, while the dimension is the size of a smallest covering set of $\mathcal{C}(P)$. Theorem 19, stated in detail in Section 3, essentially gives an embedding of the order complex $\Delta(\text{Dis}(P))$ as a subcomplex of $\mathcal{C}(P)$ and a map from $\mathcal{C}(P)$ into $\Delta(\text{Irr}(P))$ which respects the face structure. Theorem 22, also in Section 3, gives what is essentially an isomorphism between $\Delta(\text{Irr}(P))$ and $\mathcal{C}(P)$, in the case when P is dissective.

The reader interested primarily in order-dimension may wish to skip Sections 4 through 6 on the first reading, and can consider Sections 7 through 9 to be an extended example.

Lattice Properties for Posets. The third theme of this paper is taking definitions that apply to finite lattices and finding the "right" generalization to finite posets. We propose that given a lattice property A, the right generalization is the poset property A' such that a poset P has the property A' if and only if the MacNeille completion $L(P)$ has the property A. (The *MacNeille completion* of a finite poset P can be defined as the "smallest" lattice $L(P)$ containing P, in the sense that any lattice containing P as a subposet contains $L(P)$ as a subposet.) For example, the following is [21, Theorem 2.8]. We give a different proof.

Theorem 7. *For a finite poset P, the following are equivalent:*

(i) *P is dissective.*
(ii) *The MacNeille completion $L(P)$ is a distributive lattice.*
(iii) *The MacNeille completion $L(P)$ is $J(\text{Irr}(P))$.*

The strong Bruhat orders on finite Coxeter groups of types A, B, H and I provide interesting examples of dissective posets. In Section 4, we explore the extent to which dissective posets have analogous properties to distributive lattices. The most striking case is, of course, Theorem 3.

Similarly, given a structure on a lattice, we propose that the "right" generalization of the structure to posets should respect the MacNeille completion. For example, in Section 5 we define a notion of poset congruence with the following property:

Theorem 8. *Let P be a finite poset with MacNeille completion $L(P)$, and let Θ be an equivalence relation on P. Then Θ is a congruence on P if and only if there is a congruence $L(\Theta)$ on $L(P)$ which restricts exactly to Θ, in which case*

(i) *$L(\Theta)$ is the unique congruence on $L(P)$ which restricts exactly to Θ, and*
(ii) *The MacNeille completion $L(P/\Theta)$ is naturally isomorphic to $L(P)/L(\Theta)$.*

The notion of exact restriction is the usual restriction of relations, with an extra condition.

A closely related example is the problem of defining homomorphisms of posets in the right way so as to make them analogous to lattice homomorphisms. Chajda and Snášel [7] give definitions of poset homomorphisms and congruences which correspond to each other in the usual way. The same correspondence holds (by the same proof) between our poset congruences and order-morphisms, which both differ in a trivial way from the definitions in [7]. In light of Theorem 8, order-morphisms are the right generalization of lattice homomorphisms.

The reader interested primarily in lattice theory may wish to skip Section 3 on the first reading, and can consider Sections 7 through 9 to be an extended example.

Outline. This paper is structured as follows: Section 2 establishes notation, defines join-irreducibles of a non-lattice, dissectors and dissective posets, and concludes with a proof of Theorem 6. In Section 3, the critical complex is defined and Theorem 19, a geometrical version of Theorem 6, is stated and proved. Dissective posets are characterized in Section 4, which also contains a description of the critical complex of a dissective poset, and a comparison of the properties of dissective posets and distributive lattices. Poset congruences and order-quotients are defined in Section 5 and shown to behave well with respect to join-irreducibles and dissectors. Section 6 is devoted to the MacNeille completion, and the proofs of Theorems 7 and 8. Section 7 provides a short summary of Bruhat order on a Coxeter group, while Section 8 contains the proof of Theorem 4 and a calculation of the width of $\mathrm{Irr}(A_n)$. Section 9 is a brief discussion of the other types, and Section 10 contains further questions and directions for future research.

2. Posets, Distributive Lattices and Dissectors

In this section, we establish terminology and notation, and provide background information about join-irreducibles and dissectors. We finish the section with a proof of Theorem 6.

The terminology and notation used in this paper should be clear to readers familiar with partially ordered sets and order dimension, and generally agrees with [32] or [33]. Proofs of the basic results quoted here can also be found in [32] or [33]. The letter P denotes a finite poset. Throughout the paper all posets and lattices are assumed to be finite. Some notation and terminology that is less commonly used is as follows: Incomparability of two elements $x, y \in P$ will be denoted $x \parallel y$. Given $x \in P$, define

$$\begin{aligned} D(x) &:= \{y \in P : y < x\} \\ U(x) &:= \{y \in P : y > x\} \\ D[x] &:= \{y \in P : y \leq x\} \\ U[x] &:= \{y \in P : y \geq x\}. \end{aligned}$$

If x is comparable to every element of P, it is called a *pivot* (elsewhere called a *bottleneck*). Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$ and also denote the linear order on these elements. The symbol $[k, n]$ denotes the set $\{k, k+1, k+2, \ldots, n\}$.

The *order dimension* $\dim(P)$ is the smallest number d such that P is the intersection of d linear extensions of P. Equivalently the order dimension is the smallest d so that P can be embedded as a subposet of $\mathbb{N}^d$ with componentwise partial order.

The "standard example" of a poset of dimension n is the set of subsets of $[n]$ of cardinality 1 or $n-1$, ordered by inclusion.

Joins and meets, typically encountered in the context of lattices, can also be defined in general posets. Given x and y, if $U[x] \cap U[y]$ has a unique minimal element, this element is called the *join* of x and y and is written $x \vee^P y$ or simply $x \vee y$. If $D[x] \cap D[y]$ has a unique maximal element, it is called the *meet* of x and y, $x \wedge_P y$ or $x \wedge y$. Given a set $S \subseteq P$, if $\cap_{x\in S} U[x]$ has a unique minimal element, it is called $\vee S$. The join $\vee\emptyset$ is $\hat{0}$ if P has a unique minimal element $\hat{0}$, and otherwise $\vee\emptyset$ does not exist. If $\cap_{x\in S} D[x]$ has a unique maximal element, it is called $\wedge S$. The meet $\wedge\emptyset$ exists if and only if a unique maximal element $\hat{1}$ exists, in which case they coincide. The notation, $x \vee y = a$ means "x and y have a join, which is a," and similarly for other statements about joins and meets.

The description of an element as "join-irreducible" is usually heard in the context of lattices. However, it is useful to apply the definition to general posets, as in [16], [21] and [29]. An element a of a poset P is *join-irreducible* if there is no set $X \subseteq P$ with $a \notin X$ and $a = \vee X$. If P has a unique minimal element $\hat{0}$, then $\hat{0}$ is $\vee\emptyset$ and thus is not join-irreducible. In a lattice, a is join-irreducible if and only if it covers exactly one element. Such elements are also join-irreducible in non-lattices, but an element a which covers distinct elements $\{x_i\}$ is join-irreducible if $\{x_i\}$ has an upper bound incomparable to a. A minimal element of a non-lattice is also join-irreducible, if it is not $\hat{0}$. It is easily checked that if $x \in P$ is not join-irreducible, then $x = \vee D(x)$. The subposet of P induced by the join-irreducible elements is denoted $\mathrm{Irr}(P)$. In [16] and [21], the set $\mathrm{Irr}(P)$ is called the *base* of P. The subposet of meet-irreducibles does not figure strongly in this paper, and that perhaps excuses the cumbersome notation $\mathrm{MeetIrr}(P)$ for this subposet. For $x \in P$, let I_x denote $D[x] \cap \mathrm{Irr}(P)$, the set of join-irreducibles weakly below x in P.

Proposition 9. *Let P be a finite poset, and let $x \in P$. Then $x = \vee I_x$.*

Proof. By induction on the cardinality of $D[x]$. The result is trivial if $D[x]$ has one element. If x is join-irreducible, then $x \in I_x$, and every other element of I_x is below x. Thus $x = \vee I_x$. If not, then write $x = \vee D(x)$. By induction, each y in $D(x)$ has $y = \vee I_y$, or in other words $\cap_{i\in I_y} U[i] = U[y]$. Then $U[x] = \cap_{y\in D(x)} U[y] = \cap_{y\in D(x)} \cap_{i\in I_y} U[i] = \cap_{i\in I_x} U[i]$, or in other words, $x = \vee I_x$. □

For a finite poset P, define $J(P)$ to be the lattice of order ideals of P, ordered by inclusion. The Fundamental Theorem of Finite Distributive Lattices states that a finite distributive lattice L has $L \cong J(\mathrm{Irr}(L))$, and that for any finite poset P, $J(P)$ is distributive with $\mathrm{Irr}(J(P)) \cong P$.

The proofs of the following two propositions are easy.

Proposition 10. [21] *An element $x \in P$ is join-irreducible if and only if there exists a $y \in P$ such that x is minimal in $P - D[y]$.* □

Proposition 11. *If x is join-irreducible (or dually meet-irreducible) in a lattice L, then $L - \{x\}$ is a lattice.* □

While $L - \{x\}$ is a subposet of L and a lattice, it is not usually a sublattice.

