

# Rotating Black Holes

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## Abstract

This paper discusses the Kerr solution to the Einstein equations and its physical interpretation as a rotating black hole. We first discuss the singularities of the Kerr solution, and the interpretation of the associated parameters contained therein. Particle trajectories in the equatorial plane around a Kerr black hole are discussed by means of an effective potential, leading to a discussion of the fascinating Penrose process and the various limits of energy extraction from a black hole. Closely related is the phenomenon of super-radiance, a process of wave-amplification by a rotating black hole, which is discussed subsequently. Finally the appendix gives a short discussion on the uniqueness of the Kerr solution.

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## 1 Introduction

It is reasonable to expect that most black holes in nature are to a good approximation described by the Kerr metric. This metric presents the unique [7, 6] axisymmetric solution to Einstein's equations in the vacuum, containing the Schwarzschild solution. Black holes in nature are expected to form primarily due to stellar collapse. Since virtually all stars have angular momentum - the dipole which stars cannot rid themselves of through gravitational radiation - one expects that the stationary endstate of gravitational collapse of a sufficiently massive star would be a so-called Kerr black hole. This paper will discuss this vacuum solution of Einsteins equations in some detail, following a more or less standard path of discussing the solution and its caveats, and then moving on to the 'special effects' this solution has in store. Firstly the metric and its singularities, both coordinate- and curvature-, will be discussed. Then a region of considerable interest outside the outer event horizon, the ergosphere, will be examined, followed by a discussion of radial particle trajectories around a rotating black hole, the Penrose process and, closely related, superradiance. The related topic of black hole formation will not be covered here, for this we would like to refer the interested reader

to the paper by Michiel Bouwuis, [1]. Throughout this paper we will use geometric units;  $c = 1$ ,  $G = 1$ , unless explicitly stated otherwise.

## 2 The Kerr-Newman Metric

Relatively shortly after Schwarzschild's discovery, his solution was extended so as to be a solution of the coupled Einstein-Maxwell equations, giving the Reissner-Nordström solution, in 1916. The solution for a merely axisymmetric stationary spacetime was not found till 1963, when it was discovered more or less by accident by R.P. Kerr [3]. Analogously to how Schwarzschild was extended to incorporate electric and magnetic charge, the Kerr-metric can be extended to the so called Kerr-Newman metric, which in Boyer-Lindquist coordinates reads

$$ds^2 = - \frac{(\Delta - a^2 \sin^2 \theta)}{\Sigma} dt^2 - 2a \sin^2 \theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\ + \left( \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2 \cos^2 \theta \\ \Delta &= r^2 - 2Mr + a^2 + e^2 \\ e^2 &= Q^2 + P^2. \end{aligned}$$

The coordinates  $t$ ,  $r$ ,  $\theta$ , and  $\phi$  have the familiar interpretation of spherical coordinates in Minkowski space in the limit of  $M, e, a \rightarrow 0$ . The interpretation of  $Q$ ,  $P$ ,  $a$ , and  $M$  merit a bit more discussion.

As it turns out this metric describes a rotating charged black hole, presenting a three<sup>1</sup> parameter solution to the Einstein-Maxwell equations with the vector potential (one form) given by,

$$A = \frac{Qr(dt - a \sin^2 \theta d\phi) - P \cos \theta [adt - (r^2 + a^2)d\phi]}{\Sigma}.$$

The interpretation of  $e$  is none other than in electrodynamics (extended to include magnetic monopole charges), except that now one is dealing with a

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<sup>1</sup>Depending on your point of view one could reasonably argue it is a four parameter solution, however it is not commonly denoted as such in the literature.

curved background. Regardless the story remains the same; as can be verified by explicit calculation for any two-sphere in the asymptotic region,

$$\frac{1}{2} \int_{S^2} dS_{\mu\nu} F^{\mu\nu} = 4\pi e.$$

So we see that  $e$  represents the combined electric and magnetic (monopole) charges of the black hole, denoted  $Q$  and  $P$  respectively in the above. The other two parameters  $M$  and  $a$ , represent the Schwarzschild mass and the angular momentum per unit mass respectively, as will be shown below. To show this in a generic and elegant fashion, we need the fact that this metric possesses the following two Killing vector fields which will be denoted  $k$  and  $m$ ,

$$k^\mu = \left( \frac{\partial}{\partial t} \right)^\mu \quad \text{and} \quad m^\nu = \left( \frac{\partial}{\partial \phi} \right)^\nu.$$

The fact that these are Killing vector fields is clear by inspection of the metric.

## 2.1 Komar integrals

As indicated above there are three parameters in the Kerr-Newman family. The task is now to determine what their meaning is. To do so, it serves well to introduce so-called Komar-integrals, which represent (time-)conserved charges associated to conserved currents coming from Killing vector fields a spacetime possesses. This is done as follows: to each Killing vector field ( $\xi$ ) associate the following integral ( $dS_{\mu\nu}$  is the area element on the boundary of the spacelike hypersurface),

$$Q_\xi(V) = \frac{c}{16\pi} \oint_{\partial V} dS_{\mu\nu} \nabla^\mu \xi^\nu = \frac{c}{8\pi} \int_V dS_\mu \nabla_\nu \nabla^\mu \xi^\nu, \quad (2.1)$$

which can alternatively be written as,

$$Q_\xi(V) = \int dS_\mu J^\mu(\xi),$$

for,

$$J^\mu(\xi) = c \left( T^\mu{}_\nu \xi^\nu - \frac{1}{2} T \xi^\mu \right).$$

Equality of the two formulae is clear upon using the identity  $\nabla_\nu \nabla_\mu \xi^\nu = R_{\mu\nu} \xi^\nu$  for a Killing vector field  $\xi$ . This current is in fact conserved since,

$$\begin{aligned}
\nabla_\mu J^\mu &= c \underbrace{\left( T^{\mu\nu} \nabla_\mu \xi_\nu - \frac{1}{2} T \nabla_\mu \xi^\mu \right)}_{0 \text{ by Killing's equation}} - \frac{c}{2} \xi^\mu \partial_\mu T \\
&= \frac{c}{2} \xi^\mu \partial_\mu R \quad \text{by Einstein's equations} \\
&= 0 \quad \text{for } \xi \text{ is a Killing vector field.}
\end{aligned}$$

Hence we are dealing with bona fide conserved quantities, which in certain limits may be given familiar meanings, as we shall do now.

By explicit calculation of the relevant Christoffel symbols and evaluating the corresponding integral one finds that the Komar-charge associated with the Killing vector field  $k$ , is  $M(+\frac{e^2}{r})$ , as expected. A multipole expansion in  $M$  gives  $M$  the role of the Newtonian gravitational mass at first order, hence it can be viewed as the mass of the black hole.

