Patterns in the Generalized Fibonacci Word, Applied to Games

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Abstract

We analyze a natural generalization, $\mathcal{W}$, of the infinite Fibonacci word over the alphabet $\Sigma = \{a, b\}$. We provide tools to represent explicitly the set $\{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s) = b,\ \mathcal{W}(s + x) = a\}$ for any fixed positive integer x. We show how this representation can be used to analyze the preservation of P-positions of any game whose P-positions are a pair of complementary Beatty sequences, in particular a certain generalization of Wythoff Nim [13], [8].

Keywords: generalized Fibonacci word, Wythoff Nim, P-positions, Invariance

1 Introduction

1.1 The generalized Fibonacci word

Consider the alphabet $\Sigma = \{a, b\}$. Starting with the letter a, and iteratively replacing a with ab and b with a ($a \to ab \to aba \to abaab \to \cdots$), one obtains the well known infinite Fibonacci word, $\mathcal{F} = \mathcal{F}(0)\mathcal{F}(1)\mathcal{F}(2)\cdots = abaababaabaab\cdots$. See, for example, [19, ch. 1].

It is known that the positions of the a's are given by $\{\lfloor \alpha n \rfloor - 1 : n \geq 1\}$ and the positions of the b's are given by the complementary sequence $\{\lfloor \beta n \rfloor - 1 : n \geq 1\}$, where here α is the golden ratio and $\beta = \alpha + 1$.

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The infinite Fibonacci word can be generalized to any irrational $\alpha > 1$: The two sequences $\{\lfloor \alpha n \rfloor - 1 : n \geq 1\}$ and $\{\lfloor \beta n \rfloor - 1 : n \geq 1\}$, where $1/\alpha + 1/\beta = 1$, are a pair of complementary Beatty sequences (see [2], [8], [10]). Thus, there is a unique infinite word, $\mathcal{W} = \mathcal{W}[\alpha]$, for which the positions of the a's are given by the first sequence and the positions of the b's are given by the second. Throughout the paper, α and β will always be positive and irrational, and will satisfy the equality $1/\alpha + 1/\beta = 1$.

In this paper we consider simple patterns that appear in $\mathcal{W}$. For example, consider the set, S_x, of indices s such that $\mathcal{W}(s) = b$ and $\mathcal{W}(s + x) = a$ for some fixed x. We present tools that enable us to write explicit formulas for such sets, and specifically use them to study S_x.

Below we introduce an infinite number of combinatorial games, which we dub Beatty games, which constitute the motivation to study the set S_x.

1.2 Beatty games

In his paper from 1907 (see [21]), Wythoff describes a two-player game played on two piles of tokens, based on the well known game Nim: In each turn, a player is allowed to remove any positive amount of tokens from a single pile (Nim move) or the same amount of tokens from both piles (diagonal move). The player making the last move wins. For the uninitiated, we point to the following three papers, from among the extensive literature on Wythoff Nim: [5], [8], [22].

In [13] and in [8] the authors suggest a natural generalization of this game, called k-Wythoff Nim: A player is allowed, in addition to the Nim moves, to remove x tokens from one pile and y tokens from the other, provided that $|x - y| < k$, where $k \in \mathbb{Z}_{\geq 1}$ is a parameter of the game.

A well known result of Combinatorial Game Theory[1] states that in every finite game, every position is either an N-position – a position from which the **N**ext player can win, no matter what the opponent does, or a P-position – a position from which the **P**revious player can win. The set of P-positions is denoted $\mathcal{P}$.

The two papers [13] and [8] show that for k-Wythoff Nim, the set of P-positions is given by $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where $\alpha = (2-k+\sqrt{k^2+4})/2$ and $\beta = 1 + 1/(\alpha - 1) = \alpha + k$.

[1]For general references on Combinatorial Game Theory see, for example, [4], [3], [1] and [20].

A game in which every move can be played from any position (as long as the number of tokens is never negative) is called an *invariant* game. k-Wythoff Nim is an example of such a game. In [7] it was conjectured that for *all* irrationals $1 < \alpha < 2 < \beta$ such that $1/\alpha + 1/\beta = 1$, there exists an invariant game for which the set of P-positions is given by $\mathcal{P} = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$. The conjecture was proven in [16] using the $\star$-operator: This operator takes an invariant game to the invariant game in which the moves are the P-positions of the original game (except for $(0, 0)$). The authors prove that applying the $\star$-operator to a game whose *moves* are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 1}\}$ results in the desired invariant game. For more information about the $\star$-operator, see [15]. So here is an infinity of games whose P-positions are known, but their rulesets are not given by a simple explicit formula! We dub all games whose set of P-position is of this form *Beatty games*. Note that from the previous paragraph, it follows that k-Wythoff Nim is a Beatty game.

The question of preservation of $\mathcal{P}$ under the operation of adding invariant moves to Wythoff Nim ($k = 1$) was introduced in [6]. The authors provided an efficient algorithm which determines whether adding a specific move of the form: take x tokens from one pile and y from the other, to the game changes $\mathcal{P}$.

The present study generalizes this question to all Beatty games. We will see in Section 7 that we can use the analysis of the set S_x to obtain formulas for the pattern of those subtractions that are "forbidden" because adding them changes $\mathcal{P}$. In fact, the main advance of this paper over [6] is twofold: (i) Extension from the special case of the golden ratio to any irrational in $(1, 2)$. (ii) Given integers x, y, [6] provided a polynomial algorithm to decide whether (x, y) can be added as an invariant move to Wythoff Nim. In the present paper, the pattern of all y's for a fixed x is determined polynomially as shown in Section 6.3.

1.3 Outline

The paper is structured as follows:

Section 2 gives two different definitions of $\mathcal{W}[\alpha]$: the first is the definition that appears at the beginning of this introduction, and the second is based on morphisms of words. We prove their equivalence, and then list basic properties of $\mathcal{W}[\alpha]$.

In order to derive an explicit formula for S_x, we will need to define some tools useful for the analysis of $\mathcal{W}[\alpha]$ – this will be done in Section 3. Specifically, consider the partition of the natural numbers, based on the positions of a and b in $\mathcal{W}[\alpha]$. We will see that, in some sense, not all the a's and not all the b's are "the same". This will lead to another partition, finer than the previous one. In fact, we will get an infinite sequence of partitions, each finer than its predecessor. Proposition 3 will give an explicit formula for the sets in each of these partitions.

Sections 4-6 show how to write S_x as a finite union of sets from these partitions. Together with the explicit formula for these sets, we obtain an explicit formula for S_x, and an efficient algorithm (Section 6.3), that given x, outputs this formula. In fact, S_x will turn out to be

$$\dot{\bigcup}_{i=1}^{k}\{A_i\lfloor\gamma_i n\rfloor + B_i n + C_i : n \in \mathbb{Z}_{\geq 1}\} \tag{1}$$

for some $k \in \mathbb{Z}_{\geq 1}$, $A_i, B_i, C_i \in \mathbb{Z}$ and $\gamma_i \in \mathbb{R}$ (the algorithm will output these values for any given x), where $\dot{\cup}$ denotes disjoint union. To attain this representation, we employ four steps:

1. Solve a simpler problem for specific values of x (Section 4.1).
2. Generalize to an arbitrary x (Section 4.2).
3. Find a formula for S_x, similar to (1), except for the fact that the disjoint union is replaced by a symmetric difference (Section 5).
4. Finally, convert the symmetric difference to a disjoint union (Section 6).

Sections 7-8 contain an application of the study of S_x, to the preservation of P-positions in Beatty games under the operation of adding moves.

1.4 Notation

For a set $A \subseteq \mathbb{Z}$ and fixed integer x, let $A - x = \{a - x : a \in A\}$ and $A \dot{-} x = (A - x) \cap \mathbb{Z}_{\geq 0}$.

Let A_i be a finite family of sets. The symmetric difference $\triangle_i A_i$ is the set of all elements which are in an odd number of sets A_i. The disjoint union $\dot{\cup}_i A_i$ is defined if and only if the sets A_i are pairwise disjoint, in which case it equals the union of the sets. Write $A_1 \triangle A_2$ and $A_1 \dot{\cup} A_2$ for the case of two sets.

Let $y \in \mathbb{R}$. Denote its integer part by $\lfloor y \rfloor$ and its fractional part by $\{y\}$, so $y = \lfloor y \rfloor + \{y\}$, $\lfloor y \rfloor \in \mathbb{Z}$ and $\{y\} \in [0, 1)$.

Every continued fraction alluded to in the sequel is a *simple* continued fraction (with numerators 1, denominators positive integers): $[a_0; a_1, a_2, \ldots] = a_0 + 1/(a_1 + 1/(a_2 + \cdots))$. See [12, ch. 10].

Let Σ be a finite alphabet of letters. Then, Σ^* is the free monoid over Σ and its elements are the finite words over Σ. Let $\varepsilon \in \Sigma^*$ denote the empty word. For $w \in \Sigma^*$, let $|w|$ denote the length of w, counting multiplicities, and let $|w|_\sigma$ denote the number of occurrences of the letter $\sigma \in \Sigma$ in w. We refer to the i-th letter of w by $w(i)$ and we use the index 0 for the first letter. In other words, $w = w(0)w(1)\cdots w(|w|-1)$. We say that $\varphi : \Sigma^* \to \Sigma^*$ is a morphism of words if and only if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Sigma^*$. General references about words and morphisms of words are [18], [19].