An element $x \in P$ is called a(n) *(upper) dissector* of P if $P - U[x] = D[\beta(x)]$ for some $\beta(x) \in P$. In other words, P can be dissected as a disjoint union of the principal order filter generated by x and the principal order ideal generated by $\beta(x)$. By the same token, call $\beta(x)$ a *lower dissector.* From now on, however, the term dissector refers to an upper dissector. Thus for each result about dissectors, there is a dual result about lower dissectors which is not stated. The subposet of dissectors of P is called $\mathrm{Dis}(P)$. In the lattice case the definition of dissector coincides with the notion of a prime element. An element x of a lattice L is called *prime* if whenever $x \leq \vee Y$ for some $Y \subseteq L$, then there exists $y \in Y$ with $x \leq y$.

Proposition 10 implies:

Proposition 12. *If x is a dissector of P then x is join-irreducible.* □

The converse is not true in general, and the reader can find a 5-element lattice to serve as a counterexample. A poset P in which every join-irreducible is a dissector is called a *dissective* poset. In [21] this property of a poset is called "clivage."

We now give a proof of Theorem 6. Notice in particular, that this proof actually constructs an embedding of P into $\mathbb{N}^w$, where $w = \mathrm{width}(\mathrm{Irr}(P))$.

First Proof of Theorem 6. Let $C_1, C_2, \ldots, C_w$ be a chain decomposition of $\mathrm{Irr}(P)$. For each $m \in [w]$, and $x \in P$, let $f_m(x) = |I_x \cap C_m|$. By Proposition 9, $x \leq y$ if and only if $I_x \subseteq I_y$ if and only if $f_m(x) \leq f_m(y)$ for all $m \in [w]$. Thus $x \mapsto (f_1(x), f_2(x), \ldots, f_w(x))$ is an embedding of P into $\mathbb{N}^w$.

For the lower bound, consider an antichain A in $\mathrm{Dis}(P)$. Each $a \in A$ has $P - U_P[a] = D_P[\beta(a)]$ for some $\beta(a) \in P$. In particular $A - \{a\} \subseteq D_P[\beta(a)]$. So the subposet of P induced by $A \cup \beta(A)$ is a "standard example" of size $|A|$. Thus $\dim(P) \geq \dim(A \cup \beta(A)) = |A|$. □

This proof relies on knowing the order dimensions of the "standard examples." One way to find the order dimension of the standard examples is to notice that they are dissective posets whose join irreducibles form an antichain. However, to avoid this circular reasoning, one can easily compute the order dimension of the standard examples directly, or by the method of the next section.

3. The Critical Complex of a Poset

In this section, we give the definition of the critical complex, relate the critical complex to join-irreducibles and dissectors, and give another proof of Theorem 6. The simple proofs of some propositions are omitted.

A *critical pair* in a poset P is (a, b) with the following properties:

(i) $a \parallel b$,
(ii) $D(a) \subseteq D(b)$, and
(iii) $U(b) \subseteq U(a)$.

As motivation, note that properties (ii) and (iii) hold for a related pair $a \leq b$. If (a, b) is a critical pair, the partial order $\leq$ can be extended to a new partial order $\leq'$ by putting $x \leq' y$ if $x \leq y$ or if $(x, y) = (a, b)$. So in some sense, a critical pair (a, b) is "almost" a related pair $a \leq b$. The set of critical pairs of P is denoted $\mathrm{Crit}(P)$.

Say an extension E of P *reverses* a critical pair (a, b) if $b < a$ in E. The following fact is due to I. Rabinovitch and I. Rival [25]:

Proposition 13. *If $L_1, L_2, \ldots, L_n$ are linear extensions of P then $P = \cap_{i \in [n]} L_i$ if and only if for each critical pair (a, b), there is some L_i for which $b < a$ in L_i.* □

The *critical digraph* $\mathcal{D}(P)$ of P is the directed graph whose vertices are the critical pairs, with directed edge $(a, b) \to (c, d)$ whenever $b \geq c$. The next proposition follows from Lemma 6.3 of Chapter 1 of [33].

Proposition 14. *Let S be any set of critical pairs of P. Then there is a linear extension of P reversing every critical pair in S if and only if the subgraph of $\mathcal{D}(P)$ induced by S is acyclic.* □

Proposition 14 motivates the definition of the *critical complex* $\mathcal{C}(P)$ of P, an abstract simplicial complex whose vertices are the critical pairs of P, and whose faces are the sets of vertices which induce acyclic subgraphs of $\mathcal{D}(P)$. A set of faces $\{F_i\}$ of a simplicial complex $\mathcal{C}$ with vertex set V is a *covering set* if $\cup_i F_i = V$. Propositions 13 and 14 imply that (when P is not a total order) the order dimension of P is the size of a smallest covering set of $\mathcal{C}(P)$.

A similar (and in some sense dual) construction to the critical complex is given by Felsner and Trotter [12]. Their *hypergraph H_P^c of critical pairs* is exactly the hypergraph whose vertices are critical pairs and whose edges are minimal non-faces of $\mathcal{C}(P)$. They also define the *graph G_P^c of critical pairs* whose vertices are the critical pairs and whose edges are the edges of cardinality 2 of H_P^c. The size of a smallest covering set of $\mathcal{C}(P)$ is exactly the chromatic number $\chi(H_P^c)$. The following is [12, Lemma 3.3]:

$$\dim(P) = \chi(H_P^c) \geq \chi(G_P^c) \tag{1}$$

The easy proofs of the following propositions are omitted. Proposition 15 was noticed by Rabinovitch and Rival [25] in the context of distributive lattices.

Proposition 15. *If (a, b) is a critical pair, then a is join-irreducible.* □

Proposition 16. *Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ be critical pairs in P with $a_1 \leq a_2 \leq \cdots \leq a_k$. Then $\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$ is a face of $\mathcal{C}(P)$.* □

Proposition 17. *Let $a \in P$ be a non-pivot dissector. Then $(a, \beta(a))$ is a critical pair. Furthermore the only critical pair (a, b) is the pair with $b = \beta(a)$.* □

Proposition 18. *Let $a, x \in P$ be dissectors with $a \parallel x$. Then $\{(a, \beta(a)), (x, \beta(x))\}$ is not a face in $\mathcal{C}(P)$.* □

Since the width of a poset P is the size of a smallest covering set of the order complex $\Delta(P)$, one might expect that Theorem 6 follows from some relationships between the order complexes $\Delta(\mathrm{Dis}(P))$ and $\Delta(\mathrm{Irr}(P))$ and the critical complex $\mathcal{C}(P)$. The following theorem explains such a relationship. Write $\mathrm{Dis}(P)_{nonpiv}$ for the subposet of $\mathrm{Dis}(P)$ consisting of non-pivots and $\mathrm{Dis}(P)_{piv}$ for the subposet of pivots. Similarly $\mathrm{Irr}(P)_{nonpiv}$ and $\mathrm{Irr}(P)_{piv}$.

Given two abstract simplicial complexes A and B, let $A * B$ be the *join* of A and B, a simplicial complex whose vertex set is the disjoint union of the vertices of A and of B, and whose faces are exactly the sets $F \cup G$ for all faces F of A and G of B. It is evident that $\Delta(\mathrm{Dis}(P)_{piv})$ is a simplex and that

$$\Delta(\mathrm{Dis}(P)) \cong \Delta(\mathrm{Dis}(P)_{piv}) * \Delta(\mathrm{Dis}(P)_{nonpiv}).$$

Similarly $\Delta(\mathrm{Irr}(P)_{piv})$ is a simplex and

$$\Delta(\mathrm{Irr}(P)) \cong \Delta(\mathrm{Irr}(P)_{piv}) * \Delta(\mathrm{Irr}(P)_{nonpiv}).$$

In light of Propositions 15 and 17, we have well defined set maps:

$$\begin{array}{cc} i : \mathrm{Dis}(P)_{nonpiv} \to \mathrm{Crit}(P), & p : \mathrm{Crit}(P) \to \mathrm{Irr}(P)_{nonpiv} \\ a \overset{i}{\mapsto} (a, \beta(a)) & (a, b) \overset{p}{\mapsto} a \end{array}.$$

Theorem 19. *The set map i induces a simplicial map $i : \Delta(\mathrm{Dis}(P)_{nonpiv}) \to \mathcal{C}(P)$ which embeds $\Delta(\mathrm{Dis}(P)_{nonpiv})$ as a vertex-induced subcomplex of $\mathcal{C}(P)$. Also, if F is any face in the image of p, then $p^{-1}(F)$ is a face of $\mathcal{C}(P)$.*

Saying that i embeds $\Delta(\mathrm{Dis}(P)_{nonpiv})$ as a vertex-induced subcomplex of $\mathcal{C}(P)$ means that i is one-to-one, maps faces of $\Delta(\mathrm{Dis}(P)_{nonpiv})$ to faces of $\mathcal{C}(P)$ and for any face F of $i(\Delta(\mathrm{Dis}(P)_{nonpiv}))$, $i^{-1}(F)$ is a face of $\Delta(\mathrm{Dis}(P)_{nonpiv})$. If F is a face of $\mathcal{C}(P)$, then $p(F)$ need not be a face of $\Delta(\mathrm{Irr}(P)_{nonpiv})$. For example, let P be an antichain $\{a, b, c\}$ and let F be $\{(a, b), (b, c)\}$.

