Approximation Algorithms for the Earth Mover's Distance Under Transformations Using Reference Points

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1 Introduction

Reference points have been introduced in [2] and [1] to construct approximation algorithms for matching compact subsets of $\mathbb{R}^d$ under a given class of transformations. Also a general discussion of reference point methods for matching according to the Hausdorff-distance has been given in [1]. Another distance measure used for shape matching is the Earth Mover's Distance (EMD) for weighted point sets ([7]). Here we will extend the definition of reference points to weighted point sets and get fast constant factor approximation algorithms for matching weighted point sets under translations, rigid motions and similarity operations with respect to the Earth Mover's Distance. A first iterative algorithm to solve this problem has been given by Cohen ([3]). Thus we want to find algorithms where $EMD^{apx}(A,B) \leq \varepsilon EMD^{opt}(A,B)$. Under this assumption ε is called the loss factor of the approximation algorithm. Unless stated otherwise, the results given in this paper are independent of the distance measure on the ground set, therefore the results are widely applicable. Additionally, all theorems hold in arbitrary dimension d. For a full version of this extended abstract see [5].

2 Basic Definitions

In this chapter we provide all definitions required.

Definition 1 (Weighted Point Set) *([4]) Let $A = \{a_1, a_2, ..., a_n\}$ be a weighted point set such that $a_i = (p_i, \alpha_i)$ for $i = 1, ..., n$, where p_i is a point in $\mathbb{R}^d$ and $\alpha_i \in \mathbb{R}^+ \cup \{0\}$ its corresponding weight. Let $W^A = \sum_{i=1}^n \alpha_i$ be the total weight of A. Let $\mathbb{W}^d$ be the set of all weighted point sets in $\mathbb{R}^d$ and $\mathbb{W}^{d,G}$ be the set of all weighted point sets in $\mathbb{R}^d$ with total weight $G \in \mathbb{R}^+$.*

In the following we will use transformations on weighted point sets. By this we mean to transform the point set and leave the weights unchanged.

A point related to each weighted point set is the center of mass. This point will play an important role in our approximation algorithms. Note that this point can be computed in linear time and therefore does not affect the runtime of the presented algorithms.

Definition 2 (Center of Mass) *The center of mass of a weighted point set $A = \{(p_i, \alpha_i)_{i=1,...,n}\} \in \mathbb{W}^d$ is defined as*

$$C(A) = \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i p_i.$$

Definition 3 (Reference Point) *([1]) Let $\mathcal{K}$ be a subset of $\mathbb{W}^d$ and $\delta : \mathcal{K} \to \mathbb{R}$ be a distance measure on $\mathcal{K}$. A mapping $r : \mathcal{K} \to \mathbb{R}^d$ is called a δ-reference point for $\mathcal{K}$ with respect to a set of transformations $\mathcal{T}$ on $\mathcal{K}$, if the following two conditions hold:*

1. *Equivariance with respect to $\mathcal{T}$: For all $A \in \mathcal{K}$ and $T \in \mathcal{T}$ we have*
$$r(T(A)) = T(r(A)).$$
2. *Lipschitz-continuity: There is a constant $c \geq 0$, such that for all $A, B \in \mathcal{K}$,*
$$||r(A) - r(B)|| \leq c \cdot \delta(A, B).$$

We call c the quality of the reference point r.

Later, when we want to construct an approximation algorithm for similarities, we will have to rescale at least one of the weighted point sets. Unfortunately, rescaling in a way that the diameters of the underlying point sets are equal, does not work. Please note again that we will keep the weights of the points unchanged.

The key for a working algorithm is to rescale in a way that the normalized first moments with respect to their reference points coincide. Here we give the definition of the normalized first moment of a weighted point set with respect to an arbitrary point $p \in \mathbb{R}^d$. Note that this point can be computed in linear time.

Definition 4 (Normalized First Moment) *The normalized first moment of a weighted point set $A = \{(p_i, \alpha_i)_{i=1,...,n}\} \in \mathbb{W}^d$ with respect to a point $p \in \mathbb{R}^d$ is defined as*

$$m_p(A) = \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i ||p_i - p||.$$

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Next we will introduce the Earth Mover's Distance (EMD,[7]), a commonly known distance measure on weighted point sets.