2 A generalization of the Fibonacci word

Start by considering the following natural generalization of the infinite Fibonacci word to any irrational $\alpha > 1$:

Definition 1. For $\alpha \in (1, \infty) \setminus \mathbb{Q}$, $\mathcal{W}[\alpha]$ is the infinite word over $\{a, b\}$, for which the positions of the a's are given by $\lfloor \alpha n \rfloor - 1$ ($n \in \mathbb{Z}_{\geq 1}$), and the positions of the b's are given by $\lfloor \beta n \rfloor - 1$ ($n \in \mathbb{Z}_{\geq 1}$), where $1/\alpha + 1/\beta = 1$.

Notice that the two sequences: $\{\lfloor \alpha n \rfloor - 1 : n \in \mathbb{Z}_{\geq 1}\}$ and $\{\lfloor \beta n \rfloor - 1 : n \in \mathbb{Z}_{\geq 1}\}$ are a pair of complementary Beatty sequences (see [2]) and therefore partition $\mathbb{Z}_{\geq 0}$, and so $\mathcal{W}[\alpha]$ is well-defined. Also, this is indeed a generalization of the infinite Fibonacci word, as $\mathcal{F} = \mathcal{W}[(1+\sqrt{5})/2]$.

Example 1.

$$\begin{aligned}\mathcal{W}[[1;1,1,1,\ldots]] &= abaababaabaababaababaabaababaabaababaaba\cdots = \mathcal{F},\\ \mathcal{W}[[1;1,2,3,\ldots]] &= ababaababaababaababababaababaababaababababaa\cdots.\end{aligned}$$

The latter word can be computed by determining the first few convergents of $\alpha = [1; 1, 2, 3, \ldots]$, specifically, $\alpha \approx p_5/q_5 = 382/225 \approx 1.6977778$ which gives α with an error $< 1/225^2 = 1/50,625$. Another method of computation is indicated in a remark following Proposition 1.

The following two definitions enable us to characterize $\mathcal{W}[\alpha]$ based on morphisms of words.

Definition 2. Let $t \in \mathbb{Z}_{\geq 1}$. The morphism $\varphi_t : \{a, b\}^* \to \{a, b\}^*$ is defined by:

$$\varphi_t(a) = a^t b, \qquad \varphi_t(b) = a.$$

Definition 3. Let $\tau_1, \tau_2, \ldots$ be an infinite sequence of morphisms such that for each i, $\tau_i(a)$ begins with an a. Define their *infinite product* $\tau_1 \tau_2 \cdots (a)$ to be the word:

$$\lim_{n \to \infty} \tau_1 \tau_2 \cdots \tau_n(a).$$

Note that since $\tau_1 \cdots \tau_n(a)$ is a prefix of $\tau_1 \cdots \tau_{n+1}(a)$, the limit in the previous definition is well-defined. If $\tau_i(\sigma) \neq \varepsilon$ and $|\tau_i(a)| > 1$ for every i and σ, then $\tau_1 \tau_2 \cdots (a)$ is an *infinite* word.

The following theorem demonstrates an interesting connection between $\mathcal{W}[\alpha]$, the continued fraction of α and the morphism defined in Definition 2.

Theorem 1. *If* $\alpha = [1; t_1, t_2, t_3, \ldots]$ *then* $\mathcal{W}[\alpha] = \varphi_{t_1} \varphi_{t_2} \varphi_{t_3} \cdots (a)$.

To prove this theorem we will need the following lemma:

Lemma 1. *Let* $\alpha \in (1, \infty) \setminus \mathbb{Q}$ *and* $t \in \mathbb{Z}_{\geq 1}$. *Then,*

$$\varphi_t(\mathcal{W}[\alpha]) = \mathcal{W}[1 + 1/(\alpha - 1 + t)].$$

Proof. Define two morphisms: let μ be the morphism that sends $a \mapsto b$ and $b \mapsto a$ and let ν_t be the morphism that sends $a \mapsto b^t a$ and $b \mapsto b$.

We show that

$$\mu(\mathcal{W}[\alpha]) = \mathcal{W}[1 + 1/(\alpha - 1)], \qquad \nu_t(\mathcal{W}[\alpha]) = \mathcal{W}[\alpha + t].$$

Let $\beta = 1 + 1/(\alpha - 1)$ so that $1/\alpha + 1/\beta = 1$. The sequences $\{\lfloor n\alpha \rfloor - 1\}_{n=1}^{\infty}$, $\{\lfloor n\beta \rfloor - 1\}_{n=1}^{\infty}$ partition the set $\mathbb{Z}_{\geq 0}$. The elements of $\{\lfloor n\alpha \rfloor - 1\}_{n=1}^{\infty}$ are the positions of the a's of $\mathcal{W}[\alpha]$. Therefore, the elements of $\{\lfloor n\beta \rfloor - 1\}_{n=1}^{\infty}$ are the positions of the a's of $\mu(\mathcal{W}[\alpha])$, and thus $\mu(\mathcal{W}[\alpha]) = \mathcal{W}[\beta]$.

For ν_t, notice that the positions of the a's of $\mathcal{W}[\alpha + t]$ are given by $\lfloor (\alpha + t)n \rfloor - 1 = \lfloor \alpha n \rfloor - 1 + nt$. So in order to go from $\mathcal{W}[\alpha]$ to $\mathcal{W}[\alpha + t]$ we have to insert b^t to the left of each a. This is exactly the morphism ν_t.

As $\varphi_t = \mu \nu_t$, we obtain,

$$\varphi_t(\mathcal{W}[\alpha]) = \mu \nu_t(\mathcal{W}[\alpha]) = \mu(\mathcal{W}[\alpha + t]) = \mathcal{W}[1 + 1/(\alpha - 1 + t)]. \qquad \square$$

Notation. Let $\alpha = [1; t_1, t_2, t_3, \ldots]$. For $n \in \mathbb{Z}_{\geq 0}$, define

$$\alpha_n = [1; t_{n+1}, t_{n+2}, \ldots]. \tag{2}$$

Note that $\alpha_0 = \alpha$.

Proof of Theorem 1. It is easy to see that $\alpha_{n-1} = 1 + 1/(\alpha_n - 1 + t_n)$ for $n \geq 1$. Together with the previous lemma, it follows that

$$\varphi_{t_n}(\mathcal{W}[\alpha_n]) = \mathcal{W}[\alpha_{n-1}],$$

and therefore, by induction on n,

$$\mathcal{W}[\alpha] = \mathcal{W}[\alpha_0] = \varphi_{t_1}\varphi_{t_2}\cdots\varphi_{t_n}(\mathcal{W}[\alpha_n]).$$

Since a is a prefix of $\mathcal{W}[\alpha_n]$, $\varphi_{t_1}\varphi_{t_2}\cdots\varphi_{t_n}(a)$ is a prefix of $\mathcal{W}[\alpha]$. Letting $n \to \infty$, we get the desired result. $\square$

Note that substituting α_n in Theorem 1 gives: $\mathcal{W}[\alpha_n] = \varphi_{t_{n+1}}\varphi_{t_{n+2}}\cdots(a)$.

Throughout the remainder of this paper, fix an irrational $1 < \alpha = [1; t_1, t_2, \ldots] < 2$, and denote $\mathcal{W} := \mathcal{W}[\alpha]$. Define a sequence of *finite* words: $W_{-1} := b$, $W_0 := a$ and $W_n := \varphi_{t_1}\cdots\varphi_{t_n}(a)$ for $n \geq 1$. Notice that for every $n \geq 1$, W_n is a prefix of $\mathcal{W}$ and thus

$$\mathcal{W} = \lim_{n\to\infty} W_n.$$

The following proposition describes the basic properties of the sequence W_n. These are the natural generalizations of known properties of the (finite) Fibonacci words. For formulating the proposition, we will introduce the following:

Notation. For any finite word w of length ≥ 2, write $w = w^{(0)}w^{(1)}$ where $|w^{(1)}| = 2$.

Proposition 1.

