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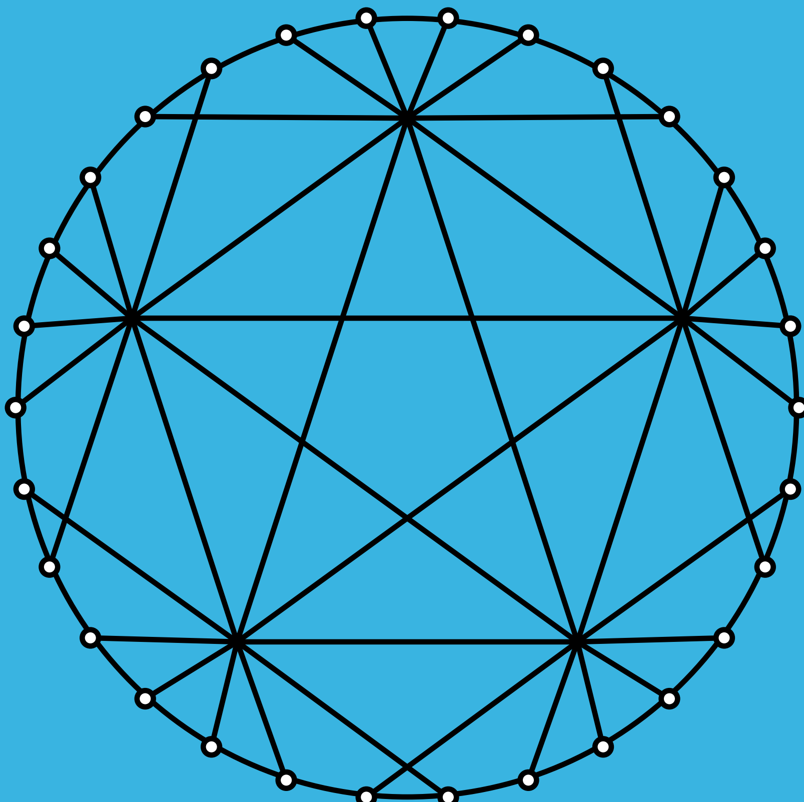

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Configurations representing a skew perspective; a classification of $(15_4\,20_3)$-configurations reflecting abstract properties of a perspective between tetrahedrons

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Abstract

A combinatorial object representing schemas of, possibly skew, perspectives, called *a configuration of skew perspective* is defined. Some classifications of skew perspectives are presented.

Introduction

A project to characterize and classify binomial partial Steiner triple systems via the arrangement of their free complete subgraphs was started in [18]. In particular, we know that if a configuration $\mathfrak{K}$ contains the maximal admissible number (with respect to its parameters, i.e. $= m+2$, where m is the rank of a point in $\mathfrak{K}$) of free K_{m+1}-subgraphs then $\mathfrak{K}$ is a combinatorial Grassmannian (cf. [13]) and if $\mathfrak{K}$ contains m free K_{m+1}-subgraphs then it

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is a multi Veblen configuration (cf. [15], [16]). One of the most fruitful observation used to obtain a required classification is quoted in Fact 1.2 after [18]:

> *a configuration $\mathfrak{K}$ with two free subgraphs K_{m+1} can be considered as a schema of an abstract perspective between these graphs.*

Let us stress on the words *schema* and *abstract*: 'ordinary' projections, as used and investigated e.g. in [6], [7], or [8] are geometric objects belonging to spaces embeddable into desarguesian projective spaces. They can be considered as examples (realizations) of our perspectives, but configurations considered in this paper do not necessarily have any realization in a desarguesian projective space. The class of configurations considered in the paper is introduced in Construction 1.1; they are called (*skew*) *perspectives.*

The above observation enables us to reduce the problem to a classification of 'line perspectives' (maps between edges of graphs, we call them also '*skews*') and a classification of 'axial configurations' (defined on intersection points of lines containing perspective edges); these axial configurations have vertices with point rank smaller by 2. If $\mathfrak{K}$ has three free K_{m+1} subgraphs, a similar technique involving a triple perspective can be used; for $m = 4$ the complete classification was given in [12]. Even in this small case $m = 4$ there are, generally, $10! \geq 3 \cdot 10^6$ admissible perspectives. One has to look for ways to distinguish some more regular and interesting among them.

In the paper we introduce two classes of more regularly behaving skews. The first class consists of skews which map concurrent edges onto concurrent. Skews in the second class map one bundle of concurrent edges onto a bundle with the same vertex, a next bundle with the first deleted onto analogous family, and so on .... Both two types of skews are associated with some natural embeddings between permutation groups: permutations (sequences of permutations) of the vertices of graphs determine permutations of edges. In Section 1 we gather together a few general properties of the introduced perspectives, in particular two criterions (Proposition 1.3 and Lemma 1.5), whose specifications appear crucial in further procedures of classifications. Here we also define the abovementioned classes of skews. General properties of perspectives with skews in the corresponding classes are presented in Sections 2 and 3. They contain (technical) tools that will be used in classifying (abstract schemes of) perspectives between tetrahedrons. In a very particular case, though, we could obtain here a complete classification of a, rather wide, class of configurations (Proposition 2.6). The final Section 5 in which we present classifications of perspectives between tetrahedrons (cf. Theorems 5.1, 5.8 and 5.10, and details in their proofs) is preceeded by a (quite technical) Section 4. In this section we discuss questions concerning labelling of the points of the Veblen-Pasch configuration by some 2-sets; this appears important because an axis of a perspective between tetrahedrons is the Veblen configuration.

1 Underlying ideas and basic definitions

Let us start with introducing some, standard, notation. Let X be an arbitrary set. The symbol $\mathcal{S}_X$ stands for the family of permutations of X. Let k be a positive integer; we write $\wp_k(X)$ for the family of k-element subsets of X. Then $K_X = \langle X, \wp_2(X)\rangle$ is the complete graph on X; K_n is K_X for any X with $|X| = n$. Analogously, $\mathcal{S}_n = \mathcal{S}_X$.

A *partial linear space* is an incidence structure with blocks (*lines*) pairwise intersecting in at most a point. A $(\nu_r\, b_\kappa)$-*configuration* is a partial linear space with ν points, each of rank r, and b lines, each of rank (size) κ. A *partial Steiner triple system* (in short: a PSTS) is a partial linear space with all the lines of size 3. A $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration with arbitrary integer $n \geq 3$ is a partial Steiner triple system, it is called a *binomial partial Steiner triple system.*

We say that a graph $\mathcal{G}$ is *freely contained* in a configuration $\mathfrak{B}$ iff the vertices of $\mathcal{G}$ are points of $\mathfrak{B}$, each edge e of $\mathcal{G}$ is contained in a line $\overline{e}$ of $\mathfrak{B}$, the above map $e \mapsto \overline{e}$ is an injection, and lines of $\mathfrak{B}$ which contain disjoint edges of $\mathcal{G}$ do not intersect in $\mathfrak{B}$. We say, shortly: G *is free in* $\mathfrak{B}$. If $\mathfrak{B}$ is a $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration and $\mathcal{G} = K_X$ then $|X| + 1 \leq n$. Consequently, a maximal complete graph that may be freely contained in a binomial $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration is K_{n-1}. Further details of this theory are presented in [18], relevant results will be quoted in the text, when needed.

In the paper we aim to develop a theory of configurations which characterize abstract properties of a perspective between two complete graphs.

Construction 1.1. Let I be a nonempty finite set, $n := |I|$. In most of the time, without loss of generality, we assume that $I = I_n = \{1, \ldots, n\}$.

Let $A = \{a_i \colon i \in I\}$ and $B = \{b_i \colon i \in I\}$ be two disjoint n-element sets, let $p \notin A \cup B$.
Then we take a $\binom{n}{2}$-element set $C = \{c_u \colon u \in \wp_2(I)\}$ disjoint with $A \cup B \cup \{p\}$. Set

$$\mathcal{P} = A \cup B \cup \{p\} \cup C.$$

Let us fix a permutation σ of $\wp_2(I)$ and write

$$\begin{array}{lcl} \mathcal{L}_p & := & \big\{\{p, a_i, b_i\} \colon i \in I\big\}, \\ \mathcal{L}_A & := & \big\{\{a_i, a_j, c_{\{i,j\}}\} \colon \{i,j\} \in \wp_2(I)\big\}, \\ \mathcal{L}_B & := & \big\{\{b_i, b_j, c_{\sigma^{-1}(\{i,j\})}\} \colon \{i,j\} \in \wp_2(I)\big\}. \end{array}$$

Finally, let $\mathcal{L}_C$ be a family of 3-subsets of C such that $\mathfrak{N} = \langle C, \mathcal{L}_C \rangle$ is a $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration. Set

$$\mathcal{L} = \mathcal{L}_p \cup \mathcal{L}_A \cup \mathcal{L}_B \cup \mathcal{L}_C \quad \text{and} \quad \mathbf{\Pi}(n, \sigma, \mathfrak{N}) := \langle \mathcal{P}, \mathcal{L} \rangle.$$

The structure $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$ will be referred to as a *skew perspective* with the *skew* σ and the *axis* (or *axial configuration*) $\mathfrak{N}$. $\square$

We frequently shorten $c_{\{i,j\}}$ to $c_{i,j}$. In essence, the names "a_i", "$c_{i,j}$" are – from the point of view of mathematics – arbitrary, and could be replaced by any other labelling (cf. analogous problem of labelling in [16, Constr. 3, Repr. 3] or in [12, Rem. 2.11, Rem. 2,13], [16, Exmpl. 2]). Formally, one can define $J = I \cup \{a, b\}$, $x_i = \{x, i\}$ for $x \in \{a, b\} =: p$ and $i \in I$, and $c_u = u$ for $u \in \wp_2(I)$. After this identification $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$ becomes a structure defined on $\wp_2(J)$. Then, it is easily seen that

$$\mathbf{\Pi}(n, \sigma, \mathfrak{N}) \text{ is a } \left(\binom{n+2}{2}_n \binom{n+2}{3}_3\right) \text{ configuration.}$$

In particular, it is a partial Steiner triple system, so we can use standard notation: $\overline{x, y}$ stands for the line which joins two collinear points $x, y \in \mathcal{P}$, and then we define the partial operation $\oplus$ by the following requirements: $x \oplus x = x$, $\{x, y, x \oplus y\} \in \mathcal{L}$ whenever $\overline{x, y}$ exists. Observe that

$$\mathbf{G}_2(n+2) = \mathbf{G}_2(J) = \langle \wp_2(J), \wp_3(J), \subset \rangle \cong \mathbf{\Pi}(n, \mathrm{id}_{I_n}, \mathbf{G}_2(I_n))$$

(cf. [13, Eq. (1), the definition of *combinatorial Grassmannian* $\mathbf{G}_2(n)$]).

It is clear that $A^* = A \cup \{p\}$ and $B^* = B \cup \{p\}$ are two K_{n+1}-graphs freely contained in $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$. Applying the results [18, Prop. 2.6 and Thm. 2.12] we immediately obtain Fact 1.2.

Fact 1.2. *Let $N = n+2$. The following conditions are equivalent.*

(i) $\mathfrak{M}$ *is a binomial* $\left(\binom{N}{2}_{N-2}\binom{N}{3}_3\right)$*-configuration which freely contains two* K_{N-1}*-graphs.*
(ii) $\mathfrak{M} \cong \mathbf{\Pi}(n, \sigma, \mathfrak{N})$ *for a* $\sigma \in \mathcal{S}_{\wp_2(I)}$ *and a* $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$*-configuration* $\mathfrak{N}$ *defined on* $\wp_2(I)$.

Consequently, the configurations defined by Construction 1.1 are essentially known, but no *general* classification of them is known, though.

The map

$$\pi = \left(a_i \longmapsto b_i,\ i \in I\right)$$

is referred to as a point-perspective of K_A onto K_B with center p. Moreover, the map

$$\xi_\sigma = \xi = \left(\overline{a_i, a_j} \longmapsto \overline{b_{i'}, b_{j'}},\ \sigma(\{i,j\}) = \{i', j'\} \in \wp_2(I)\right)$$

is referred to as a line perspective, where $\mathfrak{N}$ is the axial configuration of our perspective. As in the case of classical (geometrical) projections for every line l in $\mathcal{L}_A$ the lines l and $\xi_\sigma(l)$ intersect (on the axis). Consequently, $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$ is a schema of a perspective of some type. Contrary to the approach of [18], following the approach of this paper we can better analyze some particular properties of the perspective (π, ξ).

Clearly, $\mathcal{S}_I$ naturally (and faithfully) acts on $\wp_2(I)$: the map

$$\mathcal{S}_I \ni \alpha \mapsto \overline{\alpha} \in \mathcal{S}_{\wp_2(I)};\quad \overline{\alpha}(\{i,j\}) = \{\alpha(i), \alpha(j)\} \text{ for every } \{i,j\} \in \wp_2(I) \tag{1}$$

is a group embedding of $\mathcal{S}_I$ into $\mathcal{S}_{\wp_2(I)}$. Let us write $\overline{\mathcal{S}}_I$ for the image of $\mathcal{S}_I$ under this embedding.

Proposition 1.3. *Let $f \in \mathcal{S}_{\mathcal{P}}$, $f(p) = p$, $\sigma_1, \sigma_2 \in \mathcal{S}_{\wp_2(I)}$, and $\mathfrak{N}_1, \mathfrak{N}_2$ be two* $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$*- configurations defined on* $\wp_2(I)$*. The following conditions are equivalent.*

(i) *f is an isomorphism of $\mathbf{\Pi}(n, \sigma_1, \mathfrak{N}_1)$ onto $\mathbf{\Pi}(n, \sigma_2, \mathfrak{N}_2)$.*
(ii) *There is $\varphi \in \mathcal{S}_I$ such that one of the following holds*

$$\overline{\varphi}(\textit{comp.}((1))\textit{is an isomorphism of } \mathfrak{N}_1 \textit{ onto } \mathfrak{N}_2, \tag{2}$$

$$f(x_i) = x_{\varphi(i)},\ x = a, b,\ f(c_{\{i,j\}}) = c_{\{\varphi(i), \varphi(j)\}},\quad i, j \in I, i \neq j, \tag{3}$$

$$\overline{\varphi} \circ \sigma_1 = \sigma_2 \circ \overline{\varphi}, \tag{4}$$

or

$$\sigma_2^{-1}\overline{\varphi} \text{ is an isomorphism of } \mathfrak{N}_1 \text{ onto } \mathfrak{N}_2, \tag{5}$$

$$\left.\begin{array}{l} f(a_i) = b_{\varphi(i)},\ f(b_i) = a_{\varphi(i)}, \\ f(c_{\{i,j\}}) = c_{\sigma_2^{-1}\{\varphi(i),\varphi(j)\}},\ i,j \in I, i \neq j, \end{array}\right\} \tag{6}$$

$$\overline{\varphi} \circ \sigma_1 = \sigma_2^{-1} \circ \overline{\varphi}. \tag{7}$$

Proof. Write $\mathfrak{M}_l = \mathbf{\Pi}(n, \sigma_l, \mathfrak{N}_l)$ for $l = 1, 2$.

Assume (i). Since exactly two free K_{n+1} subgraphs of $\mathfrak{M}_l$ ($l = 1, 2$) pass through p (cf. [18, Prop.'s 2.6, 2.7]), one of the following holds

(a) $f(A) = A$ and $f(B) = B$, or
(b) $f(A) = B$ and $f(B) = A$.

