PI MU EPSILON JOURNAL

VOLUME 10 SPRING 1997 NUMBER 6

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DEDICATED TO J. SUTHERLAND FRAME

(continued on inside back cover)

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PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF THE

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The **PI MU EPSILON JOURNAL** is published twice a year - Fall and Spring. One volume consists of five years (ten issues).

This issue of the

PI MU EPSILON JOURNAL

is dedicated to

J. Sutherland Frame

1907 - 1997

A Personal Reflection on J. Sutherland Frame
by Eileen L. Poiani, Saint Peter's College

Dr. J. Sutherland Frame - quintessential research mathematician and professor; the humanizer of mathematics. It was Dr. Frame who, at every opportunity, recognized and challenged all Pi Mu Epsilon students, whether from a large prestigious university or a small local college. As he wrote in a letter to me in August 1993: "Almost all of our future research mathematicians start their lives of discovery as students and should be encouraged to present their ideas in a public forum."

Throughout my years as Pi Mu Epsilon Councilor, beginning in 1972, and as president, Dr. Frame was always there to encourage and to mentor. He was responsible for encouraging many of us to remain an active part of the PME network. How pleased he was when, on the 75th anniversary of PME, the American Mathematical Society gave special recognition to PME, as the Mathematical Association of America had done earlier.

The writings of great communicators invite pleasant reading and rereading. Always a great communicator, Dr. Frame leaves us with a legacy of rich research articles published throughout his lifetime and with thoughtful letters often written on vintage Pi Mu Epsilon stationery and, more recently, on email.

His sharpness and precision spanned broad interests. Early this year, he even wrote to me about a Sports Illustrated article in which the President of Saint Peter's College was quoted. And just three years ago, Dr. Frame wrote and reminded me again that I knew him well enough to call him "Sud."

Attending almost every Joint and Summer Mathematics Meeting, usually with his beloved spouse, Emily, Sud must have captured the mileage record for driving to meetings. I was always amazed at the lengths of his motor journeys.

For this strong and gentle educator and friend, I - like so many in the mathematics community - have enormous admiration and gratitude. I shall continue to safeguard the C. C. MacDuffee Distinguished Service plaque awarded to Sud in 1964 and passed on to me in 1995, and to enjoy rereading those beautiful letters from a legend.

J. Sutherland Frame

DEDICATION

This issue of the *Pi Mu Epsilon Journal* is dedicated to "Sud" Frame. James Sutherland Frame, Pi Mu Epsilon President 1957 - 66 and Secretary 1951 - 54, died on February 27, 1997. He was 89 years old. He is survived by his wife of 58 years, Emily, a daughter, Barbara, three sons, Paul, Roger and Larry, and seven grandchildren.

In 1949 he was instrumental in the founding of the *Pi Mu Epsilon Journal*. He promoted the growth of Pi Mu Epsilon, personally installing more than fifty Chapters, and in creating and developing in 1952 the highly successful Pi Mu Epsilon Summer Student Paper Conferences in conjunction with the American Mathematical Society and the Mathematical Association of America. He was widely respected and known as "Dr. Pi Mu Epsilon." At the 1975 summer AMS/MAA meetings he presented the first J. Sutherland Frame Lecture, named in his honor. Professor Frame faithfully attended the student presentations, raising questions and making personal comments to most of the young members on how to improve their papers and on the future directions their research might follow. His primary interest had always been in his students, their professional growth and their eventual success.

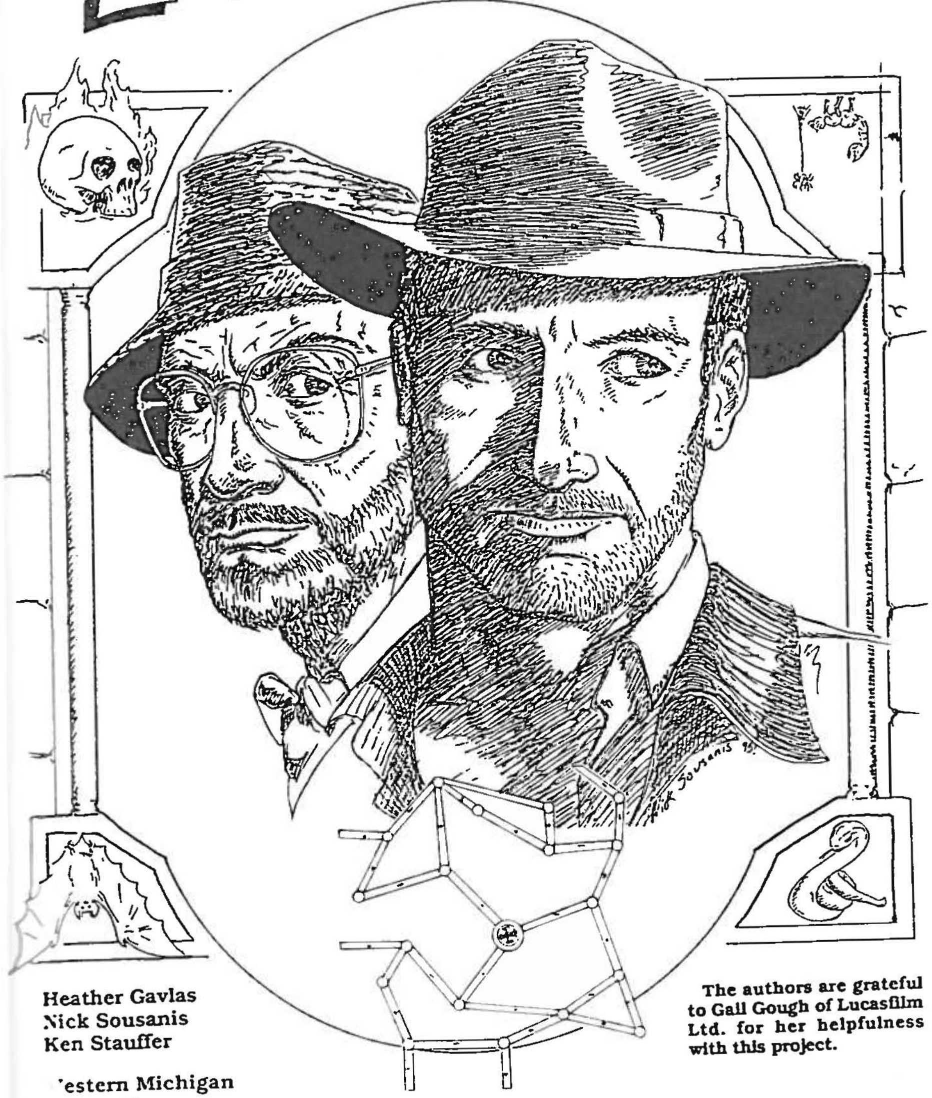
"Sud" earned his Bachelor's (1929), Master's (1930), and Doctorate (1933) at Harvard; his main research interest was in the theory of representation of finite groups, a field where he published over forty of his more than one hundred papers. He taught at Harvard University, Brown University, Allegheny College, and Michigan State University and had appointments at the Institute for Advanced Study and as a Consultant for Graduate Mathematics Programs in Thailand. He has served at the local, regional, national and international level holding positions on the Board of Governors of the MAA and as chair or member of many, many scholarly and civic organizations ranging from the Presidency of the Michigan Academy of Arts and Letters to membership on The East Lansing Board of Education and The National Council of the AAUP. Membership in Phi Beta Kappa, inclusion in Who's Who in America, and the Senior Research Award of Sigma Xi at Michigan State are among his many honors.

In 1994, J. Sutherland Frame was awarded the Yueh-Gin Gung and Dr. Charles Y. Hu Award for Distinguished Service, the highest honor given by the Mathematical Association of America. A quote (with a few modified verbs) from the award narrative, written by David Ballew, Western Illinois, in *The American Mathematical Monthly*, Volume 101, Number 2, February, 1994, is an excellent conclusion to this dedication:

"Almost every summer, "Sud" attended as many of the student paper presentations as possible. He immediately grasped the student's message and afterwards spoke to each student in a non-intimidating way to help the student gain deeper insight to improve his or her results. The students were amazed that this grand old gentleman could see to the core of the problem so quickly and then see further and clearer than they, who worked so long and so hard. As one student said to David Ballew after the 1992 Summer Conference, 'That man really loves students, doesn't he?' At the bottom, that says it all. J. Sutherland Frame simply loved students and had spent his life proving that."

"Sud" will be missed as a mathematician, as an inspiring teacher and mentor, and as a friend.

INDIANA JONES™ and the QUEST for ANTICONNECTED DIGRAPHS*



Heather Gavlas
Nick Sousanis
Ken Stauffer

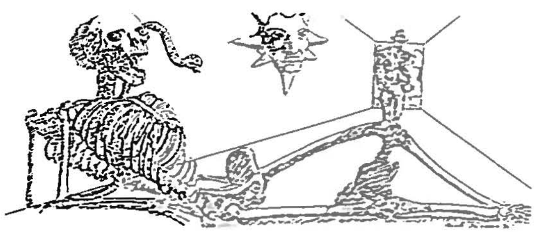
'estern Michigan
niversity

The authors are grateful to Gail Gough of Lucasfilm Ltd. for her helpfulness with this project.

Stepping carefully over the skeletal remains of a failed treasure hunter, Professor Indiana Jones made his way deeper into the ancient African temple, his goal nearly at hand. He stopped and peered at the carving on the nearby wall. Although the inscription was of a form not seen on Earth for the past two millenia, Dr. Jones was able to decipher its meaning. Reaching out for an indentation in the wall, Jones pushed and the floor gave out below him. Sliding, tumbling, plunging through the void his fall came to a jarring halt at a cold, stone floor deep beneath the temple. Looking for his extinguished torch, Indy realized the room was already filled with the light of several torches. He spotted the resting spot of his prize. Suddenly, out of the darkness behind him came a familiar voice, "About time you made your way here." Whirling about, Indiana came face to face with Professor Henry Jones, "Dad!?! What the heck are you doing here?"

"Looking for the same thing you are," the elder Jones replied. "And now that you're finally here we need to find our way out."

Indiana was accustomed to the sharp tone of his father's speech, but he knew that his dad was much more comfortable digging for relics in a museum than procuring them out in the field. The presence of his son was more of a relief than he would have cared to make known. "Well, Dad, according to the documents I brought back from Egypt, we should be at the center of a great maze. Apparently, each pathway in this labyrinth lies between rotating, stone doors. Each segment of the maze has the capability to be filled with deadly traps, but these doors trigger on/off switches. We pass through a door, and any passageways that were in the active deadly mode now become safe to cross, while those previously safe paths become unpassable. The doors only open in one direction so once we're in a section we can't turn back, which means we can't choose the wrong path at any time. "Of course, according to this," and now Indy brought forth his map of the maze, "there is only one correct sequence of paths to get us out safely."



"And I suppose you have already figured out that route before trapping yourself in here?" the older Jones queried.

"Now, Dad, you didn't even know about the maze. What was your plan to escape?" But the look on his father's face said it all. Right or wrong Indy knew it was useless to argue with Henry Jones. "All right Dad, here's the layout. We're at this spot," pointing to a large circular area in the center of the map.

The elder Jones frowned and glanced at the map. He drew out his note pad and pencil, "So what do you make of it, Junior?"

Indy had absolutely no idea of what to make of it, when inspiration hit him. "Dad, remember how we used to discuss mathematics and we talked some about the field of Graph Theory?"

"Junior, what do charts and tables have to do with the situation you've gotten us into?" Henry Jones asked his son.

"No, Dad! Graph Theory was started by Euler and has since become a subject all its own," Dr. Jones was beginning to lose his patience with his father.

"Ah yes, exactly! I remember it as if it were yesterday! Now if I could only find my notebook, we could jot down some ideas to refresh our memory."

"It's in your hand Dad," Indy announced with a smirk.

"Ah, so it is. Well no more dawdling boy, let's get to work."

Indy sighed and began, "You explained graphs as a set of vertices with edges between them, the order and size of a graph indicating the number of vertices and edges respectively."

Henry Jones began to sketch a graph, while Indy continued. They continued for some time recalling definitions and examples they had not discussed in almost twenty years. The elder Jones carefully wrote down each definition and example as they recalled them.

So, a *graph* G is a set of vertices, denoted by $V(G)$, and a set of edges, denoted by $E(G)$, between pairs of distinct vertices where at most one edge is allowed between a pair of vertices. The number of edges incident with a vertex v in G is called the *degree* of v. If ab is an edge in a graph G, then the vertices a and b are *adjacent*. For the graph G of Figure 1, the degree of vertex v_1 is 2 and vertices v_1 and v_2 are adjacent. The order (number of vertices) of G is 8 while the size (number of edges) of G is 10. A graph is *complete* if every two of its vertices are adjacent. Every vertex of a complete graph of order n has degree n - 1.