(a). *For $n \geq 0$, $W_{n+1} = (W_n)^{t_{n+1}}W_{n-1}$.*

(b). *$|W_n| = p_n, |W_n|_a = q_n$ where p_n/q_n are the convergents of the continued fraction of α.*

(c). *$p_{-1} = 1, \quad p_0 = 1, \quad p_{n+1} = t_{n+1}p_n + p_{n-1}$ (for $n \geq 0$).*

(d). $q_{-1} = 0, \quad q_0 = 1, \quad q_{n+1} = t_{n+1}q_n + q_{n-1}$ *(for $n \geq 0$).*

(e). *For $n \geq -1$, $(W_n W_{n+1})^{(0)} = (W_{n+1}W_n)^{(0)}$.*

(f). *For $n \geq 1$, if $2 \mid n$, then $(W_n)^{(1)} = ba$ and if $2 \nmid n$ then $(W_n)^{(1)} = ab$.*

(g). *$(W_n)^{(0)}$ is a palindrome for $n \geq 1$.*

Proof. Items (a) and (b) follow from the definition of W_n, items (c) and (d) are known results on the convergents of continued fractions (see [12, ch. 10]). Item (e) is proven by induction. For $n = -1$, we have $(W_{-1}W_0)^{(0)} = ba^{(0)} = \varepsilon = ab^{(0)} = (W_0W_{-1})^{(0)}$. For $n \geq 0$:

$$\begin{aligned}(W_nW_{n+1})^{(0)} &= (W_n(W_n)^{t_{n+1}}W_{n-1})^{(0)} = ((W_n)^{t_{n+1}}W_nW_{n-1})^{(0)} = \\ &= (W_n)^{t_{n+1}}(W_nW_{n-1})^{(0)} = (W_n)^{t_{n+1}}(W_{n-1}W_n)^{(0)} = \\ &= ((W_n)^{t_{n+1}}W_{n-1}W_n)^{(0)} = (W_{n+1}W_n)^{(0)}.\end{aligned}$$

Item (f) follows from the fact that W_{n-1} is a suffix of W_{n+1}. It is easy to see that (g) holds for W_1 and W_2 as $(W_1)^{(0)} = a^{t_1-1}$ and $(W_2)^{(0)} = (a^{t_1}b)^{t_2-1}a^{t_1}$. For $n > 2$, let $X = W_{n-1}$ and $Y = W_{n-2}$. We have:

$$\begin{aligned}(W_n)^{(0)} &= X^{t_n}Y^{(0)} = X^{t_n-1}(XY)^{(0)} = X^{t_n-1}(YX)^{(0)} = \\ &= \left(X^{t_n-1}Y^{(0)}\right)Y^{(1)}X^{(0)} = \left(X^{t_n-2}Y^{(0)}\right)\left(Y^{(1)}X^{(0)}\right)^2 = \cdots = \\ &= Y^{(0)}(Y^{(1)}X^{(0)})^{t_n}.\end{aligned}$$

Note that $X^{(0)}$ and $Y^{(0)}$ are palindromes by the induction hypothesis and the reverse of $Y^{(1)}$ is $X^{(1)}$ by (f). Thus taking the reverse of the last expression we obtain: $(X^{(0)}X^{(1)})^{t_n}Y^{(0)} = W_n^{(0)}$. Hence $W_n^{(0)}$ is indeed a palindrome. □

Remark. Notice the analogy between finite words and continued fraction convergents, such as (a)-(d). Proposition 1(a) implies, in particular, $W_1 = a^{t_1}b$, $W_2 = (a^{t_1}b)^{t_2}a$, $W_3 = ((a^{t_1}b)^{t_2}a)^{t_3}a^{t_1}b$, etc. This is the way Example 1 and others can be constructed.

3 The recursive structure of $\mathcal{W}[\alpha]$

3.1 Motivation

Recall that α is fixed and $\mathcal{W} = \mathcal{W}[\alpha]$. We begin by analyzing the set $S_x = \{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s) = b \text{ and } \mathcal{W}(s+x) = a\}$, $x \in \mathbb{Z}_{\geq 1}$. For example, consider the case $\alpha = [1; 1, 2, 3, \ldots]$, $x = 2$. We have (see Example 1),

$$\begin{aligned} S_2 = \{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s) = b\} \cap \{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s+2) = a\} = \\ = \{1, 3, 6, 8, 11, 13, 16, 18, 20, \ldots\} \cap \{0, 2, 3, 5, 7, 8, 10, 12, 13, 15, \ldots\} = \\ = \{3, 8, 13, 20, 25, 30, 37, \ldots\}. \end{aligned}$$

In the following word, the positions of S_2 are shown as $\mathbf{B}$:

$$\mathcal{W}[\alpha] = aba\mathbf{B}aaba\mathbf{B}aaba\mathbf{B}aababa\mathbf{B}aaba\mathbf{B}aaba\mathbf{B}aababa\mathbf{B}aa\cdots.$$

Theorem 1 implies that $\mathcal{W} = \varphi_1\varphi_2(\mathcal{W}[\alpha_2])$, where $\alpha_2 = [1; 3, 4, \ldots]$ (see (2)). Thus, $\mathcal{W}$ consists of the blocks $\varphi_1\varphi_2(a) = ababa$, $\varphi_1\varphi_2(b) = ab$ and the order of the blocks is determined by $\mathcal{W}[\alpha_2] = aaabaaaba\cdots$. Notice that the $\mathbf{B}$'s above are exactly the second b's of each block $ababa$.

In order to distinguish between $\mathbf{B}$ and b, we would like to consider "higher resolutions" of $\mathcal{W}[\alpha]$. These resolutions will be represented using the sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$. We start by constructing some tools that will help us to define these sets.

3.2 Partitions and morphisms

Let w be an infinite word over some finite alphabet Σ such that all the letters of Σ appear in w. For every $\sigma \in \Sigma$, consider the set $P_w(\sigma) := \{y \in \mathbb{Z}_{\geq 0} : w(y) = \sigma\}$. Observe that the sets $P_w(\sigma)$ for $\sigma \in \Sigma$ form a partition of $\mathbb{Z}_{\geq 0}$.

Definition 4. The *partition induced by* w is $\mathscr{P}_w := \{P_w(\sigma) : \sigma \in \Sigma\}$.

Remark. In this paper, we do not allow partitions that contain the empty set. Therefore, we defined $\mathscr{P}_w$ only when all the letters of Σ appear in w.

In the next definition, we take a morphism $\tau : \Sigma^* \to \Sigma^*$ and construct from it another morphism, I_τ, taking Σ^* to a new alphabet, with the property that $|I_\tau(\sigma)| = |\tau(\sigma)|$ for every $\sigma \in \Sigma$ and each letter in $I_\tau(\sigma)$ is unique.

Definition 5. Let Σ be some finite alphabet and let $\tau : \Sigma^* \to \Sigma^*$ be a morphism. Consider the new alphabet $\Sigma_\tau := \{\sigma_i : \sigma \in \Sigma,\ 0 \leq i < |\tau(\sigma)|\}$, where σ_i is a formal symbol. The *indicator morphism* of τ is the morphism $I_\tau : \Sigma^* \to \Sigma_\tau^*$ where $I_\tau(\sigma) = \sigma_0\sigma_1\cdots\sigma_{|\tau(\sigma)|-1}$ for every $\sigma \in \Sigma$.

Example 2. Consider the example in the "Motivation" section (Section 3.1). For $\tau = \varphi_1\varphi_2$, we have $\tau(a) = ababa$, $\tau(b) = ab$, $\Sigma_\tau = \{a_0, a_1, a_2, a_3, a_4, b_0, b_1\}$, where $a \overset{I_\tau}{\longmapsto} a_0a_1a_2a_3a_4$, $b \overset{I_\tau}{\longmapsto} b_0b_1$. Note that while τ maps a to $ababa$ and there is no way to distinguish between the two b's, I_τ maps a to 5 different letters, so in a way it "separates" these two b's. Observe that if $w = I_\tau(\mathcal{W}[\alpha_2])$ then $P_w(a_3)$ is the set of the positions of the $\mathbf{B}$'s, and therefore $P_w(a_3) = S_2$.

Consider an infinite word w. The information in $I_\tau(w)$ is larger than the information in $\tau(w)$ in the sense that if we know the letter of $I_\tau(w)$ in some position, then we also know the letter of $\tau(w)$ in the same position. This is because the letters in $I_\tau(w)$ are indexed. It is stated formally in the following definition and proposition, using the notion of the induced partition.

Definition 6. Let $\mathscr{A}$ and $\mathscr{B}$ be two partitions of a set C. We say that $\mathscr{A}$ is a *refinement* of $\mathscr{B}$, and we write $\mathscr{A} \leq \mathscr{B}$, if for every set $A \in \mathscr{A}$, there exists a set $B \in \mathscr{B}$ such that $A \subseteq B$.

It is easy to see that the relation "refinement of" is a partial order over the set of partitions of C.

Proposition 2. *Let w be an infinite word and let $\tau : \Sigma^* \to \Sigma^*$ be a morphism. Then $\mathscr{P}_{I_\tau(w)} \leq \mathscr{P}_{\tau(w)}$.*

Proof. This follows from the fact that $\tau(w)$ and $I_\tau(w)$ consist of blocks of the same lengths, in the same order, and in I_τ each letter appears once. $\square$

3.3 Definition of $\mathcal{A}_i^m$ and $\mathcal{B}_i^m$

We now generalize and formalize the construction described in Example 2.

Fix $m \in \mathbb{Z}_{\geq 0}$. The morphism $\Phi_m := \varphi_{t_1}\varphi_{t_2}\cdots\varphi_{t_m}$ satisfies: $|\Phi_m(a)| = |W_m| = p_m$, $|\Phi_m(b)| = |W_{m-1}| = p_{m-1}$ (see Proposition 1(b)). Therefore, the indicator morphism of Φ_m, $\eta_m := I_{\Phi_m}$, maps: $a \overset{\eta_m}{\longmapsto} a_0a_1\cdots a_{p_m-1}$ and $b \overset{\eta_m}{\longmapsto} b_0b_1\cdots b_{p_{m-1}-1}$.

Let $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m])$ and denote the elements of the partition induced by $\mathcal{H}_m$ by: $\mathcal{A}_0^m, \mathcal{A}_1^m, \cdots, \mathcal{A}_{p_m-1}^m, \mathcal{B}_0^m, \mathcal{B}_1^m, \cdots, \mathcal{B}_{p_{m-1}-1}^m$ respectively. Formally, $\mathcal{A}_i^m = P_{\mathcal{H}_m}(a_i)$ and $\mathcal{B}_i^m = P_{\mathcal{H}_m}(b_i)$.

