Problems and results on number theoretic

properties of consecutive integers and related questions

P. Erdős

Recently two old problems on consecutive integers were settled. An old conjecture of Catalan stated that 8 and 9 are the only consecutive powers. Tijdeman just proved that for $n > 10^{10^{500}}$ it is impossible to have $n = x^{\ell}$, $n + 1 = y^{s}$ if $\ell > 1$, $s > 1$.

Another old conjecture stated that the product of consecutive integers is never a power. This conjecture was proved by Selfridge and myself [5].

I hope to convince the reader that many interesting unsolved problems remain and in fact almost all the problems are unsolved.

Denote by $P(m)$ the greatest prime factor of m and by $p(m)$ the least prime factor of m. De Bruijn and others [1] determined the density of integers for which $P(n) < n^{\alpha}$. It seems certain that the events $P(n) < n^{\alpha}$ and $P(n + 1) < (n + 1)^{\beta}$ are independent but the proof is nowhere in sight. I can not even prove that the density $d(\alpha,\beta)$ of integers n satisfying

$$P(n) < n^{\alpha}, \; P(n + 1) < n^{\beta} \tag{1}$$

exists-in fact I can not even prove that for every $\alpha > 0$, $\beta > 0$ there are infinitely many integers satisfying (1). I will outline the proof of the following

THEOREM 1. To every $\varepsilon > 0$ and $\eta > 0$ there is a $k = k(\varepsilon,\eta)$ so that the upper density of integers n for which

PROC. FIFTH MANITOBA CONFERENCE ON
NUMERICAL MATH., 1975, pp. 25-44.

$$(2) \qquad P\left(\prod_{i=1}^{k}(n+i)\right) < n^{\frac{1}{2}-\varepsilon}$$

is less than η.

There is not the slightest doubt that Theorem 1 remains true if in (2) $n^{\frac{1}{2}-\varepsilon}$ is replaced by $n^{1-\varepsilon}$.

Perhaps the proof of this conjecture is not hard and I overlook the obvious. There is no doubt that the density of integers satisfying (2) exists but I can not prove this, which explains the use of upper density in the theorem.

It is a simple exercise to prove that the density of integers with $p(n) = p_\ell$ is $\frac{1}{p_\ell} \prod_{i=1}^{\ell} (1 - \frac{1}{p_i})$.

Put

$$L(n,k) = \max_{1 \le i \le k} p(n+i) \quad .$$

It is again a simple exercise to prove that for every k and ℓ the density $\alpha(k,\ell)$ of integers n with

$$(3) \qquad L(n,k) = p_\ell$$

exists. On the other hand it is a very difficult problem to determine the least ℓ for which $\alpha(k,\ell) > 0$. Brun's method easily gives that $\alpha(k,\ell) > 0$ implies $\ell > k^c$ for a certain $c > 0$ and Rosser proved [13] that $\ell > k^{\frac{1}{2}-\varepsilon}$ for every $\varepsilon > 0$ if $k > k_0(\varepsilon)$. Probably in fact $\alpha(k,\ell) > 0$ implies $\ell > k^{1-\varepsilon}$. A result of Rankin [15] implies that $\ell < \frac{ck(\log\log\log k)^2}{\log k \log\log k \log\log\log\log k}$. This problem is intimately connected with the difference of consecutive primes and is of course enormously difficult.

On the other hand from the theorem of Mertens we obtain by a simple sieve process that if $k \to \infty$ then the density of integers for which

$$L(n,k) \leq e^{ck}$$

is $(1 + o(1))\,(1 - e^{-e^{\gamma/c}})$ where γ is Euler's constant. We do not give the details.

A well known deep result states that

$$P(n(n + 1)) > c \text{ loglog } n \qquad \text{(see [17])} \tag{4}$$

In fact more generally for every irreducible polynomial $f(x)$ of degree greater than one $P(f(n)) > c$ loglog n. The estimate (4) is very far from being best possible. On very flimsy probabilistic grounds I conjecture that for every $\varepsilon > 0$ and infinitely many n

$$P(n(n + 1)) < (\log n)^{2+\varepsilon} \quad ,$$

but for every $n > n_0(\varepsilon)$

$$P(n(n + 1)) > (\log n)^{2-\varepsilon} \quad .$$

The basis of these conjectures is that by a result of de Bruijn [1] the number $A_\alpha(x)$ of integers $m < x$, $P(m) < (\log m)^\alpha$ satisfies

$$\log A_\alpha(X) = (1 + o(1))(1 - \frac{1}{\alpha}) \log x \quad . \tag{5}$$

If the integer m satisfying $P(m) < (\log m)^\alpha$ were distributed at random our conjecture would follow.