Proof. The statement that i is one-to-one and maps faces of $\Delta(\mathrm{Dis}(P)_{nonpiv})$ to faces of $\mathcal{C}(P)$ follows immediately from Propositions 16 and 17. Proposition 18 is exactly the statement that for any face F of $i(\Delta(\mathrm{Dis}(P)_{nonpiv}))$, $i^{-1}(F)$ is a face of $\Delta(\mathrm{Dis}(P)_{nonpiv})$. The last statement of the theorem also follows immediately from Proposition 16. □

Second Proof of Theorem 6. Since $\Delta(\mathrm{Dis}(P)_{nonpiv})$ is embedded into $\mathcal{C}(P)$, any covering set of $\mathcal{C}(P)$ restricts to a covering set of$\Delta(\mathrm{Dis}(P)_{nonpiv})$. Whenever $\Delta(\mathrm{Dis}(P)_{nonpiv})$ is non-empty, a covering set of $\Delta(\mathrm{Dis}(P)_{nonpiv})$, is easily extended to a set of the same cardinality covering $\Delta(\mathrm{Dis}(P)) \cong \Delta(\mathrm{Dis}(P)_{piv}) * \Delta(\mathrm{Dis}(P)_{nonpiv})$.

Any covering set of $\Delta(\mathrm{Irr}(P))$ restricts to a covering set of $\Delta(\mathrm{Irr}(P)_{nonpiv})$, which maps by p^{-1} to a covering set of $\mathcal{C}(P)$. □

4. Dissective Posets

In this section, we study dissective posets: posets in which every join-irreducible is a dissector. In light of this definition, Theorem 3 follows trivially from Theorem 6, and the embedding given in the first proof of Theorem 6 is an optimal embedding. The dissective property is a generalization of the distributive property, in the following sense:

Proposition 20. [11, 23] *A finite lattice L is distributive if and only if every join-irreducible is prime.* □

In other words, a lattice is distributive if and only if it is dissective. This statement is strengthened in the next section. In this section we characterize dissective posets, describe the critical complex of a dissective poset and discuss the extent to which dissective posets have properties analogous to distributive lattices.

Proposition 21. *Let L be a finite distributive lattice and let P be a subposet with $\mathrm{Irr}(L) \cup \mathrm{MeetIrr}(L) \subseteq P \subseteq L$. Then P is a dissective poset and $\mathrm{Irr}(P) = \mathrm{Irr}(L)$.*

Proof. Suppose $x \in \mathrm{Irr}(L) = \mathrm{Dis}(L)$. Then there is a $\beta(x) := \vee^L\{y \in L : y \not\geq x\} = \vee^L\{y \in \mathrm{Irr}(L) : y \not\geq x\}$. But $\beta(x) \in \mathrm{MeetIrr}(L)$, so $\beta(x) \in P$. Any upper bound z for $\{y \in P : y \not\geq x\}$ is in particular an upper bound for $\{y \in \mathrm{Irr}(L) : y \not\geq x\}$ so in particular $z \geq \beta(x)$. Thus $\beta(x) = \vee^P\{y \in P : y \not\geq x\}$. Therefore $x \in \mathrm{Dis}(P)$, and so $\mathrm{Irr}(L) \subseteq \mathrm{Dis}(P) \subseteq \mathrm{Irr}(P)$.

Suppose $x \in \mathrm{Irr}(P)$ and write $x = \vee^L(D_L[x] \cap \mathrm{Irr}(L))$. By the previous paragraph, $D_L[x] \cap \mathrm{Irr}(L) \subseteq P$, and since P is a subposet, $x = \vee^P(D_L[x] \cap \mathrm{Irr}(L))$. But $x \in \mathrm{Irr}(P)$, so $x \in D_L[x] \cap \mathrm{Irr}(L)$. Therefore $\mathrm{Irr}(P) \subseteq \mathrm{Irr}(L)$. □

In light of Theorem 7, L is the MacNeille completion of P, and every dissective poset arises as in Proposition 21. For example, the "standard examples" of n-dimensional posets arise in this manner from the Boolean lattice $2^{[n]}$.

When P is dissective, Theorem 19 simplifies greatly. Since every join-irreducible is a dissector and by Proposition 17, every dissector gives rise to exactly one critical pair, the map i is a bijection with inverse p, and therefore an isomorphism of simplicial complexes.

Theorem 22. *If P be a dissective poset, then the order complex $\Delta(\mathrm{Irr}(P))$ is isomorphic to $\mathcal{C}(P) * \Delta(\mathrm{Irr}(P)_{piv})$.* □

The statement is even simpler than it looks since $\Delta(\mathrm{Irr}(P)_{piv})$ is a simplex. Similar considerations also show that for a dissective poset, the hypergraph H_P^c of critical pairs is equal to the graph G_P^c of critical pairs. Thus equality holds in Equation (1) when P is dissective.

We now list some properties of dissective posets which are analogous to familiar properties of distributive lattices. The proofs are straightforward, and are omitted.

Proposition 23. *If P is dissective, then so is the dual of P.* □

Proposition 24. *If P is a dissective poset then $\beta : \mathrm{Irr}(P) \to \mathrm{MeetIrr}(P)$ is an order isomorphism.* □

Proposition 25. *If a dissective poset P is self-dual then $\mathrm{Irr}(P)$ is self-dual.* □

Even when P is not dissective, β is an order isomorphism from $\mathrm{Dis}(P)$ to the subposet of lower dissectors, and if P is self-dual, then $\mathrm{Dis}(P)$ is also self-dual.

Finally, we mention several properties of distributive lattices which appear not to have analogues for dissective posets. The converse of Proposition 25 holds for distributive lattices, but not for dissective posets. The distributive property in a finite lattice is inherited by intervals, but the analogous property is not true of the dissective property in a finite poset. Finally, distributive lattices can be characterized by the fact that they avoid certain sublattices. No similar characterization for dissective posets is immediately apparent.

5. Order-Quotients

In this section we define order-quotients and prove that they behave nicely with respect to join-irreducibles and dissectors. The reader familiar with Bruhat order may want to keep in mind quotients with respect to parabolic subgroups as a motivating example. Let P be a finite poset with an equivalence relation Θ defined on the elements of P. Given $a \in P$, let $[a]_\Theta$ denote the equivalence class of a under Θ. The equivalence relation Θ is a *congruence* if:

(i) Every equivalence class is an interval.
(ii) The projection $\pi_\downarrow : P \to P$, mapping each element a of P to the minimal element in $[a]_\Theta$, is order-preserving.
(iii) The projection $\pi^\uparrow : P \to P$, mapping each element a of P to the maximal element in $[a]_\Theta$, is order-preserving.

The definition given here essentially coincides, when P is finite, to the notion of *poset congruence*, as defined in [7]. The difference is that in [7], $P \times P$ is by definition always a congruence. Also in [7], there is the definition of *LU-morphisms*, which we call *order-morphisms*. The definition given here differs from [7], in a way that corresponds to the difference in the definitions of congruence. A map $f : P \to Q$

for finite P and Q is an *order-morphism* if for any $x, y \in P$,

$$f(D_P[x] \cap D_P[y]) = D_{f(P)}[f(x)] \cap D_{f(P)}[f(y)]$$

and if the dual statement also holds. Congruences and order-morphisms are related in the usual way. The proof can be found in [7] and still works with the slightly modified definitions.

A congruence on a lattice L is an equivalence relation which respects joins and meets. Specifically, if $a_1 \equiv a_2$ and $b_1 \equiv b_2$ then $a_1 \vee b_1 \equiv a_2 \vee b_2$ and similarly for meets. For a finite lattice L, the two notions of congruence coincide. So from now on, the term congruence is used without specifying "lattice" or "poset." A connection is made in Section 6 between congruences on a finite poset and congruences on its MacNeille completion.

Define a partial order on the congruence classes by $[a]_\Theta \leq [b]_\Theta$ if and only if there exists $x \in [a]_\Theta$ and $y \in [b]_\Theta$ such that $x \leq_P y$. The set of equivalence classes under this partial order is P/Θ, the *quotient* of P with respect to Θ. It is convenient to identify P/Θ with the induced subposet $Q := \pi_\downarrow(P)$, as is typically done for example with quotients of Bruhat order. Such a subposet Q is called an *order-quotient* of P. It is easily seen that $\pi^\uparrow$ maps Q isomorphically onto $\pi^\uparrow(P)$. The inverse is $\pi_\downarrow$.

We wish to compare $\text{Dis}(P/\Theta)$ and $\text{Irr}(P/\Theta)$ to $\text{Dis}(P)$ and $\text{Irr}(P)$.