Observe that $\mathcal{A}_0^0$ ($\mathcal{B}_0^0$) is the set of positions of the a's (b's) of $\mathcal{W}$, and thus $S_x = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$.

Example 3. Consider again Example 2. We have $\Phi_2 = \varphi_{t_1}\varphi_{t_2} = \varphi_1\varphi_2 = \tau$. By definition of η_m and $\mathcal{H}_m$, $\eta_2 = I_\tau$ and $\mathcal{H}_2 = w$. Thus $S_2 = P_w(a_3) = \mathcal{A}_3^2$.

3.4 Properties

The following proposition gives a formula for the sets $\mathcal{A}_i^m$:

Proposition 3. *For $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq i < p_m$, we have:*

$$\mathcal{A}_i^m = \{\lfloor \alpha_m n \rfloor p_{m-1} + n(p_m - p_{m-1}) - p_m + i : n \in \mathbb{Z}_{\geq 1}\}.$$

Proof. Observe that the n-th a_i of $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m])$ is generated by the n-th a of $\mathcal{W}[\alpha_m]$. The position of this a is $\lfloor \alpha_m n \rfloor - 1$. The first $\lfloor \alpha_m n \rfloor - 1$ letters of $\mathcal{W}[\alpha_m]$ contain $(n-1)$ a's and $(\lfloor \alpha_m n \rfloor - n)$ b's. Each a generates p_m letters, and each b generates p_{m-1} letters. The claim follows. □

Example 4. In Example 3 we saw that $S_2 = \mathcal{A}_3^2$. Proposition 3 provides an explicit formula: $S_2 = \{\lfloor \alpha_2 n \rfloor p_1 + n(p_2 - p_1) - p_2 + 3 : n \in \mathbb{Z}_{\geq 1}\} = \{2\lfloor \alpha_2 n \rfloor + 3n - 2 : n \in \mathbb{Z}_{\geq 1}\}$, where $\alpha_2 = [1; 3, 4, 5, \ldots] \approx 1.30879$, $p_1 = 2$, $p_2 = 5$.

Observation 1. *Let $m \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq i < p_m$. Then, $\mathcal{A}_i^m - j = \mathcal{A}_i^m \dot{-} j = \mathcal{A}_{i-j}^m$.*

Proposition 4. $\mathscr{P}_{\mathcal{H}_0} \geq \mathscr{P}_{\mathcal{H}_1} \geq \mathscr{P}_{\mathcal{H}_2} \geq \cdots$.

Proof. Fix $m \in \mathbb{Z}_{\geq 0}$. We have to show that $\mathscr{P}_{\mathcal{H}_m} \geq \mathscr{P}_{\mathcal{H}_{m+1}}$.

Let $\tau = \varphi_{t_{m+1}}$. Notice that $|\Phi_m(w)| = |\eta_m(w)|$ for any word $w \in \{a,b\}^*$. In particular, $|\Phi_{m+1}(\sigma)| = |\eta_m(\tau(\sigma))|$ for $\sigma \in \{a,b\}$. This implies that $I_{\eta_m\tau} = I_{\Phi_{m+1}} = \eta_{m+1}$, and so $\mathcal{H}_{m+1} = I_{\eta_m\tau}(\mathcal{W}[\alpha_{m+1}])$. Using Proposition 2, we obtain that $\mathscr{P}_{\mathcal{H}_{m+1}} = \mathscr{P}_{I_{\eta_m\tau}(\mathcal{W}[\alpha_{m+1}])} \leq \mathscr{P}_{\eta_m\tau(\mathcal{W}[\alpha_{m+1}])} = \mathscr{P}_{\eta_m(\mathcal{W}[\alpha_m])} = \mathscr{P}_{\mathcal{H}_m}$. □

Observation 2. *If $m > 0$ and $y \in \mathcal{A}_i^m$ or $y \in \mathcal{B}_i^m$, then $\mathcal{W}(y) = \mathcal{W}(i)$.*

Proof. If $y \in \mathcal{A}_i^m$, the claim follows directly from the fact that $\mathscr{P}_{\mathcal{H}_m} \leq \mathscr{P}_{\mathcal{H}_0} = \{\mathcal{A}_0^0, \mathcal{B}_0^0\}$ and the fact that $y, i \in \mathcal{A}_i^m$. For $y \in \mathcal{B}_i^m$, notice that both $W_m^{t_{m+1}} W_{m-1}$ and W_{m-1} are prefixes of $\mathcal{W}$. Therefore, as $i < p_{m-1}$, $\mathcal{W}(i) = \mathcal{W}(i + t_{m+1}p_m)$ and the claim follows since $i + t_{m+1}p_m \in \mathcal{B}_i^m$. □

4 Shifts in $\mathcal{W}$

Before analyzing $S_x = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$, we begin with a simpler task: examining the set $\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x)$, where $\triangle$ denotes symmetric difference (see

Section 1.4). This is the set of y's for which $\mathcal{W}(y) \neq \mathcal{W}(y+x)$. This task is indeed easier due to the fact that $\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x)$ can be written using sets of the same form with smaller values of x. Namely, if $x = x_1 + x_2$, then

$$\begin{aligned}\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x) &= \left(\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x_1)\right) \triangle \left((\mathcal{A}_0^0 \dot{-} x_1) \triangle (\mathcal{A}_0^0 \dot{-} x)\right) = \\ &= \left(\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x_1)\right) \triangle \left(\left(\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x_2)\right) \dot{-} x_1\right).\end{aligned}$$

Notice that $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x))$.

Our goal is to represent $\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} x)$ using the basic sets $\mathcal{A}_i^m$ (for these sets we already have an explicit formula – Proposition 3).

We start with $x = p_k$ for $k \in \mathbb{Z}_{\geq 0}$ and then we generalize to an arbitrary $x \in \mathbb{Z}_{\geq 1}$.

4.1 Shifts by p_k, $k \in \mathbb{Z}_{\geq 0}$

Lemma 2. *Let $k \in \mathbb{Z}_{\geq 0}$. If $0 \leq i < p_{k+1} - 2$, then $\mathcal{W}(i) = \mathcal{W}(i + p_k)$. On the other hand, if $p_{k+1} - 2 \leq i < p_{k+1}$, then $\mathcal{W}(i) \neq \mathcal{W}(i + p_k)$.*

Proof. Notice that $W_{k+1}W_k$ is a prefix of $\mathcal{W}$. By Proposition 1(e), $(W_k W_{k+1})^{(0)}$ is also a prefix. This implies the first part. The second part follows from Proposition 1(f). □

The following proposition describes the set $\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} p_k)$. It follows from Lemma 2, Observation 2 and the fact that $\mathcal{H}_{k+1}$ consists of the blocks $a_0 a_1 \cdots a_{p_{k+1}-1}$, $b_0 b_1 \cdots b_{p_k - 1}$.

Recall that $\uplus$ denotes disjoint union (see Section 1.4).

Proposition 5. *For $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{A}_0^0 \triangle (\mathcal{A}_0^0 \dot{-} p_k) = \mathcal{A}_{p_{k+1}-1}^{k+1} \uplus \mathcal{A}_{p_{k+1}-2}^{k+1}$.*

4.2 Arbitrary $x \in \mathbb{Z}_{\geq 1}$

To answer the question for an arbitrary x, we will use the following idea: A generalization of the Zeckendorf numeration system (see [23], [8], [9]) can be used to represent x as a sum of elements from the set $\Pi := \{p_0, p_1, p_2, \ldots\}$. Then, we use Proposition 5 for each of the summands.

Apply the following algorithm on x: While $x \neq 0$, find the largest k such that $p_k \leq x$ and subtract p_k from x. Formally, define two sequences:

$$\begin{aligned}x_0 &:= x, \\ k_i &:= \max\{k \in \mathbb{Z}_{\geq 0} : p_k \leq x_{i-1}\} \quad (i \geq 1), \\ x_i &:= x_{i-1} - p_{k_i} \quad (i \geq 1).\end{aligned}$$

Notice that if $x_i = 0$ for some i, then the two sequences k_j, x_j are not defined for $j > i$. Denote this i by n. Observe that we get a representation of x as a sum of elements from Π: $x = p_{k_1} + p_{k_2} + \cdots + p_{k_n}$.