It has been conjectured by Surányi that the only non-trivial solution of

$$n! = a!\; b! \tag{6}$$

is $10! = 7!6!$ and Hickerson conjectured that the only non-trivial solutions of

$$(7) \qquad n! = \prod_{i=1}^{r} a_i! \quad ,$$

are $10! = 7!6! = 7!5!3!$, $9! = 7!3!3!2!$, $16! = 14!5!2!$ A solution of (7) is trivial if $n = a + 1 = b!$ and of (7) if $a_1 = \prod_{i=2}^{r} a_i - 1$, $n = a_1 + 1$. Hickerson observed that (7) has no other non-trivial solutions for $n \leq 410$. Later I will outline the proof of

THEOREM 2. Assume $P(n(n - 1)) > 4 \log n$. Then if $n > n_0$ (6) and (7) have only trivial solutions.

More generally one could conjecture that for $k_1 > 3$, $k_2 > 3$, $n_1 + k_1 \leq n_2$ the equation

$$(8) \qquad \prod_{i=1}^{k_1} (n_1 + i) = \prod_{j=1}^{k_2} (n_2 + j)$$

has only a finite number of solutions.

A common generalization of (8) and of our theorem with Selfridge would state as follows: For every r there is a k so that

$$(9) \qquad \prod_{i=1}^{r} \prod_{j=1}^{k_j} (n_i + j) = x^{\ell}, \; k_j \geq k, \; n_{i+1} \geq n_i + k_j$$

has no solution in positive integers.

If true (9) is no doubt very deep. I can not even prove it for $r = \ell = 2$. It also seems likely that for every rational r the number of solutions (in n_1, n_2, k_1, k_2) of

$$\prod_{j=1}^{k_1} (n_1 + j) = r \prod_{j=1}^{k_2} (n_2 + j),\ n_2 \geq n_1 + k_1,\ \min(k_1,k_2) \geq 3$$

is finite.

I further conjectured that for $\min(k_1,k_2) \geq 3$ the two integers

$$\prod_{j=1}^{k_1} (n_1 + j) \text{ and } \prod_{j=1}^{k_2} (n_2 + j),\ n_2 \geq n_1 + k_1 \tag{10}$$

can have the same prime factors only finitely often. For $k_1 \leq 5$ and $\ell_2 \leq 5$ Tijdeman found many examples where the two integers have the same prime factors. e.g. 19 20 21 22 and 54 55 56 57. I am much less sure of my conjecture now. Perhaps if the two integers in (10) have the same prime factors then $n_2 > 2(n_1 + k_1)$. I am very far from being able to show this and cannot even prove that if n_1 and n_2 are between two consecutive primes then the two integers (10) cannot have the same prime factors.

Is it true that for infinitely many r there are two integers $p_r < n_1 < n_2 < p_{r+1}$ satisfying $P(n_1) < p_{r+1} - p_r$, $P(n_2) < p_{r+1} - p_r$? I expect that the answer is affirmative but that there are very few r's with this property. In fact it is not hard to prove that the density of r's for which there is an n with $P(n) < p_{r+1} - p_r$, $p_r < n < p_{r+1}$ is 0, but $n = 2^k$ shows that there are infinitely many such r's.

A well known theorem of Sylvester and Schur States that $P(\binom{n}{k}) > k$ for $n \geq 2k$[6]. I proved [7]: for sufficiently small c_1

$$P(\tbinom{n}{k}) > \min(n - k + 1, c_1 k \log k) \ . \tag{11}$$

These is no doubt that (11) can be replaced by

$$P(\tbinom{n}{k}) > \min(n - k + 1, c_2 k^{1+c_3}) \ . \tag{12}$$

In some recent papers K. Ramachandra [14] made considerable progress towards proving (12). The final truth probably is

$$P\left(\binom{n}{k}\right) > \min(n - k + 1, e^{ck^{\frac{1}{2}}}) \tag{13}$$

or perhaps only $e^{k^{\frac{1}{2}-\varepsilon}}$. In any case I do not believe that (13) will be decided in the foreseable future since it is intimately connected with sharp estimates of the difference between consecutive primes. Here I call attention to the well-known conjecture of Cramer