Assume, first, (a). Consequently, there is a permutation $\varphi \in \mathcal{S}_I$ such that $f(a_i) = a_{\varphi(i)}$ for each $i \in I$. This yields $f(b_i) = f(p) \oplus f(a_i) = b_{\varphi(i)}$, and, finally $f(c_{i,j}) = f(a_i \oplus a_j) = \ldots = c_{\varphi(i),\varphi(j)}$. This justifies (3). Since f preserves the lines of $\mathfrak{N}$, from (3) we infer (2). Finally, the equation $c_{\overline{\varphi}(\sigma_1^{-1}(\{i,j\}))} = f(c_{\sigma_1^{-1}(\{i,j\})}) = f(b_i \oplus b_j) = f(b_i) \oplus f(b_j) = b_{\varphi(i)} \oplus b_{\varphi(j)} = c_{\sigma_2^{-1}(\{\varphi(i),\varphi(j)\})}$ justifies (4).

In case (b) the reasoning goes analogously. We only need to note that $f(c_{\{i,j\}}) = f(b_{\varphi(i)} \oplus b_{\varphi(j)}) = c_{\sigma_2^{-1}\overline{\varphi}(\{i,j\})}$, which justifies the last condition in (6) and yields (5).

Conversely, if (ii) is assumed we directly verify that $f(x \oplus y) = f(x) \oplus f(y)$ holds for all $x, y \in (A \cup B)$, which proves (i). □

Let us note here the following immediate consequence of Proposition 1.3

Lemma 1.4. *Let $\mathfrak{N}$ be a $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration defined on $\wp_2(I)$ and let $\sigma \in \mathcal{S}_{\wp_2(I)}$. The σ^{-1}-image $\sigma^{-1}(\mathfrak{N})$ of $\mathfrak{N}$ is a $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration. Then the (involutory) map*

$$\mathfrak{J}_n = \mathfrak{J} \colon \left\{ \begin{array}{ccc} a_i & b_j & c_u \\ \downarrow & \downarrow & \downarrow \\ b_i & a_j & c_{\sigma^{-1}(u)} \end{array} \right. \qquad \text{for all } i,j \in I, \quad u \in \wp_2(I) \tag{8}$$

is an isomorphism of $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$ onto $\mathbf{\Pi}(n, \sigma^{-1}, \sigma^{-1}(\mathfrak{N}))$.

Lemma 1.5. *Let* $\mathfrak{M} = \mathbf{\Pi}(n, \sigma, \mathfrak{N})$ *for a* $\sigma \in \mathcal{S}_{\wp_2(I)}$.

(i) *Assume that* $\mathfrak{M}$ *freely contains a complete* K_{n+1}*-graph* $G \neq K_{A^*}, K_{B^*}$. *Then there is* $i_0 \in I$ *such that* $\mathrm{S}(i_0) = \{c_u \colon i_0 \in u \in \wp_2(I)\}$ *is a collinearity clique in* $\mathfrak{N}$ *freely contained in it and* σ *satisfies*

$$i_0 \in u \implies i_0 \in \sigma(u) \text{ for every } u \in \wp_2(I). \tag{9}$$

Moreover,

$$G = G_{(i_0)} \ := \ \{a_{i_0}, b_{i_0}\} \cup \mathrm{S}(i_0). \tag{10}$$

(ii) *Conversely, if* $\mathrm{S}(i_0)$ *is a collinearity clique freely contained in* $\mathfrak{N}$ *for some* $i_0 \in I$ *such that* (9) *holds then the set* $G_{(i_0)}$ *defined by* (10) *is a complete free* K_{n+1} *subgraph of* $\mathfrak{M}$.

Proof. Let $G \neq K_{A^*}, K_{B^*}$ be a complete K_{n+1}-graph freely contained in $\mathfrak{M}$. Then p, $G \cap A$, and $G \cap B$ form a triple of collinear points (cf. [18, Prop. 2.7]). So, there is $i_0 \in I$ such that $a_{i_0}, b_{i_0} \in G$. And $G \setminus \{a_{i_0}, b_{i_0}\} \subset C$. The set of points in C which are collinear with a_{i_0} is exactly $\mathrm{S}(i_0)$; it contains G. More, its cardinality is $n-1$, and therefore $G = G_{(i_0)}$. Since G is a clique, we conclude with: $\mathrm{S}(i_0)$ is a clique in $\mathfrak{N}$. Clearly, it is freely contained in $\mathfrak{N}$. The condition (9) follows form Construction 1.1 and the (evident) requirement: b_{i_0}, c_u are collinear for every $u \in \mathrm{S}(i_0)$.

It is a matter of a straightforward verification to prove (ii). □

Note that if (9) holds then $\mathcal{J}$ maps $\mathrm{S}(i_0)$ onto $\mathrm{S}(i_0)$.

1.1 Two important classes of skews

Let us write $\mathcal{P}^{\mathsf{G}}_n = \overline{\mathcal{S}}_{I_n}$, where $I_n = \{1, 2, \ldots, n\}$. A geometric characterization of skews determined by the elements of $\mathcal{P}^{\mathsf{G}}_n$ is given in Lemma 1.6.

Lemma 1.6. *The map* ξ_σ *maps intersecting edges of* K_A *onto intersecting edges of* K_B *iff either*

(i) *there is a permutation* $\sigma_0 \in \mathcal{S}_I$ *such that* $\sigma = \overline{\sigma_0}$ *or*

(ii) $n = 4$ *and* $\sigma = \varkappa\overline{\sigma_0}$ *where* $\varkappa(u) = I \setminus u$ *for every* $u \in \wp_2(I)$ *and* $\overline{\sigma_0}$ *is defined by* (1), *for some* $\sigma_0 \in \mathcal{S}_I$.

In case (i), ξ_σ *preserves the (ternary) concurrency of edges, and in case* (ii), *the concurrency is not preserved.*

Proof. One can identify an edge $\{a_i, a_j\}$ of K_A with $\{i, j\} \in \wp_2(I)$; analogously we identify $\wp_2(B) \ni \{b_i, b_j\} \mapsto \{i, j\} \in \wp_2(I)$. After this identification $\sigma = \xi_\sigma \in \mathcal{S}_{\wp_2(I)}$, and ξ_σ preserves the edge-intersection iff it preserves set-intersection. The claim is just a reformulation of the folklore (cf. [9], [13, Prop. 1.5], [16, Prop. 15]). □

A more detailed analysis of the case Lemma 1.6(ii) will be presented in more details in Subsection 5.2.

Note. If $\sigma_0 \in \mathcal{S}_I$ we frequently identify σ_0, $\overline{\sigma_0}$, and the corresponding map $\xi_{\overline{\sigma_0}}$. Consequently, if $\sigma \in \mathcal{S}_I$ we write $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$ in place of $\mathbf{\Pi}(n, \overline{\sigma}, \mathfrak{N})$. □

Next, we introduce another important class of skews.

Construction 1.7. Let X be a set with $|X| = n < \infty$ and let $\Phi = (\phi_n, \ldots, \phi_3)$ be a sequence of bijections defined on subsets of X such that

$$\left.\begin{array}{l} \mathrm{Dom}(\phi_j) = \mathrm{Rng}(\phi_j), \ \text{for } j = n, \ldots, 3, \\ |X \setminus \mathrm{Dom}(\phi_n)| = 1, \\ \mathrm{Dom}(\phi_{j-1}) \subset \mathrm{Dom}(\phi_j), \ \text{for } j = n, \ldots, 4 \\ |\mathrm{Dom}(\phi_j) \setminus \mathrm{Dom}(\phi_{j-1})| = 1, \ \text{for } j = n, \ldots, 4. \end{array}\right\} \tag{11}$$

We add one item more: $\phi_2 = \mathrm{id}_{\{1\}}$.

With such a Φ we associate a permutation σ_Φ defined as follows. Note, first, that $|\mathrm{Dom}(\phi_j)| = j - 1 = |\mathrm{Rng}(\phi_j)|$. Let us order the elements of $X = (x_n, x_{n-1}, \ldots, x_2, x_1)$ so as $x_j \notin \mathrm{Dom}(\phi_j)$. Finally, we set

$$\sigma_\Phi \colon \wp_2(X) \ni \{x_j, x_i\} \longmapsto \{x_j, \phi_j(x_i)\} \text{ for } 1 \le i < j \le n, \tag{12}$$

Then, clearly, $\sigma_\Phi \in \mathcal{S}_{\wp_2(X)}$. □

Note that, since $|\mathrm{Dom}(\phi_2)| = 1$, ϕ_2 is the identity on $\{x_1\}$, and since $|\mathrm{Dom}(\phi_3)| = 2$, ϕ_3 is either the identity on $\{x_1, x_2\}$ or it is a transposition. So, the item ϕ_2 in Φ is useless to define σ_Φ and will be usually omitted in further investigations.

Clearly, $\Phi^{-1} = (\phi_n^{-1}, \ldots, \phi_2^{-1})$ satisfies (11) and $\sigma_{\Phi^{-1}} = \sigma_\Phi^{-1}$.

Let us pay some attention to the class of maps defined in Construction 1.7. Let Φ satisfy (11), let $(x_n, \ldots, x_1)$ be an associated ordering of X (note: the order of elements x_1, x_2 of $\mathrm{Dom}(\phi_3)$ is not uniquely determined!) and write $X_j = \{x_1, x_2 \ldots, x_{j-1}\}$; then $\mathrm{Dom}(\phi_j) = X_j$. Let $\alpha \colon X \longrightarrow Y$

be a bijection. We set $y_j = \alpha(x_j)$ for $j = n, ..., 1$, so $Y_j := \alpha(X_j) = \{y_1, \dots, y_{j-1}\}$. Consequently, $\phi'_j := \phi_j^\alpha \in \mathcal{S}_{Y_j}$ for $j = n, \dots, 2$. Write $\Phi' = (\phi'_n, \dots, \phi'_2) =: \Phi^\alpha$; clearly, Φ^α also satisfies (11).

Let $i < j$ and $u = \{x_i, x_j\} \in \wp_2(X)$. Then $\overline{\alpha}(u) = \{y_i, y_j\}$. We have $\sigma_\Phi(u) = \{\phi_j(x_i), x_j\}$ and $\overline{\alpha}\big(\sigma_\Phi(u)\big)=\{\alpha(\phi_j(x_i)), \alpha(x_j))\}$. Next $\phi'_j(\overline{\alpha}(u)) = \{y_j, \phi'_j\big(\alpha(y_i)\big)\}$. Since $\alpha \circ \phi_j = \phi' \circ \alpha_j$ we conclude with the Lemma 1.8.

Lemma 1.8. *If α is a bijection of X onto Y then $(\sigma_\Phi)^{\overline{\alpha}} = \sigma_{\Phi^\alpha}$.*

In a consequence of Proposition 1.3 and Lemma 1.8, without loss of generality in most parts we shall restrict to the case

$$X = I_n, \Phi = (\phi_n, \phi_{n-1}, \dots, \phi_3) \text{ where } \phi_j \in \mathcal{S}_{I_{j-1}} \text{ for } j = n, n-1, \dots, 3; \tag{13}$$

then (12) assumes the form $\sigma_\Phi(\{i,j\}) = \{\phi_j(i), j\}$ for $i < j$.

Note that, in fact, the map

$$\mathcal{S}_{I_{n-1}} \oplus \dots \mathcal{S}_{I_2} \ni \Phi \mapsto \sigma_\Phi, \quad \Phi \text{ satisfies } (13)$$

is a group embedding of $\mathcal{S}_{(<n)} := \mathcal{S}_{I_{n-1}} \oplus \dots \oplus \mathcal{S}_{I_2}$ into $\mathcal{S}_{\wp_2(I_n)}$; we write $\mathcal{P}_n^{\mathsf{V}}$ for the image of $\mathcal{S}_{(<n)}$ under this embedding i.e. for the class of all the skews σ_Φ with Φ determined by (13). Two types of such maps are crucial:

each ϕ_j is a symmetry: $\phi_j(i) = j - i$; we denote the associated skew σ_Φ by $\zeta = \zeta_n$; clearly, $\zeta = \zeta^{-1}$.
each ϕ_j is a cycle: $\phi_j = (1, 2, \dots, j-1)$.

If every ϕ_j is the identity on I_{j-1} then σ_Φ is the identity on $\wp_2(I)$. For $n > 3$ the class of skews in $\mathcal{P}_n^{\mathsf{V}}$ is essentially distinct from the skews in $\mathcal{P}_n^{\mathsf{G}}$.

Lemma 1.9. *Let $n > 3$, $\alpha \in \mathcal{S}_{I_n}$, and a sequence Φ of permutations defined on I_j $(j = n-1, .., 2)$ satisfy (13). The following conditions are equivalent.*

(i) *The equality $\sigma_\Phi = \overline{\alpha}$ holds.*
(ii) *Either*

  a) $\alpha = \mathrm{id}_{I_n}$ *and* $\phi_j = \mathrm{id}_{I_{j-1}}$ *for* $j = n, \dots, 2$ *(then* $\overline{\alpha} = \mathrm{id}_{\wp_2(I_j)}$*),* *or*
  b) $\alpha = \mathrm{id}_{I_n \setminus I_2} \cup \big(1, 2\big)$ *and* $\phi_j = \mathrm{id}_{I_{j-1}}$ *for* $j = n, n-1, \dots, 4, 2$, $\phi_3 = (1, 2)$.

The cases (a) *and* (b) *together mean: $\mathcal{P}_n^{\mathsf{G}}$ and $\mathcal{P}_n^{\mathsf{V}}$ intersect in a C_2-subgroup of $\mathcal{S}_{\wp_2(I_n)}$.*

Proof. Assume (i). Take arbitrary $j = n, \ldots, 4$ and $i < j$. Let us set $u_i = \{n, i\}$; then $\overline{\alpha}(u_i) = \{\alpha(j), \alpha(i)\}$ and $\sigma_\Phi(u_i) = u_{\phi_j(i)}$. There are at least two distinct $i', i'' < j$ and then $\{\alpha(j)\} = \overline{\alpha}(u_{i'}) \cap \overline{\alpha}(u_{i''}) = u_{\phi_j(i')} \cap u_{\phi_j(i'')} = \{j\}$. By definition, $\phi_2(1) = 1$. Now, let us pay our attention to ϕ_2: we need $\{\alpha(3), \alpha(2)\} = \{3, \phi_3(2)\}$ and $\{\alpha(3), \alpha(1)\} = \{3, \phi_3(1)\}$ which is valid when $\alpha(3) = 3$ and both when $\phi_3 = (1)(2)$ and when $\phi_3 = (1, 2)$, which is noted in (ii).

Justifying the implication (ii) $\Longrightarrow$ (i) is a matter of a simple computation similar to the above. □

The following counterpart to Lemma 1.9 and a slight restriction to Lemma 1.8 will be useful

Lemma 1.10. *Let $\Phi \in \mathcal{S}_{(<n)}$, $\alpha \in \mathcal{S}_{I_n}$. The following conditions are equivalent:*

(i) Φ^α *satisfies* (13) *(i.e. $\Phi^\alpha \in \mathcal{S}_{(<n)}$),*

(ii) $\alpha = \mathrm{id}_{I_n}$ *or α is the transposition* $(1, 2)$.