To give an interpretation of the conserved charged associated to  $m$ , let us consider a  $t = \text{constant}$  hypersurface  $V$ , and work in Cartesian coordinates in the asymptotic region. There, since  $dS_\mu m^\mu = 0$ , we obtain (for  $c = 1$ )

$$Q_m(V) \approx \epsilon_{3jk} \int_V d^3x x^j T^{k0}.$$

This expression is none other than what one would call the third component of angular momentum in ordinary Minkowski space, and so it not unreasonable to agree that the Komar integral, (2.1), for the Killing vector field  $m$  and  $c = 1$  gives a conserved charge that we would like to call angular momentum. Evaluating this integral explicitly for the Kerr-solution<sup>2</sup> gives

$$\begin{aligned}
J &:= \frac{1}{16\pi} \oint_{S^2} dS_{\mu\nu} \nabla^\mu m^\nu = \frac{1}{16\pi} \oint_{S^2} dS_{\mu\nu} g^{\mu\alpha} \Gamma_{\alpha\beta}^\nu m^\beta \\
&= \frac{1}{16\pi} \oint_{S^2} dS_{\mu\nu} g^{\mu\alpha} \Gamma_{\alpha 3}^\nu \\
&= \int_0^{2\pi} \int_0^\pi d\theta d\phi \left\{ \right. \\
&\quad \left. - \frac{4aM (a^4 - 3r^2 a^2 + (a^2 - r^2) \cos(2\theta) a^2 - 6r^4) \left( (\cos(2\theta) a^2 + a^2 + 2r^2)^2 \sin^2(\theta) \right)^{3/2}}{(\cos(2\theta) a^2 + a^2 + 2r^2)^5} \right\} \\
&= aM.
\end{aligned}$$

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<sup>2</sup>The following expression is less insightfull in the case  $e \neq 0$ .

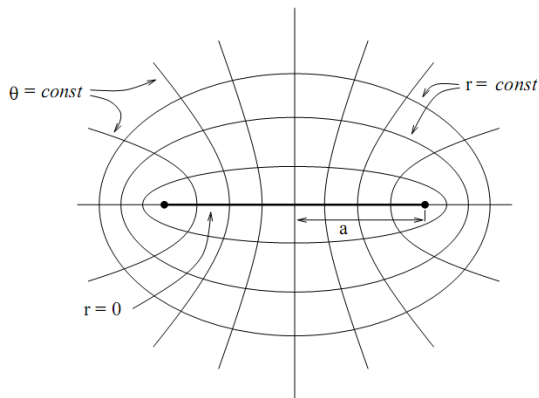


Figure 1: Ellipsoidal coordinates in the  $(r, \theta)$  plane [2]

Thus we see that we can interpret the parameter  $a$  in the Kerr-Newman metric as angular momentum per unit mass.

## 2.2 Singularities of the Kerr Metric

From this point on we will consider the Kerr-metric ( $e = 0$ ) since most of the interesting phenomena persist, while keeping things more concise. The Kerr metric has both curvature and coordinate singularities, where analogously to the Schwarzschild case, the coordinate singularities play an interesting physical role as well.

As it turns out, the singularity at  $\Sigma = 0$ , i.e.  $r = 0$  and  $\theta = \frac{\pi}{2}$ , is a curvature singularity. This might seem a bit odd at first glance, but the coordinates  $r$ ,  $\theta$ , and  $\phi$  are not the familiar spherical coordinates, and hence the interpretation of this singularity is changed.

Setting  $M$  to zero in the Kerr metric, gives regular Minkowski space, except in ellipsoidal coordinates, illustrated in the figure below (fig.1).

The nature of this singularity becomes more apparent when working in Kerr-Schild coordinates, which are defined by,

$$\begin{aligned}
 x + iy &= (r + ia) \sin \theta \exp \left[ i \int \left( d\phi + \frac{a}{\Delta} dr \right) \right] \\
 z &= r \cos \theta \\
 \tilde{t} &= \int \left( dt + \frac{r^2 + a^2}{\Delta} dr \right) - r.
 \end{aligned}$$

Expressed in these coordinates the metric becomes

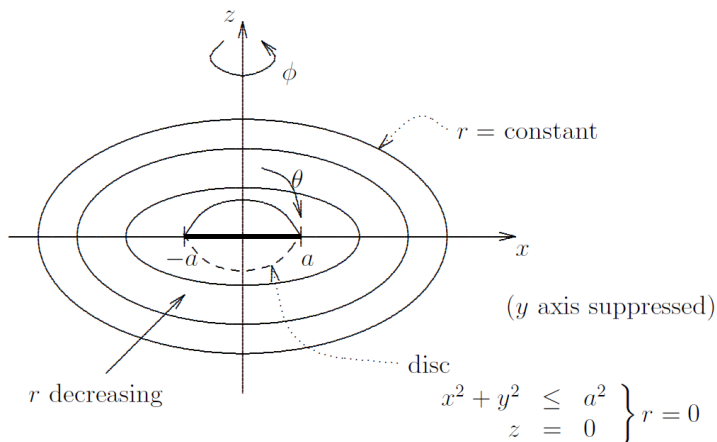


Figure 2: Confocal ellipsoids and the disc at  $r = 0$  [6]

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2z^2} \left[ \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r} + d\tilde{t} \right]^2.$$

This makes manifest the Minkowski nature of the spacetime for  $M = 0$ . In terms of these coordinates, the surfaces of constant  $\tilde{t}$  and  $r$  become confocal ellipsoids, which for  $r = 0$  degenerate into the disc  $z = 0$ ,  $x^2 + y^2 \leq a^2$ , as is illustrated in fig.2.

In these coordinates,  $\theta = \frac{\pi}{2}$  corresponds to the edge of this disc. So we see that the singularity at  $r = 0$  and  $\theta = \frac{\pi}{2}$  lies on the boundary of this disc; a Kerr black hole has a ring singularity. We mention here in passing that when  $M^2 < a^2$  this singularity is a naked one, and hence the cosmic censorship hypothesis would indicate that this case does not correspond to a physically relevant solution of the Einstein equations; slightly more details on cosmic censorship will follow in the appendix in the short discussion of uniqueness theorems and cosmic censorship. From this point on then, we shall implicitly assume that  $M^2 > a^2$ , unless explicitly stated otherwise<sup>3</sup>.

<sup>3</sup>The case  $M^2 = a^2$ , an extremal Kerr black hole, is just the limiting case of what we will consider, additionally it is unstable and would thus in a realistic situation quickly change to become of the nonextremal form. It is perhaps interesting to mention that in the case of an extremal Kerr black hole in fact all the mass of the black hole arises due to its angular momentum; inspection of (3.16) and below shows this explicitly, so until then just keep this in mind.