Definition 5 (Earth Mover's Distance) *Let* $A = \{(p_i, \alpha_i)_{i=1,...,n}\}$, $B = \{(q_i, \beta_i)_{i=1,...,m}\} \in \mathbb{W}^d$ *be weighted point sets with total weights* W^A, W^B. *Let* $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *be a distance measure on the ground set* $\mathbb{R}^d$. *The Earth Mover's Distance between* A *and* B *is defined as*

$$EMD(A, B) = \frac{\min_{F \in \mathcal{F}} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} D(p_i, q_j)}{\min\{W^A, W^B\}}$$

where $F = \{f_{ij}\}$ *is a feasible flow, i.e.*

1. $f_{ij} \geq 0, i = 1, ..., n, j = 1, ..., m$
2. $\sum_{j=1}^{m} f_{ij} \leq \alpha_i, i = 1, ..., n$
3. $\sum_{i=1}^{n} f_{ij} \leq \beta_j, j = 1, ..., m$
4. $\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} = \min\{W^A, W^B\}$

For a detailed discussion of the EMD see [7], [3] and [4]. In the following, the distance measure D used in the definition of the EMD should be the same as the one used in the defintion of the reference point. If D is the Euclidean Distance, we will also use $EEMD$ as a notation for the Earth Mover's Distance.

3 Approximation Algorithms Using Reference Points

In this chapter we present approximation algorithms using reference points. Since this would be useless if there was no reference point, we provide the following theorem:

Theorem 1 *The center of mass is an EMD-reference point for weighted point sets with equal total weights with respect to affine transformations. Its quality is* 1.

The proof of the Lipschitz-continuity was already given in [7] as a proof for a lower bound for the EMD. The equivariance of the center of mass is commonly known.

The following three sections are organized as follows: In each section we consider a class of transformations, construct an approximation algorithm for matching under these transformations for general reference points and finally use the center of mass to get a concrete algorithm.

For the whole chapter let $A = \{(p_i, \alpha_i)_{i=1,...,n}\}$, $B = \{(q_i, \beta_i)_{i=1,...,m}\} \in \mathbb{W}^{d,G}$ be two weighted point sets in dimension d with positive equal total weight $G \in \mathbb{R}^+$. Please note that the following results do not hold for weighted point sets with unequal total weights. For simplicity let m be $O(n)$. Further let $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ be an EMD-reference point for weighted point sets with respect to the considered class of transformations with quality c. Let $T^{ref}(n)$ be the time to compute the reference point, $T^{EMD}(n)$ and $T^{EEMD}(n)$ be the time to compute the EMD and $EEMD$ between two weighted point sets and $T^{rot}(n)$ be the time needed to find the optimal rotation around a fixed point. An upper bound on $T^{EMD}(n)$ and $T^{EEMD}(n)$ is $O(n^4 \log n)$ using a strongly polynomial minimum cost flow algorithm by Orlin ([6]). A in practice faster algorithm can be developed by solving the linear programming problem using the simplex method.

3.1 Translations

Consider the following algorithm to get an approximation on the problem of finding a translation minimizing the EMD under translations:

Algorithm *TranslationApx*:
1. Compute $r(A)$ and $r(B)$ and move B by $r(A) - r(B)$. Let B' be the image of B.
2. Output B' as an approximately optimal solution and the approximate distance $EMD(A, B')$.

Extending the proof in [1] to weighted point sets we can prove the following:

Theorem 2 *Algorithm TranslationApx finds an approximately optimal matching for translations with loss factor* $c + 1$ *in time* $O(T^{ref}(n) + T^{EMD}(n))$.

Corollary 3 *Algorithm TranslationApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation ratio* 2. *Its runtime is* $O(T^{EMD}(n))$.

3.2 Rigid Motions

The following algorithm gives a first approximation on the EMD under rigid motions, i.e. combinations of translations and rotations:

Algorithm *RigidMotionApx*:
1. Compute $r(A)$ and $r(B)$ and move B by $r(A) - r(B)$. Let B' be the image of B.
2. Find an optimal matching of A and B' under rotations of B' around $r(A) = r(B')$. Let B'' be the image of B' under this rotation.
3. Output B'' as an approximately optimal solution together with the approximate distance $EMD(A, B'')$.