Example 5. Consider the case $\alpha = [1; 1, 2, 3, \ldots]$, $\Pi = \{1, 2, 5, 17, 73, \ldots\}$, $x = 12 = 5 + 5 + 2$. Here the algorithm yields:

i	0	1	2	3
x_i	12	7	2	0
k_i		2	2	1
p_{k_i}		5	5	2

Let $1 \le i \le n$. Denote $\mathcal{X}_i := (\mathcal{A}_0^0 \dot{-} x_{i-1}) \bigtriangleup (\mathcal{A}_0^0 \dot{-} x_i)$ and observe that $\mathcal{A}_0^0 \bigtriangleup (\mathcal{A}_0^0 \dot{-} x) = \mathcal{X}_1 \bigtriangleup \mathcal{X}_2 \bigtriangleup \cdots \bigtriangleup \mathcal{X}_n$. Proposition 5 implies that

$$\mathcal{X}_i = (\mathcal{A}_0^0 \bigtriangleup (\mathcal{A}_0^0 \dot{-} p_{k_i})) \dot{-} x_i = (\mathcal{A}^{k_i+1}_{p_{k_i+1}-1} \uplus \mathcal{A}^{k_i+1}_{p_{k_i+1}-2}) \dot{-} x_i.$$

The fact that $x_i = x_{i-1} - p_{k_i} \le p_{k_i+1} - 1 - p_{k_i} \le p_{k_i+1} - 2$ and Observation 1 imply that $\mathcal{X}_i = \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-1} \uplus \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-2}$. Therefore,

$$\mathcal{A}_0^0 \bigtriangleup (\mathcal{A}_0^0 \dot{-} x) = \mathop{\bigtriangleup}_{i=1}^{n} (\mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-1} \uplus \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-2}).$$

Example 6. For the case in the previous example, we get:

$$\mathcal{A}_0^0 \bigtriangleup (\mathcal{A}_0^0 \dot{-} 12) = (\mathcal{A}_9^3 \uplus \mathcal{A}_8^3) \bigtriangleup (\mathcal{A}_{14}^3 \uplus \mathcal{A}_{13}^3) \bigtriangleup (\mathcal{A}_4^2 \uplus \mathcal{A}_3^2).$$

5 The set S_x

For $x = 1$, since each b of $\mathcal{W}$ is followed by an a, $\mathcal{B}_0^0 \subseteq (\mathcal{A}_0^0 \dot{-} 1)$ and so $S_1 = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} 1) = \mathcal{B}_0^0 = \mathcal{A}^1_{t_1}$.

We now assume $x > 1$. Notice that $\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathcal{B}_0^0 \cap \big[\mathcal{A}_0^0 \bigtriangleup (\mathcal{A}_0^0 \dot{-} x)\big]$. Continue with the notation of the previous section. We have:

$$S_x = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathop{\bigtriangleup}_{i=1}^{n} \big[(\mathcal{B}_0^0 \cap \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-1}) \cup (\mathcal{B}_0^0 \cap \mathcal{A}^{k_i+1}_{p_{k_i+1}-x_i-2})\big].$$

Observation 2 implies that $\mathcal{B}_0^0 \cap \mathcal{A}_i^m$ is $\mathcal{A}_i^m$ if $\mathcal{W}(i) = b$ and $\emptyset$ otherwise. We now investigate $\mathcal{W}(p_{k_i+1} - x_i - z)$ for $z \in \{1, 2\}$.

Observation 3. *If $x_i - z' \geq 0$ for $z' \in \{1,2\}$, then $\mathcal{W}(x_i - z') = \mathcal{W}(x - z')$.*

Proof. By induction on i:

The claim holds trivially for $i = 0$.

For $i > 0$, if $x_i - z' \geq 0$ then also $x_{i-1} - z' \geq 0$. Notice that $x_{i-1} - z' = (x_i - z') + p_{k_i}$ and $x_i - z' \leq x_i - 1 \leq x_{i-1} - 2 < p_{k_i+1} - 2$. By Lemma 2 and the induction hypothesis, $\mathcal{W}(x_i - z') = \mathcal{W}(x_{i-1} - z') = \mathcal{W}(x - z')$. □

Observation 4. *If $x_i + z \geq 3$ for $z \in \{1,2\}$, then $\mathcal{W}(p_{k_i+1} - x_i - z) = \mathcal{W}(x + z - 3)$.*

Proof. Proposition 1(g) implies that $\mathcal{W}(p_{k_i+1} - x_i - z) = \mathcal{W}(x_i + z - 3)$ and by the last observation (for $z' = 3 - z$), we get: $\mathcal{W}(p_{k_i+1} - x_i - z) = \mathcal{W}(x + z - 3)$. □

We now consider three cases: (1) $\mathcal{W}(x-1) = b$, (2) $\mathcal{W}(x-2) = b$ and (3) $\mathcal{W}(x-1) = \mathcal{W}(x-2) = a$.

Consider the first case: For $1 \leq i < n$ we have $x_i \geq 1$ and by Observation 4,

$$\mathcal{W}(p_{k_i+1} - x_i - 2) = \mathcal{W}(x-1) = b.$$

Notice that $b = \mathcal{W}(x-1) = \mathcal{W}(x_{n-1}-1) = \mathcal{W}(p_{k_n} - 1)$. This means that $2 \nmid k_n$ (see Proposition 1(f)). Therefore, $\mathcal{W}(p_{k_n+1} - x_n - 2) = \mathcal{W}(p_{k_n+1} - 2) = b$.

Hence, for $1 \leq i \leq n$, $\mathcal{W}(p_{k_i+1} - x_i - 2) = b$. Since $\mathcal{W}$ does not contain bb as a factor, we get that $\mathcal{W}(p_{k_i+1} - x_i - 1) = a$. This implies

$$\mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x) = \mathop{\triangle}_{i=1}^{n} \mathcal{A}_{p_{k_i+1}-x_i-2}^{k_i+1}.$$

The other cases are analyzed similarly. Formulas for the x's of each case can be obtained by considering the blocks of $\mathcal{H}_1$. The following table summarizes the three cases.

Case	$\mathcal{W}(x-2), \mathcal{W}(x-1)$	$x - 2 \in$	$S_x = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$
1	a, b	$\mathcal{A}_{t_1-1}^1$	$\triangle_{i=1}^{n} \mathcal{A}_{p_{k_i+1}-x_i-2}^{k_i+1}$
2	b, a	$\mathcal{A}_{t_1}^1 = \mathcal{B}_0^0$	$\triangle_{i=1}^{n} \mathcal{A}_{p_{k_i+1}-x_i-1}^{k_i+1}$
3	a, a	$\mathcal{A}_i^1$ $(i < t_1 - 1)$, $\mathcal{B}_0^1 = \mathcal{A}_{(t_1+1)t_2}^2$	$\mathcal{A}_{t_1}^1 = \mathcal{B}_0^0$

Example 7. For the case described in Example 6, we have $\mathcal{W}(12-1) = b$ and therefore this is Case 1. This implies $S_{12} = \{3, 20, 37, 54, 71, \ldots\} = \mathcal{A}_8^3 \triangle \mathcal{A}_{13}^3 \triangle \mathcal{A}_3^2$.

6 S_x as a disjoint union of basic sets

Our goal now is to represent S_x as a disjoint union of sets of the form $\mathcal{A}_i^m$, instead of taking their symmetric difference as we did in Section 5. Such a representation seems to be much better. However, in order to attain this, we will have to develop a better understanding of the structure formed by the sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$.

6.1 The structure of $\mathcal{A}_i^m$, $\mathcal{B}_i^m$

Notice that $\mathcal{H}_m = \eta_m(\mathcal{W}[\alpha_m]) = \eta_m\varphi_{t_{m+1}}(\mathcal{W}[\alpha_{m+1}])$, so both $\mathcal{H}_m$, $\mathcal{H}_{m+1}$ consist of blocks of lengths p_{m+1}, p_m in an order determined by $\mathcal{W}[\alpha_{m+1}]$.

By considering these blocks we obtain:

$$\mathcal{A}_i^m = \mathcal{A}_i^{m+1} \cup\!\!\!\cdot\; \mathcal{A}_{i+p_m}^{m+1} \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\; \mathcal{A}_{i+(t_{m+1}-1)p_m}^{m+1} \cup\!\!\!\cdot\; \mathcal{B}_i^{m+1}, \quad \mathcal{B}_i^m = \mathcal{A}_{i+t_{m+1}p_m}^{m+1}.$$

Therefore,

$$\mathcal{A}_i^m = \mathcal{A}_i^{m+1} \cup\!\!\!\cdot\; \mathcal{A}_{i+p_m}^{m+1} \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\; \mathcal{A}_{i+(t_{m+1}-1)p_m}^{m+1} \cup\!\!\!\cdot\; \mathcal{A}_{i+t_{m+2}p_{m+1}}^{m+2}. \tag{3}$$

Definition 7. A *partition tree* of a set $C \neq \emptyset$ is a tree, in which every node is a subset of C, the root is C, and for every node A, which is not a leaf, the set of children of A forms a partition of A.

Consider the tree of all the sets $\mathcal{A}_i^m \subseteq \mathcal{B}_0^0$, where there is an edge from $\mathcal{A}_i^m$ to each of the sets in the right-hand side of (3). We denote this tree by $\mathscr{T}_\alpha$. Notice that the root of the tree is $\mathcal{A}_{t_1}^1 = \mathcal{B}_0^0$. Let $\mathbf{pr}\,A$ denote the parent of a set A in the tree. If A is the root, we define $\mathbf{pr}\,A := A$. Notice that $\mathscr{T}_\alpha$ is a partition tree.

Example 8. Figure 1 shows the tree $\mathscr{T}_\alpha$ for $\alpha = [1; 1, 2, 3, \ldots]$. For example, $\mathbf{pr}\,\mathcal{A}_{16}^3 = \mathcal{A}_1^1$ and $\mathbf{pr}\,\mathcal{A}_1^3 = \mathcal{A}_1^2$.