The coordinate singularities of the Kerr-metric in Boyer-Lindquist coordinates occur at  $\theta = 0$  and at  $\Delta = 0$ . The coordinate singularity at  $\Delta = 0$ , can be made evident upon writing  $\Delta$  as

$$\Delta = (r - r_+)(r - r_-),$$

where

$$r_{\pm} = M \pm \sqrt{M^2 - a^2},$$

making evident the singularities at  $r = r_{\pm}$ . Removing these singularities can be done by a change of coordinates to Kerr coordinates<sup>4</sup>,

$$\begin{aligned} dv &= dt + \frac{(r^2 - a^2)}{\Delta} dr \\ d\chi &= d\phi + \frac{a}{\Delta} dr, \end{aligned}$$

giving the Kerr solution in Kerr coordinates,

$$\begin{aligned} ds^2 &= - \frac{(\Delta - a^2 \sin^2 \theta)}{\Sigma} dv^2 + 2dvdr - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dvd\chi \\ &\quad - 2a \sin^2 \theta d\chi dr + \frac{(r^2 + a)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\chi^2 + \Sigma d\theta^2, \end{aligned}$$

which is clearly nonsingular when  $\Delta$  equals zero. While these points are thus not interesting as a singularity, something physically interesting does happen at these points. Completely analogous to the Schwarzschild solution which has a coordinate singularity at  $r = 2M$ , which turns out to be a surface of considerable physical interest as it is the event horizon of the black hole, the surfaces  $r = r_{\pm}$  turn out to be event horizons of a Kerr black hole as well. These will henceforth be referred to as the inner and outer event horizon, corresponding to  $r = r_-$  and  $r = r_+$  respectively. Note that this goes over smoothly into the Schwarzschild case when  $a$  approaches zero;  $r_-$  then goes to zero, and  $r_+$  approaches  $2M$  as it should.

Inside the outer event horizon, but outside the inner, the  $r$  coordinate becomes spacelike in such a way that an observer has to move in the direction of decreasing  $r$ , just like in the Schwarzschild case, as is clear by inspection of the metric. As soon as this observer passes the inner event horizon however,  $r$  becomes spacelike again, and the observer is no longer required to move towards the singularity.

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<sup>4</sup>These are the analogue of ingoing Eddington-Finkelstein coordinates for the Schwarzschild solution



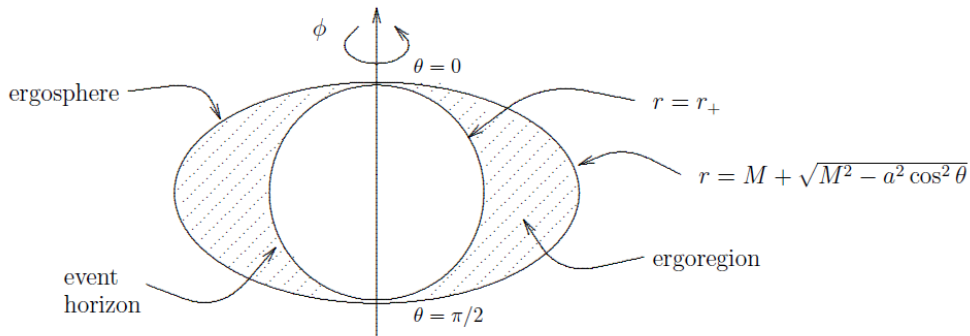


Figure 3: The ergosphere around a Kerr black hole [6]

### 3 The ergosphere

The ergosphere is a region of considerable physical interest, which starts just outside the outer event horizon of the rotating black hole. Within this region the Killing vector field  $k$  ( $\frac{\partial}{\partial t}$ ) becomes spacelike, which is a very different situation from the Schwarzschild case, where this only happens *inside* the event horizon. Of course this means another direction must have become timelike, and so as it turns out, inside the ergosphere any observer has to move with a certain minimum (nonzero) coordinate angular velocity.

One way of defining the ergosphere is the region for which the Killing vector field  $k$  is spacelike. That is, if an observer inside the ergosphere would like to follow an orbit of  $k$ , it would have to move faster than light; an observer inside the ergosphere cannot remain stationary, eventhough she/he finds himself *outside* of the event horizon. Since

$$k^2 = g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} = -\left(1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right),$$

this yields the ergosphere as the region illustrated in figure 3,

$$r_+ \leq r \leq M + \sqrt{M^2 - a^2 \cos^2 \theta}.$$

Now let us look at this nonstationarity in a bit more detail. We could already guess, since the observer is still outside the event horizon, and given the symmetry of the solution, that the nonstationarity can in some sense only involve the  $\phi$  direction. But let us make this a bit more explicit by considering the following inequality for massive observers (i.e. for the tangent vectors  $u^\alpha$  for any timelike curve)[8],

$$g_{\mu\nu}u^\mu u^\nu < 0. \quad (3.1)$$

It is easy to convince oneself that all terms on the left-hand side of equation (3.1) are manifestly positive except the term  $2g_{t\phi}u^t u^\phi = 2g_{t\phi}(dt/d\tau)(d\phi/d\tau)$ . It is also quickly verified that  $\nabla^\mu t$  is (still) past directed timelike in the ergosphere, and thus that  $dt/d\tau = u^\kappa \nabla_\kappa t > 0$  in the ergosphere. So we find that in the ergosphere, since  $g_{t\phi} < 0$  there, we must have

$$d\phi/d\tau > 0,$$

for any and all timelike curves in the ergosphere; observers in the ergosphere are forced to rotate in the direction of the black hole. This can be viewed as an extreme case of the Lense-Thirring (or frame-dragging) effect, providing a dramatic example of how some aspects of Mach's principle are incorporated in general relativity<sup>5</sup>.

To give a more quantitative result, consider an observer moving along a constant  $r, \theta$  worldline with a uniform angular velocity. Such an observer will see an unchanging space-time geometry and is thus a stationary observer. The angular velocity of such an observer, measured asymptotically, is

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t}.$$

We also know that the four-velocity of a stationary observer is proportional to a Killing vector field, giving rise to an expression which could have almost been postulated without further motivation

$$u = u^t \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) = \frac{k + \Omega m}{\|k + \Omega m\|}.$$

Of course we should insist that this four velocity is timelike;

$$g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} > 0.$$

As usual let us look at the idealized limiting case where the left-hand side vanishes. This happens for

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<sup>5</sup>Before having fully developed the theory of general relativity, Einstein already found (an example of) the Lense-Thirring effect, with which he was so satisfied he wrote the following in a letter to Mach expressing this: "it... turns out that inertia originates in a kind of interaction between bodies, quite in the sense of your considerations on Newton's pail experiment... If one rotates [a heavy shell of matter] relative to the fixed stars about an axis going through its center, a Coriolis force arises in the interior of the shell; that is, the plane of a Foucault pendulum is dragged around (with a practically unmeasurably small angular velocity)." [9]

$$\Omega = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}.$$

Now let  $\omega = -g_{t\phi}/g_{\phi\phi}$ , which will of course turn out to have an interesting interpretation. Then we have<sup>6</sup>

$$\begin{aligned}\Omega_{\min} &= \omega - \sqrt{\omega^2 - g_{tt}/g_{\phi\phi}} \\ \Omega_{\max} &= \omega + \sqrt{\omega^2 - g_{tt}/g_{\phi\phi}},\end{aligned}$$

with

$$\omega = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}.$$

The interpretation of  $\omega$  is that it is the angular velocity for stationary observers that are nonrotating with respect to local freely falling test particles, that have been dropped in radially from infinity. These observers are known as Bardeen or locally nonrotating observers[5]<sup>7</sup>. To see this, note that the angular momentum ( $J := p_\mu m^\mu$ ) of such test particles vanishes, and so for these observers  $u_\nu m^\nu = 0$ , i.e.  $(k + \Omega m)_\mu m^\mu = 0$ , which is none other than the requirement  $\Omega = \omega$ .