Proposition 26. *Suppose Q is an order-quotient of P. If $x = \vee^Q Y$ for some $Y \subseteq Q$, then $x = \vee^P Y$. If $x = \vee^P Y$ for some $Y \subseteq P$, then $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$.*

Proof. Suppose $x = \vee^Q Y$ for $Y \subseteq Q$ and suppose $z \in P$ has $z \geq y$ for every $y \in Y$. Then $\pi_\downarrow(z) \geq \pi_\downarrow(y) = y$ for every $y \in Y$. Therefore $z \geq \pi_\downarrow(z) \geq x$. Thus $x = \vee^P Y$.

Suppose $x = \vee^P Y$ for $Y \subseteq P$, and suppose that for some $z \in Q$, $z \geq \pi_\downarrow(y)$ for every $y \in Y$. Then $\pi^\uparrow(z) \geq \pi^\uparrow(y) \geq y$ for every $y \in Y$, and so $\pi^\uparrow(z) \geq x$. Thus also $\pi_\downarrow(\pi^\uparrow(z)) \geq \pi_\downarrow(x)$, but $\pi_\downarrow(\pi^\uparrow(z)) = z$, and so $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$. □

Proposition 27. *Suppose Q is an order-quotient of P and let $x \in Q$. Then x is join-irreducible in Q if and only if it is join-irreducible in P, and x is a dissector of Q if and only if it is a dissector of P. In other words,*

$$\text{Irr}(Q) = \text{Irr}(P) \cap Q \textit{ and,} \tag{2}$$

$$\text{Dis}(Q) = \text{Dis}(P) \cap Q. \tag{3}$$

In particular, if P is dissective, then so is any order-quotient. Also, for any P with order-quotient Q such that $Q \cap \text{Irr}(P) \subseteq \text{Dis}(P)$, Q is dissective.

Proof. Suppose $x \in Q$ is join-irreducible in Q. Then by Proposition 10, there is some $y \in Q$ so that x is minimal in $Q - D_Q[y]$. Then x is also minimal in $P - D_P[\pi^\uparrow(y)]$, so x is join-irreducible in P. Conversely, suppose $x \in Q$ is join-irreducible in P, and suppose $x = \vee^Q Y$ for some $Y \subseteq Q$. Then by Proposition 26, $x = \vee^P Y$, so $x \in Y$. Thus x is join-irreducible in Q.

Suppose $x \in Q$ is a dissector of Q. Then there is some $\beta^Q(x) \in Q$ such that $Q - U_Q[x] = D_Q[\beta^Q(x)]$. Then $\pi^\uparrow(\beta^Q(x)) \not\geq x$ because otherwise $\beta^Q(x) \geq \pi_\downarrow(x) = x$. Furthermore, for any $z \not\geq x$, necessarily $\pi_\downarrow(z) \not\geq x$, and therefore $\pi_\downarrow(z) \leq \beta^Q(x)$. So $z \leq \pi^\uparrow(z) \leq \pi^\uparrow(\beta^Q(x))$. Thus x is a dissector of P with $P - U_P[x] = D_P[\pi^\uparrow(\beta^Q(x))]$. Conversely, suppose $x \in Q$ is a dissector of P, or in other words, there is some $\beta^P(x) \in P$ such that $\beta^P(x) = \vee^P(P - U_P[x])$. Then by Proposition 26, $\pi_\downarrow(\beta^P(x)) = \vee^Q \pi_\downarrow(P - U_P[x]) = \vee^Q(Q - U_Q[x])$, so x is a dissector of Q. □

Quotients of strong Bruhat order with respect to parabolic subgroups are order-quotients (Proposition 31). There are also several examples in the literature relating to weak Bruhat order. A. Björner and M. Wachs [6, Section 9] show that the Tamari lattices are quotients of the weak order on A_n. R. Simion [31, Section 4] defines a congruence on the Coxeter group B_n under the weak order, such that the resulting quotient is the weak order on A_n.

6. The MacNeille Completion

In this section we define the MacNeille completion of a finite poset and point out that it preserves join-irreducibles, dissectors and critical pairs. We strengthen the assertion that the dissective property generalizes the distributive property. This leads in particular to a different proof of Theorem 3. We also strengthen the assertion that congruences on posets are a generalization of congruences on lattices.

The *MacNeille completion* (also known as the *MacNeille-Dedekind completion*, *completion by cuts* or *enveloping lattice*) of a poset P generalizes Dedekind's construction of the reals from the rationals. One construction of the completion is due to MacNeille [22] and more information can be found in [33, Section 2.5] and [2, Section V.9]. Here we confine our attention to the MacNeille completion of a finite poset. For a finite poset P, the MacNeille completion $L(P)$ is the "smallest" lattice containing P, in the sense that any lattice containing P as an induced subposet contains $L(P)$ as an induced subposet. One way to obtain $L(P)$ for a finite poset is as the smallest collection of subsets which contains P and $U_P[x]$ for each $x \in P$ and which is closed under intersection [21]. The partial order on $L(P)$ is reverse-inclusion, the join is intersection, and $x \mapsto U_P[x]$ is an embedding of P as a subposet of $L(P)$. Whatever joins exist in P are preserved by MacNeille completion: If $x = \vee^P S$ for some $S \subseteq P$, then $U_P[x] = \cap_{y \in S} U_P[y]$, or in other words, $x = \vee^{L(P)} S$. Conversely, if $x \in P$ is $\vee^{L(P)} S$ for some $S \subseteq P$, then $x = \vee^P S$. This construction also shows that any element of $L(P)$ is a join of elements of P.

The construction is seen to coincide with its dual construction (by order ideals) as follows: For $x \in L(P)$, define P_x to be the elements of P below x in $L(P)$. Then $x = \cap_{y \in P_x} U_P[y]$, the partial order on $L(P)$ is inclusion of the sets P_x and the meet is intersection of sets P_x. It also follows that $x = \vee^{L(P)} P_x$.

Proposition 28. *Let P be a finite poset and let $L(P)$ be its MacNeille completion. Then*

(i) $\mathrm{Irr}(P) = \mathrm{Irr}(L(P))$
(ii) $\mathrm{Dis}(P) = \mathrm{Dis}(L(P))$*, and*
(iii) $\mathrm{Crit}(P) = \mathrm{Crit}(L(P))$.

Remark 29. Assertion (i) is implicit in the proof of [21, Théorème 2.8]. Although we were unable to find (iii) in the literature, it is closely related to [33, Exercise 2.5.7].

Proof of Proposition 28. Suppose that $x \in \mathrm{Irr}(L(P))$. Then by Proposition 11, $L(P) - \{x\}$ is a lattice. Thus by the definition of $L(P)$, necessarily $x \in P$. Since joins are preserved in $L(P)$, join-irreducibility of x in $L(P)$ implies join-irreducibility in P. Conversely, suppose that $x \in \mathrm{Irr}(P)$ and write $x = \vee^{L(P)} S$ for some $S \subseteq \mathrm{Irr}(L(P))$. Since we have just shown that every join-irreducible in $L(P)$ is join-irreducible in P, $S \subseteq \mathrm{Irr}(P)$. Since joins are preserved, $x = \vee^P S$, and therefore $x \in S$. Thus $x \in \mathrm{Irr}(L(P))$.

Let x be a dissector in P and $\beta_P(x) = \vee^P(P - U_P[x])$. Because joins are preserved, $\beta_P(x) = \vee^{L(P)}(P - U_P[x])$. But each element of the set $(L(P) - U_{L(P)}[x])$ can be written as the join of elements of $P - U_P[x]$, so $\beta_P(x) = \vee^{L(P)}(L(P) - U_{L(P)}[x])$. Thus x is a dissector in $L(P)$ with $\beta_{L(P)}(x) = \beta_P(x)$. Conversely, suppose x is a dissector in $L(P)$ and $\beta_{L(P)}(x) = \vee^{L(P)}(L(P) - U_{L(P)}[x])$. Then $\beta_{L(P)}(x)$ is meet-irreducible, so by the dual of assertion (i), $\beta_{L(P)}(x) \in P$. Since $\beta_{L(P)}(x)$ is an upper bound for $P - U_P[x]$, and $\beta_{L(P)}(x)$ is also contained in $P - U_P[x]$, we have $\beta_{L(P)}(x) = \vee^P(P - U_P[x]) = \beta_P(x)$.

Let (a, b) be a critical pair of P. Then a is join-irreducible in P by Proposition 15 and thus join-irreducible in $L(P)$ by assertion (i). Let x be the single element of $L(P)$ covered by a. But x is the join of all the join-irreducibles weakly below it, and b is above all these join-irreducibles because they are below a. Thus $x \leq b$ and condition (ii) holds for (a, b) to be critical in $L(P)$. Condition (iii) is dual, and condition (i) holds because P is an induced subposet of $L(P)$. Conversely, let (a, b) be a critical pair of $L(P)$. By Proposition 15, a is join-irreducible, and so $a \in P$. Dually, $b \in P$. Conditions (i), (ii) and (iii) for (a, b) to be critical in P follow easily because P is an induced subposet of $L(P)$. □

Theorem 7, due to Lascoux and Schützenberger [21], follows easily from Proposition 20 and assertions (i) and (ii) of Proposition 28.