Corollary 1. *Consider the node $\mathcal{A}_i^m$ in $\mathscr{T}_\alpha$, where $\mathcal{A}_i^m$ is not the root. We have*

$$\mathbf{pr}\,\mathcal{A}_i^m = \mathcal{A}_{i \bmod p_{m-1}}^{\overline{m}}, \quad \text{where} \quad \overline{m} = \begin{cases} m-1, & i < p_{m-1} \cdot t_m \\ m-2, & i \geq p_{m-1} \cdot t_m \end{cases}.$$

Proof. This follows directly from (3). □

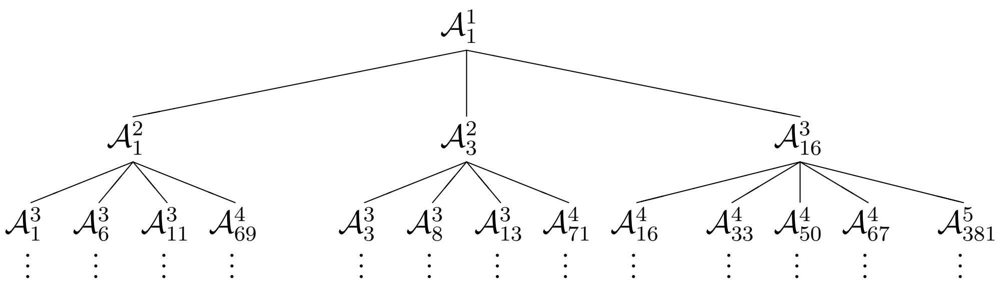

Figure 1: $\mathscr{T}_\alpha$ for $\alpha = [1; 1, 2, 3, \ldots]$

6.2 The chain proposition

Notice that for Case 3 (see table on page 14) we have $S_x = \mathcal{A}^1_{t_1}$. So we focus on the first two cases. Let $Z = 2$ for Case 1, and $Z = 1$ for Case 2. Denote $r_i := p_{k_i+1} - x_i - Z$. Then, $S_x = \bigtriangleup_{i=1}^{n} \mathcal{A}^{k_i+1}_{r_i}$.

Proposition 6. *For $1 \leq i < n$, $\mathbf{pr}\,\mathcal{A}^{k_i+1}_{r_i} \subseteq \mathbf{pr}\,\mathcal{A}^{k_{i+1}+1}_{r_{i+1}}$.*

Note that this means that the parents of the sets $A^{k_i+1}_{r_i}$ are on one branch of $\mathscr{T}_\alpha$.

In order to prove Proposition 6, we first prove the following two lemmas:

Lemma 3. *Let $1 \leq k \leq m$, $m \equiv k \pmod 2$, $1 \leq i \leq p_k$. Then, $\mathcal{A}^m_{p_m-i} \subseteq \mathcal{A}^k_{p_k-i}$.*

Proof. By Equation (3), we have that $\mathcal{A}^k_{p_k-i} \supseteq \mathcal{A}^{k+2}_{p_{k+1}\cdot t_{k+2}+(p_k-i)} = \mathcal{A}^{k+2}_{p_{k+2}-i}$. Similarly, $\mathcal{A}^{k+2}_{p_{k+2}-i} \supseteq \mathcal{A}^{k+4}_{p_{k+4}-i}$ and we get the following sequence:

$$\mathcal{A}^k_{p_k-i} \supseteq \mathcal{A}^{k+2}_{p_{k+2}-i} \supseteq \mathcal{A}^{k+4}_{p_{k+4}-i} \supseteq \cdots .$$

Clearly $\mathcal{A}^m_{p_m-i}$ is one of the elements of this sequence and so $\mathcal{A}^m_{p_m-i} \subseteq \mathcal{A}^k_{p_k-i}$. $\square$

Lemma 4. *Let $k \geq 2$, $0 \leq i < p_k - p_{k-1}$. If both $\mathcal{A}^k_i$, $\mathcal{A}^k_{i+p_{k-1}}$ are nodes of $\mathscr{T}_\alpha$, then $\mathbf{pr}\,\mathcal{A}^k_i \subseteq \mathbf{pr}\,\mathcal{A}^k_{i+p_{k-1}}$.*

Proof. Corollary 1 implies that $\mathbf{pr}\,\mathcal{A}^k_i = \mathcal{A}^{k_1}_j$, $\mathbf{pr}\,\mathcal{A}^k_{i+p_{k-1}} = \mathcal{A}^{k_2}_j$ for some j, where $k_1, k_2 \in \{k-1, k-2\}$. Since $i < i + p_{k-1}$, we have $k_2 \leq k_1$.

If $k_1 = k_2$, then the claim holds. Otherwise, $k_1 = k - 1$, $k_2 = k - 2$. This implies $j < p_{k-2}$, and so $\mathbf{pr}\,\mathcal{A}^k_i = \mathcal{A}^{k-1}_j \subseteq \mathbf{pr}\,\mathcal{A}^{k-1}_j = \mathcal{A}^{k-2}_j = \mathbf{pr}\,\mathcal{A}^k_{i+p_{k-1}}$. $\square$

Proof of Proposition 6. We use the following notation:

$$\begin{aligned} a &:= x_i + Z, & k &:= k_i + 1, \\ b &:= x_{i+1} + Z, & \ell &:= k_{i+1} + 1. \end{aligned}$$

In this notation, we have to show: $\mathbf{pr}\,\mathcal{A}^k_{p_k-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-b}$.

We have $p_{\ell-1} < a \leq p_\ell + 1$ and $p_\ell - b = p_\ell + p_{\ell-1} - a$. Note that all the sets that are mentioned in the proof are subsets of $\mathcal{B}^0_0$ and therefore they are nodes in $\mathscr{T}_\alpha$.

Consider the following 4 cases: (a) $\ell = 1$, (b) $k \equiv \ell+1 \pmod 2$, (c) $a \leq p_\ell$ and $k \equiv \ell \pmod 2$, (d) $a = p_\ell + 1$ and $k \equiv \ell \pmod 2$.

(a) is trivial. We show here the proof of (c). (b), (d) are proven similarly using applications of Lemma 3, Lemma 4 and Corollary 1.

Suppose that $a \leq p_\ell$ and $k \equiv \ell \pmod 2$. Lemma 3 implies that $\mathcal{A}^k_{p_k-a} \subseteq \mathcal{A}^\ell_{p_\ell-a}$. Therefore, $\mathbf{pr}\,\mathcal{A}^k_{p_k-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-a}$. Lemma 4 implies that

$$\mathbf{pr}\,\mathcal{A}^k_{p_k-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-a} \subseteq \mathbf{pr}\,\mathcal{A}^\ell_{p_{\ell-1}+p_\ell-a} = \mathbf{pr}\,\mathcal{A}^\ell_{p_\ell-b}. \qquad \square$$

6.3 A disjoint union

Proposition 6 implies that the sets that participate in the symmetric difference satisfy the following property:

$$\mathbf{pr}\,\mathcal{A}^{k_1+1}_{r_1} \subseteq \mathbf{pr}\,\mathcal{A}^{k_2+1}_{r_2} \subseteq \mathbf{pr}\,\mathcal{A}^{k_3+1}_{r_3} \subseteq \cdots \subseteq \mathbf{pr}\,\mathcal{A}^{k_n+1}_{r_n}. \tag{4}$$

Theorem 2. *The set S_x can be written as a disjoint union of $O(\Sigma_{i=1}^{k_1+1} t_i)$ sets of the form $\mathcal{A}^m_i$.*

Notice that if $t_i < T$ for all $i \in \mathbb{Z}_{\geq 1}$, then the number of sets is $O(T \log x)$.

Proof. Define a *partition subtree* to be a subtree which is also a partition tree. In other words, every node of the subtree which is not a leaf, should have the same set of children as the same node in the original partition tree.

Consider the minimal partition subtree of $\mathscr{T}_\alpha$ that contains the node $\mathcal{A}^{k_1+1}_{r_1}$. Denote it by T_x. This tree consists of the nodes $\mathbf{pr}^i\,\mathcal{A}^{k_1+1}_{r_1}$ ($i \in \mathbb{Z}_{\geq 1}$) and their children. Notice that (4) guarantees that all the sets $\mathcal{A}^{k_i+1}_{r_i}$ are nodes in the tree. The tree has at most $k_1 + 1$ layers, so the number of nodes is at most $\sum_{i=1}^{k_1+1}(t_i + 1)$. It is easy to see that in every finite partition tree, each element of the algebra (of sets) generated by the nodes, is a disjoint union of leaves. $\square$

Notice that Theorem 2 can be used to write an algorithm that given x outputs a list of sets $\mathcal{A}_i^m$, whose disjoint union is S_x: Compute the tree T_x and mark the sets $\mathcal{A}_{r_i}^{k_i+1}$ in it. Visit the nodes of the tree, starting from the root, and if an internal node is marked, replace its mark with its children. Then, output the marked leaves.

The complexity of this algorithm is $O(\Sigma_{i=1}^{k_1+1} t_i)$ ($k_1 = O(\log x)$). In the common case where t_i is bounded by T, the complexity is $O(T \log x)$. However, if t_i goes to infinity fast enough, the complexity might exceed $O(x)$[2]. For these rare cases, one can revise the algorithm to work with T_x implicitly (and thus work with ranges of the form $\mathcal{A}_i^m \cup \mathcal{A}_{i+p_{m-1}}^m \cup \cdots \cup \mathcal{A}_{i+bp_{m-1}}^m$) and then the output will take a shorter but somewhat more complicated form.