Proof. It suffices to observe that we need $\alpha(I_{j-1}) = I_{j-1}$ for $j = n, \ldots, 3$. □

In what follows we shall examine, first, in some details, perspectivities with skews in $\mathcal{P}^{\mathsf{G}}_n$ and with skews in $\mathcal{P}^{\mathsf{V}}_n$. After that we shall present classifications of the obtained structures for $n = 4$.

2 Perspectivities associated with permutations of indices: general properties

In this section we shall analyse in some detail configurations with the skews in $\mathcal{P}^{\mathsf{G}} = \bigcup\{\mathcal{P}^{\mathsf{G}}_n \colon n = 1, 2, 3, \ldots\}$.

Note 2.1. Let $\mathfrak{M} = \mathbf{\Pi}(n, \sigma, \mathfrak{N})$ be a skew perspective with $\sigma \in \mathcal{S}_{I_4}$. If $n = 1$ then $\mathfrak{M}$ is a single line. If $n = 2$ then $\mathfrak{N}$ is a single point and $\sigma = \mathrm{id}_{\wp_2(I_2)}$, and then $\mathfrak{M}$ is the Veblen configuration $\mathbf{G}_2(I_4)$ (the configuration in question is also frequently called the Pasch configuration, cf. e.g. [10]). If $n = 3$ then $\mathfrak{N}$ is a single 3-line L. The configurations $\mathbf{\Pi}(3, \sigma, L)$ were determined and characterized in [11]; these are exactly

- the *Desargues configuration* $\mathbf{\Pi}(3, \mathrm{id}_{I_3}, L)$,
- the *fez configuration* $\mathbf{\Pi}(3, (1,2,3), L)$, and
- the *Kantor configuration* $\mathbf{\Pi}(3, (1)(2,3), L)$;

cf. [11, Repr. 2.6]

In this section we consider structures $\mathbf{\Pi}(n, \sigma, \mathfrak{M})$ where $\sigma \in \mathcal{S}_n$ and $n > 3$.

In case when $\sigma = \overline{\sigma_0}$ with $\sigma_0 \in \mathcal{S}_I$, condition (9) used in Lemma 1.5 to characterize complete free K_{n+1} subgraphs of $\mathbf{\Pi}(n, \sigma, \mathfrak{N})$ can be specified as follows:

$$i_0 \in \mathrm{Fix}(\sigma_0).$$

Analogously, we can specify conditions of Proposition 1.3 characterizing isomorphisms between $\mathbf{\Pi}(n, \overline{\sigma_1}, \mathfrak{N}_1)$ and $\mathbf{\Pi}(n, \overline{\sigma_2}, \mathfrak{N}_2)$ with $\sigma_1, \sigma_2 \in \mathcal{S}_I$: namely, we substitute $\varphi\sigma_1 = \sigma_2\varphi$ in place of (4) and $\varphi\sigma_1 = \sigma_2^{-1}\varphi$ in place of (7).

By the results contained in [18] (cf. Proposition 2.6), in case considered in Lemma 1.5 there is a permutation of the edges of $K_{A^*\setminus\{a_{i_0}\}}$ such that $\mathfrak{M} \cong \mathbf{\Pi}(n, \sigma', \mathfrak{N}')$ for an adequate configuration $\mathfrak{N}'$: $\mathfrak{M}$ *is* a skew perspective of $K_{A^*\setminus\{a_{i_0}\}}$ onto $G_{(i_0)}$. In that case we frequently say "$\mathfrak{M} \cong \mathbf{\Pi}(n, \sigma', \mathfrak{N}')$ and a_{i_0} *is the perspective center in* $\mathbf{\Pi}(n, \sigma', \mathfrak{N}')$". However, σ' need not to be determined by a permutation of the vertices (cf. Lemma 1.6) and also σ and σ' are not necessarily conjugate (cf. Proposition 1.3).

Proposition 2.2. *Let* $\mathrm{S}(i_0)$ *be a clique in* $\mathfrak{N}$ *for some* $i_0 \in \mathrm{Fix}(\sigma)$, $\sigma \in \mathcal{S}_I$, $|I| = n + 1 \geq 4$ *(cf. Lemma 1.5). The following conditions are equivalent.*

(i) $\mathbf{\Pi}(n+1, \sigma, \mathfrak{N}) \cong \mathbf{\Pi}(n+1, \sigma', \mathfrak{N}')$, *for a* $\sigma' = \overline{\sigma_0'}$, $\sigma_0' \in \mathcal{S}_{n+1}$ *and a suitable configuration* $\mathfrak{N}'$, *where* a_{i_0} *is the perspective center in* $\mathbf{\Pi}(n+1, \sigma', \mathfrak{N}')$ *of the graphs* $G_{(i_0)}$ *and* $K_{A^*\setminus\{a_{i_0}\}}$.

(ii) *There is* $\tau \in \mathcal{S}_{I\setminus\{i_0\}}$ *such that*

$$c_{\{i_0,\tau(i)\}} \oplus c_{\{i_0,\tau(j)\}} = c_{\{i,j\}} \tag{14}$$

for all $i, j \in I$, $i, j \neq i_0$.

Proof. Assume (i). Without loss of generality we can assume that $i_0 = 0$ and $I = \{0, 1, \ldots, n\}$. So, we re-label the points of $\mathbf{\Pi}(n+1, \sigma, \mathfrak{N}) =: \mathfrak{M}$ so as $q = a_0$ becomes a perspective center and $a_i : i = 1, \ldots, n+1$

and $d_i : i = 1, \dots, n+1$ will be the complete subgraphs that are in the respective perspective. Finally, we take $e_{i,j} = a_i \oplus a_j$ for $\{i,j\} \in \wp_2(T)$, $T = \{1, \dots, n+1\}$. So, we obtain

$$\begin{aligned} a_{n+1} = p,\ d_i = q \oplus a_i = c_{0,i} \text{ for } i \in T, i \neq 0,\ d_{n+1} = q \oplus a_{n+1} = b_0, \\ e_{i,j} = c_{0,i} \oplus c_{0,j} \text{ (computed in } \mathfrak{N}) \text{ for } i,j \in T, i,j \neq 0, \\ e_{i,n+1} = b_i \text{ for } i \in T, i \neq 0. \end{aligned} \tag{15}$$

Let $\tau \in \mathcal{S}_T$ be the corresponding skew i.e. assume that

$$a_i \oplus a_j = e_{i,j} = d_{\tau(i)} \oplus d_{\tau(j)}$$

for all $\{i,j\} \in \wp_2(T)$. In particular, this yields for $i \in T$, $i \neq n+1$ the following: $a_i \oplus a_{n+1} =$

$$b_i = d_{\tau(i)} \oplus d_{\tau(n+1)} = \begin{cases} c_{0,\tau(i)} \oplus c_{0,\tau(n+1)} & or \\ c_{0,\tau(i)} \oplus b_0 & \tau(n+1) = n+1, \tau(i) \neq n \\ b_0 \oplus c_{0,\tau(n+1)} & \tau(i) = n+1, \tau(n+1) = 0 \end{cases}.$$

Since $\mathfrak{M}$ does not contain any line with exactly one point in B and two points in C, the first possibility is inconsistent. So, we end up with $\tau(n+1) = n+1$ and therefore, $\tau \in \mathcal{S}_n$. If so, we obtain $c_{i,j} = a_i \oplus a_j = e_{i,j} = d_{\tau(i)} \oplus d_{\tau(j)} = c_{0,\tau(i)} \oplus c_{0,\tau(j)}$ for distinct $1 \leq i,j \leq n$. This justifies (14).

The converse reasoning consists in a simple computation: the reasoning above defines, in fact, a required isomorphism. It also defines the configuration $\mathfrak{N}'$: the formulas $e_{i,n+1} \oplus e_{j,n+1} = b_i \oplus b_j = c_{\sigma^{-1}(i),\sigma^{-1}(j)} = e_{\sigma^{-1}(i),\sigma^{-1}(j)}$ for $1 \leq i,j \leq n$ and $e_u \oplus e_v = e_y$ iff $c_u \oplus c_v = c_y$ for $u,v,y \in \wp_2(T \setminus \{n+1\})$ determine the lines of $\mathfrak{N}'$. $\square$

2.1 Particular case: $\mathfrak{N}$ is a generalized Desargues configuration

In the class of skew perspectives one type of them seems "most similar to the classical geometrical perspective": when the perspective axis is a generalized Desargues configuration i.e. when $\mathfrak{N} = \mathbf{G}_2(n)$ (cf. [3], [4]). So, in this subsection we set

$$\mathfrak{M} = \mathbf{\Pi}(n, \sigma, \mathbf{G}_2(n)),\ \sigma \in \mathcal{S}_I,\ n \geq 4.$$

Proposition 2.3. *Either* $\mathfrak{M} = \mathbf{G}_2(n+2) = \mathbf{\Pi}(n, \mathrm{id})$ *and then each point of* $\mathfrak{M}$ *can be chosen as a center of a skew perspective, or* $\mathfrak{M}$ *does not contain any point* $q \neq p$ *such that* $\mathfrak{M} \cong \mathbf{\Pi}(n, \sigma', \mathfrak{B}) =: \mathfrak{M}'$ *for a suitable configuration* $\mathfrak{B}$*, such that* q *is the perspective center in* $\mathfrak{M}'$*.*

Proof. Assume that $\sigma \neq \mathrm{id}_I$. Suppose that such a point q exists, then – comp. Proposition 2.2 and Lemma 1.5 – there is $i_0 \in I$ such that $\sigma(i_0) = i_0$. Moreover, in view of (14), there is a permutation τ such that $c_{i_0,\tau(i)} \oplus c_{i_0,\tau(j)} = c_{i,j}$ for all $i, j \in I$, $i, j \neq i_0$. On the other hand, in $\mathbf{G}_2(I)$ we have $c_{i_0,\tau(i)} \oplus c_{i_0,\tau(j)} = c_{\tau(i),\tau(j)}$ for all i, j as above. This, finally, gives $\{i, j\} = \{\tau(i), \tau(j)\}$, from which we deduce $\tau = \mathrm{id}$ and then $\mathfrak{M}' \cong \mathbf{G}_2(n+2)$. $\square$

Corollary 2.4. *Assume that* $\mathcal{S}_I \ni \sigma_1 \neq \mathrm{id}_I$*. If* f *is an isomorphism between* $\mathbf{\Pi}(n, \sigma_1, \mathbf{G}_2(n))$ *and* $\mathbf{\Pi}(n, \sigma_2, \mathbf{G}_2(n))$ *then* $f(p) = p$ *and* $\sigma_2 \neq \mathrm{id}_I$*. Moreover,* f *is determined by a permutation* $\varphi \in \mathcal{S}_I$ *so as either* f *fixes* A *and* B *and then* $\sigma_2 = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_1^{\varphi}$*, or* f *interchanges* A *and* B *and* $\sigma_2^{-1} = \sigma_1^{\varphi}$ *(see Proposition 1.3).*

Let us recall a few facts from the folklore of group theory. Let $\sigma \in \mathcal{S}_I$, then σ has a unique (up to an order) decomposition $\sigma = \sigma_1 \circ \ldots \circ \sigma_k$ where $\sigma_1, \ldots, \sigma_k$ are pairwise disjoint cycles. Let x_i be the length of σ_i, then $n = \sum_{i=1}^{k} x_i$. Without loss of generality we can assume that $x_1 \leq \ldots \leq x_k$ and we can set $C(\sigma) := (x_1, \ldots, x_k)$. So, $C(\sigma)$ is an unordered partition of the integer n into k components (see e.g. [5, Ch. 4], [1]). The following is known:

Fact 2.5. σ_1 *and* σ_2 *are conjugate in* $\mathcal{S}_I$ *(i.e.* $\sigma_2 = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_1^{\varphi}$ *for a* $\varphi \in \mathcal{S}_I$*), iff* $C(\sigma_1) = C(\sigma_2)$*.*

In particular, σ *and* σ^{-1} *are conjugate for every* $\sigma \in \mathcal{S}_I$*.*

Permutations σ *and* id_I *are conjugate iff* $\sigma = \mathrm{id}_I$*.*

As an immediate consequence of Fact 2.5 and Corollary 2.4 we obtain

Proposition 2.6. *Let* $\sigma_1, \sigma_2 \in \mathcal{S}_I$*.* $\mathbf{\Pi}(n, \sigma_1, \mathbf{G}_2(n)) \cong \mathbf{\Pi}(n, \sigma_2, \mathbf{G}_2(n))$ *iff* σ_1 *and* σ_2 *are conjugate.*

Consequently, there are $P(n) = \sum_{k=1}^{n} P(n,k)$ *types of the skew perspectives whose axial configurations are the generalized Desargues configuration, where* $P(n,k)$ *is the number of unordered partitions of* n *into* k *components.*

3 Perspectivities generalizing Veronese configurations

Perspectivities with the skews in $\mathcal{P}_n^{\mathsf{G}}$ can be found among well known configurations like combinatorial Grassmannians. Let us show that also skews in $\mathcal{P}_n^{\mathsf{V}}$ have already appeared in mathematics.

Example 3.1. Let $|X| = 3$, $X = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$. Then the combinatorial Veronesean $\mathbf{V}_k(X) =: \mathfrak{M}$ is a $\left(\binom{k+2}{2}_k \binom{k+2}{3}_3\right)$-configuration; its point set is the set $\mathfrak{y}_k(X)$ of the k-element multisets with elements in X and the lines have form eX^s, $e \in \mathfrak{y}_{k-s}(X)$. $\mathbf{V}_1(X)$ is a single line, $\mathbf{V}_2(X)$ is the Veblen configuration, and $\mathbf{V}_3(X)$ is the known Kantor configuration (comp. [17, Prop's. 2.2, 2.3], [11, Repr. 2.7]). Consequently, we assume $k > 3$. The following was noted in [18, Fct. 4.1]:

The K_{k+1} graphs freely contained in $\mathbf{V}_k(X)$ are the sets
$X_{\mathfrak{a},\mathfrak{b}} := \mathfrak{y}_k(\{\mathfrak{a}, \mathfrak{b}\})$, $X_{\mathfrak{b},\mathfrak{c}} := \mathfrak{y}_k(\{\mathfrak{b}, \mathfrak{c}\})$, and $X_{\mathfrak{c},\mathfrak{a}} := \mathfrak{y}_k(\{\mathfrak{c}, \mathfrak{a}\})$.