Theorem 7. *For a finite poset P, the following are equivalent:*

(i) *P is dissective.*
(ii) *The MacNeille completion $L(P)$ is a distributive lattice.*
(iii) *The MacNeille completion $L(P)$ is $J(\mathrm{Irr}(P))$.* □

Assertion (iii) of Proposition 28 implies the known fact [33, Exercise 2.5.7], [2, Section V.9] that order dimension is preserved by MacNeille completion. Thus Theorem 7 combines with Theorem 5 to give a different (more complicated) proof of Theorem 3.

Given a finite lattice L with a subposet P, a congruence Θ on L *restricts exactly to P* if every congruence class $[x, y]$ of Θ has either $x, y \in P$ or $[x, y] \cap P = \emptyset$. The next proposition follows immediately from the definitions.

Proposition 30. *If a congruence Θ on L restricts exactly to P, then the restriction (as a relation) $\Theta|_P$ of Θ to P is a congruence, and L/Θ is a lattice containing $P/(\Theta|_P)$ as a subposet.* □

Thus we also say Θ restricts exactly to $\Theta|_P$.

Theorem 8. *Let P be a finite poset with MacNeille completion $L(P)$, and let Θ be an equivalence relation on P. Then Θ is a congruence on P if and only if there is a congruence $L(\Theta)$ on $L(P)$ which restricts exactly to Θ, in which case*

(i) *$L(\Theta)$ is the unique congruence on $L(P)$ which restricts exactly to Θ, and*
(ii) *The MacNeille completion $L(P/\Theta)$ is naturally isomorphic to $L(P)/L(\Theta)$.*

Proof. The "if" direction is Proposition 30.

Conversely, suppose Θ is a congruence on P and $x \in L(P)$. Let P_x/Θ be the set of equivalence classes in Θ which have non-empty intersection with P_x. Define $L(\Theta)$ to be the equivalence relation which sets $x \equiv y$ if and only if $P_x/\Theta = P_y/\Theta$. More simply, P_x/Θ is determined by Q_x, the set of elements of Q weakly below x. Here Q is the order-quotient associated to Θ, as in Section 5. Thus $x \equiv y$ if and only if $Q_x = Q_y$. Notice that for any $x, y \in L(P)$, $Q_{x \wedge y} = Q_x \cap Q_y$. Suppose $x_1 \equiv_{L(\Theta)} x_2$ and $y_1 \equiv_{L(\Theta)} y_2$ in $L(P)$. Then

$$Q_{x_1 \wedge y_1} = Q_{x_1} \cap Q_{y_1} = Q_{x_2} \cap Q_{y_2} = Q_{x_2 \wedge y_2},$$

and a dual argument shows that $L(\Theta)$ respects joins. Given any congruence class $[a, b]_P$ in Θ, the element a of $L(P)$ is minimal among elements x of $L(P)$ with $Q_x = Q_a$, and dually, b is maximal. Thus there is a congruence class $[a, b]_{L(P)}$ in $L(\Theta)$. Any element of P is in some Θ-class, and so $L(\Theta)$ restricts exactly to P.

Since $P/\Theta \cong Q$, the natural isomorphism between $L(P/\Theta)$ and $L(P)/L(\Theta)$ is easily seen by identifying the elements of each lattice with order ideals in Q. The lattice $L(Q)$ consists of Q and intersections $\cap_{q \in S} D_Q[q]$ for $S \subseteq Q$, and S may as well be an order filter. Elements of $L(P)/L(\Theta)$ are $X \cap Q$, where $X = P$ or $X = \cap_{y \in T} D_P[y]$, where T is an order filter in P. But then $X \cap Q$ is Q or $\cap_{y \in T}(D_P[y] \cap Q) = \cap_{y \in T \cap Q} D_Q[y]$, and $T \cap Q$ ranges over all order filters in Q.

Let Φ be a congruence on $L(P)$ which restricts exactly to Θ. Proposition 30 says that $L(P)/\Phi$ is a lattice containing P/Θ as a subposet. Since $L(P)/L(\Theta)$ is the MacNeille completion of P/Θ, $L(P)/\Phi$ contains $L(P)/L(\Theta)$ as a subposet, and therefore $\Phi \subseteq L(\Theta)$ (as relations). Suppose $x \equiv_{L(\Theta)} y$, or in other words $P_x/\Theta = P_y/\Theta$. Write $x = \vee^{L(P)} P_x$ and $y = \vee^{L(P)} P_y$. Since Φ restricts to Θ on P and respects joins, $x \equiv_\Phi y$, and thus $\Phi = L(\Theta)$. $\square$

7. Strong Bruhat Order on a Coxeter Group

In this section, we give some necessary background information about Coxeter groups, and about type A in particular. The reader should refer to [3] or [19] for proofs and details.

A *Coxeter group* is a group W given by generators S, and relations $s^2 = 1$ for all $s \in S$ and the *braid relations* $(st)^{m(s,t)} = 1$ for all $s \neq t \in S$. Each $m(s, t)$ is an integer greater than 1, or is ∞, where $x^\infty = 1$ by convention. Important examples of Coxeter groups include finite reflection groups and Weyl groups. Each element of W can be written (in many different ways) as a word with letters in S. A word a for an element w is called *reduced* if the length (number of letters) of a is minimal among words representing w. The length of a reduced word for w is called the *length* $l(w)$ of w.

The strong Bruhat order on a Coxeter group can be defined in several ways. One way is by the *subword property*. Given $u, w \in W$, say that $u \leq w$ if some reduced word for w contains as a subword some reduced word for u (in which case any reduced word for w contains a reduced word for u). A finite Coxeter group W has an element w_0 of maximal length which is an involution, and which gives rise an anti-automorphism $w \mapsto w_0 w$ of strong Bruhat order. The map $w \mapsto w_0 w w_0$ is an automorphism which permutes the generators S of W.

When J is any subset of S, the subgroup of W generated by J is another Coxeter group, called the *parabolic subgroup* W_J. When the generators of a Coxeter group are denoted as s_i, use shorthand notations such as $J = \{1, 2, 4\}$ to denote the subset $\{s_1, s_2, s_4\} \subseteq S$. The following proposition defines and proves the existence of *two-sided quotients* ${}^J W^K$, where $J, K \subseteq S$, and shows that such quotients are order-quotients. The more widely used one-sided quotients can be obtained by letting $J = \emptyset$.

Proposition 31. *For any $w \in W$ and $J, K \subseteq S$, the double coset $W_J w W_K$ has a unique Bruhat minimal element ${}^J w^K$. If W is finite, the subset ${}^J W^K$ consisting of the minimal coset representatives is an order-quotient of W.*

Proof. The proof of the first statement can be found in [14, Proposition 8.3], where it is also shown that w can be factored (non-uniquely) as $w_J \cdot {}^J w^K \cdot w_K$, where $w_J \in W_J$ and $w_K \in W_K$, such that $l(w) = l(w_J) + l({}^J w^K) + l(w_K)$. Let $\pi_\downarrow : W \to {}^J W^K$ be the projection $w \mapsto {}^J w^K$ onto minimal double coset representatives. We must show that $\pi_\downarrow$ is order-preserving: Suppose $v \leq w$, and write $w = w_J \cdot {}^J w^K \cdot w_K$. Choose reduced words a, b and c for w_J, ${}^J w^K$, and w_K respectively. Since $l(w) = l(w_J) + l({}^J w^K) + l(w_K)$, abc is a reduced word for w. By the subword property, there is a subword of abc which is a reduced word for v. This subword breaks into a', b' and c', which are subwords of a, b and c, respectively. Let x, y, and z be the respective elements represented by a', b' and c'. Thus $v = xyz$, and since $x \in W_J$ and $z \in W_K$, we have $y \in W_J v W_K$. In particular, $y \geq {}^J v^K$, and by the subword property, $y \leq {}^J w^K$, so ${}^J v^K \leq {}^J w^K$.

If W is finite, then multiplication on the left by w_0 is an anti-automorphism of W. If $x \in W_J w W_K$, write $x = w_J w w_K$. Then $w_0 x = w_0 w_J w_0 w_0 w w_K$, and $w_0 w_J w_0 \in W_{w_0 J w_0}$, and so $w_0 x \in W_{w_0 J w_0} w_0 w W_K$. Conversely, if $x \in W_{w_0 J w_0} w_0 w W_K$, then $w_0 x \in W_J w W_K$. Thus left-multiplication by w_0 acts as a Bruhat anti-isomorphism $W_J w W_K \overset{w_0 \cdot}{\mapsto} W_{w_0 J w_0} w W_K$. The maximal element of $W_J w W_K$ is $w_0 m$, where m is the minimal element of $W_{w_0 J w_0} w W_K$. The projection $\pi^\uparrow$ onto the maximal element is order-preserving because it is $w \mapsto w_0 \pi_\downarrow(w_0 w)$. □

In [16, 21] it is shown that join-irreducibles in the strong Bruhat order are always *bigrassmannians*. That is, any join-irreducible x in W is contained in ${}^{S-\{s\}}W^{S-\{t\}}$ for some (necessarily unique) choice of $s, t \in S$. Thus Proposition 27 can be used to simplify the task of finding join-irreducibles and dissectors in W. Abbreviate ${}^{S-\{s\}}W^{S-\{t\}}$ by ${}^s W^t$.