Finally, note that  $\Omega_{\min} = 0$  if and only if  $g_{tt} = 0$ , i.e. when  $k$  changes its nature from timelike to spacelike, exactly at the boundary of the ergosphere, as of course it should be. An observer can thus only be static (with respect to the "fixed stars") outside the static limit,  $r = r_+$ . At the static limit only lightlike observers can be static.

### 3.1 Particle trajectories

While solving the full geodesic equations can be extremely complicated, we can analyze the radial trajectories of test particles around a Kerr black hole with relative ease. This analysis is completely analogous to the way one finds an effective radial potential for a Schwarzschild black hole, except that this potential has not two but three parameters. The analysis is based on the fact that the two Killing vector fields that our metric possesses have two associated quantities that are conserved under geodesic motion. The one encountered

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<sup>6</sup>An equivalent, but more insightful definition of the ergosphere as the region for which  $\Omega_{\min} > 0$ , can now be adopted as well.

<sup>7</sup>Yet they rotate.

just above is the angular momentum,  $J$ , associated with  $m$ , while the second is the energy,  $E$ , associated with  $k$

$$J := p_\mu m^\mu = \left(1 - \frac{2Mr}{\Sigma}\right) \dot{t} + \frac{2Mar \sin^2 \theta}{\Sigma} \dot{\phi}, \quad (3.2)$$

$$E := -p_\nu k^\nu = -\frac{2Mar \sin^2 \theta}{\Sigma} \dot{t} + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta \dot{\phi}, \quad (3.3)$$

where  $\dot{x}^\mu = dx^\mu/d\tau$ . Additionally we have the normalization of the four velocity along geodesics

$$g_{\alpha\beta} u^\alpha u^\beta = \epsilon, \quad (3.4)$$

where  $\epsilon = -1, 0$  for timelike and null geodesics respectively. We now use equations (3.2) and (3.3) to eliminate  $\dot{t}$  and  $\dot{\phi}$  for  $E$  and  $J$ , which can then be substituted into equation (3.4). *For simplicity we will from now on consider motion in the equatorial plane of the black hole ( $\theta = \pi/2$ ), this yields*

$$\frac{1}{2} \dot{r}^2 + V(E, J, r) = 0 \quad (3.5)$$

for the effective potential

$$V(E, J, r) = \epsilon \frac{M}{r} + \frac{J^2}{2r^2} + \frac{1}{2}(-\epsilon - E^2) \left(1 + \frac{a^2}{r^2}\right) - \frac{M}{r^3} (J - aE)^2.$$

Thus the radial motion of a test particle with given energy and angular momentum is now reduced to a problem of nonrelativistic motion in one dimension. Several radial plots of this effective potential for various values of energy and angular momentum are given in the appendix in figure 6.

As an application of this effective potential consider the last stable circular orbit around a Kerr black hole. To find the location of this orbit, we find the location of the stable and unstable circular orbit, those being the simultaneous solutions to  $V = 0$  and  $dV/dr = 0$ . Since for the rest of this paper the massive case is the most relevant, we shall take  $\epsilon = -1$  in the following. The maximum and minimum of the effective potential occur at

$$r = \frac{(1 - E^2) a^2 + J^2 - \sqrt{(J^2 - a^2 (E^2 - 1))^2 - 12(J - aE)^2 M^2}}{2M}, \quad (3.6)$$

and

$$r = \frac{(1 - E^2) a^2 + J^2 + \sqrt{(J^2 - a^2 (E^2 - 1))^2 - 12(J - aE)^2 M^2}}{2M},$$

corresponding to an unstable and a stable circular orbit respectively. Like in the Schwarzschild case, the stable circular orbit moves away with increasing  $J$ , while the unstable circular orbit asymptotically approaches  $r = 3M$ . The last stable circular orbit then occurs when the stable and unstable circular orbit coincide, i.e. when the discriminant of the square root in equation (3.6) vanishes. This condition, being the root of a fourth order polynomial has in principle four solutions, but we are interested in the one that gives the smallest radius. This then gives the last stable circular orbit at

$$r = -\sqrt{3}aE + 3M - \sqrt{3(E^2 - 1)a^2 - 6\sqrt{3}EMa + 9M^2} := r_{\text{lsc}},$$

where  $E$  is still to be determined from the condition  $V = 0$ . This again has four possible solutions, where two turn out to be complex, and one gives rise to negative energies. This leaves us with one physical solution for  $E$  which is still a function of  $a$ . This function will not be presented here because it is not very insightful due to its length. It attains its minimal value in the limit  $a \rightarrow M$ , where we find  $E = 1/\sqrt{3}$ .

The binding energy per unit rest mass, of the last stable circular orbit is then

$$E_B = 1 - E = 1 - 1/\sqrt{3} \approx 0.42. \quad (3.7)$$

Now we should not forget that a (realistic) particle orbiting in this geometry will emit gravitational radiation. Because of this the motion will deviate slightly from geodesic motion, but the above should still provide a good estimate. A particle initially in a circular orbit with  $r \gg M$  (hence with  $E \approx 1$ ) should slowly spiral in towards the black hole as it loses energy by emitting gravitational radiation, as it comes to the radius of last stable orbit,  $r_{\text{lsc}}$ . From this point onward the orbit will really become unstable and the particle should now rapidly move past the outer event horizon. So during the time in which the particle spirals inward towards  $r = r_{\text{lsc}}$ , according to equation (3.7) about 42% of the original mass-energy of the particle will be radiated away. This is to be compared to about 6% in the Schwarzschild case. In both cases we see that, even though the emission of gravitational radiation is typically weak, in astrophysically reasonable processes large amounts of energy can be converted to gravitational radiation.