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\log n)^2} = 1 \quad . \tag{14}$$

This is clearly hopeless with the techniques which are at our disposal at present (and perhaps for the next few hundred or thousand years). Put

$$A(m;p_1,\ldots,p_r) = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad \text{where} \quad p_i^{\alpha_i} \| m \quad .$$

Mahler proved that for every r, k and $\varepsilon > 0$ if $n > n_0(r,k,\varepsilon)$ then

$$A\left(\prod_{j=1}^{k} (n + j);\ p_1,\ldots,p_r\right) < n^{1+\varepsilon} \quad . \tag{15}$$

The proof of (15) uses the p-adic Thue-Siegel Theorem and is not effective. It would be very desirable to make (15) effective and also to replace ε by a function tending to 0 as $n \to \infty$. Of course, there are limits to such an improvement. It is easy to see that for infinitely many n

$$A(n(n + 1);\ 2,3) > c\, n\log n$$

and I expect that

(16) $$\limsup_{n=\infty} A(n(n+1);\, 2,3)/_{n\log n} = \infty \ .$$

Perhaps the proof of (16) will not be very difficult but at the moment I do not have a proof.

Let $n = \prod_i p_i^{\alpha_i}$, $S_r(n) = \prod_{\alpha_i \leq r} p_i^{\alpha_i}$, $Q_{r+1}(n) = n/S_r(n)$. S. Golomb called the integers with $n = Q_h(n)$ powerful numbers and stated many interesting problems about them [12]. Denote the integers with $n = Q_r(n)$ r-powerful. As far as I know an asymptotic formula for the r-powerful numbers was first proved in [8]. Denote by $u_1^{(r)} < u_2^{(r)} < \cdots$ the sequence of r-powerful numbers (all their prime factors occur with an exponent $\geq r$). I asked Mahler nearly fourty years ago: Is it true $u_{k+1}^{(2)} = u_k^{(2)} + 1$ for infinitely many k i.e. are there infinitely many consecutive powerful numbers. Mahler immediately answered yes. $x^2 - 8y^2 = 1$ has infinitely many solutions and these give two consecutive powerful numbers. Is it true that the number of solutions of $u_{k+1}^{(2)} - u_k^{(2)} = 1$, $u_r < x$ is less than $(\log x)^c$? Are there infinitely many solutions which do not come from a Pellian equations? For many interesting problems on the differnece of powerful numbers, see [12].

It is very probable that $u_{k+1}^{(3)} - u_k^{(3)} = 1$ has no solutions and that in fact $u_{k+1}^{(3)} - u_k^{(3)} > (u_r^{(3)})^c$ for a certain $c > 0$. It is very hard to estimate $u_{k+2}^{(2)} - u_k^{(2)}$. I believe that $u_{k+2}^{(2)} - u_k^{(2)} > (u_k^{(2)})^c$ for some $c > 0$ but can not even prove that $u_{k+3}^{(2)} - u_k^{(2)} = 3$ does not have infinitely many solutions. Trivially $u_{k+4}^{(2)} - u_k^{(2)} > 4$ since among four consecutive integers exactly one is twice an odd number and hence is not powerful. I have only such trivial lower estimations for $u_{k+i}^{(2)} - u_k^{(2)}$. Put

$$\alpha_i = \liminf_{n=\infty}(u_{n+i}^{(2)} - u_n^{(2)}) \big/ \log u_n^{(2)}$$

As stated $\alpha_1 = 0$ but perhaps $\alpha_2 > 0$ and maybe $\alpha_2 = \alpha_3 = \cdots$.