Proposition 32. *For a finite Coxeter group W under the strong Bruhat order:*

(i) $\mathrm{Irr}(W) = \cup_{s,t \in S} \mathrm{Irr}({}^s W^t)$ *and*
(ii) $\mathrm{Dis}(W) = \cup_{s,t \in S} \mathrm{Dis}({}^s W^t)$.

□

Assertion (i) is due to Geck and Kim [16], who used it find the join-irreducibles for the infinite families of finite Coxeter groups and to write a GAP [30] program to compute the join-irreducibles of the other finite Coxeter groups.

Corollary 33. *Let W be a finite Coxeter group. The following are equivalent:*

(i) *The strong Bruhat order on W is dissective.*
(ii) *The strong Bruhat order on ${}^J W^K$ is dissective for any maximal parabolic subgroups J and K.*
(iii) *The strong Bruhat order on ${}^J W^K$ is dissective for any parabolic subgroups J and K.*

Proof. This follows from Proposition 32 and the observation that ${}^{J_1}W^{K_1} \subseteq {}^{J_2}W^{K_2}$ whenever $J_1^c \subseteq J_2^c$ and $K_1^c \subseteq K_2^c$, where $J^c := S - J$. □

We recall the classification of finite irreducible Coxeter groups, traditionally named with letters. There are infinite families A, B and D, indexed by natural numbers n. There are also the exceptional groups E_6, E_7, E_8, F_4, H_3 and H_4, and some groups $I_2(m)$ on two generators. We describe type A here, and refer the reader to [3] or [19] for the other types.

The Coxeter group A_n is isomorphic to the group S_{n+1} of permutations of $[n+1]$. A permutation x can be written in *one-line notation* $x_1x_2\cdots x_{n+1}$, meaning $i \mapsto x_i$ for each i. The generators S are the transpositions $s_i := (i\;\; i{+}1)$, which switch the elements i and $i+1$ and fix all other elements. It is easy to check that A_n is a Coxeter group with $m(s_i, s_j) = 3$ for $|i-j| = 1$ and $m(s_i, s_j) = 2$ for $|i-j| > 1$. The length of an element is the inversion number $\#\{(i,j) : i < j, x_i > x_j\}$. Multiplying a permutation on the right by a generator s_i has the effect of switching the entry x_i with the entry x_{i+1}. Multiplying on the left by s_i switches the entry i with the entry $i+1$.

Elements of $A_n^{s_i}$ are permutations whose one-line notation increases from left to right except possibly between positions i and $i+1$. Bruhat comparisons in $A_n^{s_i}$ can be made by entrywise comparison of the entries from 1 to i. The *Tableau Criterion* characterizes strong Bruhat order on A_n as follows: Let $x = x_1x_2\cdots x_{n+1}$, and form a tableau with rows $T_a(x)$ for each $a \in [n]$, such that $T_a(x) = (T_{a,1}, T_{a,2}, \ldots, T_{a,a})$ is the increasing rearrangement of $\{x_i : i \in [a]\}$.

Proposition 34. *$x \le y$ if and only if $T_{a,b}(x) \le T_{a,b}(y)$ for every $1 \le b \le a \le n$.* □

The tableau $T(x)$ is a special case of a *monotone triangle.* A monotone triangle of size n is a tableau of staircase shape, with n rows and n columns, with entries from $[n+1]$, such that rows are strictly increasing, columns are weakly decreasing and elements are weakly increasing in the southeast (↘) direction. The permutations are exactly the monotone triangles such that for every $1 \le b \le a < n$, either $T_{a,b} = T_{a+1,b}$ or $T_{a,b} = T_{a+1,b+1}$. The tableau criterion states that strong Bruhat order is the restriction to permutations of componentwise order on monotone triangles. There is a simple bijection between monotone triangles and *alternating sign matrices* [28].

Given a permutation $x = x_1x_2\cdots x_{n+1}$, form $\alpha(x) = y_1y_2\cdots y_{n+1}$ according to $y_i = n + 2 - x_i$. It is easily checked that α is the anti-automorphism $w \mapsto w_0w$ of the strong Bruhat order. The operation of α on tableaux is to replace each entry a by $n + 2 - a$, and to reverse the order of entries within the rows.

For the purposes of order dimension there is a much better tableau criterion [3, Exercise 2.13] than Proposition 34. Given a permutation in $A_n^{s_i}$ the entrywise comparison of the entries from 1 to i is dual to the entrywise comparison of the entries from $i + 1$ to $n + 1$. Given $x \in A_n$ define a pair (L, R) of tableaux of staircase shape, where L is the increasing rearrangements of the initial segments of x of lengths $\le \left\lfloor \frac{n+1}{2} \right\rfloor$, and R is the increasing rearrangements of the final segments of x of lengths $\le n - \left\lfloor \frac{n+1}{2} \right\rfloor$.

Proposition 35. *$x \le y$ if and only if $L(x) \le L(y)$ and $R(x) \ge R(y)$ componentwise.* □

Note that the existence of this "two-tableau criterion" is related to the existence of the symmetry $w \mapsto w_0ww_0$ in the Coxeter group A_n.

The total number of entries in (L, R) is

$$\binom{\left\lfloor \frac{n+1}{2} \right\rfloor + 1}{2} + \binom{n + 1 - \left\lfloor \frac{n+1}{2} \right\rfloor}{2} = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor.$$

Thus the bound $\dim(A_n) \leq \frac{(n+1)^2}{4}$ was already known. However, the proof of Theorem 1 does not explicitly use Proposition 35. In light of Theorem 1, from the viewpoint of order dimension this two-tableaux criterion is an optimal encoding of Bruhat order. Whether this embedding is actually the fastest way to compute Bruhat order is not quite the same question. Another simplification of the tableau criterion is given in [4]. Here it is shown that for a given x, one need only consider certain rows of $T(x)$ and $T(y)$, depending on the descents of x and y, to compare x to y. This simplification does not affect order dimension, but may speed up computations.

An element $w \in A_n$ is called *321-avoiding* if any of the following equivalent [1, Theorem 2.1] conditions holds:

(i) Let w correspond to a permutation with one-line form $w_1 w_2 \cdots w_{n+1}$. There exist no i, j, k with $1 \leq i < j < k \leq n+1$ such that $w_i > w_j > w_k$.

(ii) Let a be a reduced word for w. For all $i \in [n]$, between any two instances of s_i in a, the letters s_{i-1} and s_{i+1} occur. In particular, s_1 and s_n each occur at most once.

(iii) Any two reduced words for w are related by commutations.

The following is immediate by characterization (iii):

Proposition 36. *Let w be a 321-avoiding element of A_n, let $s_1 s_2 \cdots s_k$ be a reduced word for w, and let $s_{i_1} s_{i_2} \cdots s_{i_j}$ be a subword with $|i_m - i_{m+1}| = 1$ for every $m \in [j-1]$. Then $s_{i_1} s_{i_2} \cdots s_{i_j}$ occurs as a subword of every reduced word for w.* □

8. Order Dimension of Strong Bruhat Order on Type A

In [21], Lascoux and Schützenberger show that strong Bruhat order on Coxeter groups of type A or B is dissective and identify the join-irreducibles of A_n. In this section, we review their results for type A, determine the partial order induced on $\mathrm{Irr}(A_n)$ and determine its width. We then apply Theorem 3 to determine the order dimension of the strong Bruhat order of type A_n, and of all one-sided quotients.

For any $1 \leq b \leq a \leq n$, and $b \leq c \leq n - a + b + 1$, define $J_{a,b,c}$ to be the componentwise smallest monotone triangle such that the a, b entry is $\geq c$. It is easily checked that the permutation

$$1 \cdots (b-1)c \cdots (c+a-b)b \cdots (c-1)(c+a-b+1) \cdots (n+1)$$

gives rise to a tableau which fits the description of $J_{a,b,c}$ (where "$i \cdots j$" is $i(i+1) \cdots (j-1)j$). If $b = c$, then $J_{a,b,c}$ is the tableau associated to the identity permutation. A monotone triangle T is the join of $\{J_{a,b,T_{a,b}} : 1 \leq b \leq a \leq n\}$. Thus

$$\mathrm{Irr}(A_n) \subseteq \{J_{a,b,c} : 1 \leq b \leq a \leq n, b < c \leq n - a + b + 1\}.$$

Given $1 \leq b \leq a \leq n$, and $b < c \leq n - a + b + 1$, define $M_{a,b,c}$ to be the componentwise largest monotone triangle whose a, b entry is $< c$. The tableau $M_{a,b,c}$ can be found by applying the anti-symmetry $w \mapsto w_0 w$ to $J_{a,a-b+1,n+3-c}$. It is the permutation

$$(n+1) \cdots (n-a+b+2)(c-1) \cdots (c-b)(n-a+b+1) \cdots c(c-b-1) \cdots 1.$$

Thus each $J_{a,b,c}$ for $1 \leq b \leq a \leq n$, and $b < c \leq n - a + b + 1$ is a dissector with $\beta(J_{a,b,c}) = M_{a,b,c}$. As a result,

$$\mathrm{Irr}(A_n) = \{J_{a,b,c} : 1 \leq b \leq a \leq n, b < c \leq n - a + b + 1\},$$

A_n is dissective, and by Theorem 7, the MacNeille completion of A_n is the distributive lattice of monotone triangles [21]. Thus the order-dimension of A_n under the strong order is equal to the order dimension of the lattice of monotone triangles of the same size.