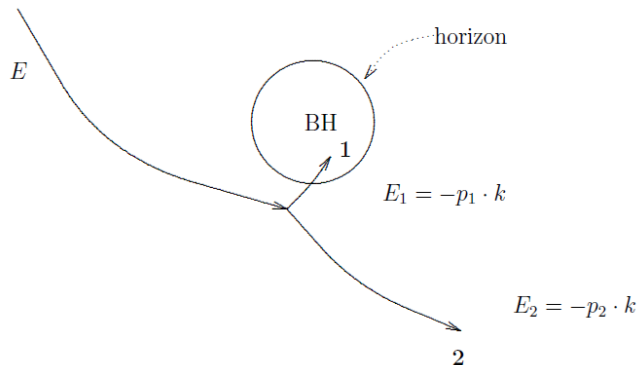


Figure 4: The decay of a particle in the Penrose process [6]

### 3.2 The Penrose Process

The Penrose process is a means of extracting energy from a rotating black hole; a very interesting consequence of the presence of an ergosphere. While the technical difficulties associated with it may still prevent an actual application in the foreseeable future, the principle remains remarkably elegant.

Suppose we have a particle that decays into two others, one of which falls into the black hole while the other escapes to infinity, illustrated in figure 4. For a decay in the ergosphere of a rotating black hole, it is possible to arrange this in such a way that the energy of the escaped particle is larger than the energy of the original particle before decay, thus providing a means of extracting energy from the black hole. Denoting the energy of the original particle by  $E$ , the energy of particle that falls in by  $E_1$  and that of the escaping one by  $E_2$ , we would normally have  $E_2 < E$  since  $E_1 > 0$  normally. In the case where the decay is in the ergosphere however, it is possible to have  $E_1 < 0$  because  $k$  is spacelike there, and so we find that we may be able to attain  $E_2 > E$  in this process, thus providing a means of extracting energy from a black hole. In the following exact analysis we will closely follow the discussion in [4].

### 3.3 Energy efficiency of the individual Penrose process

We will now analyse the constraints that arise from the conditions of conservation of momentum at the decay, that the particle has to reach the point of decay, denoted  $r_0$ , that one of the resulting particles has to fall in towards the black hole with negative energy and finally that the other has to escape to infinity. We would expect this process to depend on the distribution of the

parent particle's momentum over the decay products, especially in relation to their rest-mass, and so this will be our starting point.

Of course we have conservation of momentum,

$$p^\mu = p_1^\mu + p_2^\mu, \quad (3.8)$$

which is in this case most usefully recast in the following three relations

$$\tilde{E} = \mu_1 \tilde{E}_1 + \mu_2 \tilde{E}_2 \quad (3.9)$$

$$\tilde{J} = \mu_1 \tilde{J}_1 + \mu_2 \tilde{J}_2 \quad (3.10)$$

$$\tilde{r} = \mu_1 \tilde{r}_1 + \mu_2 \tilde{r}_2. \quad (3.11)$$

In the above we have introduced the rest-mass of the particles,  $\mu_i$ , ( $i = 1, 2$ ), where we have set the rest-mass of the parent particle,  $\mu$ , equal to one, as we expect the ratios of the masses, not the individual masses, to be the relevant parameters in this problem. This is then also the reason for introducing  $\tilde{E}$ ,  $\tilde{J}$ , and  $\tilde{r}$ ; these all have the rest-mass factored out compared to before, e.g.  $\mu \tilde{E} = E$ .

It turns out that for high efficiency of the process, which is what we are of course really interested in, it serves well to choose  $\tilde{r}_1 = 0$  at the point of split, which makes immediate intuitive sense. Now we can proceed with the analysis with the help of the effective potential,  $V$ , introduced in the previous section. With  $\tilde{r}_1 = 0$ , we have  $\tilde{r} = \mu_2 \tilde{r}_2$ . Solving this in terms of the effective potential for both particles we find the following expression for the energy of the original particle

$$\tilde{E} = \frac{\left( R\tilde{E}_1^2 - 4a\tilde{E}_1^2\tilde{J}_1^2 - (r-2)\tilde{J}_1^2 \right) \mu_1^2 + r\Delta(1-\mu_2^2)}{2\mu_1 \left( R\tilde{E}_1^2 - 2a\tilde{J}_1 \right)}, \quad (3.12)$$

where

$$R = (r(r^2 + a^2) + 2a^2).$$

The fact that the original particle came in from infinity means  $E = \tilde{E} \geq 1$ , which in turn means we can reduce equation (3.12) to the inequality,

$$\mu_1^2 \left( R\tilde{E}_1^2 - 4a\tilde{E}_1^2\tilde{J}_1^2 - (r-2)\tilde{J}_1^2 \right) + r\Delta(1-\mu_2^2) - 2\mu_1(R\tilde{E}_1 - 2a\tilde{J}_1) \geq 0.$$

This inequality can be analysed in the  $(\mu_1, \mu_2)$  -plane where the boundary is given by the equality sign above. This boundary turns out to be a hyperbola given by

$$\mu_1 = \frac{(R\tilde{E}_1 - 2a\tilde{J}_1) \pm \sqrt{(R\tilde{E}_1 - 2a\tilde{J}_1)^2 - r\Delta(1 - \mu_2^2)(R\tilde{E}_1^2 - 4a\tilde{E}_1\tilde{J}_1 - (r-2)\tilde{J}_1^2)}}{R\tilde{E}_1^2 - 4a\tilde{E}_1\tilde{J}_1 - (r-2)\tilde{J}_1^2}. \quad (3.13)$$

However by squaring the momentum conservation, equation (3.8), we also have

$$\mu_1^2 + \mu_2^2 < 1. \quad (3.14)$$

The first inequality above means that  $\mu_1$  is greater than the larger root, or less than the smaller root given in equation (3.13), combining this with the constraint (3.14), and the fact that  $\mu_1$  and  $\mu_2$  have to be greater than zero gives the allowed region for the parameters thus far. Now we have ensured that the original particle makes it to the point of decay, and that one of the decay products falls into the black hole with negative energy; it remains to make sure that the other particle bounces back from the black hole, and escapes to infinity, i.e.

$$\begin{aligned} \mu_2\tilde{E}_2 &< V_2 & \text{for } r_0 > r > r_+ \\ \mu_2\tilde{E}_2 &> V_2 & \text{for } r > r_0. \end{aligned}$$

Unfortunately the further analysis can no longer be done analytically, but numerical computations have been done that show that for  $0 \leq \mu_2 < 1$ , for small values of  $\mu_2$  the particle will not escape to infinity, while for  $\mu_2$  close to the hyperbolic boundary the particle always escapes. The key point is that there exists some critical value, call it  $\mu_{2c}$ , giving a *nonempty* region  $\mu_2 > \mu_{2c}$  for which the particle escapes to infinity [4]. Now that we have shown it is in fact possible to arrange the particle trajectories in such a way as to extract energy from the black hole, let us see what the maximum efficiency of this process is.