An alternating sequence of vertices and edges of a graph G beginning with a vertex u and ending with a vertex v such that every edge joins the vertex immediately preceding it to the vertex immediately following is called a *u-v walk*. If a u-v walk does not repeat any vertex, then it is called *u-v path*. A u-u walk containing at least three edges and not repeating any vertex other than u is called a *cycle*. For example, v_1, e_1, v_2, e_9, v_6, e_9, v_2 is a v_1-v_2 walk in the graph G of Figure 1 that is not a v_1-v_2 path. A v_1-v_2 path is v_1, e_1, v_2. Since the edges of a walk are completely determined by the vertices of the walk, we will list only the vertices of the walk. So, v_1,v_2,v_3,v_4,v_8,v_1 is a v_1-v_1 cycle in the graph G of Figure 1. A graph G is said to be *connected* if for every pair u,v of vertices, there is a u-v path in G. (If any such paths are absent, then G is *disconnected*.) A graph G is *hamiltonian* if a cycle exists in G containing every vertex of G. This cycle is known as an *hamiltonian cycle*. Thus the graph G of Figure 1 is connected and hamiltonian since $v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_1$ is a hamiltonian cycle.

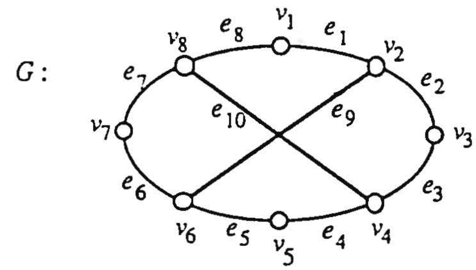


Figure 1

A *directed graph* or *digraph* is a set of vertices where connections between vertices are represented with directed edges or arcs. Two vertices u and v of a digraph D are *adjacent* if D contains at least one of the arcs

(directed edges) (u,v) and (v,u). If (u,v) is an arc of D, then u is *adjacent to* v while v is *adjacent from* u. The *underlying graph* G of a digraph D has the vertices of D as its vertex set and if an arc (u,v) is present in D, then the edge uv is present in G. A digraph D is *connected* if the underlying graph of D is connected. An *orientation* of a graph G is an assignment of directions to the edges of G. A *tournament* T is a digraph resulting from assigning directions to the edges of a complete graph.

After some time, Indy interjected, "Dad, I enjoy this, but we have only so long to survive in here. We need to make some progress towards figuring out our escape and do it quickly."

"Ah, Junior, you always were the impatient one." Henry Jones had become so excited by this rediscovery of mathematics that he too had forgotten the situation they were in. But admitting this to his son was just not his style. And so he remarked to Indy, "I think if we recall the work of our friend and colleague Branko Grünbaum we may find the clue that we will need."

"I see where you're headed Dad! From this point we can establish a great many conjectures about graphs in general that may help us get out of here in a hurry! Let's put together what we know about digraphs in general."

For vertices u and v in a digraph D, a *u-v semipath* P is an alternating sequence $u = v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k = v$ of vertices and arcs such that the vertices $v_0, v_1, \ldots, v_k$ are distinct and either $e_i = (v_{i-1}, v_i)$ or $e_i = (v_i, v_{i-1})$ for each i $(1 \le i \le k)$. If $e_i = (v_{i-1}, v_i)$ for each such i, then P is a *(directed) u-v path*, while if every pair of consecutive arcs of P are oppositely directed, then P is an *antidirected u-v path* or a *u-v antipath*.

"Grünbaum proved that, with the exception of three tournaments (see Figure 2), *every tournament has an antipath containing all vertices of the tournament (a hamiltonian antipath)," Indy remarked.*

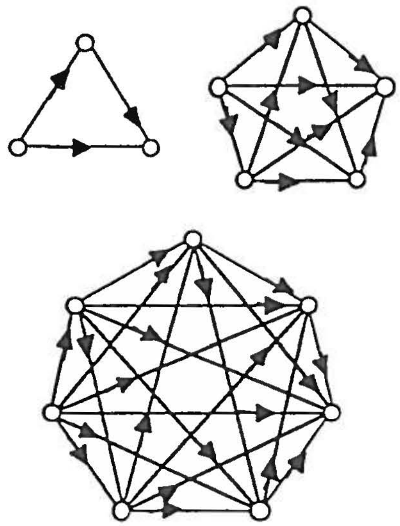


Figure 2

Henry Jones added, "A digraph is antihamiltonian if it contains a hamiltonian anticycle. It is not difficult to see, Junior, that every antihamiltonian tournament has even order. Grünbaum found two non-antihamiltonian tournaments (of orders 6 and 8) and conjectured that every tournament of even order $n \ge 10$ is antihamiltonian. The brilliant Danish graph theorist Carsten Thomassen later proved Grünbaum's conjecture for tournaments of order at least 50. The following year, Moshe Rosenfeld further decreased this bound to 28."

A digraph D is *anticonnected* if for every pair u, v of vertices of D, there exists a u-v antipath in D. If a digraph contains a u-v antipath, then it also contains a v-u antipath. Thus, if D is an anticonnected digraph, then its *converse* $\vec{D}$ (obtained by replacing every arc (x,y) with (y,x)) is also anticonnected. Recall that an *orientation* of a graph G is an assignment of

directions to the edges of G. If an orientation of a graph is anticonnected, then the orientation is referred to as an *anticonnected orientation*. In [2] it was shown that although every connected graph has an anticonnected orientation, it is not the case that *every* orientation of a connected graph is anticonnected. However, it was also shown in [2] that if G is a graph of order $n \ge 3$ such that $\deg v \ge (3n - 1)/4$ for every vertex v of G, then *every* orientation of G is anticonnected. Furthermore, this bound is sharp.

For a fixed positive integer n, the *complete symmetric digraph* K_n^* is that digraph where for every pair u,v of vertices the arcs (u,v) and (v,u) are present. Every digraph K_n^* is anticonnected. Thus if the size of a digraph D of order n is sufficiently large, then D is anticonnected.

"Okay Dad, I think we can state a formal theorem giving the precise bound now."

"Good Junior, you really must have paid attention at some time while we were studying."

Theorem 1. If D is a digraph of size $m \ge (n-1)(n-2) + 1$ and order $n \ge 3$ then D is anticonnected.

Proof. Let D be a digraph of order $n\ (\ge 3)$ and size m that is not anticonnected. Then there exist vertices u and v of D for which there is no u-v antipath. Certainly neither (u,v) nor (v,u) are arcs of D. Also for every vertex $w\ (\ne u, v)$ of D, the arcs (u,w) are (v,w) are not both present in D, and similarly (w,u) and (w,v) are not both arcs of D. Since the maximum size of D is $n(n - 1)$, it follows that $m \le n(n - 1) - 2(n - 1) = (n-1)(n - 2)$. □

Theorem 1 is best possible since for $n \ge 3$, the digraph $K_{n-1}^* \cup K_1$ is of order n and size $(n - 1)(n - 2)$, but is not anticonnected.

"Remember we have already mentioned that a connected digraph need not be anticonnected. However, some connected digraphs are closer to being anticonnected than others. Now let's look at two measures of anticonnectedness for connected digraphs."

The (anticonnected) *reversal number* $\mathrm{ar}(D)$ of a connected digraph D is the minimum number of arcs the reversal of the directions of which produces an anticonnected digraph.

Henry Jones continued, "We must first show that ar(D) *exists for every connected digraph* D *and then present a bound on this parameter. Before continuing, we'll need the following definitions."*

A *tree* is a connected graph containing no cycles. A *bipartite graph* is a graph G whose vertex set can be partitioned into two sets U and V such that every edge joins a vertex of U to a vertex of V. It is well-known (see [1], for example) that a tree of order n has size $n - 1$ and that every tree is bipartite. A *spanning tree* of a connected graph G is a tree that is a subgraph of G and has the same vertex set as G. Every connected graph has a spanning tree.

Theorem 2. For a connected digraph D of order n, the (anticonnected) reversal number $\mathrm{ar}(D) \le \lfloor (n-1)/2 \rfloor$.

Proof. Let D be a connected digraph of order p and let T be a spanning directed tree of D. Then T is bipartite with partite sets U and V. Suppose that T has m_1 arcs from U to V and m_2 arcs from V to U. Then $m = \min\{m_1, m_2\} \le \lfloor (n-1)/2 \rfloor$. Reversing the direction of these m arcs will produce an anticonnected directed tree and hence an anticonnected digraph. Thus, $\mathrm{ar}(D) \le \lfloor (n-1)/2 \rfloor$. □

"Now notice Dad, that the bound in Theorem 2 is sharp in the sense that for every positive integer n, the directed path of order n has (anticonnected) reversal number $\lfloor (n-1)/2 \rfloor$*."*

For a connected graph G, the *(anticonnected) reversal number* $\mathrm{ar}(G)$ is defined as $\mathrm{ar}(G) = \max\{\mathrm{ar}(D)\}$, where the maximum is taken over all orientations D of G. So, by Theorem 2, for a connected graph G of order n, the reversal number $\mathrm{ar}(G) \le \lfloor (n-1)/2 \rfloor$. It was already noted that the directed path of order n has reversal number $\lfloor (n-1)/2 \rfloor$. Thus, for the path P_n of order n, it follows that $\mathrm{ar}(P_n) = \lfloor (n-1)/2 \rfloor$. It was also noted that if G is a graph of order $n \ge 3$ such that $\deg v \ge (3n-1)/4$ for every vertex v of G, then every orientation of G in anticonnected. Hence $\mathrm{ar}(G) = 0$ for each such graph G.

"With what we know now we should be able to determine the reversal number of a tree." Henry Jones continued to scrawl out theoretical results.

Theorem 3. For a tree T of order $n \ge 3$, the reversal number $\mathrm{ar}(T) = \lfloor (n-1)/2 \rfloor$.

Proof. Let T be a tree of order n. Then T is bipartite with partite sets U and V. Obtain an orientation D of T by directing $\lceil (n-1)/2 \rceil$ edges of T from U to V and the remaining $\lfloor (n-1)/2 \rfloor$ edges from V to U. It will be shown that $\mathrm{ar}(D) = \lfloor (n-1)/2 \rfloor$ so that $\mathrm{ar}(T) = \lfloor (n-1)/2 \rfloor$. If the $\lfloor (n-1)/2 \rfloor$ arcs from V to U are reversed, then the resulting digraph is anticonnected. So, $\mathrm{ar}(D) \le \lfloor (n-1)/2 \rfloor$. To see that $\mathrm{ar}(D) \ge \lfloor (n-1)/2 \rfloor$, suppose, to the contrary, that $\mathrm{ar}(D) < \lfloor (n-1)/2 \rfloor$. Let D' be an anticonnected digraph obtained from D by reversing fewer than $\lfloor (n-1)/2 \rfloor$ arcs. Since there are at least $\lfloor (n-1)/2 \rfloor$ arcs from U to V and from V to U and since D' was obtained from D by reversing at most $\lfloor (n-1)/2 \rfloor - 1$ arcs, there exist arcs (u_1, v_1) and (v_2, u_2) of D' such that $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Since D' is an orientation of T, either $u_1 \ne u_2$ or $v_1 \ne v_2$. Suppose that $u_1 \ne u_2$. (The proof is similar if $v_1 \ne v_2$.) Since D' is anticonnected, there is a $u_1 - v_2$ antipath $P: u_1 = w_0, w_1, \ldots, w_k = v_2$. Also since D' is bipartite and $u_1 = w_0$, , if i $(1 < i \le k)$ is even, then $w_i \in U$ while if i $(1 \le i \le k)$ is odd, then $w_i \in V$. Thus, since $w_k = v_2$, it follows that k is odd. We consider two cases, depending on whether $u_1 = w_0$ is adjacent to or from w_1.

Case 1. Suppose that (w_0, w_1) is an arc of D'. Then for i odd $(1 \le i < k)$, (w_{i-1}, w_i) and (w_{i+1}, w_i) are arcs of D'. For $i = k$, the arc (w_{k-1}, w_k) is present in D'. Thus $w_{k-1} \ne u_2$ and furthermore u_2 is not a vertex of P for otherwise P together with (v_2, u_2) produces a cycle in T. So, P together with (v_2, u_2) is a u_1-u_2 semipath Q that is not an antipath. Since D' is an orientation of a tree, Q is the unique u_1-u_2 semipath in D'. Thus, there is no u_1-u_2 antipath in D', producing a contradiction.