In particular, $\mathfrak{M}$ freely contains two complete subgraphs $X_{\mathfrak{a},\mathfrak{b}}$, $X_{\mathfrak{c},\mathfrak{a}}$, which cross each other in $p = \mathfrak{a}^k$. We shall present $\mathfrak{M}$ as a perspective between these two graphs. Let us re-label the points of $\mathbf{V}_k(X)$:

$$c_i = \mathfrak{b}^i\mathfrak{a}^{k-i},\ b_i = \mathfrak{c}^i\mathfrak{a}^{k-i},\ i \in \{1, \dots, k\} =: I,\ e_{i,j} = c_i \oplus c_j,\ \{i, j\} \in \wp_2(I).$$

Assume that $i < j$, then $\overline{c_i, c_j} = \mathfrak{a}^{k-j}\mathfrak{b}^i X^{j-i}$, so $e_{i,j} = \mathfrak{a}^{k-j}\mathfrak{b}^i\mathfrak{c}^{j-i}$. Clearly, $p \oplus c_i = b_i$ so, the map $\big(c_i \mapsto b_i,\ i \in I\big)$ is a point-perspective. After routine computation we obtain $b_i \oplus b_j = e_{\zeta(\{i,j\})}$ whenever $i < j$; moreover, in this representation the axial configuration consists of the points in $\mathfrak{b}\mathfrak{c}\mathfrak{y}_{k-2}(X)$ so, it is isomorphic to $\mathbf{V}_{k-2}(X)$. Consequently,

Fact 3.2. $\mathbf{V}_k(X) \cong \mathbf{\Pi}(k, \zeta, \mathbf{V}_{k-2}(X))$. □

In case when $\Phi \in \mathcal{S}_{(<n)}$ and $\sigma = \sigma_\Phi$, condition (9) of Lemma 1.5 can be specified as follows:

$$i_0 < j \implies \phi_j(i_0) = i_0, \text{ for all } j \in I_n.$$

Note. $G_{(n)}$ is a 'new' complete free subgraph in $\mathfrak{M}$ iff $\mathrm{S}(n)$ is a collinearity clique in $\mathfrak{N}$. □

Note. Assume that $\mathrm{Fix}(\phi_n) = \emptyset$, or $\mathrm{Fix}(\phi_n) \cap \mathrm{Fix}(\phi_{n-1}) = \emptyset$ and $n - 1 \notin \mathrm{Fix}(\phi_n)$ (this holds, in particular, when $\mathrm{Fix}(\phi_j) \neq \emptyset$ only for $j = 2$). Then

$G_{(n)}$ is a 'unique new' complete free subgraph of $\mathfrak{M}$ iff $\mathrm{S}(n)$ is a collinearity clique in $\mathfrak{N}$. $\square$

Then as a consequence of Lemma 1.5 we obtain

Corollary 3.3. *The structure* $\mathbf{\Pi}(n, \zeta_n, \mathbf{G}_2(I_n))$ *freely contains exactly three* K_{n+1}*-graphs.*

In essence, in many places, $\mathfrak{J}$ is the unique non identity automorphism of $\mathbf{\Pi}(n, \zeta, \mathfrak{N})$ (when $\mathfrak{N} = \zeta\mathfrak{N}$). An analogue of Proposition 2.2 formulated for skews in $\mathcal{P}_n^{\vee}$ can be also proved; we shall skip over this task because we shall not need it to present respective configurations as perspectives with a center changed in a general setting.

Next, a technical result.

Lemma 3.4. *Let* $\mathfrak{M} = \mathbf{\Pi}(n, \sigma_\Phi, \mathfrak{N})$ *for a* $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$*-configuration* $\mathfrak{N}$ *and a sequence* Φ *satisfying* (13)*. Next, let* $n > 3$ *and* $k \in I_n$, $k > 3$*. The following conditions are equivalent:*

(i) *the formula*

$$\forall i \neq k\ \exists j \neq k\ \ \overline{a_k, a_i}\ \textit{crosses}\ \overline{b_k, b_j} \qquad \mathsf{Cross}(k)$$

holds in $\mathfrak{M}$*;*

(ii) $n = k$ *or* $k < n$ *and* $\phi_j(k) = k$ *for all* $j > k$*.*

Proof. It is evident that (ii) implies (i): we take: in $\mathsf{Cross}(k)$ $j = \phi_k(i)$ when $i < k$ and $j = \phi_i(k)$ when $k < i$.

Suppose that $\mathsf{Cross}(k)$ holds for $3 < k < n$. Note that $\mathsf{Cross}(k)$ means, in fact the following

for all $i \neq k$ it holds $\{k, i\} = \sigma_\Phi(\{k, j\})$ for some $j \neq k$ i.e. $k \in \sigma_\Phi^{-1}(\{k, i\})$.

As in the proof of Lemma 1.5 we derive from it $\phi_i^{-1}(k) = k$ for $k > i$, which proves our claim. $\square$

As a corollary to Lemma 3.4 we obtain the following rigidity property:

Proposition 3.5. *Let $\mathfrak{M}$ be as in Lemma 3.4 with $n > 3$ such that there is no integer k such that $\phi_j(k) = k$ for all $j > k$. Assume that $f \in \mathrm{Aut}(\mathfrak{M})$ with $f(p) = p$. Then either $f = \mathrm{id}$ or $f = \mathfrak{J}$ and $\mathfrak{N} = \sigma_\Phi(\mathfrak{N})$.*

Proof. Evidently, either $f(A) = A$ or $f(A) = B$ (in the notation of Construction 1.1). From Lemma 3.4 we obtain $f(a_n) = a_n$. Then, let us restrict $\mathfrak{M}$ to points with indices in I_{n-1}; in this structure $\mathsf{Cross}(n-1)$ holds and therefore $f(a_{n-1}) = a_{n-1}$ as well. Step by step we get $f(a_i) = a_i$ for $3 < i \leq n$. Next, we look at $c_{4,2} = a_4 \oplus a_2$, it goes under f onto $a_4 \oplus a_{\alpha(2)} = c_{4,\alpha(2)}$ for a permutation $\alpha \in \mathcal{S}_{I_3}$. Simultaneously, $c_{4,2} = b_4 \oplus b_2$ and thus $c_{4,\alpha(2)} = c_{4,4-\alpha(2)}$ which gives $\alpha(2) = 2$. Similarly we compute $\alpha(3) = 3$ and $\alpha(1) = 1$.

If $f(A) = B$ the reasoning is provided analogously; we obtain $f(a_i) = b_i$ for $3 < i \leq n$ and then $f(a_i) = b_i$ for *all* $i \in I_n$. $\square$

4 On labelling of the Veblen Configuration

Let $Y \in \wp_3(I)$, we set $\mathrm{T}(Y) = \wp_2(Y)$; if $i_0 \in I$ we write $\mathrm{S}(i_0) = \{u \in \wp_2(I) \colon i_0 \in u\}$ and $\mathrm{T}(i_0) = \mathrm{T}(I \setminus \{i_0\})$. A set of the form $\mathrm{S}()$ is called *a star*, and of the form $\mathrm{T}()$ *a top*.

All the labelings of $\wp_2(I_4)$ which yield a Veblen configuration are known (cf. [11, Fct. 2.5, Fig. 2]). This paper is not a right place to quote all the definitions that are needed to define respective labelings. On the other hand, they can be recognized on the figures 1-3, namely: observe the lines which join the points $c_{i,j}$. These are the structures named

$$\mathbf{G}_2(I_4) \text{ (Fig. 1)}, \quad \mathbf{I\!B}(2) \text{ (Fig. 2)}, \quad \mathcal{V}_5(\text{ Fig. 3}). \tag{16}$$

Moreover, the image of any of the three structures in the list (16) under the map $\varkappa$ is again the Veblen configuration, to be denoted as follows, respectively.

$$\varkappa(\mathbf{G}_2(I_4)) = \mathbf{G}_2^*(I_4), \quad \varkappa(\mathbf{I\!B}(2)) = \mathcal{V}_4, \quad \varkappa(\mathcal{V}_5) = \mathcal{V}_6. \tag{17}$$

In what follows we shall also frequently identify subsets of C with the corresponding subsets of $\wp_2(I)$: $\mathrm{T}(Y) = \{c_u \colon u \in \mathrm{T}(Y)\}$ and $\mathrm{S}(i_0) = \{c_u \colon u \in \mathrm{S}(i_0)\}$.

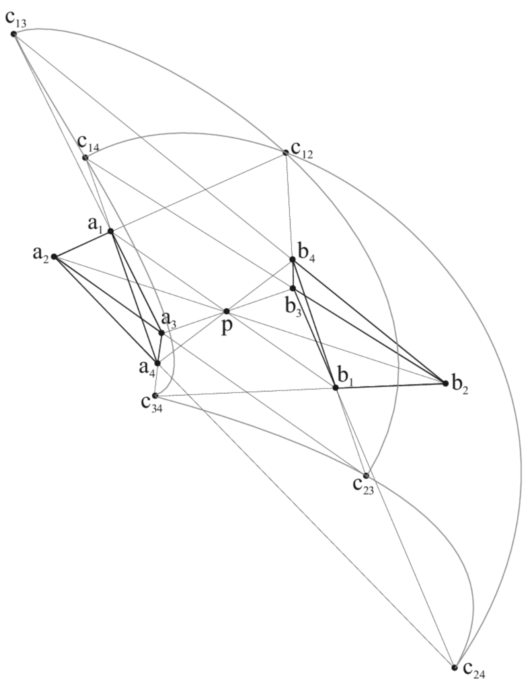


Figure 1: The configuration $\mathbf{\Pi}(p, \mathrm{id}, \mathbf{G}_2(4))$

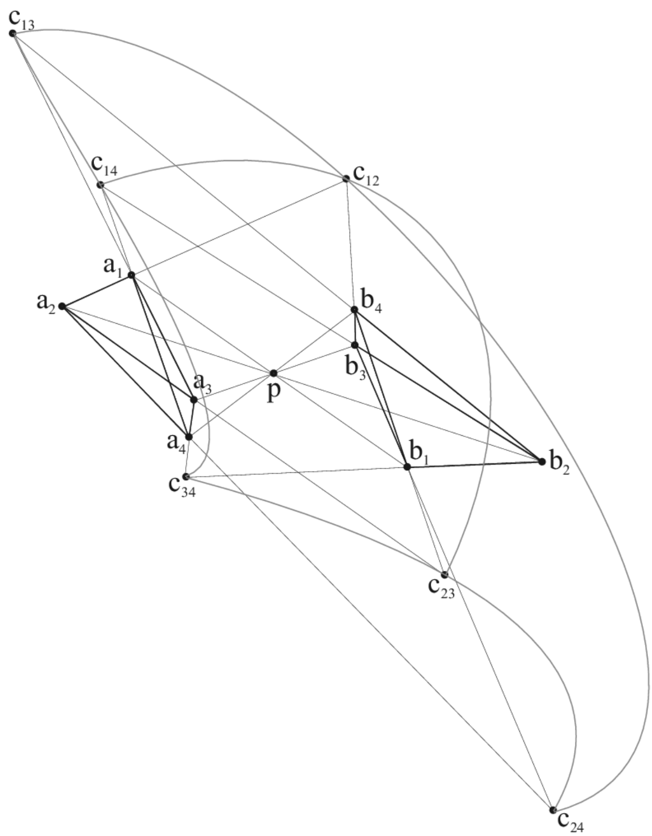


Figure 2: The configuration $\mathbf{\Pi}(p, \mathrm{id}, \mathbf{I\!B}(2))$

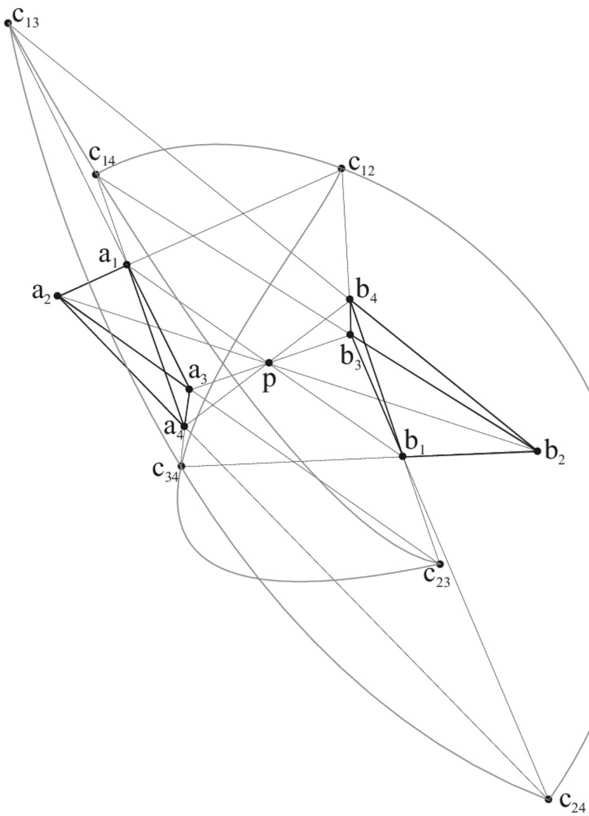


Figure 3: The configuration $\mathbf{\Pi}(p, \mathrm{id}, \mathcal{V}_5)$

Fact 4.1 ([11, Fct. 2.5])**.** *For every Veblen configuration $\mathfrak{V}$ defined on $\wp_2(I_4)$ there is a permutation $\alpha \in \mathcal{S}_{I_4}$ such that either $\overline{\alpha}$ or $\varkappa\overline{\alpha}$ is an isomorphism of $\mathfrak{V}$ onto $\mathfrak{V}_0$, where $\mathfrak{V}_0$ is a one among* (16)*.*

In other words, the lists (16) and (17) present all (up to a permutation of indices in I_4) the possible labellings of the Veblen configuration by the elements of $\wp_2(I_4)$.

In general, the autmorphisms of the structures in the list (16)&(17), quoted in Fact 4.1 are also known. Evidently, they are in $\mathcal{S}_{\wp_2(I_4)}$.

Fact 4.2. *Let $\varphi \in \mathcal{S}_{I_4}$ and let $\mathfrak{V}_0$ be one in the list* (16)*&*(17)*. Then $\overline{\varphi} \in \mathrm{Aut}(\mathfrak{V}_0)$ iff*

$\mathfrak{V}_0 = \mathbf{G}_2(I_4)$ or $\mathfrak{V}_0 = \mathbf{G}_2^*(I_4)$: *$\varphi \in \mathcal{S}_{I_4}$ is arbitrary. In this case* $\mathrm{Aut}(\mathfrak{V}_0) = \overline{\mathcal{S}_{I_4}}$.

$\mathfrak{V}_0 = \mathbb{B}(2)$ or $\mathfrak{V}_0 = \mathcal{V}_4$: *$\varphi \in \mathcal{S}_{I_4}$ fixes two sets $\{1,2\}$ and $\{3,4\}$.*

$\mathfrak{V}_0 = \mathcal{V}_5$ or $\mathfrak{V}_0 = \mathcal{V}_6$: *$\varphi$ fixes an element $i_0 \in I_4$ (e.g. $i_0 = 3$), so, in fact, $\varphi \in \mathcal{S}_{I_4\setminus\{3\}}$.*

Proof. Note that each line of $\mathbf{G}_2(I)$ has the form $\mathrm{T}(Y)$ with $Y \in \wp_3(I)$ and each line of $\mathbf{G}_2^*(I)$ has the form $\mathrm{S}(i)$ with $i \in I$. In the reasoning below we consider only the structures in the list (16). The dual case (the list (17)) runs analogously, with $\mathrm{T}()$ replaced by $\mathrm{S}()$.

(a) Let $\mathfrak{V}_0 = \mathbf{G}_2(I_4)$. Then, clearly, $\overline{\varphi} \in \mathrm{Aut}(\mathfrak{V}_0)$ for each $\varphi \in \mathcal{S}_{I_4}$.

(b) Let $\mathfrak{V}_0 = \mathbb{B}(2)$. Then $\mathfrak{V}_0$ contains exactly two lines of the form $\mathrm{T}(Y)$, say these are $\mathrm{T}(\{1,3,4\}) = \mathrm{T}(2)$ and $\mathrm{T}(\{2,3,4\}) = \mathrm{T}(1)$. So, φ fixes the set $\{1,2\}$ and $\{3,4\}$. From the definition of $\mathbb{B}(2)$, a permutation which fixes these two sets is an automorphism of $\mathfrak{V}_0$.