Theorem 4 *Algorithm RigidMotionApx finds an approximately optimal matching for rigid motions with loss factor* $c + 1$ *in* $O(T^{ref}(n) + T^{EMD}(n) + T^{rot}(n))$.

Since the position of the reference point as rotation center is fixed, several degrees of freedom have been eliminated and the problem is easier than the one finding the optimal rigid motion itself. Unfortunately, even for this problem no efficient algorithm is known so far. Therefore it would be nice to have at least an approximation algorithm for this problem. We will show one based on the following lemma. Please note that this lemma is only proven if we take the Euclidean distance on the ground set.

Lemma 5 *Let $p \in \mathbb{R}^d$ be some point. Let $Rot(p)$ be the set of all rotations around p. Then there is a rotation $R' \in Rot(p)$ such that*

$$EEMD(A, R'(B)) \leq 2 \cdot \min_{R \in Rot(p)} EEMD(A, R(B)),$$

where R' aligns p and at least one point of each set A and B.

Therefore, given a fixed point $p \in \mathbb{R}^d$ we get a 2-approximation on the problem of finding an optimal rotation of B around p by the following algorithm:

Algorithm *RotationApx*

1. Compute the minimum $EEMD$ over all possible alignments of p and at least one point of each set A and B.

The runtime of this algorithm is $O(n^2 T^{EEMD}(n))$. Using this algorithm combined with reference points we now get an easy to implement and fast approximation algorithm for rigid motions. Unfortunately, the increased efficiency must be paid by the increased approximation ratio $(2c + 2)$.

Algorithm *RigidMotionApxUsingRotationApx*

1. Compute $r(A)$ and $r(B)$ and move B by $r(A) - r(B)$. Let B' be the image of B.
2. Find a best matching of A and B' under rotations of B' around $r(A) = r(B')$ where $r(A)$ and at least one point in A and B' are aligned. Let B'' be the image of B' under this rotation.
3. Output B'' as an approximately optimal solution together with the approximate distance $EEMD(A, B'')$.

Theorem 6 *RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with loss factor $2c + 2$ in time $O(T^{ref}(n) + n^2 T^{EEMD}(n))$. This holds for the Euclidean distance on the ground set.*

In the next two corollaries we apply the center of mass to the last two theorems:

Corollary 7 *RigidMotionApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation ratio 2 in time $O(T^{rot}(n) + T^{EMD}(n))$.*

Corollary 8 *RigidMotionApxUsingRotationApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation ratio 4. Its runtime is $O(n^2 T^{EEMD}(n))$. This holds for the Euclidean distance on the ground set.*

3.3 Similarities

In this section we present approximation algorithms for matching two given weighted point sets under similarity transformations, i.e. combinations of translations, rotations and scalings. More precisely, we want to compute $\min_S EMD(A, S(B))$, where the minimum is taken over all similarity operations S. Note that exchanging A and B makes a difference.

Algorithm *SimilarityApx*:

1. Compute $r(A)$ and $r(B)$ and move B by $r(A) - r(B)$. Let B' be the image of B.
2. Determine the normalized first moments $m_{r(A)}(A)$ and $m_{r(B')}(B')$ and scale B' by $\frac{m_{r(A)}(A)}{m_{r(B')}(B')}$ around the center $r(A) = r(B')$. Let B'' be the image of B' under this scaling.
3. Find an optimal matching of A and B'' under rotations of B'' around $r(A) = r(B'')$. Let B''' be the image of B'' under this rotation.
4. Output B''' as an approximately optimal solution together with the approximate distance $EMD(A, B''')$.

