SURPRISING QUANTUM EFFECTS

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We consider circumstances wherein a quantum-mechanical system is subjected to a varied sequence of measurements, some of which are substantially more precise than others. Such systems are shown to exhibit paradoxical behavior. The resolution of this paradox turns out to involve bizarre interference effects in the measuring apparatus. The possibilities of observing such behavior in the laboratory are briefly considered.

Von Neumann's famous account of the operations of quantum-mechanical measuring devices runs, roughly, like this: in order to measure some given observable A of a quantum-mechanical system S, what is required is that one produce a hamiltonian of interaction, between S and a measuring device, which has the form [1]:

$$H_{\rm int} = -g(t)qA, \tag{1}$$

where q is an internal variable of the measuring device, and $g(t)$ is a time-dependent coupling function which is non-zero only during a short interval $t_0<t<t_1$, when the measuring device is "switched on". Then the measurement is accomplished as follows: the Heisenberg equation for π, where π is defined to be the canonical momentum conjugate to the canonical coordinate q of the measuring device, reads

$$d\pi/dt = g(t)A, \tag{2}$$

and so, if π is initially set, say, at zero, and if the value of

$$\int_{t_0}^{t_1} g(t)\,dt$$

is known, then the value of A at $t \simeq t_0 \simeq t_1$ can be read off from the value of π after t_1 (and π is therefore often referred to as the "pointer variable") [#1].

The fact that any precise measurement of A must necessarily and uncontrollably disturb the values of observables which fail to commute with A can be traced, within this account, to the fact that a precise measurement of A requires that the value of π be precisely fixed prior to t_0, and consequently that the uncertainty in q during the measurement interaction described in eq. (1) (and hence, as well, the possible *strength* of that interaction) is unbounded.

On the other hand, it emerges quite clearly within this account that if one is willing to accept uncertainties in the initial value of π, and the resultant inaccuracies in the measurement of A, then the uncertainties in the value of q during the measurement interaction, and hence the possible *strength* of that interaction, and the disturbance caused by it to variables of system S which fail to commute with A, *can* be bounded and controlled. We shall refer here to such a trading-off, to the sacrificing of the accuracy of measurements of A in order to gain some control of the disturbances caused by such measurements to variables which fail to commute with A, as

[#1] This, for example, is precisely how a Stern–Gerlach spin measuring device works, wherein the position-coordinate of the particle being measured (which here plays the role of q in (1)) is effectively coupled to its spin (which plays the role of A) by means of an externally applied magnetic field (whose gradient plays the role of g).

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a *weakening* of the measurement of A; and our concern in the present note shall be to point out a most extraordinary statistical property of such weakened measurements, which we have recently discovered.

Consider a system of N spin-$\frac{1}{2}$ particles (the hamiltonian of which we shall suppose, for simplicity, to be zero), and suppose that at time t_i a precise measurement of the total angular momentum of this N-particle system in the x-direction (J_x) is carried out, and that this measurement produces the (largest possible) result $J_x=N$ (we take $\hbar=2$); and suppose, furthermore, that at time t_f ($t_f>t_i$) a precise measurement of J_y is carried out on this system, and that *this* measurement happens to produce the result $J_y=N$ (such pairs of results, when N is large, will of course be rare, but they are nonetheless always *possible*; and we should like to confine our attention here to a system wherein such a pair of results happens to have emerged). If we are later informed that another precise measurement of J_x, say, were carried out at time t_1, with $t_i<t_1<t_f$, then (as is well known) we could assert with certainty that the result of that measurement *must* have been $J_x=N$ (since otherwise, the result of the measurement at t_i could not have been what it was). Similarly, if we are later informed that a precise measurement of J_y were carried out at t_2, with $t_i<t_2<t_f$, we would be in a position to assert with certainty that the result of that measurement must have been $J_y=N$; and indeed it is even the case that if we were later informed that a precise measurement of J_x were carried out at t_1 *and* a precise measurement of J_y were carried out at t_2, with $t_i<t_1<t_2<t_f$, then we should be in a position to say with certainty that the result of the measurement at t_1 was $J_x=N$ *and* the result of the measurement at t_2 was $J_y=N$. But it should be carefully noted that in this last case the time-order of the two intermediate measurements is vitally important. These two measurements, after all, being precise, will uncontrollably disturb one another; and so in the event that $t_i<t_2<t_1<t_f$, there will, in general, be no correlation whatever between the results of the measurements at t_i and t_1, nor between the results of those at t_f and t_2.