(c) Let $\mathfrak{V}_0 = \mathcal{V}_5$. Then $\mathfrak{V}_0$ contains exactly one line of the form $\mathrm{T}(Y)$; let it be $\mathrm{T}(\{1,2,4\}) = \mathrm{T}(3)$. Consequently, $\varphi(3) = 3$. each permutation $\varphi \in \mathcal{S}_{I_4\setminus\{3\}}$ extended by the condition $\varphi(3) = 3$ determines the automorphism $\overline{\varphi}$ of $\mathfrak{V}_0$.

This completetes our proof. □

Again, it is a simple task to determine Lemma 4.3.

Lemma 4.3. *The following are the star-triangles in corresponding Veblen configuration* $\mathfrak{V}_0$*.*

$\boldsymbol{\mathfrak{V}_0 = \mathbf{G}_2(I_4)}$**:** $\mathrm{S}(i)$*,* $i \in I_4$*; 4 star-triangles.*
$\boldsymbol{\mathfrak{V}_0 = \mathbb{B}(2)}$**:** $\mathrm{S}(1)$*,* $\mathrm{S}(2)$*; 2 star-triangles.*
$\boldsymbol{\mathfrak{V}_0 = \mathcal{V}_5}$**:** $\mathrm{S}(3)$*; a unique star-triangle.*
$\boldsymbol{\mathfrak{V}_0 = \mathbf{G}^*_2(I_4), \mathcal{V}_4, \mathcal{V}_6}$**:** *no* $i \in I_4$ *such that* $\mathrm{S}(i)$ *is a triangle in* $\mathfrak{V}_0$*.*

Let $\varphi, \psi \in \mathcal{S}_{I_4}$, we write $\varphi \sim \psi$ when φ and ψ are conjugate i.e. when $\varphi = \psi^\alpha = \alpha\psi\alpha^{-1}$ for an $\alpha \in \mathcal{S}_{I_4}$. Set $C(\varphi) = (x_1, \ldots, x_k)$ when φ can be decomposed into disjoint cycles $c_1, \ldots, c_k$ such that x_i is the length of c_i, and $x_1 \leq \ldots \leq x_k$. Clearly, $\sum_{i=1}^k x_i = 4$. It is a folklore that $\varphi \sim \psi$ is equivalent to $C(\varphi) = C(\psi)$.

If $H \subset \mathcal{S}_{I_4}$ then $\varphi \sim_H \psi$ denotes that $\varphi = \psi^\alpha$ for an $\alpha \in H$.

Analogously, it needs only an elementary computation to determine, with the help of Lemma 4.2 the following Lemma 4.4; this determining may take some time and sweat, though.

Lemma 4.4. *Let* $\mathfrak{V}_0$ *be a one of* (16)*&*(17)*. The following* β*'s are representatives of the equivalence classes of* $\sim_{\mathrm{Aut}(\mathfrak{V}_0)}$ *in the group* $\mathcal{S}_{I_4}$*.*

$\boldsymbol{\mathfrak{V}_0 = \mathbf{G}_2(I_4)}$ **or** $\boldsymbol{\mathfrak{V}_0 = \mathbf{G}^*_2(I_4)}$**:**
$\beta = \mathrm{id}$, $(1)(2,3,4)$, $(1,2)(3,4)$, $(1)(2)(3,4)$, $(1,2,3,4)$.
$\boldsymbol{\mathfrak{V}_0 = \mathbb{B}(2)}$ **or** $\boldsymbol{\mathfrak{V}_0 = \mathcal{V}_4}$**:** $\boldsymbol{C(\beta) = (1,1,1,1)}$, then $\beta = \mathrm{id}$*;*
$\boldsymbol{C(\beta) = (1,1,2)}$, then $\beta = (1)(2)(3,4)$, $(1,2)(3)(4)$, $(1)(2,3)(4)$*;*
$\boldsymbol{C(\beta) = (1,3)}$, then $\beta = (1)(2,3,4)$, $(4)(1,2,3)$*;*
$\boldsymbol{C(\beta) = (2,2)}$, then $\beta = (1,2)(3,4)$, $(1,4)(2,3)$*;*
$\boldsymbol{C(\beta) = (4)}$, then $\beta = (1,2,3,4)$, $(1,3,2,4)$.
$\boldsymbol{\mathfrak{V}_0 = \mathcal{V}_5}$ **or** $\boldsymbol{\mathfrak{V}_0 = \mathcal{V}_6}$**:** $\boldsymbol{C(\beta) = (1,1,1,1)}$, then $\beta = \mathrm{id}$*;*
$\boldsymbol{C(\beta) = (1,1,2)}$, then $\beta = (1)(3)(2,4)$, $(1)(2)(3,4)$*;*
$\boldsymbol{C(\beta) = (1,3)}$, then $\beta = (1)(2,3,4)$, $(3)(1,2,4)$*;*
$\boldsymbol{C(\beta) = (2,2)}$, then $\beta = (1,2)(3,4)$*;*
$\boldsymbol{C(\beta) = (4)}$, then $\beta = (1,2,3,4)$.

Actually, dealing with the elements of $\mathcal{K}^{\mathsf{V}}_4$ we need to determine *all* labellings of the Veblen configuration by the elements of $\wp_2(I_4)$. By a way of an example on Figure 4 we show drawings presenting the schemas of $\mathbf{\Pi}(4, \zeta_4, \mathbf{G}^*_2(I_4))$ and of $\mathbf{\Pi}(4, \zeta_4, \zeta_4(\mathbf{G}^*_2(I_4)))$; but remember that these are

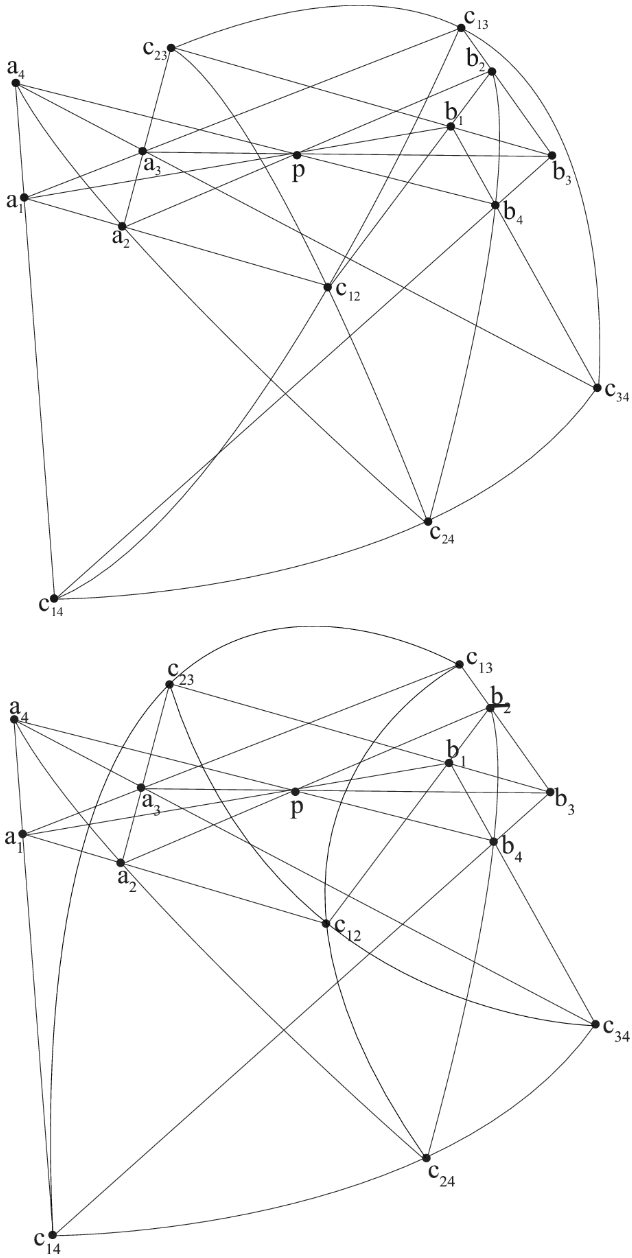


Figure 4: The structures $\mathbf{\Pi}(4,\zeta_4,\mathbf{G}_2^*(I_4)) = \mathbf{\Pi}(4,\zeta_4,\mathcal{V}_6(\mathrm{id}))$ (top) and $\mathbf{\Pi}(4,\zeta_4,\zeta_4(\mathbf{G}_2^*(I_4))) = \mathbf{\Pi}(4,\zeta_4,\mathcal{V}_6\big((1,2,3)(4)\big))$ (bottom) (cf. definition of $\mathcal{V}_s(\mu)$ in Equation (19)).

merely two among eighteen associated with a single ζ_4, cf. details in Theorem 5.10!

Below we introduce a procedure of enumerating respective labellings in a way convenient for our analysis. Suppose that a Veblen configuration $\mathfrak{V}$ defined on $\wp_2(I_4)$ contains $\mathrm{S}(i_0)$ as a triangle, and then it contains $\mathrm{T}(i_0)$ as a line. Let us introduce a numbering of the sides of $\mathrm{S}(i_0)$ and of points of $\mathrm{T}(i_0)$, invariant under permutations in $\mathcal{S}_{I_4\setminus\{i_0\}}$:

$$\overline{c_{i,i_0}, c_{j,i_0}} \sim k \text{ and } c_{i,j} \sim k \text{ iff } \{i,j,k\} = I_4 \setminus \{i_0\}.$$

Then the definition of $\mathfrak{V}$ corresponds to a $\mu \in \mathcal{S}_{I_4\setminus\{i_0\}}$ with the following rule

$$k \sim \overline{c_{i,i_0}, c_{j,i_0}} \text{ yields } c_{i,i_0} \oplus c_{j,i_0} = c_{i',j'} \sim \mu(k). \tag{18}$$

Next, suppose that $\mathfrak{V}$ contains $\mathrm{T}(i_0)$ as a triangle, and then it contains $\mathrm{S}(i_0)$ as a line. Analogously to the above we introduce a numbering of the sides of $\mathrm{T}(i_0)$ and of points of $\mathrm{S}(i_0)$:

$$\overline{c_{i,k}, c_{j,k}} \sim k \text{ and } c_{k,i_0} \sim k \text{ iff } \{i,j,k\} = I_4 \setminus \{i_0\}.$$

Then the definition of $\mathfrak{V}$ corresponds to a $\mu \in \mathcal{S}_{I_4\setminus\{i_0\}}$ with the following rule

$$k \sim \overline{c_{i,k}, c_{j,k}} \text{ yields } c_{i,k} \oplus c_{j,k} = c_{i',j'} \sim \mu(k). \tag{19}$$

Let $\mu \in \mathcal{S}_{I_4\setminus\{i_0\}}$; we write $\mathcal{V}_5(\mu)$ for the Veblen configuration defined by (18): it has $\mathrm{T}(i_0)$ as a line, and $\mathcal{V}_6(\mu)$ for the Veblen configuration defined by (19): it has $\mathrm{S}(i_0)$ as a line.

In accordance with the rules above, $\mathfrak{V} = \mathcal{V}_s(\mu)$ ($\mu \in \mathcal{S}_{I_4\setminus\{i_0\}}$) has another star-triangle $\mathrm{S}(i_0')$ ($s = 5$) or another top-triangle $\mathrm{T}(i_0')$ ($s = 6$) iff $\mu(i_0') = i_0'$. In other words, $\mu = (i_0')(j_1, j_2)$. It is easy to compute that then $\mathfrak{V} = \mathcal{V}_s((i_0)(j_1, j_2))$.

Since under every labelling by the elements of $\wp_2(I_4)$ the Veblen configuration contains either at least one top-line or at least one star-line, each Veblen configuration has either the form $\mathcal{V}_5(\mu)$ or $\mathcal{V}_6(\mu)$ for some $\mu \in \mathcal{S}_{I_4\setminus\{i_0\}}$ and $i_0 \in I_4$. So, each Veblen configuration $\mathfrak{V}$ can be uniquely associated with a permutation $\mu \in \mathcal{S}_4$ with at least one fixed point (not a derangement of I_4) and a 'switch' $s \in \{5, 6\}$ so as $\mathfrak{V} = \mathcal{V}_s(\mu)$. We have:

$$\begin{array}{ccc} \mathcal{V}_5(\mathrm{id}) = \mathbf{G}_2(I_4), & \mathcal{V}_5((3)(4)(1,2)) = \mathbb{B}(2), & \mathcal{V}_5((4)(1,2,3)) = \mathcal{V}_5, \\ \mathcal{V}_6(\mathrm{id}) = \mathbf{G}_2^*(I_4), & \mathcal{V}_6((3)(4)(1,2)) = \mathcal{V}_4, \quad \text{and} & \mathcal{V}_6((4)(1,2,3)) = \mathcal{V}_6. \end{array}$$

Let us note the following tricky observation, justified on Figure 5.

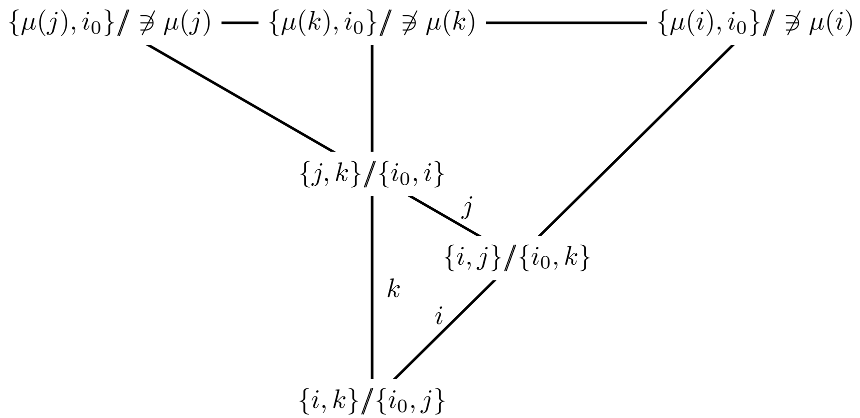


Figure 5: Comparing $\mathcal{V}_6(\mu)$ and its $\varkappa$-image, $\mu(i_0) = i_0$. $I_4 = \{i,j,k,i_0\}$. Points on the diagram are denoted following the convention: value-of-u/value-property-of-$\varkappa(u)$ with $u \in \wp_2(I_4)$, where the 'starting' structure $\mathcal{V}_6(\mu)$ has the line $\mathrm{S}(i_0)$ and the triangle $\mathrm{T}(i_0)$.

Fact 4.5. *For every $s \in \{5,6\}$ and $\mu \in \mathcal{S}_4$ with $\mathrm{Fix}(\mu) \neq \emptyset$ the following holds*

$$\varkappa(\mathcal{V}_s(\mu)) = \mathcal{V}_{11-s}(\mu).$$

The following is evident.