The partial order induced on $\text{Irr}(A_n)$ is studied using the subword definition of strong Bruhat order. It is convenient to fix a particular word for the maximal element of A_n, and also to write the word as an array:

$$w_0 = \begin{array}{llllll} s_1 & & & & & \\ s_2 & s_1 & & & & \\ s_3 & s_2 & s_1 & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ s_n & s_{n-1} & \cdot & \cdot\ \cdot & s_2 & s_1 \end{array}$$

Reading the array in the standard order for reading English text gives a word for $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots$. Elements of A_n are in bijection with left-justified subsets of the array. It is easily seen [16, 21] that $\text{Irr}(A_n)$ consists of left-justified rectangles in the array. That is, an element is join-irreducible if and only if its left-justified form is:

$$\begin{array}{ccccc} s_j & s_{j-1} & s_{j-2} & \cdot\ \cdot\ \cdot & s_{j-i+1} \\ s_{j+1} & s_j & s_{j-1} & \cdot\ \cdot\ \cdot & s_{j-i+2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ s_{j+k-1} & s_{j+k-2} & s_{j+k-3} & \cdot\ \cdot\ \cdot & s_{j+k-i} \end{array}$$

for some $1 \leq i \leq j \leq n$, and $k \leq n-j+1$. Counting such rectangles shows that there are $\binom{n+2}{3}$ join-irreducibles in A_n. Refer to these rectangles and the corresponding irreducibles by the triples (i,j,k). A triple (i,j,k) corresponds to the tableau $J_{j-i+k,j-i+1,j+1}$.

Example 37. The monotone triangle $J_{5,3,5}$ in A_7 and the corresponding rectangle $(2,4,3)$ are shown below. The corresponding permutation is 12567348.

$$\begin{array}{llllllll} 1 \\ 1 & 2 \\ 1 & 2 & 5 \\ 1 & 2 & 5 & 6 \\ 1 & 2 & 5 & 6 & 7 \\ 1 & 2 & 3 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \qquad \begin{array}{ll} s_4 & s_3 \\ s_5 & s_4 \\ s_6 & s_5 \end{array}$$

A criterion is given in [21] for deciding whether a given permutation is above a given join-irreducible. For now, we are interested in an easy criterion for comparing two join-irreducibles. A subrectangle of (i,j,k) is a rectangle that can be obtained by deleting columns from the left and/or right of (i,j,k) and/or deleting rows from the top and/or bottom of (i,j,k).

Proposition 38. *Join-irreducibles $u = (i,j,k)$ and $v = (i',j',k')$ in A_n have $u \leq v$ if and only if (i,j,k) is a sub-rectangle of (i',j',k').*

Proof. The "if" direction follows immediately from the subword property.

Suppose $u \leq v$. The subword property requires that some reduced word for the rectangle (i,j,k) be a subword of the rectangle-word for (i',j',k'). It is easily verified that (i,j,k) stands for a 321-avoiding element of A_n. Notice also that the rectangle form for (i,j,k) has a subword

$$s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1} s_{j-i+2} \cdots s_{j-i_k-1} s_{j-i+k}$$

which satisfies the hypotheses of Proposition 36. Therefore, the subword of (i',j',k') which is a reduced word for (i,j,k) must itself contain the same subword. For the word given by (i',j',k') to contain the letters $s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1}$ in that order, in particular, it must contain the letter s_{j-i+1} somewhere after an occurrence of s_j. Thus there is a row in the rectangle for v containing $s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1}$. For the letters $s_{j-i+2} s_{j-i+1} s_{j-i+2} \cdots s_{j-i_k-1} s_{j-i+k}$ to occur afterwards, there must be at least $k-1$ more rows. □

There are four types of covers in $\mathrm{Irr}(A_n)$, corresponding to striking the left or right column or the top or bottom row from a rectangle. Of course, a column can only be deleted if there is more than one column present, and similarly for rows. Thus a rectangle (i,j,k) covers the following rectangles:

$$\begin{aligned} (i-1,j,k) &\quad \text{if } i>1, \\ (i-1,j-1,k) &\quad \text{if } i>1, \\ (i,j+1,k-1) &\quad \text{if } k>1, \\ (i,j,k-1) &\quad \text{if } k>j. \end{aligned}$$

The minimal rectangles are $(1,j,1)$ for $j \in [n]$, and the maximal elements are $(i,i,n-i+1)$ for $i \in [n]$. Also, $\mathrm{Irr}(A_n)$ is ranked by $r(i,j,k) = i+k-1$, with the lowest rank being 1 and the highest rank being n. A diagram of $\mathrm{Irr}(A_4)$ is given in Example 39 below.

The rank number $R_r(\mathrm{Irr}(A_n))$ is determined by counting the number of ways to choose i, j and k subject to the constraints: $1 \leq i \leq j \leq n$, $1 \leq k \leq n-j+1$, and $i+k-1=r$. Necessarily, $i \in [r]$ (otherwise, $i+k-1 \geq i > r$) and j must be chosen so that $1 \leq i \leq j \leq n$, and $1 \leq r-i+1 \leq n-j+1$, or equivalently, so that $i \leq j \leq n+i-r$. Thus,

$$R_r(\mathrm{Irr}(A_n)) = r(n-r+1). \tag{4}$$

The maximum rank number is $R_{\lfloor \frac{n+1}{2} \rfloor} = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$. Thus, in order to verify the statement in Theorem 1, it only remains only to prove Theorem 4, which asserts that $\mathrm{Irr}(A_n)$ has a symmetric chain decomposition. In particular, Theorem 4 implies that $\mathrm{Irr}(A_n)$ is Sperner, so its width is equal to its maximum rank number.

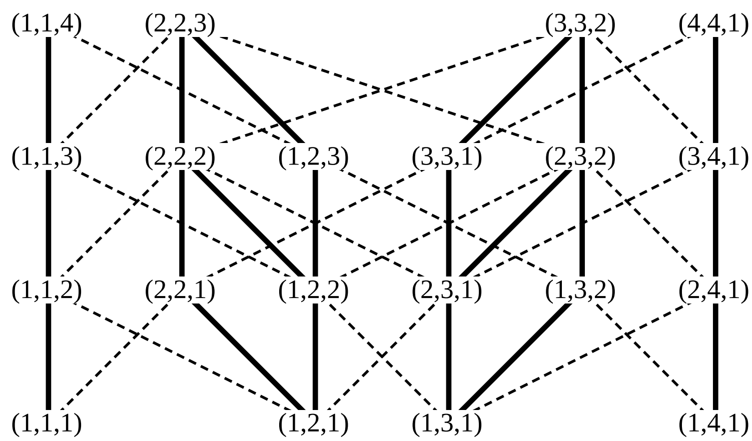
Proof of Theorem 4. Restrict to a weaker order on $\mathrm{Irr}(A_n)$, by allowing a rectangle (i,j,k) to cover only

$$\begin{aligned} (i-1,j,k) &\quad \text{if } i>1 \text{ or,} \\ (i,j,k-1) &\quad \text{if } k>j. \end{aligned}$$

In other words, restrict the covers by only allowing the rightmost column or the bottom row to be deleted. Call this weaker order $\mathrm{Irr}'(A_n)$. Then $\mathrm{Irr}'(A_n)$ consists of n disjoint components, each of which is isomorphic to a product of chains: For each $j \in [n]$, there is a maximal element $(j,j,n-j+1)$ in $\mathrm{Irr}(A_n)$ and the interval below $(j, j, n-j+1)$ in $\mathrm{Irr}'(A_n)$ is isomorphic to the product of chains $[j]\times[n-j+1]$. Thus $\mathrm{Irr}'(A_n)$ has a symmetric chain decomposition. Since $\mathrm{Irr}'(A_n)$ is ranked with the same rank function as $\mathrm{Irr}(A_n)$, the symmetric chain decomposition is inherited by $\mathrm{Irr}(A_n)$. □

Incidentally, the covers in $\mathrm{Irr}'(A_n)$ are exactly the covers in $\mathrm{Irr}(A_n)$ which are order relations in the right weak Bruhat order.

Example 39. A diagram of $\mathrm{Irr}(A_4)$. The dotted lines are covers which are not in $\mathrm{Irr}'(A_4)$.