Case 2. Suppose that (w_1, w_0) is an arc of D'. For i odd $(1 \le i < k)$, the arcs (w_i, w_{i-1}) and (w_i, w_{i+1}) are present in D'. When $i = k$, the arc (w_k, w_{k-1}) is present in D'. Then $w_1 \ne v_1$ and, as before, v_1 is not a vertex of P for otherwise P together with (u_1, v_1) produces a cycle in T. So, P together with (v_1, u_1) is the unique v_1-v_2 semipath Q in D'; yet Q is not an antipath. Thus D' is not anticonnected.

Therefore $\mathrm{ar}(D) \ge \lfloor (n-1)/2 \rfloor$ and thus $\mathrm{ar}(D) = \lfloor (n-1)/2 \rfloor$. Hence

$\mathrm{ar}(T) = \lfloor (n-1)/2 \rfloor$. □

Although a general lower bound for ar(G) has not been found when G is a connected graph, such a bound does exist if G has cut-vertices. Let G be a connected graph containing a cut-vertex v. Suppose that $G - v$ has k components, where $k \geq 2$, say $G_1, G_2, \ldots, G_k$. Obtain an orientation D of G by directing all edges from $V(G_i)$ to v for $1 \leq i \leq \lfloor k/2 \rfloor$, directing from v to $V(G_j)$ for $\lfloor k/2 \rfloor < j \leq k$, and directing the remaining edges of G arbitrarily.

Let v_i be a vertex of G_i for $i = 1,2,\ldots,k$. Then for $i = 1,2,\ldots,\lfloor k/2 \rfloor$ and $j = \lfloor k/2 \rfloor + 1, \ldots, k$, every $v_i - v_j$ semipath contains v. Thus at least one arc from $V(G_i)$ to v or from v to $V(G_j)$ must be reversed. Therefore, $\mathrm{ar}(D) \geq \lfloor k/2 \rfloor$ and hence $\mathrm{ar}(G) \geq \lfloor k/2 \rfloor$. Maximizing $k(G - v)$, the number of components of $G - v$, we have the following theorem.

Theorem 4. Let v be a cut-vertex of a connected graph G such that $k(G - v)$ is maximum. Then $\mathrm{ar}(G) \geq \lfloor k(G-v)/2 \rfloor$.

Indy interjected, "Let's look at another measure of anticonnectedness."

For a digraph D, the (anticonnected) *addition number* ad(D) is the minimum number of arcs that can be added to D to produce an anticonnected digraph. Certainly if every arc not present in D is added to D, the resulting digraph is anticonnected. Consequently, ad(D) is well-defined for every digraph D.

For a given connected digraph D, let D' be some anticonnected digraph obtained from D by reversing ar(D) arcs. Then the digraph D'', obtained from D by adding the ar(D) reversed arcs in D', that is $E(D'') = E(D) \cup E(D')$, is anticonnected. Therefore $\mathrm{ad}(D) \leq \mathrm{ar}(D)$. For a connected graph G, the (anticonnected) *addition number* ad(G) is defined as $\mathrm{ad}(G) = \max\{\mathrm{ad}(D)\}$, where the maximum is taken over all orientations D of G. So, since $\mathrm{ad}(D) \leq \mathrm{ar}(D)$ for every digraph D, it follows that $\mathrm{ad}(G) \leq \mathrm{ar}(G) \leq \lfloor (n-1)/2 \rfloor$.

"From this we can show that for every positive integer $n \geq 3$, there exists a connected graph G of order n such that $\mathrm{ar}(G) = \mathrm{ad}(G) = \lfloor (n-1)/2 \rfloor$."

"Obviously, Junior. Now let's write this one out."

Theorem 5. For every positive integer $n \geq 2$,

$$\mathrm{ad}(K_{1,n}) = \mathrm{ar}(K_{1,n}) = \lfloor n/2 \rfloor.$$

Proof. By Theorem 3, $\mathrm{ar}(K_{1,n}) = \lfloor n/2 \rfloor$. It remains to show that $\mathrm{ad}(K_{1,n}) = \lfloor n/2 \rfloor$. Let $G = K_{1,n}$ and let v be the vertex of degree n and $v_1, v_2, \ldots, v_n$ be the remaining vertices of G. Obtain an orientation D of G by directing the edges v_iv from v_i to v for $i = 1,2,\ldots,\lfloor n/2 \rfloor$ and the edges v_jv from v to v_j for $j = \lfloor n/2 \rfloor + 1,\ldots,n$. We show that $\mathrm{ad}(D) = \lfloor n/2 \rfloor$. Let D' be a digraph obtained from D by adding fewer than $\lfloor n/2 \rfloor$ arcs and suppose, to the contrary, that D' is anticonnected. Then there exist integers i ($1 \leq i \leq \lfloor n/2 \rfloor$) and j($\lfloor n/2 \rfloor + 1 \leq j \leq n$) such that (v_i,v) and (v,v_j) are the only arcs of D' incident to v_i and v_j respectively. Since D' is anticonnected, there exists a v_i-v_j antipath $P : v_i = w_0, w_1, \ldots, w_{k-1}, w_k = v_j$. Since $\deg v_i = \deg v_j = 1$, it follows that $w_1 = v$ and $w_{k-1} = v$, so necessarily $P : v_i, v, v_j$. Hence P is not an antipath, producing a contradiction. Therefore $\mathrm{ad}(D) \geq \lfloor n/2 \rfloor$ and hence $\mathrm{ad}(D) = \lfloor n/2 \rfloor$. Thus $\mathrm{ad}(K_{1,n}) = \lfloor n/2 \rfloor$. □

"Now Junior, we have seen that $\mathrm{ad}(D) \leq \mathrm{ar}(D)$ for every connected digraph D and that we have an infinite class of graphs for which the addition number equals the reversal number. Next let's show that the addition number and the reversal number can be arbitrarily far apart."

Theorem 6. For every positive integer k, there exists a connected digraph D such that $\mathrm{ar}(D) - \mathrm{ad}(D) \geq k$.

Proof. For $k = 1$, consider the directed tree D_1 shown in Figure 3. By the proof of Theorem 3, it follows that $\mathrm{ar}(D_1) = 3$. Let D_1' be the digraph shown in Figure 3, which is obtained from D_1 by adding the two arcs indicated in Figure 3. Since D_1' is anticonnected, $\mathrm{ad}(D_1) \leq 2$ and hence $\mathrm{ar}(D_1) - \mathrm{ad}(D_1) \geq 1$.

For $k = 2$, consider the digraph D_2 shown in Figure 4. Now D_2', also shown in Figure 4, is obtained from D_2 by adding four arcs. Since D_2' is anticonnected, $\mathrm{ad}(D_2) \leq 4$. Let $V_1 = \{v_1, v_2, v_3, v_4, v_7, v_{10}, v_{11}, v_{12}, v_{13}\}$ and let $V_2 = \{v_5, v_6, v_8, v_9\}$. Then there are six arcs from V_1 to V_2 and six arcs from V_2 to V_1. As in the proof of Theorem 3, to produce an anticonnected directed tree, we must reverse all arcs of D_2 from V_1 to V_2 or all arcs of D_2 from V_2 to V_1. Thus $\mathrm{ar}(D) = 6$ and so $\mathrm{ar}(D_2) - \mathrm{ad}(D_2) \geq 2$.

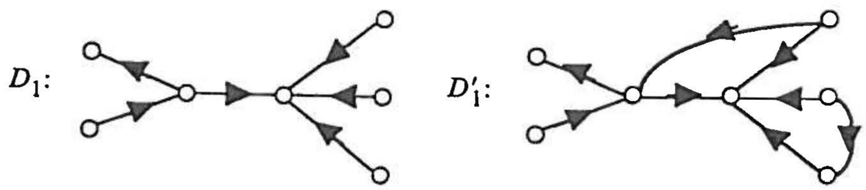


Figure 3

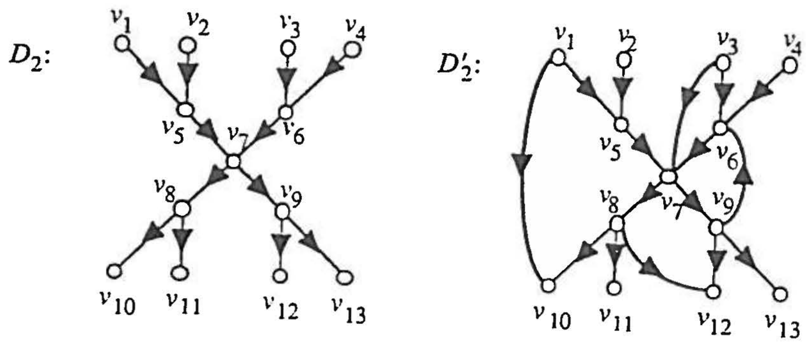


Figure 4

Finally, let $k \geq 2$ be a positive integer and consider the digraph D_k obtained from D_2 by adding the vertices $u_1, u_2, \ldots, u_{k-2}, w_1, w_2, \ldots, w_{k-2}$ and for $i = 1, 2, \ldots, k-2$, adding the arcs (u_i, v_5) and (v_9, w_i). As before, let $V_1 = \{v_1, v_2, v_3, v_4, v_7, v_{10}, v_{11}, v_{12}, v_{13}, u_1, u_2, \ldots, u_{k-2}, w_1, w_2, \ldots, w_{k-2}\}$ and let $V_2 = \{v_5, v_6, v_8, v_9\}$. There are $k + 4$ arcs from V_1 to V_2 and $k + 4$ arcs from V_2 to V_1. Hence, $\mathrm{ar}(D_k) = k + 4$. Also the digraph D'_k obtained from D_k by adding the four arcs (v_1, v_{10}), (v_3, v_7), (v_8, v_{12}), and (v_9, v_6) is anticonnected. Thus $\mathrm{ad}(D_k) \leq 4$ and so $\mathrm{ar}(D_k) - \mathrm{ad}(D_k) \geq k$. □

"This has been great Dad, but I think we really need to make use of it all and get out of here."

"What's your hurry Junior? The maze has been here three thousand years and I am not that hungry yet." The elder Jones still had visions of more theorems that the two could work out.

"Well," Indiana responded sheepishly, "I didn't want to mention it before, but the map mentions that the maze will eventually fill with water once its most valued possession is removed, and I just started to notice a slight trickle from the walls."

"How convenient! We're just about to get to the part of our work that I have been looking forward to the most and you decide we've got to get out of here."

"Dad, I promise I will work with you until we get this finished after we get out of here."

"All right," Henry Jones grumbled, "pay attention now, and we will ascertain the correct solution to our predicament."

Returning to Indy's map of the maze, the two men scoured over the document, searching for a connection to their mathematics and a path to freedom. "Look what happens, Dad, if we reduce the paths of the maze to oriented edges on a digraph. Knowing the initial conditions of the labyrinth should allow us to deduce the correct antipath to safety."

"You've done it Indiana! All my years of educating you have paid off!" Henry Jones was beaming with pride towards his son.

Indiana began placing arrows on the graph in directions matching the symbol "+" for a safe path and "-" for a deadly path. "Now from this Dad, we would be able to find an antipath of alternating directed edges so that we will always be on a segment of the path when it is safe to do so."

"Exactly! We start on a dangerous path, which means when we open the first door to leave this room the maze will be reversed and it will be safe to pass. The next path we take will originally have been traversable but will then be currently dangerous. But as we open the next door the maze will switch again and we will be able to pass freely." Henry could barely contain his excitement.

"Right Dad. We just continue in this manner of alternating safe and deadly paths and we can map out a route that will lead us to an exit." They made short work of deciphering the rest of the path from the maze and plotted a course to their freedom.

"Well here goes nothing."

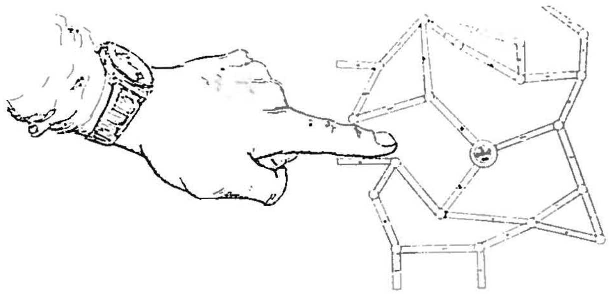
"I have the utmost confidence in you son."

"So you'd like to go first?" Indiana jokingly asked.

"Harumph." Henry made a half-laugh, half-cough sound and replied, "Ach, Junior, you've come so far, but still some things ..."

"Never change. I know Dad. And you either. Let's go."

Following their prescribed route, with grumbling and bickering along the way, after what seemed an eternity, they found themselves once again under the African blue sky. Henry Jones piped up, "Ah, I can't wait to get back home and we continue to study and formalize our results on graphs. You know, it would be nice to write up the results we found into a paper and submit it for publication to the Pi Mu Epsilon Journal so that many math students can see what we've done and perhaps even find more theorems."