Fact 4.6. *Let $\mu, \alpha \in \mathcal{S}_{I_4}$, with $\mathrm{Fix}(\mu) \neq \emptyset$, $s \in \{5,6\}$ and let $\Phi = (\phi_4, \phi_3)$ determine a suitable skew. Then*

$$\overline{\alpha}\big(\mathcal{V}_s(\mu)\big) = \mathcal{V}_s(\mu^\alpha).$$

Consequently, if $\alpha = \mathrm{id}_{I_4}$ or $\alpha = (1,2)(3)(4)$ (cf. Lemma 1.10) then

$$\overline{\alpha}\big(\mathbf{\Pi}(4, \sigma_\Phi, \mathcal{V}_s(\mu))\big) = \mathbf{\Pi}(4, \sigma_{\Phi^\alpha}, \mathcal{V}_s(\mu^\alpha)).$$

Note. So, enumerating all the σ_Φ-perspectives along admissible sequences Φ and after that along *all the admissible* μ one can cancel one item in every pair $(\Phi, \Phi^{(1,2)(3)(4)})$. □

Note. If, additionally, $\Phi = \Phi^{(1,2)(3)(4)}$ then in the list of considered μ one can cancel one item in every pair $(\mu, \mu^{(1,2)(3)(4)})$. □

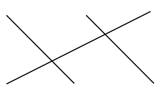
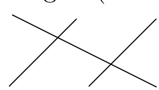
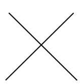
Note. Analogously, in view of Lemma 1.4 we can cancel one item in every pair (Φ, Φ^{-1}). $\square$

5 Classification of perspectivities between tetrahedrons

In this section we will present a classification of the classes

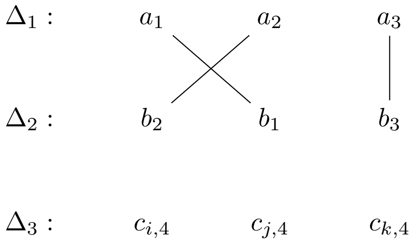
$$\mathcal{K}^{\mathsf{G}} := \big\{\mathbf{\Pi}(4,\delta,\mathfrak{V})\colon \delta \in \mathcal{P}_4^{\mathsf{G}},\ \mathfrak{V} \in \boldsymbol{\mathfrak{B}}\big\},$$
$$\mathcal{K}^{\mathsf{G}'} := \big\{\mathbf{\Pi}(4,\delta,\mathfrak{V})\colon \varkappa\delta \in \mathcal{P}_4^{\mathsf{G}},\ \mathfrak{V} \in \boldsymbol{\mathfrak{B}}\big\}, \text{ and}$$
$$\mathcal{K}^{\mathsf{V}} := \big\{\mathbf{\Pi}(4,\delta,\mathfrak{V})\colon \delta \in \mathcal{P}_4^{\mathsf{V}},\ \mathfrak{V} \in \boldsymbol{\mathfrak{B}}\big\}.$$

where $\boldsymbol{\mathfrak{B}}$ is the class of all (suitably labelled by the elements of $\wp_2(I_4)$) Veblen Configurations. Some other classes related to skews in $\mathcal{P}_4^{\mathsf{V}}$ are also discussed. In this section we slightly change our notation and we write $\mathbf{\Pi}(p,\delta,\mathfrak{V})$ instead of $\mathbf{\Pi}(4,\delta,\mathfrak{V})$ to emphasise the role of the perspective center p.

If a perspective $\mathfrak{M} = \mathbf{\Pi}(p,\sigma,\mathfrak{V})$ freely contains three K_5 then, in accordance with [12], $\mathfrak{M}$ can be presented as a so called *system of triangle perspectives* (STP, in short). Let us start with a slight reminder of this representation technique of [12]. Suppose (e.g.) that $\mathrm{S}(4)$ is a triangle in $\mathfrak{V}$. We arrange the vertices of three triangles of $\mathfrak{M}$: $\Delta_1 = \{a_1, a_2, a_3\}$, $\Delta_2 = \{b_1, b_2, b_3\}$, and $\Delta_3 = \mathrm{S}(4)$ in three rows of a 3×3-matrix in such a way that when we join in pairs points in the same two columns, the obtained lines of $\mathfrak{M}$ have a common point. So obtained three common points form the line $\mathrm{T}(4)$. After that we join points in distinct rows when there is a line in $\mathfrak{M}$ which joins them: these lines for every pair of rows should meet in a common point. Schemas of types of perspectives between the triangles (labelled by symbols $\rho, \rho^{-1}, \sigma_x, \sigma_y$: ρ for  , ρ^{-1} for  , and σ for one of ) were used in [12] to determine suitable types of the respective STP's.

On Figure 6 we visualize a schema of this procedure.

It is known that after such a representation the obtained structures are



$$a_4 \in \overline{a_i, c_{4,i}},\ i = 1, 2, 3.$$
$$b_4 \in \overline{b_2, c_{2,4}},\ \overline{b_1, c_{3,4}},\ \overline{b_3, c_{1,4}}.$$

$$c_{1,2} \in \overline{a_1, a_2},\ \overline{b_2, b_1},\ \overline{c_{i,4}, c_{j_4}},$$
$$c_{2,3} \in \overline{a_2, a_3},\ \overline{b_1, b_3},\ \overline{c_{j,4}, c_{k_4}},$$
$$c_{1,3} \in \overline{a_1, a_3},\ \overline{b_2, b_3},\ \overline{c_{i,4}, c_{k_4}}.$$

Figure 6: A schema of the (uncompleted yet) diagram of the line $\mathrm{T}(4)$ in $\mathbf{\Pi}(p, \zeta_4, \mathfrak{V})$, $\{i, j, k\} = \{1, 2, 3\}$. $\mathfrak{V}$ is a labelling of the Veblen configuration which contains a free triangle $\mathrm{S}(4)$ and, consequently, $\mathrm{T}(4)$ as a line.

(with a few exceptions) isomorphic when the associated diagrams are isomorphic (can be mapped one onto the other by a permutation of rows and columns). From Figure 6 we read that the diagram is determined by the permutation $\{i, j\} \mapsto \{i', j'\}$: $c_{i',j'} = c_{i,4} \oplus c_{j,4}$ with $1 \le i, j \le 3$.

5.1 Perspectivities associated with permutations of indices

In this section we concentrate upon the case when $\delta = \overline{\sigma}$, $\sigma \in \mathcal{S}_{I_4}$; we write, for short, $\delta = \sigma$. So, let us fix

$$\mathfrak{M} = \mathbf{\Pi}(p, \sigma, \mathfrak{V}_0), \quad \sigma \in \mathcal{S}_{I_4},\ \mathfrak{V}_0 \text{ among listed in (16)\&(17).}$$

It was already remarked that $\mathfrak{M}$ (freely) contains at least two K_5 graphs: A^* and B^*.

On the other hand all the $(15_4\, 20_3)$-configurations which freely contain at least three K_5 were classified in [12]. So, our classification is separated in two ways:

(A) $\mathrm{Fix}(\sigma) \ni i_0$ and $\mathrm{S}(i_0)$ is a triangle in $\mathfrak{V}_0$ for some $i_0 \in I_4$

- then $\mathfrak{M}$ must be identified among those defined in [12].

(B) There is no $i_0 \in I_4$ such that $\sigma(i_0) = i_0$ and $\mathrm{S}(i_0)$ is a triangle in $\mathfrak{V}_0$

- then each isomorphism type of $\mathfrak{M}$'s is determined by a conjugacy class of σ wrt. $\sim_{\mathrm{Aut}(\mathfrak{V})}$ and we obtain a class of 'new' configurations.

Finally, we are now in a position to formulate our main theorem classifying perspectives determined by permutations of indices.

Theorem 5.1.

(i) *There are exactly 33 isomorphism types of configurations in* $\mathcal{K}^{\mathsf{G}}$ *which freely contain exactly two* K_5*-graphs.*

(ii) *Let* $\mathfrak{M} \in \mathcal{K}_4^{\mathsf{G}}$. *Then* $\mathfrak{M}$ *freely contains at least three* K_5*-graphs iff it is isomorphic to one of 10* STP*'s of* [12]*: those enumerated in* [12, Classification 2.8] *with (iv), (vi), (vii), (viii), (ix), (x), and (xiii) excluded and (i) and (ii) in* [12, Remark 2.10].

(iii) *The quasi Grassmannian* $\mathfrak{N}_4$ *of* [14] *with two free* K_5 *graphs belongs to* $\mathcal{K}_4^{\mathsf{G}}$.

Proof. Let $\mathfrak{M} = \mathbf{\Pi}(p, \sigma, \mathfrak{V})$, where $\sigma \in \mathcal{S}_{I_4}$ and $\mathfrak{V}$ is a Veblen configuration defined on $\wp_2(I_4)$. Direct verification involving [14, Representation 2.4] proves that with $\sigma = (1,2)(3,4)$ and $\mathfrak{V} = \mathbf{G}_2(I_4)$ we obtain $C(\sigma) = (2,2)$ and $\mathfrak{M} \cong \mathfrak{N}_4$; so, (iii) is proved.

In view of Lemmas 1.5, 4.2, 4.4, and 4.3 it only remains to justify that in the cases when $i \in \mathrm{Fix}(\sigma)$ and $\mathrm{S}(i)$ is a triangle of $\mathfrak{V}$ the obtained $\mathfrak{M}$ is of the appropriate type of [12, Prop.'s 2.8, 2.10]. At the beginning we note that if $\mathfrak{M}$ contains more than 3 free K_5 graphs then it is a so called multiveblen configuration of [16] (see also [15]) and it freely contains 4 or 6 copies of K_5. All these multiveblen configurations are known and in this case one can directly find skews determining such $\mathfrak{M}$.

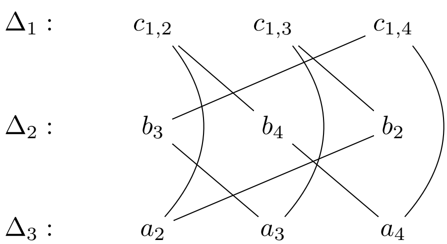


Figure 7: The diagram of the line $\{c_{2,3}, c_{2,4}, c_{3,4}\}$ in $\mathfrak{M}$ considered in Theorem 5.1, case $\sigma = (1)(2,3,4)$, $\mathfrak{V} = \mathbf{G}_2(I_4)$.

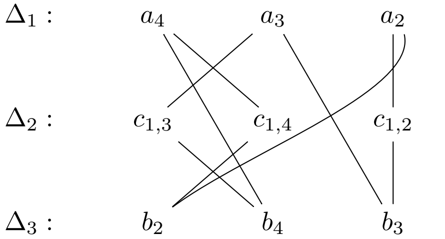


Figure 8: The diagram of the line $\{c_{2,3}, c_{2,4}, c_{3,4}\}$ in $\mathfrak{M}$ considered in Theorem 5.1, case $\sigma = (1)(2,3,4)$, $\mathfrak{V} = \mathbf{IB}(2)$.

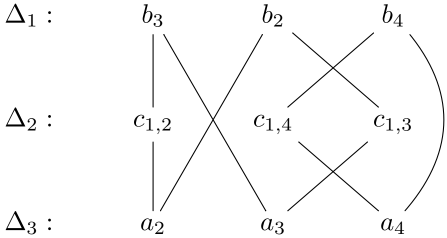


Figure 9: The diagram of the line $\{c_{2,3}, c_{2,4}, c_{3,4}\}$ in $\mathfrak{M}$ considered in Theorem 5.1, case $\sigma = (1)(4)(2,3)$, $\mathfrak{V} = \mathbf{IB}(2)$.

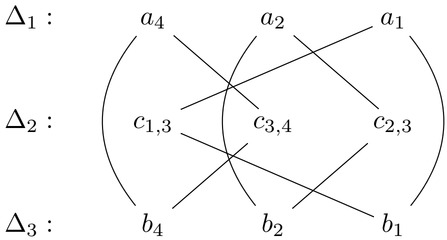


Figure 10: The diagram of the line $\{c_{1,2}, c_{1,4}, c_{2,4}\}$ in $\mathfrak{M}$ considered in Theorem 5.1, case $\sigma = \mathrm{id}$, $\mathfrak{V} = \mathcal{V}_5$.

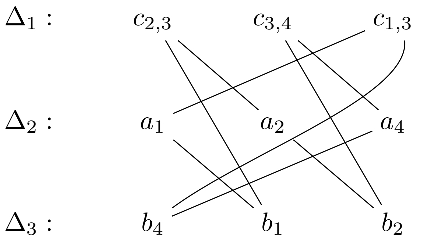


Figure 11: The diagram of the line $\{c_{1,2}, c_{2,3}, c_{2,4}\}$ in $\mathfrak{M}$ considered in Theorem 5.1, case $\sigma = (3)(1,2,4)$, $\mathfrak{V} = \mathcal{V}_5$.

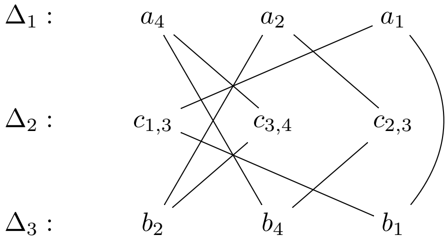


Figure 12: The diagram of the line $\{c_{1,2}, c_{1,4}, c_{2,4}\}$ in $\mathfrak{M}$ considered in Theorem 5.1, case $\sigma = (3)(1)(2,4)$, $\mathfrak{V} = \mathcal{V}_5$.

Next, we determine perspectives containing exacly 3 free K_5: they ere STP's. The representation technique involving labels of diagrams of STP's was used in [12] to determine suitable types. Here, we indicate these labels, but we do not quote precise definitions, the more we do not cite the whole theory of [12]. Namely, the cases that appear when we study skews in $\mathcal{P}_4^{\mathsf{G}}$ are the following:

$\mathfrak{V} = \mathbf{G}_2(I_4)$: $C(\sigma) = (1,1,1,1))$, $\sigma = \mathrm{id}$ – $\mathfrak{M} = \mathbf{G}_2(6)$, it is a generalized Desargues Configuration (or Cayley-Simson configuration; cf. e.g. [2]).

$C(\sigma) = (1,3)$, $\sigma = (1)(2,3,4)$ – $\mathfrak{M}$ has the type 2.8(ii) of [12]; observe Figure 7 and note that
$c_{2,3} \in \overline{c_{1,2}, c_{1,3}},\ \overline{b_3, b_4},\ \overline{a_2, a_3}$, $c_{3,4} \in \overline{c_{1,3}, c_{1,4}},\ \overline{b_4, b_2},\ \overline{a_3, a_4}$, and $c_{2,4} \in \overline{c_{1,2}, c_{1,4}},\ \overline{b_3, b_2},\ \overline{a_2, a_4}$.
b_1 is the centre of Δ_1 and Δ_2, p is the centre of Δ_2 and Δ_3, and a_1 is the centre of Δ_1 and Δ_3 (lines in the diagram join points which correspond each to other under respective perspective).
In accordance with the notation of [12], $\mathfrak{M}$ is determined by the sequence $(\rho, \rho, \mathrm{id})$.

$C(\sigma) = (1,1,2)$, $\sigma = (1)(2)(3,4)$ – $\mathfrak{M}$ has the type 2.10(iii) of [12], it is also isomorphic to the multiveblen configuration

$$\mathbb{MV}^p_{I_4 \rhd L_4} \mathbf{G}_2(I_4),$$

and to the skew perspective

$$\mathbf{\Pi}(p, \mathrm{id}, \mathbb{B}(2))$$

(in this case the corresponding $\mathbb{B}(2)$ has lines $\mathrm{T}(2)$ and $\mathrm{T}(3)$),

$\mathfrak{V} = \mathbb{B}(2)$: $C(\sigma) = (1,1,2)$, $\sigma = (1)(2)(3,4)$ – $\mathfrak{M}$ is the multiveblen configuration $\mathbb{MV}^p_{I_4 \rhd N_4} \mathbf{G}_2(I_4)$, it has the type 2.10(ii) of [12].

$C(\sigma) = (1,3)$, $\sigma = (1)(2,3,4)$ – $\mathfrak{M}$ has the type 2.8(xii) of [12]; observe Figure 8. $\mathfrak{M}$ is determined by the sequence $(\sigma_x, \sigma_x, \rho)$.