By Corollary 33, any one-sided or two-sided quotient of A_n is a dissective poset. The same symmetric chain decomposition proves the following:

Theorem 40. *The order dimension of a one-sided quotient A_n^J of A_n, is:*

$$\dim(A_n^J) = \sum_{j\in(S-J)} \min(j, n-j+1).$$

Proof. The symmetric chain decomposition given for $\mathrm{Irr}(A_n)$ arises from symmetric chain decompositions of the components of $\mathrm{Irr}'(A_n)$. Each such component is $\mathrm{Irr}(A_n)\cap {}^{[n]-j}W$ for some j. Thus the same symmetric chain decomposition can be given to $\mathrm{Irr}({}^JW)$ for any J. The quotients ${}^JW$ and W^J are isomorphic by the map which takes w to w^{-1}. □

9. Other Types

Type B. The Coxeter group B_n is the group of *signed permutations.* Signed permutations are permutations x of $\pm[n] := [-n, n] - \{0\}$ subject to the condition that $x(-a) = -x(a)$ for each $a \in [n]$. The generators S are the transpositions $s_i := (i\ \ i{+}1)$ for each $i \in [n-1]$, and the transposition $s_0 := (-1\ \ 1)$. There is a Signed Tableau Criterion for B_n which associates to each signed permutation in B_n a *signed monotone triangle* of size n: A tableau of staircase shape, with n rows and n columns, with entries from $\pm[n]$, with $+i$ and $-i$ never occurring in the same signed triangle. Also, the rows are required to be strictly increasing, columns weakly decreasing, and elements weakly increasing in the southeast (↘) direction. The signed permutations are exactly the signed monotone triangles such that for every $1 \le b \le a < n$, either $T_{a,b} = T_{a+1,b}$ or $T_{a,b} = T_{a+1,b+1}$. The Signed Tableau Criterion for B_n states that strong Bruhat order is dual to the restriction of componentwise order on signed monotone triangles. The two-tableaux criterion in type A was related to the symmetry $w \mapsto w_0ww_0$. Since $w \mapsto w_0ww_0$ is the identity on B_n, one might not expect to find a great improvement over the Signed Tableau Criterion. And indeed, the order dimension of B_n is not much lower than the upper bound given by the Signed Tableau Criterion.

A characterization of the join-irreducibles of B_n as reduced words is in [16, 21]. In [27] they are characterized by tableaux in a manner similar to the $J_{a,b,c}$ in type A. The calculation of the width of $\text{Irr}(B_n)$ is complicated by the fact that $\text{Irr}(B_n)$ is not graded. For example, in $\text{Irr}(B_3)$ there are maximal chains of lengths 5 and 6 sharing the same top and bottom element. The width is calculated by exhibiting a chain decomposition and an antichain of the same size. The details of this calculation, which proves the statement about $\dim(B_n)$ in Theorem 2, will appear in [27]. More specifically, it is proven that

Theorem 41. *The order dimension of a one-sided quotient B_n^J of B_n, is:*

$$\dim(B_n^J) = \begin{cases} \sum_{j \in J^c} n - j & \text{if } 0 \in J \\ \lfloor \frac{n+1}{2} \rfloor + \sum_{j \in J^*} n - j & \text{if } 0 \notin J, 1 \in J \\ n + \sum_{j \in J^*} n - j & \text{if } \{0,1\} \cap J = \emptyset \end{cases}$$

Here $J^* = J^c \cap [2, n-1]$.

Type H. Type H contains two groups H_3 and H_4, the symmetry groups of the icosahedron and the 600-cell respectively. Since H_3 and H_4 are dissective, their order dimensions can be calculated as the width of their subposet of irreducibles. We used the `GAP` [30] program `brbase` [16] and the package `CHEVIE` [15] to find $\text{Irr}(H_3)$ and $\text{Irr}(H_4)$. Then we used a program written in Prolog to calculate widths, obtaining the results in Theorem 2. For details of this calculation, see [27].

Type I. Type I consists of the dihedral groups, each with two generators s and t. The m in $I_2(m)$ is $m(s,t)$. In $I_2(m)$, every element except the identity and w_0 is a dissector. The group $I_2(6)$ is also called G_2.

Types D, E and F. Theorem 6 enables the computer to set bounds on the order dimensions of some groups of types D, E and F. We used `brbase` to find the bigrassmannians of several groups, and then a Prolog program to find Irr and Dis and calculate widths. The results are:

$$\begin{array}{rcl} 6 \le & \dim(D_4) & \le 9 \\ 10 \le & \dim(D_5) & \le 14 \\ 14 \le & \dim(D_6) & \le 22 \\ 18 \le & \dim(D_7) & \end{array}$$

$$\begin{aligned} 14 \leq \quad & \dim(E_6) \quad \leq 26 \\ 18 \leq \quad & \dim(E_7) \\ \\ 10 \leq \quad & \dim(F_4) \quad \leq 12. \end{aligned}$$

Further width calculations were beyond the ability of a fast computer to perform even for run times of about two weeks. Also, it appears that the bounds obtained in this way continue to worsen with increasing numbers of generators, because the number of join-irreducibles appears to grow more rapidly than the number of dissectors.

10. Further questions

Strong Bruhat Order and Alternating Sign Matrices.

(1) Determine the order dimension of
  (a) The strong Bruhat order on the other finite Coxeter groups. Give a uniform treatment, independent of the classification.
  (b) Intervals in the strong Bruhat order.
  (c) Two-sided quotients of types A, B and H. These are all dissective by Corollary 33. All of the two-sided quotients by maximal parabolic subgroups in type A are one-dimensional.
  (d) The weak Bruhat order on a finite Coxeter group. The lower bound given by Theorem 6 is just the number of generators, and the upper bound of Theorem 6 appears to be much larger than the known upper bound—the number of reflections (cf. [3, Exercise 3.2]). (**Note added in proof:** For type A, the order dimension of the weak Bruhat order was determined by Flath [13] in 1993. Recently, using methods from the study of hyperplane arrangements, the author determined the order dimension of the weak Bruhat order for types A and B [26]. For both of these types, the order dimension is equal to the number of generators.)

(2) Some ideals in $\mathrm{Irr}(A_n)$ correspond to elements of A_n and some do not. Give a purely order-theoretic characterization of the order ideals which are elements of A_n. A necessary but not sufficient condition on an ideal $I \subseteq \mathrm{Irr}(A_n)$ is that $\max(I) \cup \min(I^c)$ is an antichain.

(3) What statistic on permutations is $|I_x|$, the number of join-irreducibles below $x \in A_n$? The distributions for $n = 1, 2$ and 3 are:

$$1 + q,$$

$$1 + 2q + 2q^3 + q^4,$$

$$1 + 3q + q^2 + 4q^3 + 2q^4 + 2q^5 + 2q^6 + 4q^7 + q^8 + 3q^9 + q^{10}.$$

(4) The MacNeille completion $L(A_n)$ is the componentwise order on monotone triangles, which biject with alternating sign matrices. Does $L(B_n)$ have any connection to alternating sign matrices with symmetry conditions? Since the set of signed monotone triangles is not closed under entrywise meet, $L(B_n)$ is larger than the set of all signed monotone triangles. Okada [24] has type-B and type-C Weyl denominator formulas which are expressed in terms of alternating sign matrices with half-turn symmetry. However, the

numbers of such matrices do not agree with the number of order ideals in $\mathrm{Irr}(B_n)$.

(5) Find a bijection between order ideals in $\mathrm{Irr}(A_n)$ and descending plane partitions [28].

(6) Study the lattice quotients induced on the componentwise order on monotone triangles by quotients of the strong Bruhat order on A_n, as in Theorem 8. This is not as simple as one might guess. For example, the congrence on A_3 obtained from the subgroup $\{1, s_1\}$ induces a lattice congruence on the lattice of monotone triangle which has 15 congruence classes, rather than the 12 one would expect.

Order Dimension.

(1) Is there any condition weaker than requiring that a poset P be dissective, that would imply $\dim(P) = \mathrm{width}(\mathrm{Irr}(P))$? Is there any condition that would imply $\dim(P) = \mathrm{width}(\mathrm{Dis}(P))$?

(2) Develop efficient algorithms for finding the critical complex of a poset. If this can be done, covering sets, and thus order dimension, can in principle be determined or approximated by linear programming [18].

Lattice Properties for Posets.

(1) Find other naturally occurring examples of dissective posets. One possibility is the "strong Bruhat order" on complete matchings on $[2n]$ defined in [8], which is dissective at least for $n \leq 3$.

(2) What is the right generalization of modularity to posets [20]? In other words, is there a simple order-theoretic condition on P that is equivalent to requiring that $L(P)$ be modular?

(3) Find other naturally occurring examples of congruences and quotients of non-lattices.

(4) The lattice $\mathrm{Con}(L)$ of congruences of a lattice L has been studied extensively [17]. Similar questions can be asked about the poset $\mathrm{Con}(P)$ of a poset. Under what conditions is it a lattice? a meet-semilattice? How is $\mathrm{Con}(P)$ related to $\mathrm{Con}(L(P))$?

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