Taking  $E = \tilde{E} = 1$ , the four velocity of the original particle at the point of split comes out as

$$u = f(1, 0, 0, \Omega), \quad (3.15)$$

where

$$\begin{aligned} f &= -(g_{tt} + g_{t\phi}\Omega)^{-1} \\ \Omega &= \frac{-g_{t\phi}(1 + g_{tt}) + \sqrt{-(g_{tt}g_{\phi\phi} - g_{t\phi}^2)(1 + g_{tt})}}{g_{t\phi} + g_{\phi\phi}}. \end{aligned}$$



Here  $\Omega$  is the asymptotic angular velocity of the incident particle, and  $f$  is there for normalization. The angular velocities of particles 1 and 2 after the split, are as always constrained, since  $\Omega_- < \Omega_i < \Omega_+$ . Now it turns out the maximal output will be gained when letting  $\Omega_1 \rightarrow \Omega_-$  and  $\Omega_2 \rightarrow \Omega_+$ , in line with an intuitive picture of "pushing off" of the black hole.

Using momentum conservation once more, eq.(3.8), we find

$$\mu_2 \tilde{E}_2 = \left( \frac{\Omega - \Omega_-}{\Omega_+ - \Omega_-} \right) \left( \frac{g_{tt} + g_{t\phi} \Omega_+}{g_{tt} + g_{t\phi} \Omega} \right).$$

Defining then the efficiency,  $\eta$  of the process as expected (gain in energy per input of energy), we find that since  $\tilde{E} = 1$

$$\eta = \frac{\mu_2 \tilde{E}_2 - \tilde{E}}{\tilde{E}} = \mu_2 \tilde{E}_2 - 1.$$

Now in the limit where the split point tends to  $r_+$ <sup>8</sup>

$$\mu_2 \tilde{E}_2 = \frac{\sqrt{1 + g_{tt}} + 1}{2}.$$

Then looking at the case of an extreme Kerr black hole ( $a^2 = M^2$ ), we have at  $r = r_+$

$$g_{tt} = 1,$$

which then gives as the maximal efficiency of the process

$$\eta = \frac{\sqrt{2} - 1}{2} \approx 0.207.$$

In fact it turns out that having a charged black hole reduces this efficiency for a Penrose process with uncharged particles, but also that this efficiency has no limit when the particles are charged [4]; this means that a particle can come out with many times the energy the original particle had, not just 20.7% more. Regardless, we have found that in the case of a non-charged extremal Kerr black hole, the Penrose process allows for a particle to escape with roughly 20.7% more energy than the original came in with. However, this does not answer the question of how much energy can maximally be extracted from the black hole itself; this will be answered now.

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<sup>8</sup>For details see [4]

### 3.4 The maximum energy extracted from a Kerr black hole

The energy that is extracted from a Kerr black hole by means of the Penrose process can of course only come from one place: the black hole itself. Thus we expect there to be a limit to this extraction. It turns out that in line with the intuitive picture from above, it is in fact the angular momentum of the black hole that decreases, and that the limit of energy extraction lies at the point where the angular momentum of the black hole is zero. Now let us make this explicit.

Let us consider, at the event horizon, the Killing vector field

$$\xi_+ = k + \Omega_H m.$$

Since  $\xi_+$  is future-directed null and  $p$  is future-directed timelike or null on the horizon, we have

$$-p_\mu \xi_+^\mu \geq 0.$$

Hence, denoting the particles angular momentum by  $L$ , we find

$$E - \Omega_H L \geq 0.$$

Now we want to, and can have,  $E < 0$ , which thus means  $L < 0$ . Thus by having this particle fall into the black hole, we end up with a black hole with mass  $M + \delta M$ , and angular momentum  $J + \delta J$ , where  $\delta M = E$  and  $\delta J = L$ . So now we explicitly see that upon extracting energy from a black hole by means of the Penrose process, we in fact reduce its angular momentum. The relation  $\delta J \leq \delta M / \Omega_H$  can quite easily be seen to be equivalent to the following perhaps more insightful inequality

$$\delta(M^2 + \sqrt{M^4 - J^2}) \geq 0. \tag{3.16}$$

This turns out to be directly proportional to the surface area of the event horizon of the black hole, and thus we have found a special case of the second law of black hole thermodynamics (a fascinating subject which this paper will not go into), the fact that the surface area of a black hole cannot decrease in any classical process.

We can use this area to define the irreducible mass,  $M_{\text{irr}}$ , by

$$M_{\text{irr}}^2 = \frac{A}{16\pi}.$$

(The constant of proportionality relating (3.16) to  $A$  is  $8\pi$ .) Then we have that the maximum amount of energy that can be extracted from a black hole before slowing its rotation to zero is

$$M - M_{\text{irr}} = M - \frac{1}{\sqrt{2}} \left( M^2 + \sqrt{M^4 - J^2} \right)^{(1/2)}. \quad (3.17)$$

Not unexpectedly, relatively speaking the maximum amount of energy can be extracted from an extreme Kerr black hole, where we find

$$M - M_{\text{irr}} = M \left( 1 - \frac{1}{\sqrt{2}} \right). \quad (3.18)$$

In that case we can extract approximately 29% ( $(1 - 1/\sqrt{2})$ ) of the total energy. As a perhaps interesting sidenote to put this in perspective: for an extreme Kerr black hole of solar mass, *this would be enough energy to power the earth for roughly  $10^{26}$  years at current consumption rates!*

### 3.5 Super-radiance

Analogous to how the Penrose process allows for a particle to come out of a black hole with more energy than its "parent particle" had, certain wavemodes that enter and leave the ergosphere of a black hole are amplified by doing so; this process is known as superradiance. For simplicity this will be demonstrated for a scalar field, with stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\partial\Phi)^2.$$

By covariant conservation of the stress-energy tensor ( $\nabla_\mu T^\mu{}_\nu = 0$ ) it follows that (here  $k$  is once again the usual Killing vector field,  $\partial/\partial t$ )

$$\nabla_\mu (T^\mu{}_\nu k^\nu) = T^{\mu\nu} \nabla_\mu k_\nu = 0,$$

so that we can consider the following conserved current,

$$j^\mu = -T^\mu{}_\nu k^\nu = -\partial^\mu \Phi k_\nu \partial^\nu \Phi + \frac{1}{2} k^\mu (\partial\Phi)^2,$$

as the energy flux vector associated with  $\Phi$ . Now what we want to do is look at a region of spacetime, with part of its boundary on the event horizon as illustrated in figure 5, and see what conservation of this current implies.