Example 9. Consider the sets that appear in Example 7. The minimal partition subtree that contains $\mathcal{A}_8^3$ is shown in Figure 2. We have $S_{12} = \mathcal{A}_8^3 \triangle \mathcal{A}_{13}^3 \triangle \mathcal{A}_3^2 = \mathcal{A}_3^3 \cup \mathcal{A}_{71}^4$.

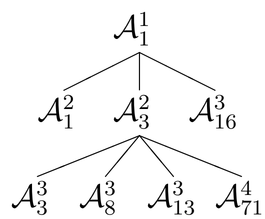

Figure 2: T_{12}

7 Applications – Beatty games

As explained in the introduction, the set S_x plays an important role when considering the question of preservation of P-positions in Beatty games. Before diving into the details on this question and its relationship with S_x, we first lay the formal settings in which we can formulate the question.

An *invariant* subtraction game (also known as an invariant take-away game), played on two piles of tokens, is a game in which every move is of the form: take x tokens from one pile and y tokens from the other, provided only that every pile retains a nonnegative number of tokens after the move.

[2]For example, if $t_i = 2^{3^i}$ then there is an infinite sequence of x's for which the complexity is $\Theta(x^2)$.

Invariant subtraction games were defined in [11]. The game positions are encoded in the form (X, Y), where X and Y are the sizes of the piles and $X \leq Y$. We restrict attention to *symmetric* invariant subtraction games. Symmetric means that whenever a move from (X, Y) to $(X-x, Y-y)$ is permitted, then a move to $(X-y, Y-x)$ is permitted too, provided the resulting position remains nonnegative. Thus, moves can be encoded in the form (x, y) with $x \leq y$. The set of moves (x, y) is called the *ruleset* of the game. References on invariant subtraction games are, for example, [11], [7], [16], [15] and [17].

In [6] the authors consider the following problem: what subtractions (x, y) can we add to the game Wythoff Nim, such that the set of P-positions will remain unchanged? A similar question, regarding the addition of new moves to Wythoff Nim, and its effect on the set of P-positions was studied in [14].

Consider the following reformulation of this question: What is the unique maximal set (with respect to $\subseteq$) of moves for which the set of P-positions is $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$? The fact that this question is well-defined, for any irrational $1 < \alpha < 2$, follows from the existence of an invariant game with these P-positions (which was proven in [16]). In fact, this set consists exactly of all subtractions that do not connect one P-position to another. The advantage of this reformulation is that it involves only the set of P-positions of the game, rather than the "original" ruleset (which we do not generally know). As such, it can be generalized to *any* irrational $1 < \alpha < 2$.

In the rest of this section we will see the relationship between this question and the set S_x.

7.1 The set E_x

Let $\mathscr{M}$ denote the set of all subtractions that would connect one P-position to another. As stated above, the complement of $\mathscr{M}$ is exactly the set answering our main question. $\mathscr{M}$ can be written as $\mathscr{M}_1 \cup \mathscr{M}_2$ where:

1. $\mathscr{M}_1 = \{(\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor) : n \geq m\}$.
2. $\mathscr{M}_2 = \{(\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor, \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor) : \lfloor \alpha n \rfloor > \lfloor \beta m \rfloor,\ m > 0\}$.

Figure 3 shows a matrix (a_{xy}) where $a_{xy} = 1$ if $(x, y) \in \mathscr{M}_1$, $a_{xy} = 2$ if $(x, y) \in \mathscr{M}_2$, $a_{xy} = 3$ if $(x, y) \in \mathscr{M}_1 \cap \mathscr{M}_2$ and $a_{xy} = 0$ otherwise, for the case $\alpha = [1; 1, 2, 3, \ldots] = 1.6977746...$, $\beta = 2.4331274....$ As an example, consider $(x, y) = (1, 3)$. Note that taking $n = 7$ and $m = 6$ gives $(1, 3) =$

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	1	3	2	0	0	2	2	0	0	2	0	0	0	2	2	0	0	0	2	0	0	2	2	0	0
2		1	1	0	0	2	0	0	0	0	2	0	0	2	0	0	0	0	2	0	0	0	2	0	0	2
3			0	1	1	2	0	0	2	0	0	0	2	0	0	0	0	2	0	0	0	2	0	0	0	2
4				1	1	0	0	2	2	0	0	2	2	0	0	2	0	0	0	0	2	0	0	0	2	0
5					0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
6						0	1	1	1	1	2	0	0	2	2	0	0	2	2	0	0	0	2	0	0	0
7							0	0	1	1	0	0	0	2	0	0	0	2	0	0	0	0	0	0	0	2
8								0	0	0	0	1	3	0	0	2	0	0	0	0	2	0	0	0	2	0
9									0	0	0	1	1	0	0	2	0	0	2	2	0	0	0	2	0	0
10										0	0	0	0	1	1	0	0	0	0	0	0	0	2	0	0	0
11											0	0	0	1	1	0	1	3	0	0	2	0	2	0	0	2
12												0	0	0	0	0	1	1	0	0	2	0	0	0	0	0

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lfloor \alpha n \rfloor$	1	3	5	6	8	10	11	13	15	16	18	20	22
$\lfloor \beta n \rfloor$	2	4	7	9	12	14	17	19	21	24	26	29	31

Figure 3: The sets $\mathscr{M}_1, \mathscr{M}_2$ for $\alpha = [1; 1, 2, 3, \ldots]$

$(11, 17) - (10, 14)$ and thus $(1, 3) \in \mathscr{M}_1$. Similarly, taking $n = 2$ and $m = 1$ gives $(1, 3) = (3, 4) - (2, 1)$ and thus $(1, 3) \in \mathscr{M}_2$.

Checking whether a specific subtraction (x, y) is in $\mathscr{M}$ is polynomial in the input size $(\log(x) + \log(y))$: For $\mathscr{M}_2$, $(x, y) \in \mathscr{M}_2$ implies that $(x, y) = (\alpha n - \beta m, \beta n - \alpha m) + (\varepsilon_1, \varepsilon_2)$ where $-1 < \varepsilon_1, \varepsilon_2 < 1$. By inverting this linear transformation, we obtain an approximation for n and m. Therefore, by checking the $O(1)$ many potential pairs (n, m), we can determine whether $(x, y) \in \mathscr{M}_2$. Polynomiality holds also for $\mathscr{M}_1$, but the explanation of this fact is more complicated.

In this paper we analyze the *pattern* of a given row in this matrix. Consider the set $\{y \geq x : (x, y) \in \mathscr{M}\}$ for some fixed x. To understand the contribution of $\mathscr{M}_1$ to this set, note that

$$\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor \in \{\lfloor \alpha(n - m) \rfloor, \lfloor \alpha(n - m) \rfloor + 1\}.$$

Hence the equation $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor$ for fixed x, implies that $n - m$ can take one of two values. This implies that $y = \lfloor \beta n \rfloor - \lfloor \beta m \rfloor$ can take one of four values. It follows that $\mathscr{M}_1$ may contribute no more than 4 elements. It remains, therefore, to analyze the set

$$E_x := \{y \geq x : (x, y) \in \mathscr{M}_2\}.$$

Let $g(n) = \lfloor \alpha n \rfloor$, $h(n) = \lfloor \beta n \rfloor$. Notice that the inverse functions are $g^{-1}(m) = \lceil m/\alpha \rceil$ (where $m = \lfloor \alpha n \rfloor \in \operatorname{Im} g$), $h^{-1}(m) = \lceil m/\beta \rceil$ (where $m = \lfloor \beta n \rfloor \in \operatorname{Im} h$).

The following proposition describes the relationship between E_x and S_x. Notice that [6] describes a simpler relationship for the case $\alpha = [1; 1, 1, \ldots]$. A similar relationship can be given also for t-Wythoff Nim ($\alpha = [1; t, t, \ldots]$, $t \in \mathbb{Z}_{\geq 1}$. See Section 8.2), but the case of an arbitrary α is more complicated.

Proposition 7. *Let $x \in \mathbb{Z}_{\geq 1}$. Then,*

$$E_x = \{hg^{-1}(s + x + 1) - gh^{-1}(s + 1) : s \in S_x\}.$$

Proof. Suppose that $y \in E_x$. Then, $y = h(n) - g(m)$ and $x = g(n) - h(m)$. Choose $s = h(m) - 1$. Then $s \in \mathcal{B}_0^0$, $s + x \in \mathcal{A}_0^0$, so $s \in \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. Moreover, $y = h(n) - g(m) = hg^{-1}g(n) - gh^{-1}h(m) = hg^{-1}(s + x + 1) - gh^{-1}(s + 1)$.

The other direction is similar. □

8 Explicit representation of the set E_x

We saw that $S_x = \biguplus_{j=1}^{n'} \mathcal{A}_{i_j}^{m_j}$ for some $n', i_1, m_1, \ldots, i_{n'}, m_{n'}$. Proposition 7 implies that $E_x = \bigcup_{j=1}^{n'} F[\mathcal{A}_{i_j}^{m_j}]$ where $F(s) = hg^{-1}(s+x+1) - gh^{-1}(s+1)$. In this section we give a more explicit representation of E_x.