The following problem seems very interesting to me: Is it true that for every $\varepsilon > 0$ and r if $n > n_0(\varepsilon, r)$ then

$$Q_2\left(\prod_{i=1}^{\ell} (n+i)\right) < n^{2+\varepsilon} \quad . \tag{16}$$

If (16) is true then it seems very difficult to prove. By Mahler's result for every $\ell > 2$

$$\limsup Q_2\left(\prod_{i=1}^{\ell} (n+i)\right) \Big/ n^2 \geq 1 \tag{17}$$

and probably for $\ell \geq 3$ the lim sup in (17) is infinite. In the opposite direction I can not even prove that for $k > 2$

$$\lim_{n=\infty} Q_2\left(\prod_{i=1}^{\ell} (n+i)\right) \Big/ n^{\ell} = 0 \quad .$$

All these questions can of course also be asked for $r > 2$. I have no idea how large $Q_r(n(n+1))$ can become for $r > 2$. In general try to determine

$$\beta_{r,\ell} = \limsup \log Q_r\left(\prod_{i=1}^{\ell} (n+i)\right) \Big/ \log n \quad .$$

It would be of interest to determine or estimate $Q_r(2^n \pm 1)$ and $Q_r(n! \pm 1)$ for $r \geq 2$. One would expect that these numbers can be powerful for only a finite number of n.

It is well known that there are infinitely many triples of squares in an arithmetic progression, but four squares never form an arithmetic many quadruples of relatively prime powerful numbers which form an arithmetic progression? Relative primeness is obviously needed. In general denote by $A(r)$ the largest integer for which there are $A(r)$ relatively prime r-powerful numbers in arithmetic progression and by $A^{\infty}(r)$ the largest integer for which infinitely many sets of $A^{\infty}(r)$ relatively prime r-powerful numbers are in arithmetic progression. On rather flimsy probabilistic grounds I conjecture $A^{\infty}(r) = 0$ for $r \geq 4$, but $A^{\infty}(3) = 3$ [9]. Incidentally is it true that $2u_i = u_{i+1} + u_{i-1}$ has only a finite number of solutions where the u_i are consecutive powerful numbers?

Again on flimsy probabilistic grounds I conjecture that $u_i^{(3)} + u_j^{(3)} = u_\ell^{(3)}$ has infinitely many solutions $(u_i,u_j,u_\ell) = 1$ but $u_i^{(4)} + u_j^{(4)} = u_\ell^{(4)}$ has no solutions (or at most finitely many solutions). More generally it is probably true that the sum of $r - 2$ r-powerful numbers is never (or at most finitely often) r-powerful. A famous conjecture of Euler stated that the sum of $k - 1$ k-th powers is never a k-th power. This has been disproved a few years ago for $k = 5$. Lander and Parkin proved (Math. Comp. 21 (1967), 101-103) that $27^5 + 84^5 + 110^5 + 133^5 = 144^5$. R.B. Killgrove found in 1964 that $1176^2 + 49^3 = 35^4$ (R.B. Killgrove, The sum of two powers is a third, sometimes).

Imitating Hardy and Littlewood denote by $g_p(r)$ the smallest integer so that every integer is the sum of $g_p(r)$ or fewer r-powerful numbers and $G_p(r)$ is the smallest integer so that every sufficiently large integer is the sum of $G_p(r)$ or fewer r-powerful numbers. I would expect that $g_p(r)$ will in general be much smaller than $g(r)$ but that $G_p(r)$ will be close, in fact often equal, to $G(r)$. A simple counting argument[1] gives $G_p(r) > r$, in fact the lower density of the integers

[1] See comment at end of paper.

which are not the sum of r or fewer r-powerful numbers is positive. Perhaps in fact this density is 1. This is easy to see for $r = 2$ but certainly will be very difficult for $r > 2$.

Landau proved that the number of integers $n < x$ which are the sum of two squares is $(c + o(1)) \frac{x}{(\log x)^{\frac{1}{2}}}$. It seems likely that the number of integers $n < x$ which are the sum of two powerful numbers satisfies a similar asymptotic formula but as far as I know this has never been proved. It is not known whether the density of integers which are the sum of three cubes is 0. Davenport proved that the number of integers not exceeding n which are the sum of three positive cubes is $\geq n^{47/54-\varepsilon}$ and as far as I know this has never been improved [2].