Suppose, however, that we were to *weaken* these two intermediate measurements in such a way as to gain some considerable control over the disturbances they cause to one another. Suppose, more particularly, that the initial state of the measuring devices are arranged in such a way as to bound the possible value of q as follows:

$$|q|<N^{-1/2-\epsilon}, \qquad (3)$$

where ϵ may be an arbitrarily small positive number. In that case, the resulting uncertainty in π will be of the order of $\sqrt{N}$, which if N is taken to be large, is small compared with the maximal possible values of J_x and J_y; so that measuring devices prepared in this way can still serve (albeit imperfectly) as reasonably informative indicators of the values of those angular momenta. On the other hand, if we set

$$\int g(t)\,\mathrm{d}t=1 \qquad (4)$$

for each of these devices, then the bound (3) on q will guarantee that measurements of J_x, say, with such devices as these, will change the value of J_y only by amounts of the order of $\sqrt{N}$, which is (as we have just seen) within the intrinsic error associated with these measurements. Such weakened measurements of J_x and J_y, then, can be expected, as it were, to "commute"; it can be expected, that is, that two such measurements will verifiably leave one another's results essentially undisturbed.

Reconsider, now, the system of N spins described above, which was measured precisely, at time t_i, to be in the state $J_x=N$, and at t_f to be in the state $J_y=N$. Suppose that we are informed later on, that a weak measurement of J_x, of the kind we have just described, was carried out at t_i ($t_i<t_1<t_f$). Then, especially if N is large, it can be asserted with a high degree of confidence that the result of this weakened measurement was $J_x=N$ (more precisely, it will be the case that if $\langle\pi\rangle=0$ before the interaction begins, then it will invariably be the case that $\langle\pi\rangle=N$ after t_1, where π is the pointer variable of the weakened J_x measuring device; and furthermore, if N is large, the uncertainties in π, both before and after the experiment, will be very small compared with this displacement in its expectation value); and, by virtue of the time-reversal-symmetric character of the statistical predictions of quantum theory [2], the same argument can be made concerning a weak measurement of J_y which may have been carried out at t_2, within that same interval. Clearly no additional complications are introduced by supposing that *both*

measurements (first the measurement of J_x and then that of J_y) are carried out within that interval, as we did above; but in the *present* case, because of the "commutative" behavior of these weak measurements, we also expect that *the order in which they are carried out will make no difference*. Indeed, it can be easily confirmed by straightforward calculation that whether $t_1 < t_2$ *or* $t_2 < t_1$, the expectation values of the pointer variables of both the J_x and the J_y measuring devices, will, in the circumstances described above, be displaced by precisely (up to corrections of the order of $\sqrt{N}$) N!

This produces something of a paradox, which runs as follows: suppose that instead (as above) of measuring the value of J_x at time t_1 and the value of J_y at time t_2, we measure, with a single device, the *sum* of those two values. Such a measurement can easily be accomplished by means of an interaction hamiltonian of the form

$$H_{\text{int}} = g_1(t) q J_x/\sqrt{2} + g_2(t) q J_y/\sqrt{2}, \tag{5}$$

where $g_1(t)$ is non-zero only in the vicinity of t_1, and $g_2(t)$ is non-zero only in the vicinity of t_2 (the factors of $1/\sqrt{2}$, as the reader shall presently see, have been inserted for the sake of convenience). Furthermore, if "weak" bounds of the form of eq. (3) are imposed on q, and in cases where J_x is precisely measured to be N at t_i and J_y is precisely measured to be N at t_f, the total displacement of the expectation value of π after both t_1 and t_2 will, by the above arguments, *always* be (up to corrections of order $\sqrt{N}$) $\sqrt{2}\,N$, whether t_1 precedes t_2, *or* t_2 precedes t_1 or, indeed, $t_1 = t_2$. But consider this last possibility. In the event that $t_1 = t_2$ (in the event, that is, that $g_1(t) = g_2(t)$) the interaction hamiltonian of eq. (5) reduces to