"Ummm..." Indy had the sheepish look again. "You were serious about that Dad? Oh, forget that, of course you were. You realize that was the kind of promise a son makes to his father when he realizes there's a good chance they may not make it out of the situation they are in, don't you? I mean you aren't really going to hold me to that?" Indy prodded his horse into motion.

"Junior!" Henry shouted, but his son was already on the go and he had naught to do but to follow. He would make Indy see the virtues of his methods yet. But probably not this day. The pair charged off across the landscape and finally disappeared as fading specks melting into the African setting sun.

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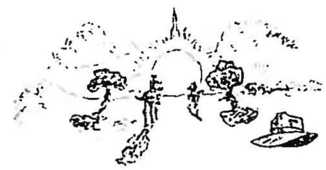

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ANALYTICAL FORMULAS FOR $\sum_{i=1}^{n}\left\lceil \frac{i}{p}\right\rceil$ AND $\sum_{i=1}^{n}\left\lfloor \frac{i}{p}\right\rfloor$

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Introduction

Given a real number x, denote by $\lceil x\rceil$ the least integer that is no less than x and denote by $\lfloor x\rfloor$ the greatest integer that is no larger than x. The functions $\lceil x\rceil$ and $\lfloor x\rfloor$ are often referred to as the ceiling and floor values of x, respectively [1,2]. In this note, we present two analytical formulas for computing the sums $\sum_{i=1}^{n}\left\lceil \frac{i}{p}\right\rceil$ and $\sum_{i=1}^{n}\left\lfloor \frac{i}{p}\right\rfloor$, where n and p are arbitrary positive integers. These summations occur in many areas of applied mathematics/engineering and to the best of the authors' knowledge, no closed-form solution exists in the open literature. The formulas derived here stem from an analysis of basic number theory and should prove interesting for undergraduate students in particular.

Discussion

For given integers p and $i \ge 1$, we write

$$i = px_i + r_i \tag{1}$$

where x_i and r_i are integers such that $x_i \ge 0$ and $1 \le r_i \le p$. Note that such a representation of i is unique. It follows that

$$\left\lceil \frac{i}{p}\right\rceil = x_i + 1 \tag{2}$$

$$\left\lfloor \frac{i}{p}\right\rfloor = \begin{cases} x_i & \text{if } 1 \le r_i \le p-1 \\ x_i + 1 & \text{if } r_i = p \end{cases} \tag{3}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\left\lceil \frac{i}{p}\right\rceil &= \sum_{i=1}^{n}(x_i+1)\\ &= n + \sum_{i=1}^{n}\frac{i-r_i}{p}\\ &= n + \frac{n(n+1)}{2p} - \frac{1}{p}\sum_{i=1}^{n} r_i.\end{aligned} \tag{4}$$

In order to evaluate the term $\frac{1}{p}\sum_{i=1}^{n} r_i$, we consider the following cases:

Case 1: n/p is an integer.

It can be observed that for $1 \le i \le p$, we have $x_i = 0$ and $r_i = i$. Hence $\sum_{i=1}^{p} r_i = \frac{p(p+1)}{2}$. Similarly, for $(p+1) \le i \le 2p$, we have $x_i = 1$ and $r_i = i-p$. So $\sum_{i=p+1}^{2p} r_i = \frac{p(p+1)}{2}$. In general, we obtain $\sum_{i=p(j-1)+1}^{pj} r_i = \frac{p(p+1)}{2}$ for $j \in \left\{1,2,\ldots,\frac{n}{p}\right\}$. Therefore

$$\begin{aligned}\frac{1}{p}\sum_{i=1}^{n} r_i &= \frac{1}{p}\left[\sum_{i=1}^{p} r_i + \sum_{i=p+1}^{2p} r_i + \cdots + \sum_{i=n-2p+1}^{n-p} r_i + \sum_{i=n-p+1}^{n} r_i\right]\\ &= \left(\frac{n}{p}\right)\frac{(p+1)}{2}\\ &= \left\lfloor \frac{n}{p}\right\rfloor \frac{(p+1)}{2}\end{aligned} \tag{5}$$

Case 2: n/p is not an integer.

$$\begin{aligned}\frac{1}{p}\sum_{i=1}^{n} r_i &= \frac{1}{p}\sum_{i=1}^{\lfloor \frac{n}{p}\rfloor p} r_i + \frac{1}{p}\sum_{i=\lfloor \frac{n}{p}\rfloor p+1}^{n} r_i\\ &= \left\lfloor \frac{n}{p}\right\rfloor \frac{(p+1)}{2} + \frac{1}{p}\sum_{i=\lfloor \frac{n}{p}\rfloor p+1}^{n} r_i.\end{aligned} \tag{6}$$

Let $i = p\left\lfloor \frac{n}{p}\right\rfloor + j$ where $j \in \left\{1,2,\ldots,n-\left\lfloor \frac{n}{p}\right\rfloor p\right\}$. Since $n - \left\lfloor \frac{n}{p}\right\rfloor p = (n \bmod p) < p$, we have $1 \le j < p$. Furthermore, since $i = px_i + r_i = p\left\lfloor \frac{n}{p}\right\rfloor + j$, the uniqueness of the representation of i as given by (1) implies that

$$x_i = \left\lfloor \frac{n}{p}\right\rfloor \text{ and } r_i = j.$$

Therefore,

$$\frac{1}{p}\sum_{i=\lfloor \frac{n}{p}\rfloor p+1}^{n} r_i = \frac{1}{p}\sum_{j=1}^{n-\lfloor \frac{n}{p}\rfloor p} j$$

$$= \frac{\left(n - \left\lfloor \frac{n}{p} \right\rfloor p\right)\left(n + 1 - \left\lfloor \frac{n}{p} \right\rfloor p\right)}{2p}. \tag{7}$$

Substituting (7) into (6), we obtain

$$\frac{1}{p}\sum_{i=1}^{n} r_i = \left\lfloor \frac{n}{p} \right\rfloor \frac{(p+1)}{2} + \frac{\left(n - \left\lfloor \frac{n}{p} \right\rfloor p\right)\left(n+1 - \left\lfloor \frac{n}{p} \right\rfloor p\right)}{2p}. \tag{8}$$

Note that the second term on the right hand side of (8) vanishes when n/p is an integer and (8) in such a case becomes (5). Hence (8) can be used to evaluate $\frac{1}{p}\sum_{i=1}^{n} r_i$ for the two discussed cases. Substituting (8) into (4), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left\lceil \frac{i}{p} \right\rceil &= n + \frac{n(n+1)}{2p} - \left\lfloor \frac{n}{p} \right\rfloor\left(\frac{p+1}{2}\right) - \frac{1}{2p}\left[\left(n - \left\lfloor \frac{n}{p} \right\rfloor p\right)\left(n+1 - \left\lfloor \frac{n}{p} \right\rfloor p\right)\right]\\ &= n + \left(n - \frac{p}{2}\right)\left\lfloor \frac{n}{p} \right\rfloor - \frac{p}{2}\left\lfloor \frac{n}{p} \right\rfloor^2. \end{aligned} \tag{9}$$

By a similar approach, one can obtain a closed formula for $\sum_{i=1}^{n}\left\lfloor \frac{i}{p} \right\rfloor$. Using (3), we can express

$$\begin{aligned}\sum_{i=1}^{n}\left\lfloor \frac{i}{p} \right\rfloor &= \sum_{j=1}^{\left\lfloor \frac{n}{p} \right\rfloor}(x_{pj}+1) + \sum_{j=1}^{\left\lfloor \frac{n}{p} \right\rfloor}\sum_{i=p(j-1)+1}^{pj-1} x_i + \sum_{i=\left\lfloor \frac{n}{p} \right\rfloor p+1}^{n} x_i\\ &= \left\lfloor \frac{n}{p} \right\rfloor + \sum_{i=1}^{n} x_i\\ &= \left\lfloor \frac{n}{p} \right\rfloor + \sum_{i=1}^{n}\frac{i - r_i}{p}\\ &= \left\lfloor \frac{n}{p} \right\rfloor + \frac{n(n+1)}{2p} - \frac{1}{p}\sum_{i=1}^{n} r_i. \end{aligned} \tag{10}$$

Substituting (8) into (10), we obtain

$$\sum_{i=1}^{n}\left\lfloor \frac{i}{p} \right\rfloor = \left(n + 1 - \frac{p}{2}\right)\left\lfloor \frac{n}{p} \right\rfloor - \frac{p}{2}\left\lfloor \frac{n}{p} \right\rfloor^2. \tag{11}$$

It can be observed that the formulas given by (9) and (11) are computationally efficient as n asymptotically increases.

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THE QUADRATIC FORMULA REVISITED AGAIN

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By "completing the square" on the quadratic equation $ax^2 + bx + c = 0$, one can derive the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where a, b, and c are real numbers with $a \neq 0$. If $b^2 - 4ac < 0$, then the quadratic equation has two nonreal complex conjugate solutions and if b^2 -

$4ac \geq 0$, then the solutions are two real numbers. The purpose of this note is to show how to derive the quadratic formula by assuming the solutions are complex numbers expressed in trigonometric form

$$x = r(\cos\theta + i\sin\theta),\ \text{where } r > 0 \text{ and } i^2 = -1.$$

Assuming that $x = r(\cos\theta + i\sin\theta)$ is a solution of $ax^2 + bx + c = 0$, then

$$a[r(\cos\theta + i\sin\theta)]^2 + b[r(\cos\theta + i\sin\theta)] + c = 0,$$

so that

$$[2a(r\cos\theta)^2 - ar^2 + b(r\cos\theta) + c] + [(r\sin\theta)(2ar\cos\theta + b)]\,i = 0 + 0i.$$

Equating the real and imaginary parts, then

$$2a(r\cos\theta)^2 - ar^2 + b(r\cos\theta) + c = 0, \tag{1}$$

and

$$(r\sin\theta)(2ar\cos\theta + b) = 0. \tag{2}$$

Now consider the two possible cases for (2).

Case 1. If $r\sin\theta = 0$, then $\sin\theta = 0$ since $r > 0$.

This yields real number solutions of $ax^2 + bx + c = 0$ and is discussed later.

Case 2. If $2ar\cos\theta + b = 0$, then

$$r\cos\theta = \frac{-b}{2a}, \tag{3}$$

and (1) becomes

$$2a\left(\frac{-b}{2a}\right)^2 - ar^2 + b\left(\frac{-b}{2a}\right) + c = 0,$$

so that

$$r^2 = \frac{c}{a}. \tag{4}$$

Since $(r\sin\theta)^2 = r^2 - (r\cos\theta)^2$, using (3) and (4), then

$$(r\sin\theta)^2 = \frac{c}{a} - \left(\frac{-b}{2a}\right)^2,$$

$$(r\sin\theta)^2 = \frac{4ac - b^2}{4a^2}, \tag{5}$$

or,

$$r\sin\theta = \frac{\pm\sqrt{4ac - b^2}}{2|a|}. \tag{6}$$

Using (3) and (6), the complex solutions of $ax^2 + bx + c = 0$ become

$$x = r\cos\theta + (r\sin\theta)\,i,$$

$$x = \frac{-b}{2a} + \frac{\pm\sqrt{4ac - b^2}}{2|a|}\,i.$$

Since $a \neq 0$, then

$$x = \frac{-b \pm \left(\sqrt{4ac - b^2}\right) i}{2a} \tag{7}$$

Since the numerator of (5) is positive, or $b^2 - 4ac < 0$, this indicates that the roots expressed in (7) are two nonreal complex conjugates. If the numerator of (5) in nonpositive, or $b^2 - 4ac \geq 0$, then the solutions of $ax^2 + bx + c = 0$ are two real numbers and (7) can be rewritten as

$$\begin{aligned} x &= \frac{-b \pm \left(\sqrt{(-1)(b^2 - 4ac)}\right) i}{2a} \\ &= \frac{-b \pm \left(\sqrt{-1}\right)\left(\sqrt{b^2 - 4ac}\right) i}{2a} \\ &= \frac{-b \pm (i)^2\left(\sqrt{b^2 - 4ac}\right)}{2a} \\ &= \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

Exercise. Derive the quadratic formula by assuming that the solutions are complex numbers expressed in rectangular form, $x = \alpha + \beta i$, where α and β are real numbers. This approach is similar to what was previously done.

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COMMUTATIVITY OF MATRIX MULTIPLICATION

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In this paper, we will discuss the importance for commutativity of matrix multiplication. We know that, for two $n \times n$ matrices A and B, it is not always true that $AB = BA$. We will show that Hirsch & Smale's method

[1, page 109] of solving the system $\mathbf{x}'=A\mathbf{x}$, where A is an n x n matrix and $\mathbf{x} = (x_1,\ldots,x_n)^T$, depends heavily on the commutativity of matrix multiplication of two appropriate matrices.