To show the correctness of this algorithm we use the following two lemmata:

Lemma 9 *Let $A \in \mathbb{W}^d$ be a weighted point set with normalized first moment $m_p(A)$ with respect to a point $p \in \mathbb{R}^d$. Let τ_1, τ_2 be scalings with center p and ratios γ_1 and γ_2, respectively. Then*

$$EMD(\tau_1(A), \tau_2(A)) \leq |(\gamma_1 - \gamma_2) m_p(A)|.$$

The next lemma gives a new lower bound for the EMD of two weighted point sets:

Lemma 10 *Let $A, B \in \mathbb{W}^{d,G}$ and $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ a reference point with quality c. Then*

$$|m_{r(A)}(A) - m_{r(B)}(B)| \leq (1 + c) EMD(A, B).$$

Using these lemmata we can prove the following:

Theorem 11 *SimilarityApx finds an approximately optimal matching for similarities with loss factor $2c+2$ in time $O(T^{ref}(n) + T^{EMD}(n) + T^{rot}(n))$.*

As for *RigidMotionApx*, *SimilarityApx* depends on finding the optimal rotation, which is impractical. Again, we make this algorithm practical and efficient by using *RotationApx* and again we have to pay by a worse approximation ratio:

Algorithm *SimilarityApxUsingRotationApx*

1. Compute $r(A)$ and $r(B)$ and move B by $r(A) - r(B)$. Let B' be the image of B.
2. Determine the normalized first moments $m_{r(A)}(A)$ and $m_{r(B')}(B')$ and scale B' by $\frac{m_{r(A)}(A)}{m_{r(B')}(B')}$ around the center $r(A) = r(B')$. Let B'' be the image of B' under this scaling.
3. Find a best matching of A and B'' under rotations of B'' around $r(A) = r(B'')$ where $r(A)$ and at least one point in each set A and B'' are aligned. Let B''' be the image of B'' under this rotation.
4. Output B''' as an approximately optimal solution and the approximate distance $EEMD(A, B''')$.

Theorem 12 *Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with loss factor* $4c + 4$ *in time* $O(T^{ref}(n) + n^2 T^{EEMD}(n))$. *This holds for the Euclidean distance on the ground set.*

Corollary 13 *Algorithm SimilarityApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation ratio* 4. *Its runtime is* $O(T^{EMD}(n) + T^{rot}(n))$.

Corollary 14 *Algorithm SimilarityApxUsingRotationApx using the center of mass as EMD-reference point induces an approximation algorithm with loss factor* 8. *Its runtime is* $O(n^2 T^{EEMD}(n))$. *This holds for the Euclidean distance on the ground set.*

3.4 Lower Bound for Algorithm TranslationApx

In Section 3.1 we presented the center of mass as an EMD-reference point with quality 1, and thus inducing an approximation algorithm for translations with ratio 2. We now show that this bound is tight:

Theorem 15 *There are sets where the upper bound for algorithm TranslationApx is assumed.*

Proof. Let $A := \{((0,0),1),((1,0),K)\}$ and $B := \{((0,0),1),((0,1),K)\}$, where $K \in \mathbb{R}^+$ is some constant. Let $EMD^C(A,B)$ be the Earth Mover's Distance, where the center of masses coincide and $EMD^{opt}(A,B)$ be the optimal distance under translation. Then $\frac{EMD^C(A,B)}{EMD^{OPT}(A,B)} \to 2$ as $K \to \infty$. This can be seen easily by using an upper bound for $EMD^{opt}(A,B)$ by matching the two thick points. □

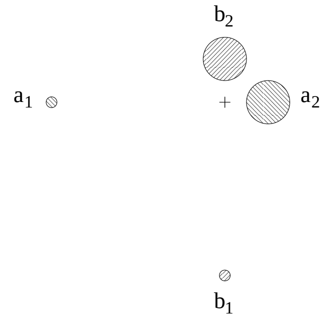

Figure 1: Matching according to center of mass

4 Conclusion

In this paper we introduced EMD-reference points for weighted point sets and constructed efficient approximation algorithms for matching under various classes of transformations. Additionally, we presented the center of mass as an EMD-reference point for weighted point sets with equal total weights. Unfortunately, the center of mass is no EMD-reference point if you consider the set of all weighted point sets, including those with different total weights. Even worse, we show in [5] that there is no EMD-reference point for all weighted point sets. A variation of the EMD is the Proportional Transportation Distance (PTD), see [4]. In [5] we also show, that the center of mass is a PTD-reference point even for weighted point sets with different total weight and all theorems and corollaries mentioned in this paper carry over. But the PTD has a couple of disadvantages against the EMD, for example it is not suitable for partial matching applications.

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