Is it true that $G_p(2) = 3$? If the answer is affirmative determine the largest integer which is not the sum of three powerful numbers. Is it true that the number of solutions of $n = u_i^{(2)} + u_j^{(2)}$ is $o(n^{\varepsilon})^2$. Clearly many more problems can be formulated here, but I leave this to the reader. Denote by $f(c,x)$ the number of integers $i < x$ for which

$$u_{i+1}^{(2)} - u_i^{(2)} < c(u_i^{(2)})^{\frac{1}{2}} \quad .$$

Then it is not difficult to prove that

$$\lim_{x=\infty} \frac{f(c,x)}{x} = K(c)$$

exists and is a continuous function of c. $K(0) = 0, K(2) = 1 - \alpha$, $\alpha > 0$. Clearly $u_{i+1}^{(2)} - u_i^{(2)} \leq 2u_i^{(2)} + 1$ with equality for infinitely many squares $u_i^{(2)} = m^2$. It would perhaps to be of some interest to calculate $K(c)$ explicitly. I do not think that this would be very difficult but I have not done so.

Denote by $h(n)$ the number of powerful numbers $u_i^{(2)}$ in $(n^2,(n+1)^2)$. It is not hard to prove that $\limsup_{n=\infty} h(n) = \infty$ and that the density d_ℓ of integers n for which $h(n) = \ell$ exists and $\sum_{k=0}^{\infty} d_\ell = 1$. It seems much harder to estimate how fast $h(n)$ can tend to infinity. I suspect that there is an α so that for all $n > n_0(\varepsilon)$, $h(n) < (\log n)^{\alpha+\varepsilon}$, but for infinitely many n, $h(n) > (\log n)^{\alpha-\varepsilon}$.

Let k be an integer. Put

$$A_k(m) = \prod_{\substack{p^\alpha \| m \\ p < k}} p^\alpha .$$

I conjectured long ago that

$$\lim_{k=\infty} \frac{1}{k} \max_{0 \leq n < \infty} \left(\min_{1 \leq r \leq k} A_k(n+r) \right) = 0 . \tag{19}$$

In other words for every $\varepsilon > 0$ and $k > k_0(\varepsilon)$ among k consecutive $n+r$, $1 \leq r \leq k$ integers there always is one for which $A_k(n+r) < \varepsilon k$. Conjecture (19) seems very difficult and I made no progress with it. More precisely: It would be of interest how fast $\max_{0 \leq n < \infty} \left(\min_{1 \leq r \leq k} A_k(n+r) \right) = f(k)$ can tend to infinity.

The following problem seems much simpler: Put

$$f(n\ ;\ k) = \left(A_k \left(\prod_{i=1}^{k} (n+i) \right) \right)^{1/k}$$

It is well known that

$$\sum_{p^\alpha < k} \frac{\log p}{p^\alpha} = \log k + c + o(1)$$

for a certain absolute constant c. It is not difficult to prove that for every $\varepsilon > 0$ and $\eta > 0$ there is a $k_0(\varepsilon,\eta)$ so that for $k > k_0(\varepsilon,\eta)$ the density of integers n for which

$$(20) \qquad (1 - \varepsilon)ck < f(n\ ;\ k)^{1/k} < (1+\varepsilon)ck$$

is greater than $1 - \eta$. The proof of (20) is by an averaging argument and second moment considerations and I hope to return to it on another occasion.

Denote by $g(n\ ;\ k)$ the number of integers $i, 1 \le i \le k$ for which $n + i \equiv 0 (\mathrm{mod}\ p^\alpha)$, $p \le k$, $p^\alpha > k$. Clearly $\max_n g(n\ ;\ k) = \pi(k)$ for every k, but it is easy to see that for $k > k_0(\varepsilon,\eta)$ the density of integers n for which

$$(1 - \varepsilon) \frac{\sqrt{K}}{\log K} < g(n,K) < (1 + \varepsilon) \frac{\sqrt{K}}{\log K}$$

is greater than $1 - \eta$. I omit the simple proof.

Before I finish the introduction (which is really the main part of the paper) I state a few miscellaneous problems and results. Put

$$\binom{n}{k} = u_k v_k$$

where all prime factors of u_k are less than or equal to k and all prime factors of v_k are greater than k. The theorem of Sylvester and Schur states that for $n \ge 2k$ $v_k > 1$ and Eggleton, Selfridge and I in a forthcoming paper hope to determine all the cases for which $u_k > v_k$. Ecklund, Selfridge and I investigated the smallest $n_k \ge 2k$ for which $u_k = 1$, our results are very far from being best possible [4]. (See also E. F. Ecklund Jr. and R.B. Eggleton, Prime factors of consecutive integers, Amer. Math. Monthly 79 (1972, 1082-1089).