Before we go further, we recall two definitions. An n x n matrix Y is said to be *diagonalizable* if there exists a nonsingular matrix X and a diagonal matrix D such that $X^{-1}YX = D$. An n x n matrix C is said to be *nilpotent* if $C^k = 0$ for some positive integer k. Now consider the system $\mathbf{x}' = A\mathbf{x}$ of differential equations, where A is an n x n matrix and $\mathbf{x} = (x_1,\ldots,x_n)^T$. Hirsch & Smale [1, page 109] indicate the following method of solving this system.

Let $A = B + C$ where B is diagonalizable, C is nilpotent, and $BC = CB$. Then the solution is $e^{At} = e^{(B+C)t} = e^{Bt}\,e^{Ct}$. Here e^{Bt} and e^{Ct} are easily determined. The following examples illustrate the importance of the condition $BC = CB$ in solving systems of equations of this form.

Example 1. Consider the system $\begin{cases} X'_1 = X_2 + X_3 \\ X'_2 = -X_2 + X_3 \\ X'_3 = -X_3 \end{cases}$.

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = C + B$ where $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Here, $B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, B is nilpotent.

Consider $e^{At} = e^{(C+B)t} = e^{Ct}e^{Bt}$.

$$e^{Ct} = I + (Ct) + (1/2!)\,(Ct)^2 + \ldots + (1/m!)\,(Ct)^m + \ldots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & -t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & (-t)^2 & 0 \\ 0 & 0 & (-t)^2 \end{bmatrix}$$

$$+ \ldots + \frac{1}{m!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & (-t)^m & 0 \\ 0 & 0 & (-t)^m \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+(-t)+(1/2!)(-t)^2+\ldots) & 0 \\ 0 & 0 & (1+(-t)+(1/2!)(-t)^2+\ldots) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}.$$

$$e^{Bt} = I + (Bt) + (1/2!)\,(Bt)^2 + 0$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & t & t+(1/2!)t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{Ct}e^{Bt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & t & t+(1/2!)t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t & t+(1/2!)t^2 \\ 0 & e^{-t} & te^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}$$

As a result, one obtains

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad X_2 = \begin{bmatrix} t \\ e^{-t} \\ 0 \end{bmatrix},\ \text{and}\ X_3 = \begin{bmatrix} t+(1/2!)t^2 \\ te^{-t} \\ e^{-t} \end{bmatrix}.$$

However, calculations show that neither X_2 nor X_3 is a solution of this system. The reason for this is that $BC \neq CB$.

Example 2. Consider the system $\begin{cases} X'_1 = X_1 + X_2 \\ X'_2 = X_2 \\ X'_3 = -2X_3 \end{cases}$.

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = B + C$ where $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Here, $C^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus C is nilpotent. Since $BC = CB$,

$$e^{At} = e^{Bt}e^{Ct} = e^{Bt}(I+Ct)$$

$$= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

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A VISUAL REPRESENTATION OF THE SEQUENCE SPACE

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Introduction

In an undergraduate course on chaotic dynamical systems based on Devaney's text [1], one encounters a strange metric space called the *sequence space*, $(\sum, d_\infty)$. This is a very difficult space to visualize. In fact, it can easily be shown that this space cannot be embedded isometrically in $\mathbb{R}^n$ for any n. In this paper we develop a geometric model of $\sum$ which gives a very good impression of the geometric structure of the space. The model consists of a subset of the Euclidean plane and an unusual, but easy

to visualize, metric on these points making the model isometric to $(\sum, d_\infty)$.

Definition 1.1. The *sequence space on two symbols* is the set

$$\sum = \{s_0s_1s_2s_3\ldots \mid s_i \in \{0,1\}\}$$

with the distance between two elements of this set defined by

$$d_\infty(\sigma,\tau) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

where $\sigma = s_0s_1s_2\ldots$ and $\tau = t_0t_1t_2\ldots$. That is, $\sum$ is the set of all sequences of zeros and ones.

While the points and distances are clearly defined, it is very difficult to visualize the geometric structure of the space as a whole. The goal of this paper is to construct a geometric model of the sequence space as an aide to visualizing its geometry.

In order to gain some insight into the geometry of $\sum$ we begin by considering a family of finite subspaces of $\sum$.

Definition 1.2. Define the family of metric spaces $(\sum_n, d_n)$ by

$$\sum\nolimits_n = \{s_0s_1s_2\ldots s_{n-1} \mid s_i \in \{0,1\}\},$$

$$d_n(\sigma,\tau) = \sum_{i=0}^{n-1} \frac{|s_i - t_i|}{2^i},$$

where $\sigma = s_0\ldots s_{n-1}$ and $\tau = t_0\ldots t_{n-1}$. That is $\sum_n$, is what we get if we consider only the first n terms of an element of $\sum$.

Clearly d_∞ and d_n are metrics and $(\sum_n, d_n)$ and $(\sum, d_\infty)$ are metric spaces.

Perhaps the simplest model one could make for a given metric space would be to embed it isometrically into some Euclidean space (preferably $\mathbb{R}$ or $\mathbb{R}^2$ or $\mathbb{R}^3$). We will show in Section Four that even the tiny subspace $\sum_2$ of $\sum$ can not be isometrically embedded in $\mathbb{R}^n$, no matter how large we choose n. Thus $\mathbb{R}^n$ with the Euclidean metric fails to contain an embedded model for $\sum$.

However, if we switch to the grid metric (often called the taxicab metric) we can construct a model of $\sum_n$ in $\mathbb{R}^n$.

Definition 1.3. Consider $\mathbb{R}^n$ with the grid metric

$$d_G(s,t) = \sum_{i=0}^{n-1} |s_i - t_i|$$

where $s = (s_0, s_1, \ldots, s_{n-1})$ and $t = (t_0, t_1, \ldots, t_{n-1})$.

Define $\gamma_n : \left(\sum_n, d_n\right) \to (\mathbf{R}^n, d_G)$ by

$$\gamma_n(a_0 a_1 \ldots a_{n-1}) = (s_0, s_1, \ldots, s_{n-1})$$

where $s_i = \frac{a_i}{2^i}$.

We will prove that γ_n is an isometry and thus that its image is a model of $\sum_n$ in $(\mathbf{R}^n, d_G)$.

For example, consider $\sum_1$. The image of $\sum_1$ contains only two points, as pictured in Figure 1(a.).

(a.) $\sum_1$ (b.) $\sum_2$ (c.) $\sum_3$

Figure 1

Now let us consider $\sum_2$. Its image consists of four points in the plane arranged as shown in Figure 1(b.). $\gamma_3\left(\sum_3\right)$ is pictured in Figure 1(c.). Continuing this way we could easily model $\sum_n$ in n-space. However this n-dimensional model is more difficult to visualize for large n, and the analogous model for $\sum$ would be a subspace of $\mathbf{R}^\infty$, hardly an easy thing to visualize. So we would like to find a way of "compressing" the entire space into three or fewer dimensions. In fact we will be able to fit it into two dimensions.

To do this we should first consider $\gamma_3\left(\sum_3\right)$. Think of it as a box sitting on the floor in front of you. The box has length one unit, width one half unit, and height one quarter unit. You can find the distance between any two points in $\sum_3$ by measuring the length of any shortest path which travels along the edges of the box between the corresponding points in $\gamma_3\left(\sum_3\right)$. Now take your Swiss Army knife out of your pocket and cut off the ½ by ¼ sides. Step on the box gently. Don't bend the sides; let it bend at the edges. You have just transformed a three dimensional model into a two dimensional model. In an analogous manner we can "crush" our higher dimensional models of $\sum_n$ into $\mathbf{R}^2$.

In this paper we develop a model which "crushes" $\sum$ into the real plane in an analogous manner. However, before we consider this model we will look at a model for a much simpler space to develop our intuition for what we mean by a "geometric model."

Modeling the Discrete Metric

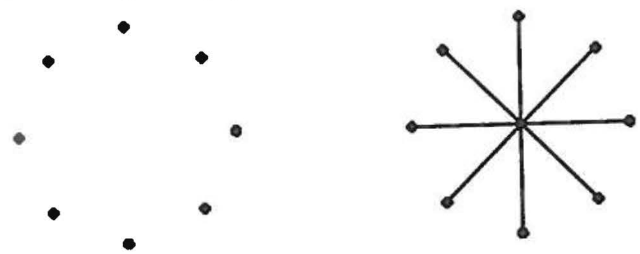
To illustrate the kind of geometric model we have in mind, let us first consider a similar model for a simpler metric space: the discrete metric on a set X. Under the discrete metric the distance between any two distinct points of X is one. Our model will have four key ingredients: 1) a set of points, P (corresponding to the points of X),2) a set of line segments (providing paths between any two points),3) a definition of a unique "proper" path between any two points, and 4) a metric induced on P (by measuring the Euclidean lengths of the line segments in the proper path between two points). In the case of the discrete metric we can define these four items as follows.

The Points

Let the set of points, $P \subset \mathbf{R}^2$, consist of the n equally spaced points on a circle of radius ½ (that is, in polar coordinates

$$P = \left\{(\tau, \theta) \mid \tau = \frac{1}{2} \text{ and } \theta = \frac{2\pi}{\kappa}, \kappa \in 0, 1, \ldots, n-1\right\})$$

as shown in Figure 2(a.) for $\kappa = 8$.



(a.) P (b.) P and S

Figure 2

The Line Segments

Define a set of line segments, S, to consist of all those segments connecting points of P to the center of the circle, as shown in Figure 2(b.).

The Paths

We define a *proper path* between the points of P, to be the shortest path (in the Euclidean metric) from one point to another along line segments of S.

The Metric

Finally, define a metric on P induced by the proper paths, namely the distance between two points is the sum of the lengths of the line segments in the unique shortest path between the points. With this metric, P is isometric to our original discrete metric space X.

We can think of this model as the spokes and hub of a wagon wheel. The points lie at the end of the spokes. To get from one point to another, one must travel down the spoke to the center and then out along the other spoke.

Having modeled the discrete metric we now turn our attention to the sequence space to develop a similar model, i.e. one which isometrically maps each sequence to a point in the real plane.

Modeling the Sequence Space

In order to model the sequence space we will follow a pattern analogous to the one we created for the discrete metric. Thus we will define points and line segments between those points, such that the lengths of certain paths using the segments induces a metric on the set of points, making it isometric to the sequence space.

The Points

Definition 3.1. For $x \in [0,2) \subset \mathbb{R}$ define $a_n(x)$ to be the coefficient of $\frac{1}{2^n}$ in the binary expansion of x. That is,

$$x = \sum_{n=0}^{\infty} a_n(x) \frac{1}{2^n}$$

and $a_n(x) \in \{0,1\}$.

[Note: If x has two binary expansions we consider only the one which ends in repeating zeros, for example 1.0000... instead of 0.1111....]

Definition 3.2. Define $P \subset [0,2) \times [0,2)$ by

$$P = \{(x,y) \mid \forall n \in \mathbb{N}, a_{2n}(x) = 0 \text{ and } a_{2n+1}(y) = 0\}.$$

The set P is pictured in Figure 3. We are now ready to define our *Crush Map*, κ, which will identify points of P with those of $\sum$.

Definition 3.3. Define the Crush Map $\kappa : \sum \to P$ by

$$\kappa(s_0 s_1 s_2 \ldots) = \left(\sum_{i \text{ odd}} \frac{s_i}{2^i}, \sum_{i \text{ even}} \frac{s_i}{2^i}\right).$$

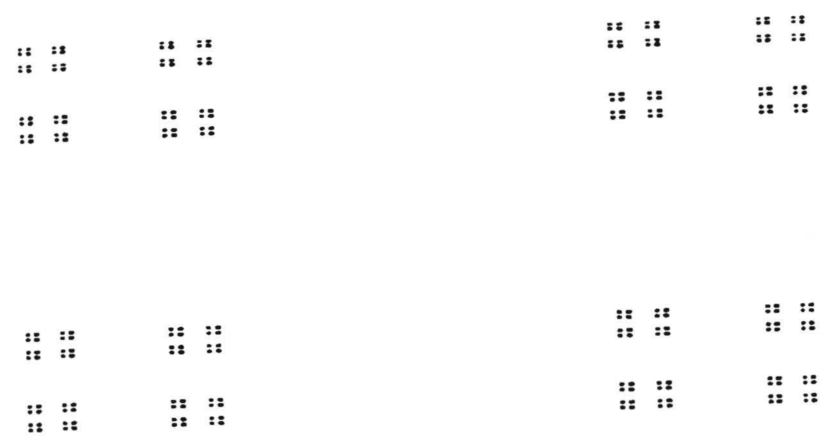


Figure 3 P

For example, consider the sequence $\sigma = 111111\overline{0} \in \sum$. The corresponding x-coordinate of $\kappa(\sigma)$ would be $0.10101\overline{0}$ and the y-coordinate would be $1.01010\overline{0}$ in base two. So in base ten the coordinates would be $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + 0 + \ldots = \frac{21}{32}$ and

$1 + \frac{1}{4} + \frac{1}{16} + 0 + \ldots = \frac{21}{16}$ respectively. So $\kappa(\sigma)$ is $\left(\frac{21}{32}, \frac{21}{16}\right)$.