$$H_{\text{int}} = g_1(t) q (J_x + J_y)/\sqrt{2}, \tag{6}$$

which is the hamiltonian required for a measurement of the projection of the total angular momentum along the $\hat{\alpha}$-axis (J_α), where α is the ray which bisects the right angle between $\hat{x}$ and $\hat{y}$. Now, we have just argued that this measurement will (within such intervals as we have just described, and so long as q is bounded in accordance with (3)) almost *invariably*, produce the result $\sqrt{2}\,N$; but this seems a most paradoxical result, since the particular measurement here in question is (looked at in another way) simply a measurement of J_α, the largest possible eigenvalue of which is the vastly smaller number N! *How can it be* that measurements of J_α, under these circumstances, and with such regularity, produce *impossible* results?

The first thing to do, it would seem, is to verify the result of our argument by more rigorous techniques, and this, happily, is not a particularly difficult task. The state of the composite system, consisting of the N spins together with the J_α measuring apparatus after the J_α-interaction is complete and supposing that J_x was found to have the value N at t_i, will be:

$$\exp[iq(J_x + J_y)/\sqrt{2}]\,|J_x = N\rangle\,|\pi \simeq 0\rangle, \tag{7}$$

wherein $|\pi \simeq 0\rangle$ represents the initial state of that device, which (in accordance with (3)) will be characterized by a gaussian distribution of π-values, of width $\sqrt{N}$, and peaked, say, about $\pi = 0$. Now, if it subsequently happens that at t_f J_y is found to have the value N, then the final state of the measuring apparatus (modulo an overall constant of normalization) will be:

$$\langle J_y = N|\exp[iq(J_x + J_y)/\sqrt{2}]\,|J_x = N\rangle\,|\pi \simeq 0\rangle, \tag{8}$$

so the time-evolution operator for the measuring apparatus through such a sequence of events is

$$\langle J_y = N|\exp[iq(J_x + J_y)/\sqrt{2}]\,|J_x = N\rangle, \tag{9}$$

and it can be rigorously shown (without too much trouble) that if q is taken to obey the bound (3), then

$$\langle J_x = N|\exp[iq(J_x + J_y)/\sqrt{2}]\,|J_y = N\rangle$$
$$\simeq \langle J_x = N|J_y = N\rangle\, \exp\{iq[\sqrt{2}\,N + O(\sqrt{N})]\} \tag{10}$$

as N becomes large. The effect of such a sequence of events, then, in this limit, is invariably to translate the initial $|\pi \simeq 0\rangle$ apparatus state by the impossible (or at least, at first sight, unreasonable) distance of $N\sqrt{2}$, rather than (what would seem more reasonable) a distance equivalent to any of the eigenvalues of J_α, precisely as our earlier (and more intuitive) argument had led us to believe.

What is happening here – albeit the demonstration is quite straightforward – is something of a miracle. The measuring apparatus state is translated, in the course of these events, by a superposition of different distances corresponding to the various possible eigenvalues of J_α; and the resultant translated states, in the end, *quantum-mechanically interfere* with one another in such a way as to produce an *effective* translation which is larger than any of them! In such sequences of events, everything in the final apparatus states save the outermost limits of the *tails* (which must necessarily exist, given (3)) of the translated π-distributions ends up cancelling itself out; the central peaks annihilate one another and disappear, and what remains is a new peak, made up of constructive interferences among the many tails, way out in the middle of nowhere, at $\sqrt{2}\,N$. Moreover, (and this is what seems genuinely miraculous) the *nature* of these anomalous interferences is precisely such as to make the J_x and J_y components of the total angular momentum (both of which have the value N) appear, as measured by our weak experiments, to add together in J_α as if they were components of a *classical* vector. These results, of course, are of the sort that would normally be construed as "errors" of the measuring-device; but that seems an inappropriate name for them here, since they are results which (given inital and final conditions on the spins such as we have postulated here) *invariably* arise, and which invariably conspire together to point to an internally consistent picture of a *classical*, rather than a quantum-mechanical, system.