Denote by $p_{n,k}$ the greatest prime factor of $\binom{n}{k}$ $(n \geq 2k)$, and by $d_{n,k}$ the greatest divisor not exceeding n of $\binom{n}{k}$. Ecklund [3] proved that if $n > 1$, $n \neq 7$ then $p_{n,k} < \frac{n}{2}$ and Selfridge and I proved $p_{n,k} < c\,\frac{n}{k^{\varepsilon}}$ and very likely if $n > k^2$, $p_{n,k} < \frac{n}{k}$ except for a finite number of exceptions. Probably $d_{n,k} > cn$ for a certain absolute constant c [16] (see also M. Faulkner, On a theorem of Sylvester and Schur, J. London Math. Soc. 41 (1966), 107-110).

Is it true that for $n > 4$, $\binom{2n}{n}$ is never squarefree? Probably for $n > n_0$ there is an odd prime p for which $p^2 \mid \binom{2n}{n}$. Probably the number of integers k, $1 < k < n$ for which $\binom{n}{k}$ is squarefree is $o(n^{\varepsilon})$. The prime factors of $\binom{2n}{n}$ where investigated in a recent paper of Graham, Ruzsa, Straus and myself [10].

Denote by $v(m)$ the number of distinct prime factors of m. Let $n = (1 + o(1))k^{1+\alpha}$. Is it true that [11]

$$v\left(\binom{n}{k}\right) = (1 + o(1))k \log(1 + \alpha) \quad ?$$

Put

$$(21) \qquad n! = \prod_{i=1}^{n} a_i,\ a_1 \leq a_2 \leq \cdots a_n \quad ,$$

where the a's are integers. It easily follows from Stirling's formula and from some elementary prime number theory that in (21) $a_1 < \frac{n}{e}\left(1 - \frac{c}{\log n}\right)$ for some $c > 0$. I conjectured that for every $\varepsilon > 0$ and $n > n_0$ $a_1 > \frac{n}{e}(1 - \varepsilon)$ is possible. Selfridge and Straus proved that for $a_1 > \frac{n}{3}$ is possible. Many further extremal problems on the representation of $n!$ as products can be stated. Let q_n be the greatest prime not exceeding n. There is a representation (21) with $a_n = q_n$. Further consider

$$n! = b_1, \dots, b_m,\ b_1 \le \cdots \le b_m \le n \quad .$$

It is not very difficult to prove that

$$\min m = n - (1 + o(1)) \frac{n}{\log n} \quad .$$

It does not seem easy to determine m very accurately. Here is another question: Put

$$n! = \prod_{i=1}^{s} d_i,\ n < d_1 < \cdots < d_s \quad .$$

Determine or estimate $\min d_s$. (s is not fixed) It is not hard to prove that $d_s \le 2n$ is impossible for $n > n_0$. $6! = 8\ 9\ 10$ and $14! = 16.\ 21.\ 22.\ 24.\ 25.\ 26.\ 27.\ 28$ shows that $d_s \le 2n$ is possible. It would not be difficult to determine all the solutions of $d_s \le 2n$ but I have not done this. It is not easy to determine $\min d_s$ but I can show $\min d_s \le 2n + o(n)$.

Now we prove Theorem 1. The proof will use Turán's method and will be very simple. Denote by $p_1 < \cdots < p_r$ the primes in the internal $(n^{\frac{1}{2}-\varepsilon}, \frac{n^{\frac{1}{2}}}{\log n})$. Denote by $f_k(m)$ the number of primes p_i, $1 < i \le r$ for which there is a t, $1 \le t \le k$, satisfying

$$m + t \equiv 0 (\bmod\ p_i) \quad .$$

To prove our theorem we only have to show that for $k > k_0(\varepsilon,\eta)$ the number of integers $m < n$ for which $f_k(m) = 0$ is less than $\eta\, n$.