Lemma 3.1. κ is a bijection.

Proof. First we will show that κ is injective. Let $\sigma = (s_0 s_1 s_2 \ldots)$, $\tau = (t_0 t_1 t_2 \ldots) \in \sum$ and assume $\kappa(\sigma) = \kappa(\tau)$. Then

$$\kappa(s_0 s_1 s_2 \ldots) = \kappa(t_0 t_1 t_2 \ldots)$$

$$\left(0 + \frac{s_1}{2^1} + 0 + \frac{s_3}{2^3} + 0 + \ldots, \frac{s_0}{2^0} + 0 + \frac{s_2}{2^2} + 0 + \ldots\right)$$
$$= \left(0 + \frac{t_1}{2^1} + 0 + \frac{t_3}{2^3} + 0 + \ldots, \frac{t_0}{2^0} + 0 + \frac{t_2}{2^2} + 0 + \ldots\right)$$

so,

$$0 + \frac{s_1}{2^1} + 0 + \frac{s_3}{2^3} + 0 + \ldots = 0 + \frac{t_1}{2^1} + 0 + \frac{t_3}{2^3} + 0 + \ldots$$

and

$$\frac{s_0}{2^0} + 0 + \frac{s_2}{2^2} + 0 + \ldots = \frac{t_0}{2^0} + 0 + \frac{t_2}{2^2} + 0 + \ldots$$

But two numbers are equal if and only if the digits of their binary expansions are equal, so

$$s_1 = t_1, s_3 = t_3, \ldots, \text{ and } s_0 = t_0, s_2 = t_2, \ldots.$$

Thus $s_i = t_i$ for all i, and so $\sigma = \tau$. Thus κ is injective.

To see that κ is surjective let $p \in P$. Then $p = (x, y)$ where $a_{2n}(x) = 0$ and $a_{2n+1}(y) = 0$ for all n. Define $\pi \in \sum$ to be $\pi = s_0 s_1 s_2 \ldots$ where

$$s_i = \begin{cases} a_i(x) & \text{if } i \text{ is even} \\ a_i(y) & \text{if } i \text{ is odd} \end{cases}.$$

Clearly $\kappa(\pi) = p$. Therefore κ is surjective. So κ is a bijection. □

Now that we have a mapping of points from $\sum$ to $P \subset \mathbb{R}^2$ we need to define the distances between those points in a way which preserves the distances in $(\sum, d_\infty)$. We begin by defining a set of line segments whose endpoints are points of P.

The Line Segments

Definition 3.4. Let S be the set of all line segments connecting those points of P whose inverse images under the crush map differ by only one digit. That is,

$$S = \{\ell(p,q) \mid \exists! n, \kappa^{-1}(p)_n \neq \kappa^{-1}(q)_n\}$$

where θ_n denotes the n^{th} term of the sequence $\theta \in \sum$, and $\ell(p,q)$ denotes the line segment joining p to q in $\mathbb{R}^2$.

Notice that every point in P is the endpoint of a unique segment of length $\frac{1}{2^\kappa}$ for all $\kappa \in \mathbb{N}$. This is easy to see by analogy in $\sum_3$. Go and get the box that you stepped on earlier. Decompress it so that it is a box again. Notice that each vertex of the box has three edges connected to it. One has length one, another has length one half, and the third has length one quarter. This can also be seen by referring to Figure 1(c.).

Lemma 3.2. Let $|\ell(p,q)|$ denote the length (in the Euclidean metric) of the line segment $\ell(p,q) \in S$. Then

$$|\ell(p,q)| = d_\infty(\kappa^{-1}(p), \kappa^{-1}(q)).$$

Proof. Since the points $\kappa^{-1}(p)$ and $\kappa^{-1}(q)$ differ by only one term, either the x or the y coordinate will be the same for both p and q. Assume they have the same y coordinate, the other case being similar.

$$|\ell(p,q)| = \left|\sum_{i \text{ odd}} \frac{\kappa^{-1}(p)_i}{2^i} - \sum_{i \text{ odd}} \frac{\kappa^{-1}(q)_i}{2^i}\right|$$
$$= \left|\sum_{i \text{ odd}} \frac{\kappa^{-1}(p)_i - \kappa^{-1}(q)_i}{2^i}\right|$$
$$= \left|0 + 0 + \ldots + \frac{\kappa^{-1}(p)_n - \kappa^{-1}(q)_n}{2^n} + 0 + 0 + \ldots\right|$$
$$= d_\infty(\kappa^{-1}(p), \kappa^{-1}(q))$$

where n is the index of the unique term in which $\kappa^{-1}(p)$ and $\kappa^{-1}(q)$ differ. Thus the length of the line segment equals the distance in $\sum$ between inverse images of p and q. □

The entire set of line segments, S, is pictured in Figure 4. Notice that wherever perpendicular line segments in the figure cross, the point of intersection is in P. Notice also that many distinct line segments in S overlap each other in the figure. For example there is a line segment

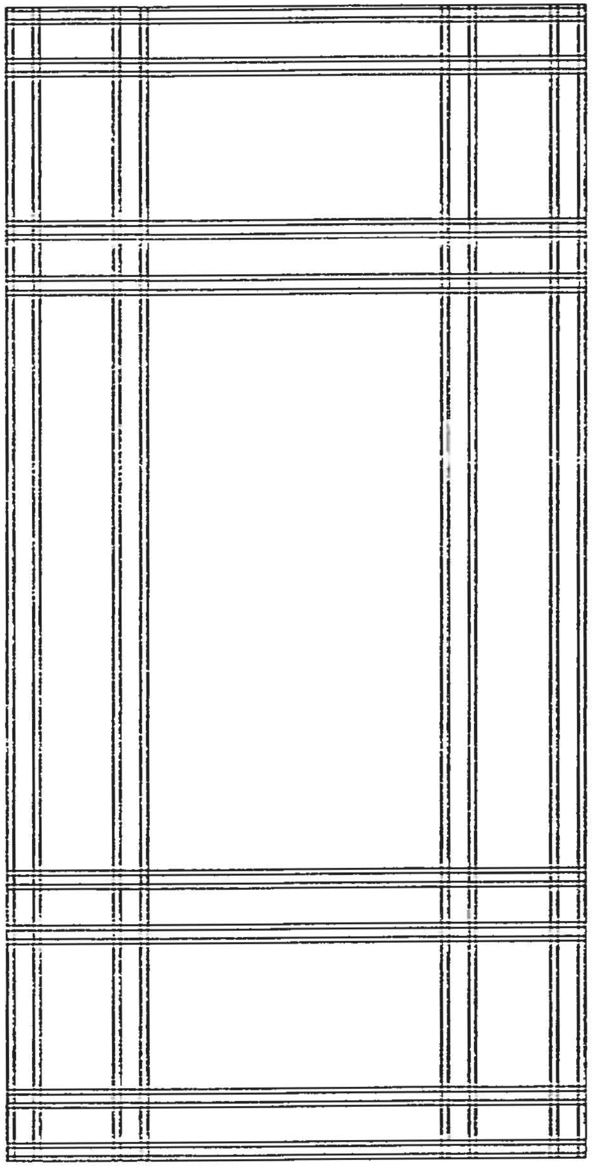


Figure 4 S

connecting $\kappa(\overline{0}) = (0,0)$ and $\kappa(1\overline{0}) = (0,1)$. There is also one connecting $\kappa(\overline{0}) = (0,0)$ and $\kappa(001\overline{0}) = \left(0, \frac{1}{4}\right)$. Further note that the segments in the figure enclose infinitely many rectangular regions. In this respect it has a fractal-like nature.

We are now ready to define the proper paths along line segments of S.

The Path

Definition 3.5. Let $p, q \in P$. A *path from p to q* is a (possibly infinite) sequence of points of P

$$p_0, p_1, p_2, p_3, \ldots$$

where $p_0 = p$, satisfying

$$\ell(p_j, p_{j+1}) \in S$$

for all j and

$$\lim_{n \to \infty} p_n = q$$

(or $p_k = q$ in the case where the sequence is finite of length $k + 1$, i.e. when $\kappa^{-1}(p)$ and $\kappa^{-1}(q)$ differ in k digits).

The *length of a path* from p to q is defined to be the sum of the lengths of its line segments, $\sum_{j=0}^{\infty} |\ell(p_j, p_{j+1})|$. A path from p to q is said to be a *proper path* if $\kappa^{-1}(p_{j-1})$ *and* $\kappa^{-1}(p_j)$ differ in the j th term that $\kappa^{-1}(p)$ differs from $\kappa^{-1}(q)$ and in no other terms.

Note that the series in the definition of length of proper path must converge since a proper path can contain a line segment of length $\frac{1}{2^j}$ at most once.

For an example consider the proper path from $\kappa(000111\overline{0})$ to $\kappa(110100\overline{0})$. The proper path is
(0.15625,0.0625),(0.15625,1.0625),(0.65625,1.0625),(0.65625,1),(0.625,1) since these points are equal to $\kappa(000111\overline{0})$, $\kappa(100111\overline{0})$, $\kappa(110111\overline{0})$, $\kappa(110101\overline{0})$, and $\kappa(110100\overline{0})$ respectively. The segments which comprise this path are shown on a graph of P in Figure 5.

As can be seen in the preceding example, traveling along a proper path and measuring the distances traveled is comparable to comparing the terms (in order) of the corresponding points in the sequence space. This is further explained in the following theorem.

Lemma 3.3. For any $p, q, \in P$, there is a unique proper path from p to q.

Proof. Suppose $\sigma = \kappa^{-1}(p) = s_0 s_1 s_2 \ldots$ and $\tau = \kappa^{-1}(q) = t_0 t_1 t_2 \ldots$.

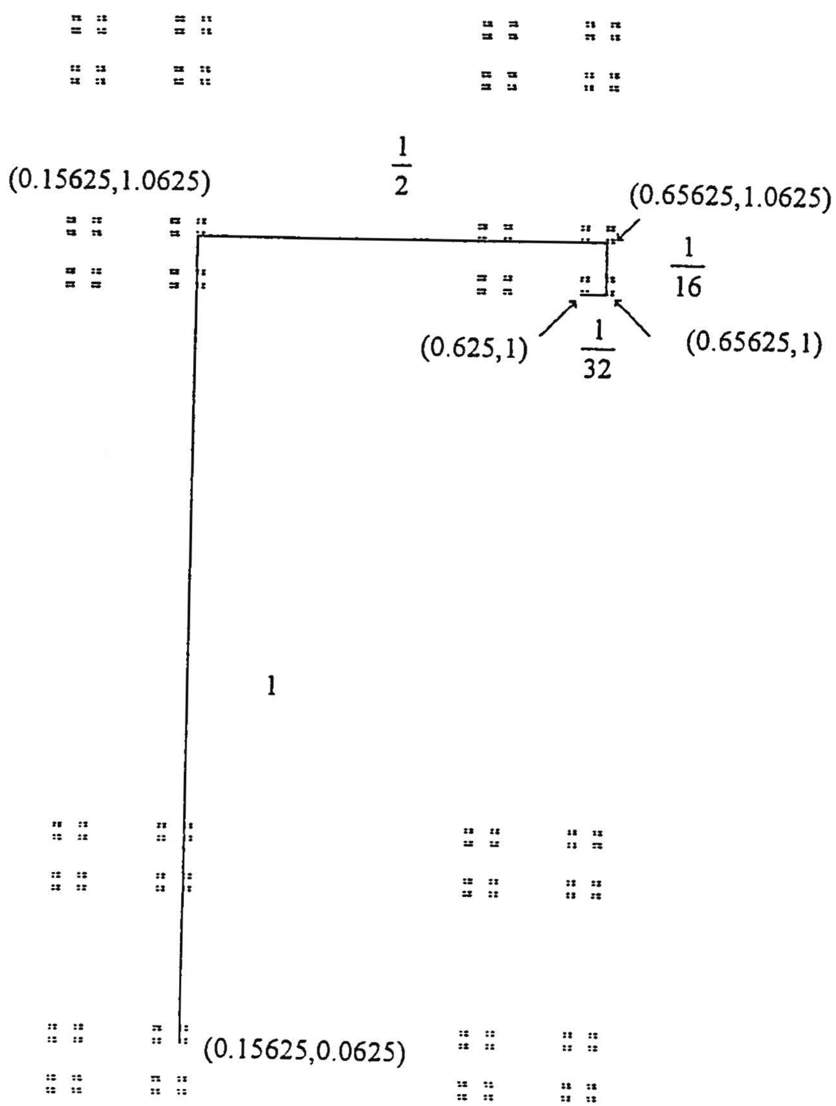


Figure 5 Proper Path from (0.15625,0.0625) to (0.625,1)

Let $p_0, p_1, p_2, \ldots$ be a proper path from p to q. We will prove the case where σ and τ differ in infinitely many terms, the finite case being similar. Let $j \geq 1$ and assume i is the index of the j^{th} term in which σ and τ differ. We show that

(1) $$\kappa^{-1}(p_j) = t_0 t_1 t_2 \ldots t_{i-1} t_i s_{i+1} s_{i+2} \ldots .$$

We proceed by induction on j.