Assuming that  $\partial\Phi = 0$  at spatial infinity ( $i_0$ ), using conservation of the current defined above we find

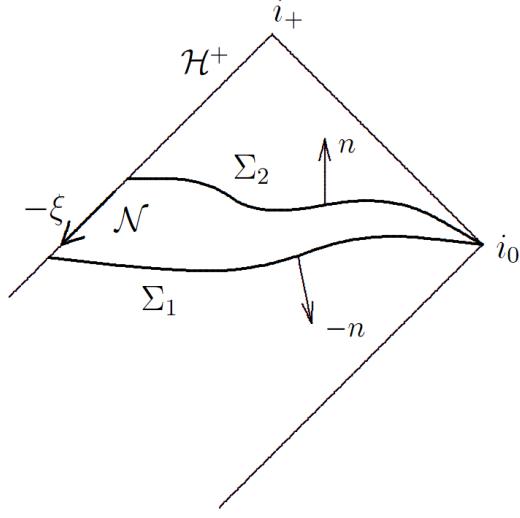


Figure 5: The region of spacetime under consideration [6]

$$\begin{aligned}
0 &= \int_S d^4x \sqrt{-g} \nabla_\mu j^\mu = \int_{\partial S} dS_\mu j^\mu \\
&= \int_{\Sigma_2} dS_\mu j^\mu - \int_{\Sigma_1} dS_\mu j^\mu - \int_{\mathcal{N}} dS_\mu j^\mu \\
&= E_2 - E_1 - \int_{\mathcal{N}} dS_\mu j^\mu,
\end{aligned}$$

where  $E_i$  stands for the energy of  $\Phi$  on the spacelike hypersurface  $\Sigma_i$ . Thus the energy flux through the horizon is given by

$$\begin{aligned}
\Delta E &= E_1 - E_2 = - \int_{\mathcal{N}} dS_\mu j^\mu \\
&= - \int_{\mathcal{N}} dA dv \xi_\mu j^\mu,
\end{aligned}$$

with  $v$  the Kerr coordinate defined earlier.

The energy flux per unit time (power) is then

$$\begin{aligned}
P &= - \int dA \xi_\mu j^\mu = \int dA (\xi^\mu \partial_\mu \Phi) (k^\mu \nabla_\mu \Phi) \\
&= \int dA \left( \frac{\partial}{\partial v} \Phi + \Omega_H \frac{\partial}{\partial \chi} \Phi \right) \left( \frac{\partial \Phi}{\partial v} \right),
\end{aligned}$$

where this used the fact that  $\xi_\mu k^\mu = 0$  on the horizon. This follows from a more general lemma for Killing vector fields on Killing horizons, however in this case it is also easily verified to be explicitly true.

Then considering a simple wavemode of frequency  $\omega$

$$\Phi = \Phi_0 \cos(\omega v - \nu\chi), \quad \nu \in \mathbb{Z} \quad (\text{angular quantum number}),$$

it is easily found that the time average lost power across the horizon is given by

$$P = \frac{1}{2} \Phi_0^2 A \omega (\omega - \nu \Omega_H),$$

with  $A$  being the area of the horizon.

Now the crucial point is that while  $P$  is positive for most values of  $\omega$ , it is in fact negative for values of  $\omega$  in the range

$$0 < \omega < \nu \Omega_H.$$

This means that a wave mode with parameters falling within this inequality is in fact *amplified* by the black hole. Note that in connection with the Penrose process this wavemode has to have a nonzero angular quantum number, as it has to take away angular momentum from the black hole.

This process is in fact quite similar to stimulated emission in atomic physics, which suggests it might be possible to have spontaneous emission, and in fact it can be shown to occur in the quantum theory, which implies any black hole with an ergoregion cannot be stable quantum mechanically [6]. Furthermore this calculation neglected the backreaction of the scalar field on the metric. Upon incorporating this, the metric remains stationary only if  $\partial\Phi/\partial v = 0$ , but then  $j^\mu = 0$  and the black hole energy remains the same[6]. So stationarity is strictly speaking incompatible with super-radiance, which is in a way to be expected.

This phenomenon of super-radiance is of course not limited to a scalar field, and can for example be extended to incorporate electromagnetic waves; for some more details on how the picture changes slightly in that case see e.g. Wald [8].

## 4 Conclusion

This paper has attempted to provide a relatively selfcontained account of the Kerr solution to the Einstein equations, and its interesting features. Amongst

the aspects mentioned were the singularities of the Kerr-solution and how they can be removed; by coordinate transformations, or interpreted; as a ring singularity in this case. Furthermore this paper discussed what the Kerr-solution in fact represents, and how the parameters in the solution are to be interpreted. Also the event horizons and the ergosphere were investigated, as they are interesting regions of this spacetime which give rise to a rich structure. This structure became manifest in discussing particle trajectories around a rotating black hole, and how this extra structure of two event horizons and an ergosphere, which is not present in the case of a familiar Schwarzschild black hole, leads to interesting phenomena of extracting energy from a black hole: the Penrose process and super-radiance. The appendix gives a short discussion of the still open problems of the exact physical nature of these singular (black hole) solutions of the Einstein equations, and the fact that, analogous to Birkhoff's theorem for the Schwarzschild case, the Kerr solution is in fact the unique axi-symmetric solution.

Of course many of the features discussed here could be discussed in much greater detail; we hope that this has presented a useful overview. The references should provide more details on most aspects where so desired.

## 5 Appendix

### 5.1 Cosmic censorship and uniqueness theorems

The cosmic censorship hypothesis lies at the basis of nearly all of the work on the collapse of stellar bodies and black holes. However reasonable it may seem, a concrete proof has been very elusive thus far, and without it hardly any conclusions can be drawn on non spherically symmetric black hole solutions and how they could be physically attained.

The approaches towards, and formulations of cosmic censorship are varied; it is a complicated issue to find an exact statement of the cosmic censorship hypothesis, which then still leads to the problem of verifying or contradicting such an exact statement. Geroch and Horowitz in [7] divide the problem of formulating 'the' cosmic censorship hypothesis into four different approaches: the causal, the asymptotic, the stability, and the evolutionary approach, where each has its specific merits. This is not the place for extensive complicated discussions of this nature, so let us be pragmatic and give an hint of an intuitive picture of the hypothesis and say what cosmic censorship is most likely (certain) to say about the case at hand: Kerr black holes.

In the case of Kerr black holes cosmic censorship would basically state

that the kind of (naked) singularities that the  $a^2 > M^2$  Kerr solutions would present us with, are unphysical<sup>9</sup>. The problem of exactly what type of naked singularities one would like to exclude is hard, but clearly excluding all naked singularities is hopeless, some are 'worse' than others; Hawking and Israel give some examples in their book, [7] p. 271. One quite intuitive picture, related to the case at hand, of why one might believe some form of cosmic censorship might be true, is the following.

Suppose that we have a spacetime which has a non-singular slice (physically: a time at which the spacetime is nonsingular), and that we would attempt to cause this spacetime to be nakedly singular to the future of this slice. This could be done for example by allowing a spherically symmetric cloud of negative-mass dust to collapse, which would form a negative-mass Schwarzschild solution at later times; or by letting a cloud of positive-mass dust with large angular momentum relative to its mass to collapse, which could give a Kerr solution with  $a^2 > M^2$ . However, the point is that in either of these cases the collapse has a tendency not to occur at all; in the first case because the gravitational effects of the negative mass are repulsive, and in the second case because the effective centrifugal effects are repulsive.