8.1 The general case

We start by computing $h^{-1}(s + 1)$ for $s \in \mathcal{A}_i^m \subseteq \mathcal{B}_0^0$. Suppose that s is the n-th element of $\mathcal{A}_i^m$. It is generated (when applying Φ_m, see Section 3.3) by the n-th a of $\mathcal{W}[\alpha_m]$. Let $j = h^{-1}(i + 1)$ be the number of b's in the first $i + 1$ letters of $\Phi_m(a)$. Since the n-th a of $\mathcal{W}[\alpha_m]$ is in position $\lfloor \alpha_m n \rfloor - 1$, there are $(n - 1)$ a's and $(\lfloor \alpha_m n \rfloor - n)$ b's before this a. Each a contributes (when applying Φ_m) $(p_m - q_m)$ b's and each b contributes $(p_{m-1} - q_{m-1})$ b's. This implies:

$$h^{-1}(s + 1) = (p_m - q_m) \cdot (n - 1) + (p_{m-1} - q_{m-1}) \cdot (\lfloor \alpha_m n \rfloor - n) + j.$$

In other words, there are constants $A, B, C \in \mathbb{Z}$ such that $h^{-1}(s + 1) = A\lfloor \alpha_m n \rfloor + Bn + C$.

In order to compute $g^{-1}(s+x+1)$ we will need the following generalization of a proposition that appears in [6] (it is proven there for the case $\alpha = [1;1,1,\ldots])$:

Proposition 8. *If bua is a factor of $\mathcal{W}$ where $n = |u|$ then $|u|_a = |w|_a$ and $|u|_b = |w|_b$ where w is the prefix of $\mathcal{W}$ of length n.*

Proof. It suffices to prove that $|u|_b = |w|_b$ as $|u| = |w|$. Denote by j the index of the first b of the bua factor.

Let $X = \{i\beta : i \in \mathbb{Z}\}$. Notice that $(z+1, z+2) \cap X \neq \emptyset$ if and only if $\mathcal{W}(z) = b$. Let $f : \mathbb{R} \to \mathbb{Z}$, $f(x) = |(x, x+n) \cap X|$. In other words, $f(x)$ is the number of points from X in the interval $(x, x+n)$. It is easy to see that f is periodic with period β and that f is increasing on the interval $[0, \beta)$.

Notice that $|u|_b = f(j+2)$ and $|w|_b = f(1)$. Since we have an a after the u it implies that $f(j+3) \leq f(j+2)$. We also know that there is a b before the u and therefore there is $r \in \mathbb{Z}$ such that $j+1 < \beta r < j+2$. Hence

$$\beta r < j+2 < \beta r + 1 < j+3 < \beta(r+1).$$

But f is increasing in the interval $[\beta r, \beta(r+1))$ and so

$$f(j+2) \leq f(\beta r + 1) \leq f(j+3) \leq f(j+2).$$

We conclude that $|w|_b = f(1) = f(\beta r + 1) = f(j+2) = |u|_b$. □

Notice that $\mathcal{W}(s-1) = a$. Using the same arguments we used for $h^{-1}(s+1)$, we may infer a similar formula for $g^{-1}(s)$. Let w be the prefix of length $x-1$. By the last proposition, we have $g^{-1}(s+x+1) = g^{-1}(s) + |w|_a + 1$ and so we obtain a formula for $g^{-1}(s+x+1)$ that has the form $A'\lfloor \alpha_m n \rfloor + B'n + C'$.

We obtain the following theorem:

Theorem 3. *The set E_x can be written as a union of sets of the form*

$$\{h\left(A'\lfloor \alpha_m n \rfloor + B'n + C'\right) - g\left(A\lfloor \alpha_m n \rfloor + Bn + C\right) : n \in \mathbb{Z}_{\geq 1}\},$$

where $A, B, C, A', B', C' \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$ are effectively computable constants that depend only on x.

Example 10. For $\alpha = [1;1,2,3,\ldots]$ we have $E_{12} = \{21, 33, 45, 58, 71, \ldots\} = F[\mathcal{A}_3^3] \cup F[\mathcal{A}_{71}^4]$ and

$$F[\mathcal{A}_3^3] = \{h(3\lfloor \alpha_3 n \rfloor + 7n) - g(2\lfloor \alpha_3 n \rfloor + 5n - 5) : n \in \mathbb{Z}_{\geq 1}\},$$

$$F[\mathcal{A}_{71}^4] = \{h(10\lfloor \alpha_4 n \rfloor + 33n + 7) - g(7\lfloor \alpha_4 n \rfloor + 23n) : n \in \mathbb{Z}_{\geq 1}\},$$

$$\alpha_3 = [1;4,5,6,\ldots] \approx 1.23845, \quad \alpha_4 = [1;5,6,7,\ldots] \approx 1.19369.$$

8.2 The case $\alpha = [1; t, t, t, \ldots]$

In turns out that in the case $\alpha = [1; t, t, t, \ldots]$, there is a simpler relationship between E_x and S_x:

Proposition 9. *Let $x \in \mathbb{Z}_{\geq 1}$. There exists $C \in \mathbb{Z}$ such that $F(s) = ts + C$ for any $s \in S_x$.*

Proof. Let $s \in S_x = \mathcal{B}_0^0 \cap (\mathcal{A}_0^0 \dot{-} x)$. Notice that $\beta = \alpha + t$ and so $h(y) = g(y) + yt$. Therefore, $h(g^{-1}(s+x+1)) = g^{-1}(s+x+1)t + s + x + 1$ and $g(h^{-1}(s+1)) = s+1-h^{-1}(s+1)t$. We also have $h^{-1}(s+1) + g^{-1}(s) = s+1$. This implies, $F(s) = hg^{-1}(s+x+1) - gh^{-1}(s+1) = x + [g^{-1}(s+x+1) - g^{-1}(s) + s + 1]\, t$. Proposition 8 implies that $g^{-1}(s+x+1) - g^{-1}(s)$ does not depend on s and this completes the proof. □

9 Wrap-up

This study began with the analysis of the set $S_x = \{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s) = b \text{ and } \mathcal{W}(s+x) = a\}$, where $\mathcal{W} = \mathcal{W}[\alpha]$ is a generalization of the infinite Fibonacci word in which the golden ratio is replaced by any irrational $1 < \alpha < 2$. In order to give an explicit formula for S_x, we defined the sets $\mathcal{A}_i^m$, $\mathcal{B}_i^m$, which form refinements of the trivial partition $\{i : \mathcal{W}(i) = a\} \cup \{i : \mathcal{W}(i) = b\}$.

Let $[1; t_1, t_2, t_3, \ldots]$ be the continued fraction representation of α and let p_i/q_i be its convergents. Section 4.2 describes a simple algorithm to write x as a sum of p_i's and in the process, two sequences are obtained: $x = x_0 > x_1 > \ldots > x_n = 0$ and $k_1 \geq k_2 \geq \ldots \geq k_n$, such that $\sum_{j=i+1}^{n} p_{k_j} = x_i$. It turns out that there are 3 different cases:

1. When $\mathcal{W}(x-1) = b$, we have $S_x = \triangle_{i=1}^{n} \mathcal{A}_{p_{k_i+1} - x_i - 2}^{k_i+1}$.
2. When $\mathcal{W}(x-2) = b$, we have $S_x = \triangle_{i=1}^{n} \mathcal{A}_{p_{k_i+1} - x_i - 1}^{k_i+1}$.
3. When $\mathcal{W}(x-1) = \mathcal{W}(x-2) = a$, we have $S_x = \mathcal{B}_0^0 = \mathcal{A}_{t_1}^1$.

For the first two cases, there exists a polynomial time algorithm that converts the symmetric difference to a disjoint union of sets of the form $\mathcal{A}_i^m$ (page 18).

As an application, we considered the maximal set of moves that defines a game with P-positions $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$ for an arbitrary irrational $1 < \alpha < 2$ and β such that $1/\alpha + 1/\beta = 1$. This set consists of all moves except for a set of forbidden subtractions given by $\mathscr{M}_1 \cup \mathscr{M}_2$. We represented this set by a matrix (a_{xy}) where a_{xy} indicates whether $(x, y) \in \mathscr{M}_1$ and whether $(x, y) \in \mathscr{M}_2$.

We examined the structure of any fixed row, x, of this matrix. The set $\mathscr{M}_1$ may contribute at most 4 elements to each row. We defined $E_x = \{y \geq x : (x, y) \in \mathscr{M}_2\}$, and we saw that E_x is related to $\mathcal{W}[\alpha]$ in the following manner: $E_x = F[S_x]$ where $F(s) = hg^{-1}(s+x+1) - gh^{-1}(s+1)$, $g(n) = \lfloor \alpha n \rfloor$ and $h(n) = \lfloor \beta n \rfloor$.

It turns out that this representation can be simplified even further: Theorem 3 shows that E_x is a finite union of sets of the form

$$\{h\,(A' \lfloor \alpha_m n \rfloor + B'n + C') - g\,(A \lfloor \alpha_m n \rfloor + Bn + C) : n \in \mathbb{Z}_{\geq 1}\}\,.$$

Examples 5, 6, 7, 9, 10 show the process for the case $\alpha = [1; 1, 2, 3, \ldots]$ and $x = 12$.

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