Let $j = 1$ so i is the index of the first term where σ and τ differ. Since the path is proper, $\kappa^{-1}(p_0)$ and $\kappa^{-1}(p_1)$ must differ in the i^{th} term, and in no other. Therefore, $\kappa^{-1}(p_1) = s_0 s_1 \ldots s_{i-1} t_i s_{i+1} \ldots$ (since $t_i \neq s_i$) and further, since the i^{th} term is the first term which they differ, $s_m = t_m$ for $m < i$. Therefore $\kappa^{-1}(p_1) = t_0 t_1 \ldots t_{i-1} t_i s_{i+1} s_{i+2} \ldots$.

Now for the induction step. Let $j > 1$. Assume (1) is true. Let i be the index of the j^{th} term in which σ and τ differ. Let I be the index of the $(j+1)^{th}$ term in which they differ. Then

$$\kappa^{-1}(p_j) = t_0 t_1 t_2 \ldots t_{i-1} t_i s_{i+1} s_{i+2} \ldots s_{I-1} s_I s_{I+1} \ldots$$

Since $t_I \neq s_I$,

$$\kappa^{-1}(p_{j+1}) = t_0 t_1 t_2 \ldots t_{i-1} t_i s_{i+1} s_{i+2} \ldots s_{I-1} t_I s_{I+1} \ldots$$

and since the I^{th} term is the next term (after the i^{th} term) in which they differ, $s_m = t_m$ for $i < m < I$. Therefore,

$$\kappa^{-1}(p_{j+1}) = t_0 t_1 t_2 \ldots t_{i-1} t_i t_{i+1} t_{i+2} \ldots t_{I-1} t_I s_{I+1} s_{I+2} \ldots$$

This completes the induction. Therefore the proper path is unique, as it is the one prescribed by (1). □

We can now use the lengths of the proper paths to induce a metric on P.

The Metric

Definition 3.6. Define $d_p : P \times P \to \mathbb{R}$ by $d_p(p, q)$ = the length of the unique proper path from p to q.

Theorem 3.4. (P, d_p) is a metric space and the Crush Map, $\kappa : (\sum, d_\omega) \to (P, d_p)$, is an isometry.

Proof. By Lemma 3.1 κ is a bijection, so it suffices to show that $d_\infty(\sigma, \tau) = d_p(\kappa(\sigma), \kappa(\tau))$.

$d_p(\kappa(\sigma), \kappa(\tau))$ = the distance of the proper path from $\kappa(\sigma)$ to $\kappa(\tau)$

$$= \sum_{j=1}^{\infty} |\ell(p_{j-1}, p_j)|$$

$$= \sum_{j=1}^{\infty} d_{\infty}(\kappa^{-1}(p_{j-1}), \kappa^{-1}(p_j))$$

Since $\kappa^{-1}(p_{j-1})$ and $\kappa^{-1}(p_j)$ differ by only one digit in i_j^{th} place, (where i_j is the index of the j^{th} term in which σ differs from τ) we have $d_{\infty}(\kappa^{-1}(p_{j-1}), \kappa^{-1}(p_j)) = \frac{1}{2^{i_j}}$ so

$$= \sum_{j=1}^{\infty} \frac{1}{2^{i_j}}$$

$$= \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{2^i}$$

$$= d_{\infty}(\sigma, \tau). \; \square$$

Thus, (P, d_p) is a geometric model of the sequence space.

Additional Theorems

We close by proving some of the claims stated in the introduction.

Theorem 4.1. There does not exist an isometric embedding from $(\sum, d_{\infty})$ into $(\mathbf{R}^n, d_E)$ where d_E denotes the Euclidean metric, $d_E(x, y) = |x - y|$.

Proof. Since $(\sum_2, d_2)$ can be considered to be a subspace of $(\sum, d_{\infty})$ it suffices to show that $\sum_2$ does not embed isometrically into $\mathbf{R}^n$. Suppose $k: \sum_2 \to \mathbf{R}^n$ is an isometric bijection. Then $d_2(\alpha, \beta) = |k(\alpha) - k(\beta)|$ for any $\alpha, \beta \in \sum_2$. Consider the points:

$$\sigma = k(00)$$
$$\tau = \kappa(01)$$
$$\rho = \kappa(10)$$
$$\nu = k(11)$$

$$|\sigma - \tau| = d_2(00,01) = \frac{1}{2}$$
$$|\tau - \nu| = d_2(01,11) = 1$$
$$|\sigma - \nu| = d_2(00,11) = \frac{3}{2}$$

Thus σ, τ, and ν are collinear as depicted in Figure 6(a.).

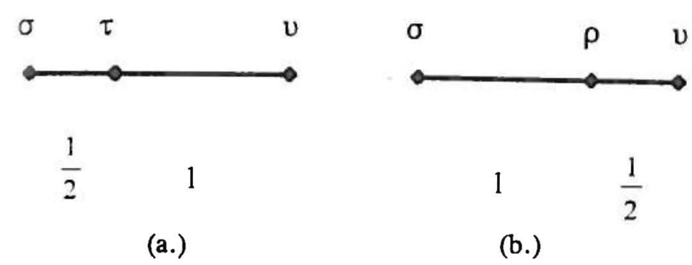


Figure 6

Now let's consider the point ρ.

$$|\sigma - \rho| = d_2(00,10) = 1$$
$$|\rho - \nu| = d_2(10,11) = \frac{1}{2}$$

Thus ρ is collinear with σ and ν, as shown in Figure 6(b.) and, therefore, collinear with τ also. Since ρ, τ, and σ are collinear

$$|\tau - \rho| = |\sigma - \rho| - |\sigma - \tau|$$
$$|\tau - \rho| = 1 - \frac{1}{2} = \frac{1}{2}$$

However $|\tau - \rho| = d_2(01,10) = \frac{3}{2} \neq \frac{1}{2}$. So we have a contradiction. Thus there does not exist an isometric embedding of $\sum_2$ into $\mathbf{R}^n$, and consequently there does not exist an isometric embedding of $\sum$ into $\mathbf{R}^n$.
$\square$

Theorem 4.2. γ_n is an isometry (between $\sum_n$ and its image).

Proof. First we shown that γ is a bijection. It is obviously surjective so now we must show that γ is injective. Assume $\gamma_n(a) = \gamma_n(b)$.

$$\gamma_n(a_0 a_1 a_2 \ldots a_{n-1}) = \gamma_n(b_0 b_1 b_2 \ldots b_{n-1})$$

$$\left(\frac{a_0}{2^0}, \frac{a_1}{2^1}, \frac{a_2}{2^2}, \ldots, \frac{a_n}{2^{n-1}}\right) = \left(\frac{b_0}{2^0}, \frac{b_1}{2^1}, \frac{b_2}{2^2}, \ldots, \frac{b_n}{2^{n-1}}\right)$$

$$\frac{a_0}{2^0} = \frac{b_0}{2^0}, \frac{a_1}{2^1} = \frac{b_1}{2^1}, \ldots, \frac{a_n}{2^{n-1}} = \frac{b_n}{2^{n-1}}$$

$$a_0 = b_0, a_1 = b_1, \ldots, a_{n-1} = b_{n-1}$$
$$(a_0a_1a_2\ldots a_{n-1}) = (b_0b_1b_2\ldots b_{n-1})$$
$$a = b$$

Thus γ is a bijection. We now must show that γ preserves distances. Consider the points a and b in $\sum_n$. By the definition of d_n

$$d_n(a_0a_1a_2\ldots a_{n-1}, b_0b_1b_2\ldots b_{n-1}) = \frac{|a_0-b_0|}{2^0} + \frac{|a_1-b_1|}{2^1} + \ldots + \frac{|a_{n-1}-b_{n-1}|}{2^{n-1}}.$$

And by the definition of the grid metric the distance between $\gamma_n(a)$ and $\gamma_n(b)$ is

$$d_G\left(\left(\frac{a_0}{2^0}, \frac{a_1}{2^1}, \ldots, \frac{a_{n-1}}{2^{n-1}}\right), \left(\frac{b_0}{2^0}, \frac{b_1}{2^1}, \ldots, \frac{b_{n-1}}{2^{n-1}}\right)\right)$$
$$= \left|\left(\frac{a_0}{2^0} - \frac{b_0}{2^0}\right)\right| + \left|\left(\frac{a_1}{2^1} - \frac{b_1}{2^1}\right)\right| + \ldots + \left|\left(\frac{a_{n-1}}{2^{n-1}} - \frac{b_{n-1}}{2^{n-1}}\right)\right|$$
$$= \frac{|a_0 - b_0|}{2^0} + \frac{|a_1 - b_1|}{2^1} + \ldots + \frac{|a_{n-1} - b_{n-1}|}{2^{n-1}}.$$

Thus γ preserves distances. So γ is an isometry between $\sum_n$ and its image. □

Acknowledgements

The material for this paper was conceived in an undergraduate course on chaotic dynamical systems taught by Dr. Ken Monks at the University of Scranton, who challenged me to come up with a geometric model for the sequence space. This paper is the result of my response to that challenge. I would like to thank him for his help with this paper and for his contributions to it.

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Since researching and writing this paper under the direction of Dr. Ken Monks, Marc Fusaro has graduated from the University of Scranton with a B.S. in mathematics and economics. He has since taken a job at the Federal Reserve Board of Governors. His hobbies include spelunking, ultimate frisbee, solo synchronized swimming, biking, and gnome hunting. Marc intends to pursue a Ph.D. in economics.

SEARCHING FOR INFINITE FAMILIES OF 2-TRANSITIVE SPACES

John Morrison (student)
St. John's University

Most everybody is familiar with the basic properties of geometric shapes such as the triangle, hexagon, octagon, and so on. However, defining these shapes as sets of points reveals some properties which are not so familiar, such as 2-transitivity. That is, shapes that can form a partition P of the edges into equal size sets so that there is a doubly transitive group of automorphisms of the shape that acts as a group of permutations on P. These shapes can often be categorized into infinite families, and this research is an introduction to identifying these families.

First we need to define the sets of points we will work with.

Definition. $(S, \equiv)$ is a set S together with a relation $\equiv$ on pairs of points. $(S, \equiv)$ is a *space* (or more formally an *equidistance space*) iff:

i) $\equiv$ is an equivalence relation on pairs of points;

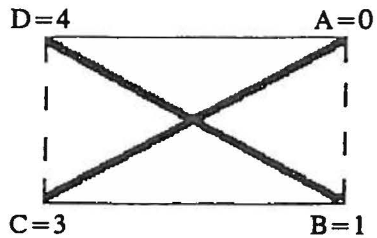
ii) for all A, B and C in S, $AB \equiv BC$ iff $A = C$;

iii) for all A and B in S, $AB \equiv BA$.

Fortunately, these spaces can be studied without knowing the actual distance between points, instead we only know the distance relation between pairs of points.

Let's look at a common space such as $(K_4, \equiv)$. The variously marked edges each represent different distances.

Example 1. K_4



LENGTHS
AB ≡ DC
AC ≡ DB
AD ≡ BC

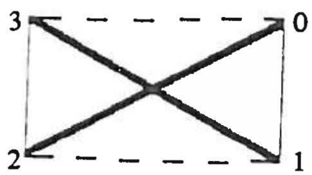
Note that the actual distance does not matter, only which edges are equivalent, such as AB and DC. These edges can also be shown numerically with AB = [0,1] and DC = [4,3]. Then the three different types of edges represented here can each be described by the following sets:

——— $= \{[2i, 2i + 1] \bmod 4 \;/\; i = 0,1,2,3\}$
▬▬▬ $= \{[2i + 1, 2i + 2] \bmod 4 \;/\; i = 0,1,2,3\}$
— — $= \{[i^2, i^2 + 2] \bmod 4 \;/\; i = 0,1,2,3\}$

Acting on these sets of points are permutations known as similarities.

Definition. A *similarity* is a permutation on a set of points which preserves the distance relations.
A similarity maps one point to another in a shape. For example, the points of a rectangle can be rotated using the similarity $f(X) = X + 1$. Remember that since we have only 4 points, we operate in mod 4.