$\boldsymbol{\sigma = (1)(4)(2,3)}$ – $\mathfrak{M}$ has the type 2.8(xi) of [12]. Observe, analogously, Figure 9. $\mathfrak{M}$ is determined by the sequence $(\sigma_x, \sigma_x, \sigma_y)$.

$\boldsymbol{\mathfrak{V} = \mathcal{V}_5,\ 3 \in \mathrm{Fix}(\sigma)\colon\ C(\sigma) = (1,1,1,1),\ \sigma = \mathrm{id}}$ – $\mathfrak{M}$ has the type 2.8(v) of [12]. Observe Figure 10. $\mathfrak{M}$ is determined by the sequence $(\rho, \rho^{-1}, \mathrm{id})$.

$\boldsymbol{C(\sigma) = (1,3),\ \sigma = (3)(1,2,4)}$ – $\mathfrak{M}$ has the type 2.8(i) of [12]. Observe Figure 11. $\mathfrak{M}$ is determined by the sequence (ρ, ρ, ρ).

$\boldsymbol{C(\sigma) = (1,1,2),\ \sigma = (3)(1)(2,4)}$ – $\mathfrak{M}$ has the type 2.8(xiv) of [12]. Observe Figure 12. $\mathfrak{M}$ is determined by the sequence $(\rho, \rho^{-1}, \sigma_x)$.

The remaining isomorphism types of the elements of $\mathcal{K}^{\mathsf{G}}$ are determined by the following permutations σ:

$\boldsymbol{\mathfrak{V} = \mathbf{G_2}(I_4)\colon}$ $C(\sigma) = (4),\ \sigma = (1,2,3,4)$.

$\boldsymbol{\mathfrak{V} = \mathbf{G}^*_2(I_4)\colon}$ $C(\sigma) = (1,1,1,1),\ \sigma = \mathrm{id},\ \ C(\sigma) = (2,2),\ \sigma = (1,2)(3,4),$ $C(\sigma) = (1,3),\ \sigma = (1)(2,3,4),\ \ C(\sigma) = (1,1,2),\ \sigma = (1)(2)(3,4),$ $C(\sigma) = (4),\ \sigma = (1,2,3,4)$.

$\boldsymbol{\mathfrak{V} = \mathbb{B}(2)\colon}$ $C(\sigma) = (2,2),\ \sigma = (1,2)(3,4),\ C(\sigma) = (2,2),\ \sigma = (1,3)(2,4),$ $C(\sigma) = (1,3),\ \sigma = (3)(1,2,4),\ C(\sigma) = (4),\ \sigma = (1,2,3,4),\ C(\sigma) = (4),\ \sigma = (1,3,2,4)$.

$\boldsymbol{\mathfrak{V} = \mathcal{V}_4\colon}$ $C(\sigma) = (1,1,1,1),\ \sigma = \mathrm{id},\ \ C(\sigma) = (2,2),\ \sigma = (1,2)(3,4),$ $C(\sigma) = (2,2),\ \sigma = (1,3)(2,4),\ \ C(\sigma) = (1,3),\ \sigma = (1)(2,3,4),$ $C(\sigma) = (1,3),\ \sigma = (4)(1,2,3),\ \ C(\sigma) = (1,1,2),\ \sigma = (1)(2)(3,4),$ $C(\sigma) = (1,1,2),\ \sigma = (1,2)(3)(4),\ \ C(\sigma) = (1,1,2),\ \sigma = (1)(2,3)(4),$ $C(\sigma) = (4),\ \sigma = (1,2,3,4),\ \ C(\sigma) = (4),\ \sigma = (1,3,2,4)$.

$\boldsymbol{\mathfrak{V} = \mathcal{V}_5\colon}$ $C(\sigma) = (2,2),\ \sigma = (1,2)(3,4),\ \ C(\sigma) = (1,3),\ \sigma = (1)(2,3,4),$ $C(\sigma) = (1,1,2),\ \sigma = (1)(2)(3,4), C(\sigma) = (4),\ \sigma = (1,2,3,4),$

$\boldsymbol{\mathfrak{V} = \mathcal{V}_6\colon}$ $C(\sigma) = (1,1,1,1),\ \sigma = \mathrm{id},\ \ C(\sigma) = (2,2),\ \sigma = (1,2)(3,4),$ $C(\sigma) = (1,3),\ \sigma = (1)(2,3,4),\ \ C(\sigma) = (1,3),\ \sigma = (3)(1,2,4),$ $C(\sigma) = (1,1,2),\ \sigma = (1)(3)(2,4),\ \ C(\sigma) = (1,1,2),\ \sigma = (1)(2)(3,4),$ $C(\sigma) = (4),\ \sigma = (1,2,3,4)$.

□

Combining Proposition 1.3, Lemma 4.2, and Lemma 4.4 one can write down explicitly the list of automorphism groups of the structures defined in Theorem 5.1; we pass over this task, as no new essential information can be obtained in this way.

5.2 Perspectivities associated with boolean complementing

Here, we shall pay attention to the structures $\mathbf{\Pi}(p, \sigma, \mathfrak{V}) \in \mathcal{K}^{\mathsf{G}'}$, i.e. to perspectivities where the axis is a $(6_2\,4_3)$-configuration (i.e. it is the Veblen configuration, suitably labelled, defined on $\wp_2(I_4)$) and the skew $\sigma \in \mathcal{S}_{\wp_2(I_4)}$ is defined as the composition of the boolean complementing in $\wp_2(I_4)$ and a map determined by a permutation of I_4.

So, let $\sigma = \varkappa\overline{\varphi}$ where $\varphi \in \mathcal{S}_{I_4}$; recall: $\varkappa\overline{\varphi} = \overline{\varphi}\varkappa$ for every $\varphi \in \mathcal{S}_{I_4}$. Let us write, for short

$$\mathbf{\Pi}(p, \overline{\varphi}\varkappa, \mathfrak{V}) =: \mathfrak{K}_{\varphi,\mathfrak{V}};$$

let $\mathfrak{V}$ be any Veblen configuration defined on $\wp_2(I_4)$. As we know, A^* and B^* are two K_5 graphs freely contained in $\mathfrak{K}_{\varphi,\mathfrak{V}}$.

Lemma 5.2. *$\mathfrak{K}_{\varphi,\mathfrak{V}}$ does not freely contain any other K_5-graph.*

Proof. Suppose that $G \neq K_{A^*}, K_{B^*}$ is a complete K_5 graph freely contained in $\mathfrak{K}_{\varphi,\mathfrak{V}}$. Arguing as in the proof of Lemma 1.5 we find $i_0 \in I_4$ such that $a_{i_0} \in A, G$, $b_{i_0} \in B, G$. Then $G \setminus \{a_{i_0}, b_{i_0}\} \subset \mathrm{S}(i_0) \subset C$. So, $c_{i_0,x}, b_{i_0}$ must colline for every $x \in I_4 \setminus \{i_0\} =: I'$; this means: for every $x \in I'$ there is $x' \in I$ such that $c_{i_0,x} = b_{i_0} \oplus b_{x'} = c_{\overline{\varkappa\varphi^{-1}}(\{i_0,x'\})}$. Write $I_4 = \{i_0, j, k, l\}$. Then we obtain

$$\{\varphi^{-1}(i_0), \varphi^{-1}(j')\} = \{k, l\},\ \{\varphi^{-1}(i_0), \varphi^{-1}(l')\} = \{k, j\}, \text{ and} \\ \{\varphi^{-1}(i_0), \varphi^{-1}(k')\} = \{j, l\}.$$

Consequently, there is no room for $\varphi^{-1}(i_0)$. □

As an immediate consequence of Lemma 5.2 we conclude with

Corollary 5.3. *$f(p) = p$ for every $f \in \mathrm{Aut}(\mathfrak{K}_{\varphi,\mathfrak{V}})$.*

Lemma 5.4. *There is no $\sigma \in \mathcal{S}_{I_4}$ such that $\mathfrak{K}_{\varphi,\mathfrak{V}} \cong \mathbf{\Pi}(p, \overline{\sigma}, \mathfrak{V}')$ for a Veblen configuration $\mathfrak{V}'$. Consequently, $\mathcal{K}^{\mathsf{G}} \cap \mathcal{K}^{\mathsf{G}'} = \emptyset$.*

Proof. Suppose that an isomorphism f exists which maps $\mathbf{\Pi}(p, \overline{\sigma}, \mathfrak{V}')$ onto $\mathfrak{K}_{\varphi,\mathfrak{V}}$. Then, by Lemma 5.2, $f(p) = p$. By Proposition 1.3, there is $\alpha \in \mathcal{S}_{I_4}$ such that $\overline{\sigma^{\alpha}} = \overline{\sigma}^{\overline{\alpha}} = \varkappa\overline{\phi}$, which is impossible. □

As a direct consequence of Proposition 1.3 and Lemma 1.4 we obtain:

Corollary 5.5. *Let $\alpha, \varphi \in \mathcal{S}_{I_4}$ and $\mathfrak{V}$ be a Veblen configuration. Then* $\mathbf{\Pi}(p, \overline{\varphi}\varkappa, \mathfrak{V}) \cong \mathbf{\Pi}(p, \overline{\varphi^{-1}}\varkappa, \varkappa\overline{\varphi^{-1}}(\mathfrak{V}))$ *and* $\mathbf{\Pi}(p, \overline{\varphi}\varkappa, \mathfrak{V}) \cong \mathbf{\Pi}(p, \overline{\varphi^{\alpha}}\varkappa, \overline{\alpha}(\mathfrak{V}))$.

In view of Fact 4.1, Corollary 5.5, Fact 4.5, and the above we obtain the following, rather rough, yet, classification.

Proposition 5.6. *For every $\varphi \in \mathcal{S}_{I_4}$ and every Veblen configuration $\mathfrak{V}$ defined on $\wp_2(I_4)$ there is $\beta \in \mathcal{S}_{I_4}$ such that $\mathfrak{K}_{\varphi,\mathfrak{V}} \cong \mathfrak{K}_{\beta,\mathfrak{V}_0}$, where $\mathfrak{V}_0$ is a structure in the list* (16).

Consequently, we only need to classify all the structures $\mathfrak{K}_{\beta,\mathfrak{V}_0}$. Recall, that there is no $\alpha \in \mathcal{S}_{I_4}$ such that $\varkappa\overline{\alpha} \in \mathrm{Aut}(\mathfrak{V}_0)$, where $\mathfrak{V}_0$ is among structures listed in (16). Consequently, from Proposition 1.3 we conclude with Lemmas 5.7.

Lemma 5.7. *Let $\beta_1, \beta_2 \in \mathcal{S}_{I_4}$, let $\mathfrak{V}_0$ be among structures defined in* (16). *Then $\mathfrak{K}_{\beta_1,\mathfrak{V}_0} \cong \mathfrak{K}_{\beta_2,\mathfrak{V}_0}$ iff β_1, β_2 are conjugate under an α such that $\overline{\alpha} \in$* $\mathrm{Aut}(\mathfrak{V}_0)$.

Finally, we obtain our final classification.

Theorem 5.8. *There are exactly 22 isomorphism types of skew perspectives in $\mathcal{K}^{\mathsf{G}'}$. No one of them appears in $\mathcal{K}^{\mathsf{G}}$.*

Proof. Let $\mathfrak{M} = \mathbf{\Pi}(p, \overline{\varphi}\varkappa, \mathfrak{V})$, where $\mathfrak{V}$ is a Veblen configuration defined on $\wp_2(I_4)$. From Lemma 5.7, substituting Lemma 4.2 and Lemma 4.4 to the definition of $\mathbf{\Pi}(p, \overline{\varphi}\varkappa, \mathfrak{V})$ we infer that $\mathfrak{M}$ is isomorphic to (exactly) one of $\mathbf{\Pi}(p, \overline{\beta}\varkappa, \mathfrak{V}_0)$, where $\mathfrak{V}_0$ is a one in (16) and the representatives β are enumerated in Lemma 4.2. Following this way we obtain 22 isomorphism types. $\square$

5.3 Perspectivities associated with skews imitating Veronese configurations

In this subsection we present a classification of perspectives $\mathbf{\Pi}(p, \delta, \mathfrak{V})$ with skews of the form $\delta = \sigma_\Phi$, $\delta = \varkappa\sigma_\Phi$, or $\delta = \sigma_\Phi\varkappa$, where $\Phi = (\phi_4, \phi_3)$, $\phi_4 \in \mathcal{S}_{I_3}$, $\phi_3 \in \mathcal{S}_{I_2}$, and $\mathfrak{V}$ is a $(6_2\, 4_3)$-configuration i.e. $\mathfrak{V}$ is the Veblen (Pasch) configuration suitably labelled. Let us note a formula evident when $\delta = \varkappa\sigma_\Phi$ as above:

$$\delta(\{1,2\}) = \{3,4\}, \delta(\{4,j\}) = \{\phi_4(i), \phi_4(k)\}, \text{ for } I_3 = \{i,j,k\},$$
$$\delta(\{3,i\}) = \{4, \phi_3(3-i)\} \text{ for } i \in I_2. \quad (20)$$

Proposition 5.9. *There is no* $\Phi \in \mathcal{P}_4^{\mathsf{V}}$ *and* $\alpha \in \mathcal{S}_{I_4}$ *such that* $\sigma_\Phi = \varkappa\overline{\alpha}$ *so,* $\mathcal{K}^{\mathsf{V}} \cap \mathcal{K}^{\mathsf{G}'} = \emptyset$.

Proof. Suppose that there are corresponding Φ and α, then $\overline{\alpha} = \varkappa\sigma_\Phi$. From (20) we read, however, that $\varkappa\sigma_\Phi$ does not preserve concurrency of the edges of K_A, which contradicts Lemma 1.6. □

The technique involving diagrams of STP's allows us to formulate a complete characterization of the structures of the form $\mathbf{\Pi}(p, \sigma_\Phi, \mathfrak{V})$ with $\Phi \in \mathcal{S}_{(<4)}$. With arduous computer-aided computations we obtain the following theorem.

Theorem 5.10.

(i) *There are 103 isomorphism types of perspectives in* $\mathcal{K}^{\mathsf{V}} \setminus \mathcal{K}^{\mathsf{G}}$ *which freely contain exactly two* K_5 *graphs.*

(ii) *Let* $\mathfrak{M}$ *be a skew perspective* $\mathbf{\Pi}(p, \sigma_\Phi, \mathfrak{V})$ *with* $\Phi = (\phi_4, \phi_3) \in \mathcal{S}_{(<4)}$ *such that* $\sigma_\Phi \neq \overline{\alpha}$ *for every* $\alpha \in \mathcal{S}_{I_4}$, *and* $\mathfrak{V}$ *being a Veblen configuration defined on* $\wp_2(I_4)$. *Then* $\mathfrak{M}$ *freely contains at least three* K_5*-graphs iff it is isomorphic to one of 11* STP*'s of* [12]*: those enumerated in* [12, Classification 2.8] *with (i), (ii), (iv), (v), and (xiii) excluded and (ii), (iii) in* [12, Remark 2.10].

Note 5.11. Let us remind, after Lemma 1.9 and Fact 4.6, that if Φ satisfies (13) and $\sigma_\Phi = \overline{\alpha}$ with $\alpha \in \mathcal{S}_{I_4}$ then we can assume that $\alpha = \mathrm{id}$ or $\alpha = (1,2)(3)(4)$, and all the perspectives of the form $\mathbf{\Pi}(p, \overline{\alpha}, \mathfrak{V})$ were already classified in Subsection 5.1.