Here, then, is a particularly bizarre prediction of quantum mechanics; something that looks like magic, and which demands to be tested. What seems frustrating in that respect is that the circumstances described above (i.e. circumstances like $J_x=N$ at t; and $J_y=N$ at t_f), wherein those bizarre effects have been shown to occur, are exceedingly, *exponentially*, improbable; so the task of actually searching out such effects in the laboratory seems hopeless. But it turns out that that improbableness is by no means a *necessary* attribute of these effects; and indeed it turns out that a very simple modification of the experimental procedure described above will suffice to guarantee that such effects are extremely common things! The trick is to do the weak J_α-measurements *separately* on each particle in the ensemble, rather than *combining* them all into a single *total* weak J_α measurement, as above. Here, in more detail, is what to do: start at t, with a large collection of electrons, all of which are in the state $|s_x=1\rangle$ (such a collection is, of course, not at all hard to come by: something like half of any randomly chosen collection of electrons, whose x-spins are measured at t_i, will form precisely such a group). Measure s_α (that is: $(s_x+s_y)/\sqrt{2}$) for each particle, weakly and *separately* (i.e. using a *separate* measuring device and obtaining a *separate* specific result, for each separate electron) at t_1. Finally, at t_f, measure the y-spins of all those electrons. Something like half of them will be found to have $s_y=1$ at t_f. Now focus on *that half* of the full collection of weak s_α *measuring devices* which happen to have interacted at t_1 with that particular ($s_y=1$) half of the original collection of *electrons*. Among *them*, for precisely the reasons described above, the same sorts of bizarre conspiracies of "error" must necessarily arise; *their* π-values (that is: the π-values of *that particular half* of the weak J_α-measuring devices), will be found to have been displaced in the course of the interaction of t_1 by an average distance $\sqrt{2}$, even though that distance seems impossibly large. These displacements will, of course, be far smaller than the original *widths* of the π-space wave packets; but if the original ensemble of electrons is sufficiently large, the average displacement can nonetheless be determined, by statistical means, with arbitrarily high precision.

The difference between this experimental procedure and the one described above, as we have already mentioned, is simply that here the s_α-devices are all stipulated to be separate and distinguishable degrees of freedom, rather than having been combined, as they were above, into a single total J_α device; and it is precisely this separateness and distinguishability which here allows us to focus, *after* t_f, on that particular *half* of those devices wherein such effects must necessarily arise, and thereby to make the apparent "improbableness" of those effects go away.

The accomplishment, in practice, of this separateness of the measuring devices presents no serious obstacle. If it could be arranged that, say, certain spatial degrees of freedom of the particles themselves were made to serve as the pointer-variables of the measuring devices (as in a Stern–Gerlach experiment, for example), that would suffice. Such an arrangement would have the additional advantage of making it very easy to ascertain the average π-displacement, since that displacement would, in this case, amount to a shift of the center of a macroscopically large beam of particles!

The details of experiments wherein these effects might be observed will be described elsewhere. Let it suffice, for now, to say that there appears to be no reason to suppose that such experiments will prove in any way beyond our present technological capacities.

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References

[1] J. von Neumann, Mathematische Gründerlagen der Quantenmechanik (Springer, Berlin, 1932) [English translation: Mathematical foundations of quantum theory and measurement (Princeton Univ. Press, Princeton, 1983)].

[2] Y. Aharonov, P.G. Bergman and J.L. Lebowitz, Phys. Rev. 134 (1964) B1410.