We evidently have by the theorem of Mertens

$$(22)\quad \sum_{m=1}^{n} f_k(m) = k \sum_{i=1}^{r} \frac{n}{p_i} + 0(\sqrt{n}) = kn(\log \tfrac{1}{2} - \log(\tfrac{1}{2} - \varepsilon)) + o(1)$$
$$= kc_\varepsilon n + o(n) \quad .$$

Further we have by a simple argument

$$(23) \qquad \sum_{m=1}^{n} f_k(m)^2 = \sum_{1\le i<j\le r} h(n\ ;\ p_i,p_j) + k \sum_{1\le i\le r} \frac{n}{p_i} + O(\sqrt{n})$$

where $h(n\ ;\ p_i,p_j)$ denotes the number of integers u satisfying,

$$(24) \qquad 1 \le u \le n,\ u + r_1 \equiv 0 (\bmod\ p_i),\ u + r_2 \equiv 0 (\bmod\ p_j)$$

for some $1 \le r_1 \le k$, $1 \le r_2 \le k$. In a complete system of residues $(\bmod\ p_i p_j)$ the number of u's satisfying (24) is clearly k^2. Thus since $p_i p_1 < \frac{n}{(\log n)^2} = o(n)$ we have

$$(25) \qquad h(n\ ;\ p_i,p_j) = k^2 \frac{n}{p_i p_j} + o\left(\frac{n}{p_i p_j}\right) \quad .$$

thus from (23) and (25)

$$(26) \qquad \sum_{m=1}^{n} f_k(m)^2 = k^2 c_\varepsilon^2 n + k c_\varepsilon n + o(n) \quad .$$

From (22) and (26) we obtain

$$(27) \qquad \sum_{m=1}^{n} (f_k(m) - kc_\varepsilon)^2 = \sum_{m=1}^{n} f_k(m)^2 - 2kc_\varepsilon \sum_{m=1}^{n} f_k(m) + nk^2 c_\varepsilon^2 =$$

$$kc_\varepsilon n + o(n) \quad .$$

Equation (27) immediately implies Theorem 1. If (2) holds then $f_k(m) = 0$ and the number of these integers is by (27) less than $(1 + \sigma(1)) \frac{nc_\varepsilon}{k}$ which proves our theorem.

I had difficulties with extending the theorem for the primes greater than $n^{\frac{1}{2}}$ since if $p_1 p_j \neq o(n)$ (25), (26) and (27) will present difficulties.

To prove Theorem 2 we need the following Lemma. Assume $n! \equiv 0 (\mathrm{mod}\ a_1! a_2!)$. Then $a_1 + a_2 < n + 3 \log n$.

The Lemma was a problem of mine in Elemente der Mathematik 1968, 111-113 It follows easily from the fact that if $a_1 + a_2 \geq n + 3 \log n$ then $a_1!\, a_2!$ is divisible by a higher power of 2 than $n!$

Assume that

$$(28) \qquad n! = \prod_{i=1}^{k} a_i,\ n - 2 \geq a_1 \geq \cdots \geq a_k \geq 2 \quad .$$

Let q_n be the greatest prime not exceeding n. Equation (28) clearly implies $a_1 \geq q_n$. From the prime number theorem $a_1 \geq n - o(n)$. Further (28) clearly implies by (28) and our assumption $P(n(n-1)) > 4 \log n$, (in (Π' the product runs over $1 \leq t \leq n - a_1$)

$$(29) \qquad a_2 \geq P(\Pi'(a_1 + t)) > 4 \log n \quad .$$

To prove Theorem 2 assume first assume first $a_1 \geq n - \log n$. By (29) and our Lemma it immediately follows that (28) cannot hold.

Assume next $a_1 \leq n - \log n$. Then $a_1 = n + o(n)$ and (11) implies

$$a_2 > c_1(n - a_1) \log(n - a_1) > c_2(n - a_1) \operatorname{loglog} n$$

and hence by our Lemma (28) cannot hold for $n > n_0$.

The proof of Theorem 2 could be improved in many ways. The prime number theorem is not needed and n_0 could be determined explicitely. We did not attempt this in view of the fact that the assumption $P(n(n - 1) > 4 \log n$ can certainly not be justified by the methods which are at our disposal at present.

Finally we outline the proof of

Theorem 3. Let $n > n_0(\varepsilon)$. Then

$$n! = \prod_{i=1}^{t} (m + i) \tag{30}$$

has no solutions for $m < (2 - \varepsilon)^n$.