Practically, explicit enumerating all the 103 configurations of Theorem 5.10(i) together with values of parameters which yield isomorphic items does not give any essential information. However, to complete the course let us briefly say a few words on the subject.

Note 5.12. Consider $\mathfrak{M} = \mathbf{\Pi}(p, \sigma_\Phi, \mathcal{V}_s(\mu))$ where $\Phi = (\phi_4, \phi_3)$ satisfies (13), $s \in \{5, 6\}$, and $\mu \in \mathcal{S}_{I_4}$ with $\mathrm{Fix}(\mu) \neq \emptyset$. Taking into account Fact 4.6 we can restrict ourselves to the following list *PHI* of permutations:

$$PHI = ([(1)(2)(3), (1)(2)],\ [(1)(2)(3), (1,2)],\ [(1)(2,3),\ (1)(2)],\ [(2)(1,3),\ (1,2)],\ [(1,2)(3),\ (1)(2)],\ [(1,2)(3)],\ (1,2)],\ [(1,2,3),\ (1)(2)],\ [(1,2,3)),\ (1,2)]). \quad (21)$$

Note that $\sigma_{PHI[1]} = \overline{\mathrm{id}_{I_4}}$, $\sigma_{PHI[4]} = \zeta_4$, $(PHI[i])^{-1} = PHI[i]$ for $i = 2, .., 6$, $\sigma_{PHI[7]}$ has order 3, and $\sigma_{PHI[8]}$ has order 6. Next, the admissible μ can be arranged into the following sequence MU of lenght 15:

$$MU = (\,(1)(2)(3)(4),\ (1,2,3)(4),\ (1,3,2)(4),\ (1,2,4)(3),\ (1,4,2)(3),\\ (1,3,4)(2),\ (1,4,3)(2),\ (1)(2,3,4),\ (1)(2,4,3),\ (1)(2)(3,4),\ (1)(2,4)(3),\\ (1)(4)(2,3),\ (1,4)(2)(3),\ (1,3)(2)(4),\ (1,2)(3)(4)\,). \quad (22)$$

Let us denote $\mathcal{M}(f,s,i) = \mathbf{\Pi}(p, \sigma_{PHI[f]}, \mathcal{V}_s(MU[i]))$ for $f = 1,...,8$, $s = 5,6$, and $i = 1,...,15$. In view of Lemma 1.9 and Fact 4.6 we can assume that $f \neq 1$.

Suppose that $\mathcal{M}(f,s,i)$ freely contains at least three K_5; then $s = 5$. Using suitable specification of Lemma 1.5 we get the following. If $G_{(4)}$ is an additional K_5 then $i \in \{1,2,3,12,14,15\}$, and f is arbitrary. If $G_{(3)}$ is a K_5 then $f \in \{1,2,5,6\}$ and $i \in \{1,4,5,11,13,15\}$. $G_{(2)}$ cannot be a free K_5 for any f. $G_{(1)})$ is a free K_5 for $f = 3$ and $i \in \{1,8,9,10,11,12\}$. Summing (and using a Maple program which says for which parameters the corresponding perspectives are pairwise isomorphic) we conclude that $\mathcal{M}(f,s,i)$ freely contains at least three K_5 iff $s = 5$ and (f,i) are among the following:

$$(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,8),(3,9),(3,15),\\ (4,1),(4,2),(4,12),(4,14),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,4),\\ (6,11),(6,12),(6,15),(7,1),(7,2),(7,12),(7,14),(7,15),(8,1),(8,2),\\ (8,3),(8,12),(8,14),(8,15). \quad (23)$$

Diagrams of the respective configurations computed by a Maple program prove (ii) of Theorem 5.10.

In the remaining cases, taking into account Lemma 1.4 and Proposition 1.3 and determining (with the help of a Maple program) when

$$\sigma_{PHI[f]}(\mathcal{V}_{s_2}(MU[i_2])) = \mathcal{V}_{s_1}(MU[i_1])$$

we get the following list of parameters of pairwise non isomorphic configurations $\mathcal{M}(f,s,i)$ with exactly two free K_5:

$\boldsymbol{f = 2}$**:** $s = 6$ and $i = 1,2,3,4,5,6,7,8,9,10$;
$\boldsymbol{f = 3}$**:** $s = 6$ and $i = 1,2,3,4,5,6,7,8,9,13$;
$\boldsymbol{f = 4}$**:** $s = 5$ and $i = 4,5,6,9,10,11,13$, $s = 6$ and $i = 1,3,5,12,14$;
$\boldsymbol{f = 5}$**:** $s = 6$ and $i = 1,2,3,4,5,6,7,8,9,10$;
$\boldsymbol{f = 6}$**:** $s = 5$ and $i = 6,7,10$, $s = 6$ and $i = 1,2,4,6,7,10,11,12,15$.

Note that for $f = 7, 8$ we cannot get $\mathcal{M}(f, s, i_i) \cong \mathcal{M}(f, s, i_2)$ for $i_1 \neq i_2$ when the respective perspectives have exactly two free K_5, so in this case pairwise non isomorphic are all $\mathcal{M}(f, 6, i)$ and $\mathcal{M}(f, 5, i)$ with (f, i) not in the list (23).

Note 5.13. If a system $\mathfrak{M}$ is of type (ii) or (vi) of [12, Classification 2.8] then $\mathfrak{M} = \mathbf{\Pi}(p, \sigma, \mathfrak{V})$, where $\sigma = \sigma_\Phi$ and Φ is defined in a way very similar to (13). With

$$\phi_1 = \begin{array}{c|c|c} 2 & 3 & 4 \\ \hline 3 & 1 & 4 \end{array} \text{ and } \phi_2 = \begin{array}{c|c} 3 & 4 \\ \hline 1 & 4 \end{array}$$

we have $\sigma(\{1, i\}) = \{2, \phi_1(i)\}$ and $\sigma(\{2, j\}) = \{3, \sigma_2(j)\}$; clearly, $\sigma(\{3, 4\}) = \{3, 4\}$. For $\mathfrak{M}$ of type (i), (iv), and (xiii) there is no such a pair Φ.

The automorphisms of the structures $\mathbf{\Pi}(p, \sigma_\Phi, \mathfrak{V})$ which freely contain three K_5 are determined in [12] so, there is no need to write them down explicitly here. Then, again using a Maple program to determine when (in the notation of Note 5.12) the equation $\sigma_{PHI[f]}(\mathcal{V}_s(MU[i])) = \mathcal{V}_s(MU[i])$ holds we obtain the following theorem.

Theorem 5.14. *Let* $\mathfrak{V} = \mathcal{V}_s(\mu)$ *be a Veblen configuration defined on* $\wp_2(I_4)$ *and* $\mathfrak{M} = \mathbf{\Pi}(p, \sigma_\Phi, \mathfrak{V})$, $s \in \{5, 6\}$, $\mu \in \mathcal{S}_{I_4}$, *and* $\Phi = (\phi_4, \phi_3)$ *satisfy* (13). *Assume that* $\mathfrak{M}$ *freely contains exactly two* K_5. *Then* $\mathrm{Aut}(\mathfrak{M})$ *is nontrivial only when*

$\mathbf{\Phi = (\,(2)(1,3),\ (1,2)\,)}$ and

- $\boldsymbol{s = 5}$**:** $\mu \in \{(4)(1,2,3),\ (1)(2,3,4),\ (1)(4)(2,3)\}$,
- $\boldsymbol{s = 6}$**:** $\mu \in \{(4)(1,3,2),\ (1)(2,4,3),\ (1)(4)(2,3)$;

$\mathbf{\Phi = (\,(1,2)(3),\ (1,2)\,)}$ and $\boldsymbol{s = 5}$ $\mu = (1)(2)(3,4)$.
- $\boldsymbol{s = 6}$, $\mu \in \{(1)(2)(3)(4),\ (1)(2)(3,4),\ (1,2)(3)(4)\}$.

If $\mathrm{Aut}(\mathfrak{M})$ *is nontrivial then* $\mathrm{Aut}(\mathfrak{M}) = \{\mathrm{id}, \mathfrak{J}\} \cong C_2$.

Proof. The claim is an immediate consequence of Proposiiton 3.5. $\square$

At the end let us say a few words on configurations in the classes

$$\mathcal{K}^{\mathsf{V}'} = \{\mathbf{\Pi}(p,\delta,\mathfrak{V})\colon \varkappa\delta \in \mathcal{P}_4^{\mathsf{V}},\ \mathfrak{V} \in \boldsymbol{\mathcal{B}}\} \text{ and}$$
$$\mathcal{K}^{\mathsf{V}''} = \{\mathbf{\Pi}(p,\delta,\mathfrak{V})\colon \delta\varkappa \in \mathcal{P}_4^{\mathsf{V}},\ \mathfrak{V} \in \boldsymbol{\mathcal{B}}\}.$$

From Proposition 5.9 we obtain immediately

$$\mathcal{K}^{\mathsf{V}'} \cap \mathcal{K}^{\mathsf{G}} = \emptyset = \mathcal{K}^{\mathsf{V}''} \cap \mathcal{K}^{\mathsf{G}}.$$

Computing, with the help of (20), formulas defining $\sigma_\Phi\varkappa(u)$ for $u \in \wp_2(I_4)$ we prove

Lemma 5.15.

(i) *No $\mathfrak{M} \in \mathcal{K}^{\mathsf{V}'} \cup \mathcal{K}^{\mathsf{V}''}$ freely contains three K_5.*

(ii) *Let $\delta^\varkappa \in \mathcal{P}_4^{\mathsf{V}}$, $i_0 \in I_4$, $\mathfrak{V} \in \boldsymbol{\mathcal{B}}$. Assume that $\mathrm{S}(i_0)$ is a collinearity clique in $\mathfrak{V}$. Then $G_{(i_0)}$, defined in Lemma 1.5, is a free K_5 subgraph in $\mathbf{\Pi}(p,\delta,\mathfrak{V})$ iff one of the following holds: $i_0 = 4$, $i_0 = 3$ and $\phi_4(3) = 3$, $i_0 = 2$ and $\phi_3(2) = 2$, or $i_0 = 1$ and $\phi_3(1) = 1$.*

Proof. As an example we prove a part of (i). Let $\delta = \varkappa\sigma_\Phi$ and $\Phi \in \mathcal{P}_4^{\mathsf{V}}$. We use Lemma 1.5 and (20): $1 \notin \delta(\{1,2\})$ so, $G_{(1)}$ is not a K_5 free clique in $\mathfrak{M} =: \mathbf{\Pi}(p,\delta,\mathfrak{V})$, $2 \notin \delta(\{1,2\})$ so, $G_{(2)}$ is not a K_5 free clique in $\mathfrak{M}$, $3 \notin \delta(\{2,3\})$ so, $G_{(3)}$ is not a K_5 free clique in $\mathfrak{M}$, $4 \notin \delta(\{1,4\})$ so, $G_{(4)}$ is not a K_5 free clique in $\mathfrak{M}$. $\square$

In an analogous way we compute a criterion which enables us to determine $\mathcal{K}^{\mathsf{V}'} \cap \mathcal{K}^{\mathsf{V}''}$.

Lemma 5.16. *Let $\Phi \in \mathcal{S}_{(<4)}$. Then $\sigma_\Phi^\varkappa \in \mathcal{P}_4^{\mathsf{V}}$ iff $\phi_4(3) = 3$. In that case $\sigma_\Phi^\varkappa = \sigma_\Psi$ where $\Psi = (\psi_4,\psi_3)$, $\psi_4(3) = 3$ and $\psi_j(i) = \phi_{7-j}(i)$ for $i \in I_2$, $j = 3,4$.*

Finally, one can classify the elements of $\mathcal{K}^{\mathsf{V}'}$.

Theorem 5.17. *There are 179 isomorphism types of configurations in $\mathcal{K}^{\mathsf{V}'} \setminus \mathcal{K}^{\mathsf{G}'}$, 179 isomorphism types of perspectives in $\mathcal{K}^{\mathsf{V}''} \setminus \mathcal{K}^{\mathsf{G}'}$, and 59 perspectives in $(\mathcal{K}^{\mathsf{V}'} \cap \mathcal{K}^{\mathsf{V}''}) \setminus \mathcal{K}^{\mathsf{G}'}$.*

Proof. We make essential use of the computer technique elaborated to justify Theorem 5.10; in particular, we use computations made in Note 5.12. From Lemma 5.15 we note that every isomorphism between structures in $\mathcal{K}^{\mathsf{V}'}$ must fix the centre p and therefore it is determined by a suitable permutation α in $\mathcal{S}_{I_4}$. Set $\mathcal{M}'(f,s,i) = \mathbf{\Pi}(p, \varkappa\sigma_{PHI[f]}, \mathcal{V}_s(MU[i]))$. $\mathcal{M}'(f,s,i) \in \mathcal{K}^{\mathsf{G}'}$ iff $f = 1$. In the list (21) we have enumerated all the representatives in $\mathcal{S}_{(<4)}$ of the conjugacy classes determined by the action of $\overline{\mathcal{S}_{I_4}}$. We see that $\mathcal{M}'(f,s,i) \cong \mathcal{M}'(f_1,s_1,i_1)$ iff either $f = f_1$, $s = s_1$, $i = i_1$ or $\mathcal{M}'(f_1,s_1,i_1) = \mathcal{J}(\mathcal{M}'(f,s,i))$ i.e. $\mathbf{\Pi}(p,\varkappa\sigma,\mathcal{V}_s(\mu)) = \mathbf{\Pi}(p,\varkappa\sigma_\Psi,\sigma_{\Phi^{-1}},\mathcal{V}_{11-s}(\mu))$ for a Ψ such that $\varkappa\sigma_\Phi = \sigma_\Psi\varkappa$ (we use Fact 4.5 here).

With Lemma 5.16 we find that the equation $\varkappa\sigma_{PHI[f_1]} = \sigma_{PHI[f_2]}\varkappa$ holds for $\{f_1,f_2\} \in \{\{2,5\},\{6\}\}$, while $\mathcal{V}_s(MU[i]) = \sigma_{PHI[6]}(\mathcal{V}_{11-s}(MU[i]))$ holds for $i = 10$ only. Evidently, the classes $\{\mathcal{M}'(2,s,i)\colon s = 5,6,\ i \in I_{15}\}$ and $\{\mathcal{M}'(5,s,i)\colon s = 5,6,\ i \in I_{15}\}$ exhaust the same isomorphism types. That way we obtain 179 pairwise nonisomorphic configurations of the form $\mathcal{M}'(f,s,i)$ for $f \neq 1$, $f \in I_8$, $i \in I_{15}$, $s = 5,6$.

Note that the function $\mathcal{J}$ maps the class $\mathcal{K}^{\mathsf{V}'}$ onto $\mathcal{K}^{\mathsf{V}''}$; this justifies that $\mathcal{K}^{\mathsf{V}''} \setminus \mathcal{K}^{\mathsf{G}'}$ contains 179 isomorphism types.

Finally note: we have shown above that isomorphism types of perspectives in $\mathcal{K}^{\mathsf{V}'} \cap \mathcal{K}^{\mathsf{V}''}$ outside $\mathcal{K}^{\mathsf{G}}$ are represented by structures $\mathcal{M}'(f,s,i)$ with $f \in \{2,5,6\}$; there are (at most) 59 among them pairwise nonisomorphic. $\square$

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