So while the Schwarzschild with  $m < 0$  and Kerr with  $a^2 > M^2$  solutions are nakedly singular spacetimes, there does not seem to be a viable physical mechanism which could lead to the creation of the objects represented by these spacetimes, within an otherwise non-singular spacetime[7, 2].

The analysis of collapse of non-spherically symmetric bodies towards a black hole is based on the cosmic censorship hypothesis; without it, it turns out to be impossible to conclude that such collapse will lead to a black hole. The cosmic censorship hypothesis is also assumed in further work, including the proof that the Kerr-solution is in fact the unique axisymmetric vacuum spacetime described by the asymptotic parameters  $M$  and  $J$ . This statement is known as the Carter-Robinson (or Robinson) theorem [6]:

**Theorem 5.1.** (Carter-Robinson Theorem) *If  $(M,g)$  is an asymptotically-flat stationary and axi-symmetric vacuum spacetime that is non-singular on and outside an event horizon, then  $(M,g)$  is a member of the two-parameter Kerr family. The parameters are the mass  $M$  and angular momentum  $J$ .*

For the exact conditions (and/or direct references to the original work) on the spacetime and some of the logical structure of the arguments we refer the reader to [7], from p. 359 onwards. This theorem thus contains the

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<sup>9</sup>"Spacetimes which, [in some sense which will not be made precise here], have the property that certain observers can detect that their spacetime is singular (i.e. can directly perceive the singular character) are said to be nakedly singular." [7]

more well known Birkhoff's theorem, which states that Schwarzschild is the unique stationary, spherically symmetric (hence actually static) solution to the Einstein equations. In short we see that in the axi-symmetric case (or in a more pragmatic sense, approximately axi-symmetric astrophysical cases) we have a unique solution, which should therefore provide a good, and *unique*, approximation to approximately axi-symmetric black holes in our physical universe. Hence this provides us with a concrete unambiguous theoretical model to lie at the basis of searches for astrophysical black holes.

Note that such a proof has not been found in the Kerr-Newman case; there the strongest result that is available at present is a specific no-hair theorem, which states that such a solution can be uniquely described by the three external parameters ( $M$ ,  $J$ , and  $e$ ), and thus that under a continuous variation of these parameters, the continuous variation of the solution remains a solution. However, a proof that the known solution (Kerr-Newman) is unique has not been found thus far.

## 5.2 Effective potentials around a Kerr black hole

Fig.(6) below is an illustration of the effective potential as first presented in section 3.1. These potentials allow for simple analysis of the radial motion of particles in the equatorial plane, reducing that problem to a problem in one-dimensional dynamics. The potentials can thus be read and analyzed as is familiar from Newtonian dynamics, although the potential itself is clearly not the Newtonian one. The potential depends on the two conserved charges  $E$  and  $J$ , the energy and angular momentum of the particle respectively. The plots cover a qualitatively representative range of these quantities and how they affect the potential.



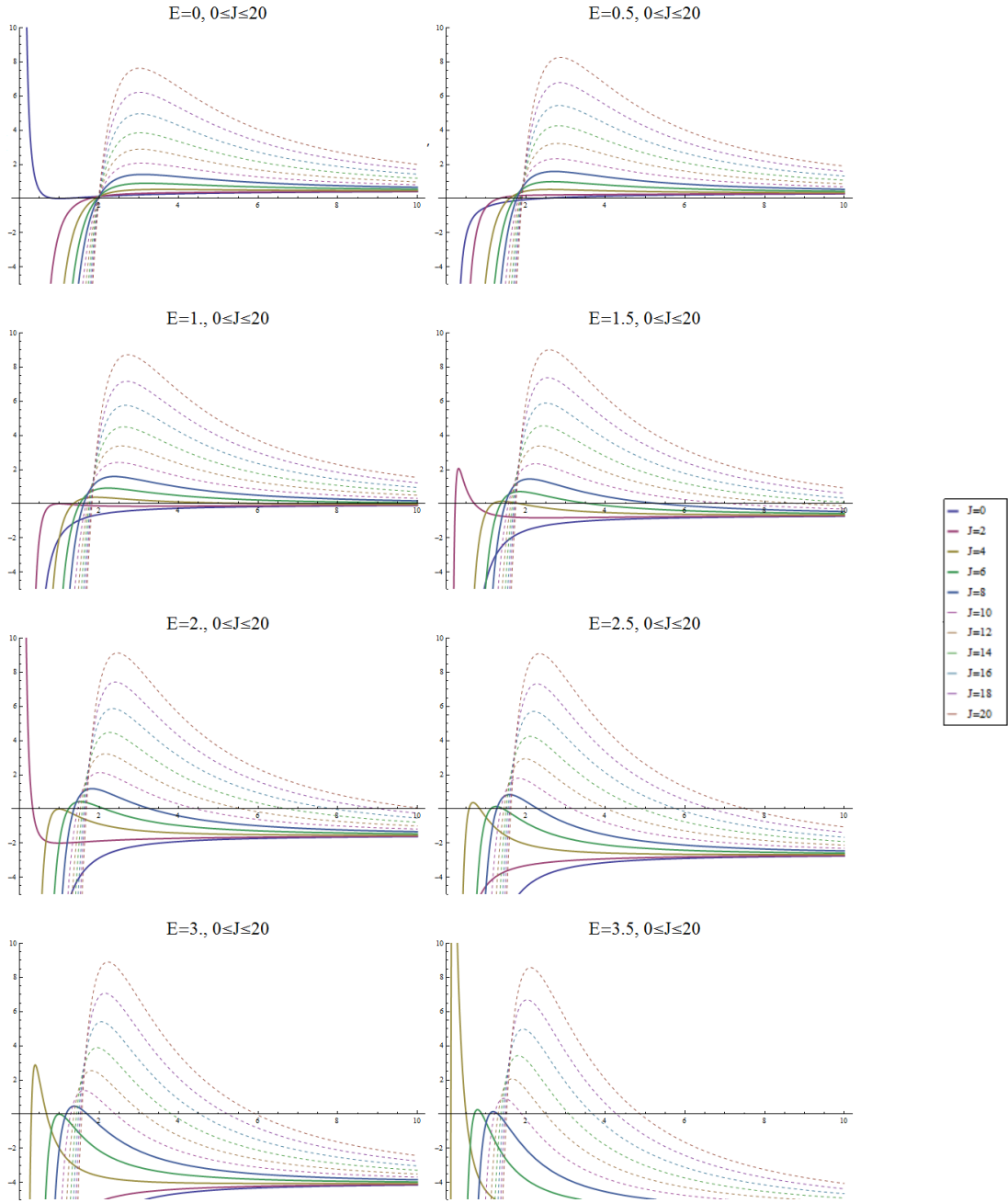


Figure 6: The effective potential in the equatorial plane for  $M = a = 1$  and various energies and angular momenta, as labelled.

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