It is very likely that the only no non-trivial $(t > 1)$ solution of (30) is $6! = 8.\ 9.\ 10.$

It is easy to see by trial and error that (30) has no other small solution (say for $n < 1000$). If (30) is solvable there can be no prime p satisfying $m < p \leq m + t$. Thus by Stirling's formula and $p_{r+1} < \frac{5}{4} p_r$ for $p_r > 29$ it is easy to see that we can assume

$$m > 4n, \ t < m - \frac{n}{\log n} \ . \tag{31}$$

Legendre's formula easily gives

$$2^{\alpha_2} \geq \frac{2^n}{n+1}, \ 2^{\beta_2} \leq (m + t)2^t/\log 2 \tag{32}$$

where $2^{\alpha_2} \| n!$, $2^{\beta_2} \| \prod_{i=1}^{t} (m + i)$. Thus if (30) holds

$$2^t(m + t)(n + 1)/\log 2 \geq 2^n \ . \tag{33}$$

(33) and (31) easily implies Theorem 3, we suppress the details.

With a little more trouble I can prove that if $n \neq 6$ then (30) has no solutions for $m < \frac{2^n}{n^3}$. At present I do not quite see how to prove that (30) has no solutions for $m \leq 2^n$. I cannot believe that this will be very hard. The following question just occured to me: Let $u_k \geq 1$ be the smallest integer for which $\prod_{i=0}^{k-1} (u_k + i) \equiv 0 (\mathrm{mod}\ n!)$

Clearly $1 = u_n \leq u_{n-1} \leq \cdots \leq u_1 = n! - 1$. The study of this sequence might lead to interesting problems.

At the meeting in Oberwolfach on number theory Nov. 2-8, 1975, Schinzel informed me that he also considered $G_p(r)$ and $g_p(r)$. Ivič made some numerical explorations and on the basis of this, now conjectures that every $n > 119$ is the sum of three powerful numbers. Schinzel also pointed out to me that I made a numerical mistake and $G_p(r) > r$ does not follow by a simple counting argument and in fact is open at present.

REFERENCES

1. N. G. de Bruijn, On the number of positive integers $\leq x$ and free of prime factors > y, Nederl. Akad. Wetcnsh Proc. 59 (1951), 50-60 (see also Indigationes Math).
2. H. Davenport, Sums of three positive cubes, J. London Math. Soc. 25 (1950), 339-343.
3. E. F. Ecklund Jr., On prime divisors of the binomial coefficient, Pacific J. Math. (29) (1969), 267-270.
4. E. F. Ecklund Jr., P. Erdös and J. L. Selfridge, A new function associated with the prime factors of $\binom{n}{k}$, Math. of Computation, 28 (1974), 647-649.
5. P. Erdös and J. L. Selfridge, The product of consecutive integers is never a power, Illinois J. Math. 19 (1975), 292-301. Tijdeman's paper will appear soon.
6. P. Erdös, A theorem of Sylvester and Schur, J. London Math. Soc. 9 (1934), 282-288.
7. P. Erdös, On consecutive integers, Nieuw Arch. Weskunde, 3 (1955), 124-128.
8. P. Erdös and G. Szekeres, Uber die Annahl der Abėlschen Gruppen gegebener ordnung und über ein verwandtes zahlentheoretishes Problem, Acta Sci Math. Szeged, 7 (1934), 94-103.
9. P. Erdös and S. Ulam, Some probabilistic remarks on Fermat's last theorem, Rocky Mountain J. Math. 1 (1971), 613-616.
10. P. Erdös, R. L. Graham, I. Z. Ruzsa and E. G. Straus, On the prime factors of $\binom{2n}{n}$, Math. Comp. 29 (1975), 83-92.

11. P. Erdős, Über die Anzahl der Prumfaktoren von $\binom{n}{k}$, Archiv der Math. 24 (1973), 53-56.

12. S. Golomb, Powerful numbers, Amer. Math. Monthly 77 (1970), 848-852.

13. Halberstam and Richert, Sieve Methods.

14. K. Ramachandra, Note on numbers with a large prime factor, J. London Math. Soc. 1 (ser 2) (1969), 303-306, see also Acta. Arith 19 (1971), 49-62.

15. R. A. Rankin, The difference between consecutive prime numbers, J. London Math. Soc. 13 (1938), 242-247.

16. A Schinzel, Sur un problème de P. Erdős, Colloq. Math. 5 (1958), 198-204.

17. S.V. Kotov, The greatest prime factor of a polynomial, (Russian) Mat. Zametki 13 (1973), 515-522.