Example 2. 1 rotation on K_4



$f(x) = X + 1 \pmod 4$
$f(0) = 1$
$f(1) = 2$
$f(2) = 3$
$f(3) = 0$

Note that while rotating the points, the similarity preserves the distance relations; that is, the horizontal tops and the bottoms remain equivalent in length, the vertical sides remain equivalent in length, and the diagonals

remain equivalent in length. This quality of preserving distance relations is essential for a space to be singly or doubly transitive.

Before we can begin to search for 2-transitive spaces we must understand the meaning of 1-transitive.

Definition. A space is said to be *1-transitive* if, given any point A in the space, there exists a similarity defined by the function f that sends A to any point, say A', in the space.

This means that if we choose any point in the space, we can map it to every other point in the space while preserving the distance relations.

If the similarity $f(X) = X + 1 \pmod 4$ is applied to the rectangle three times, each point is mapped to every other point while preserving the distance relations each time. Thus, the rectangle is 1-transitive.

We can expand this idea of 1-transitive and move two points instead of only one.

Definition. A finite set of points is said to be *2-transitive* if, given any two points (A and B) in the space and two images for these points (A' and B'), there is a similarity f such that $f(A) = A'$ and $f(B) = B'$.
That is, if it is possible to choose any pair of points and map them to any other two points in the space while preserving the distance relations, the space is 2-transitive.

Before attempting to find examples of 2-transitive shapes, there are certain conditions necessary for 2-transitivity which should be considered: (1) the number of different edge lengths in a shape must divide the total number of edges; (2) the number of different edge lengths at a point must divide the total number of edges at that point. With the use of these two conditions, it is much easier to narrow down which shapes are candidates for 2-transitivity.

Now that we know of the necessary conditions for 2-transitivity, all we need is a way to prove that something is indeed 2-transitive; we need a theorem that states when a space is 2-transitive. Hence, the following very crucial theorem developed from the definition of 2-transitivity.

Theorem. $(S, \equiv)$ is 2-transitive if and only if

i) $(S,\equiv)$ is 1-transitive.

ii) When a point A in $(S,\equiv)$ is fixed, there are similarities which map any other point B to all the remaining points of $(S,\equiv)$.

Proof. Assume that $(S,\equiv)$ is 2-transitive. Then by definition $(S,\equiv)$ is 1-transitive.

Note that given any two points in $(S,\equiv)$ and any two images for the points, there is a similarity f such that $f(A) = A'$ and $f(B) = B'$. Thus, there is a similarity such that if A is fixed, B can be mapped to any other B' in $(S,\equiv)$. Therefore both conditions are satisfied.

Assume that $(S,\equiv)$ is 1-transitive and when a point A is fixed, there are similarities which map other B to all the remaining points of $(S,\equiv)$. For $(S,\equiv)$ to be 2-transitive, we must find similarities which map any two points in S, say A and B, to any two images for these in $(S,\equiv)$, say A' and B'.

Since $(S,\equiv)$ is 1-transitive, there is a similarity defined by the function $f(A) = A'$ which maps any point A to any point A' in $(S,\equiv)$.

There are also similarities which leave A fixed and map B to any other point C in S. Let g be any similarity such that $g(A) = A$ and $g(B) = C$. Note the composition of $f \circ g$:

$$f(g(A)) = f(A) = A' \text{ and } f(g(B)) = f(C).$$

Let $f(A) = A'$. Since f is one to one and onto, there exists a P such that $f(P) = B'$. Now pick g: $g(A) = A$ and $g(B) = P$. Then $f(g(A)) = f(A) = A'$; and $f(g(B)) = f(P) = B'$.

Thus for any two points A and B in $(S,\equiv)$, the composition $f \circ g$ is a similarity which maps A and B to any two images A' and B' in S. Hence $S,\equiv)$ is 2-transitive. Therefore the theorem holds.

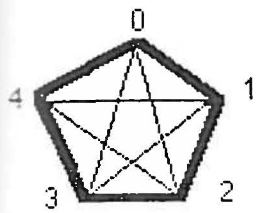
Now that we know necessary conditions for 2-transitivity and posses a theorem to prove the quality of 2-transitivity, we can begin the search for 2-transitive spaces.

The search begins with two trivial examples that will always be 2-transitive: (1) a space where all the lengths between the vertices are the same; (2) a space where all the lengths between the vertices are different. No matter how the points are arranged in each of these respective spaces, the distance relations will not change; that is, all of the lengths between the vertices will remain the same length, or all the lengths between the vertices will remain a different length. Since there can be infinitely many points in these types of spaces, they are two infinite families of 2-transitive spaces.

Now let's look at a less trivial example. Following our necessary conditions for 2-transitivity, we can draw a pentagon with two sets of distances.

Example 3. (thick line) $= \{[i,i+1] \bmod 5 / i = 0,1,2,3,4\}$

(thin line) $= \{[i,i+2] \bmod 5 / i = 0,1,2,3,4\}$



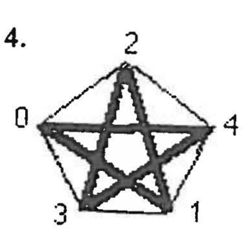
The configuration of this pentagon is notable since it has the characteristics of a regular space.

Definition. A space, $(S,\equiv)$ is said to be *regular* if each point in $(S,\equiv)$ is structurally identical. That is, each point is hit by the same sets of edges and with the same number of edges from each set.

Now we must show that the regular pentagon is indeed 2-transitive. It is clear that we can rotate the points and still retain the distance relations, so it is 1-transitive. But it is not so obvious that a similarity can take a pair of points, say [0,1], and map them to [2,4] while preserving the distance relations. Thus, we will have to work harder to show 2-transitivity.

Note that if we use this similarity and map [0,1] to [2,4], the thin edge that exists between [2,4] is replaced by the thick edge of [0,1]. If we are going to preserve the distance relations of the space, all of the thin edges must become thick and vice versa. As shown below:

Example 4.

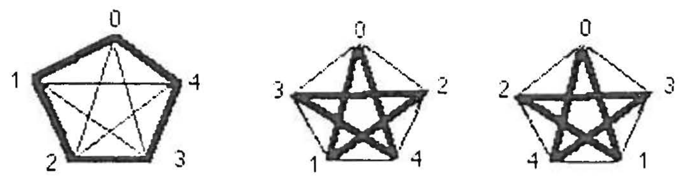


Similarity:
$f(X) = 2X + 2 \pmod 5$

This shows that a structure preserving similarity exists which can map [0,1] to [2,4]. However, it would be an extremely lengthy process if we had to do this for every pair of points to show whether or not a space is 2-transitive. Fortunately, we can use our theorem for 2-transitivity.

We have already shown that the pentagon is 1-transitive, so the first part of the theorem is satisfied. Now we need to fix a point, say 0, and show that there exist similarities which can take any other point, say 1, and map it to all of the remaining points.

Example 5.

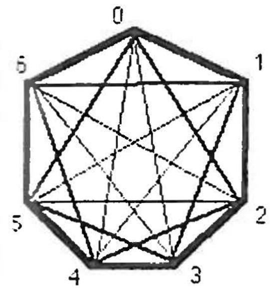


Thus, we have shown that a regular pentagon with two edge lengths is 2-transitive. With this in mind, we can discover another infinite family of 2-transitive shapes.

To find this infinite family, we need to look for qualities of the regular pentagon that may be important for its 2-transitivity. Note that the pentagon has an odd number of points, but more importantly it has a prime number of points. Also, there is one type of edge for each possible distance. Here we have a distance spanning one point represented with a thick line and a distance spanning two points with a thin line. This results in a regular space with a continuous edge type forming the perimeter. We will denote a space with these characteristics as Z_n. In this case, Z_5 is a 2-transitive space.

Since 7 is both odd and prime, Z_7 is the next logical space to check. As with Z_5, Z_7 can be drawn with one set of vertices representing each possible distance.

Example 6. Z_7



$= \{[i,i+1] \bmod 7 \ / \ i = 0,\ldots,6\}$

$= \{[i,i+2] \bmod 7 \ / \ i = 0,\ldots,6\}$

$= \{[i,i+3] \bmod 7 \ / \ i = 0,\ldots,6\}$

By using the theorem for 2-transitivity, it can be shown that this regular 7 point space is 2-transitive. Since Z_7 has the same qualities as Z_5, a pattern is developing, but it needs to be defined even further. Both 5 and 7 are odd and prime numbers, so we must determine if an odd number suffices, or if a prime number is required.

For this test we will examine Z_9. There exist four sets of edges for Z_9:

$$A = \{[i,i+1] \bmod 9 \ / \ i = 0,\ldots,8\}$$
$$B = \{[i,i+2] \bmod 9 \ / \ i = 0,\ldots,8\}$$
$$C = \{[i,i+3] \bmod 9 \ / \ i = 0,\ldots,8\}$$
$$D = \{[i,i+4] \bmod 9 \ / \ i = 0,\ldots,8\}$$

Sets A, B, and D form one cycle, while C which forms three:

$$A = \{[0,1,2,3,4,5,6,7,8]\}$$
$$B = \{[0,2,4,6,8,1,3,5,7]\}$$
$$C = \{[0,3,6]\ [1,4,7]\ [2,5,8]\}$$
$$D = \{[0,4,8,3,7,2,6,1,5]\}.$$

Note that Z_9 is 1-transitive since we can use the similarity $f(x) = x + 1$ (mod 9 to rotate all the points. However, the separate cycles in set C will prevent Z_9 from being 2-transitive because the similarities on a 2-transitive space must be able to interchange every set of edges.

For example, map an edge from set C, say [0,3], to and edge from set A, say [0,1]. For this to occur, the cycle formed by C must be analogous to the cycle formed by A. However, set C consists of three separate cycles of length three, while set A has one cycle of length nine. It is impossible to combine the cycles of set C to a single cycle of length nine and [0,3] cannot be mapped to [0,1]. Therefore, Z_9 is not 2-transitive.

So it appears that our 2-transitive family must be composed of prime numbers and further research has proven this to be true. Since there are infinitely many prime numbers, Z_p, where p is prime, is another infinite family of 2-transitive spaces.

There exist many more 2-transitive spaces than shown here, and nearly all are members of an infinite family. Other infinite families that were found with these methods include: prime numbered spaces where all parallel edges are from the same set; affine planes projected over a field; projective planes over a field. This is by no means an exhaustive list, and with the use of these basic search methods plus a little group theory, many more families of 2-transitive spaces can be found.

This paper was written by John Morrison at St. John's University during his senior year as part of an independent learning project under the guidance of professor Tom Sibley. John graduated from St. John's in 1993 with a double major in mathematics and history. He is enrolled in the College of Biological Sciences at the University of Minnesota.

MATHACROSTICS

Solution to Mathacrostics 43 by **Gerald M. Leibowitz** (Fall 1996).

Words:

A. REEFS
B. OMEGA
C. BANACH
D. ELEMENT
E. RACY
F. TOOTHPASTE
G. SHILOV
H. FIELDS
I. ADZ
J. CHAPT.W
K. EFT
L. SPACE
M. OATH
N. FOURIER
O. MILMAN
P. AWASH
Q. TOWPATH
R. HILBERT
S. EWES
T. MATCHED
U. AFFINE
V. TWEET
W. ITEM
X. COSETS
Y. SWEETS

Author and title: Roberts, *Faces of Mathematics*

Quotation: We chose the title "*Faces of Mathematics*" for two reasons. First, we wanted to emphasize the fact that mathematics was developed by human beings, real people with real faces.

Solvers: Avraham and Chana G. Adler (jointly); Thomas Banchoff; Frank P. Battles; Corine Bickley; Jeanette Bickley; Paul S. Bruckman; Charles R. Diminnie; Thomas Drucker; Victor G. Feser; Robert C. Gebhardt; F. C. Leary; Henry S. Lieberman; Rebecca Martel; Naomi Shapiro; Stephanie Sloyan; and the proposer. One incorrect solution was received.

Late solutions to Mathacrostic 42 were received from James Campbell and Victor G. Feser.

Currently, no mathacrostics are on file for publication.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 1, 1997.

Problems for Solution

901. *Proposed by the Elizabeth Andy, Limerick, Maine.*

Solve this base twelve multiplication alphametic

$$PROF \times EVES = GEOMETRY.$$

902. *Proposed by Bill Correll, Jr., Student, Denison University, Granville, Ohio.*

For all positive integers n, prove that

$$n + 1 \geq \sqrt{2^{1/n}(n^{4/3} + n^{5/3})}.$$

903. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Evaluate the indefinite integral

$$\int \frac{x \ln[x(x - 1)] - \ln(x - 1)}{x(x - 1)}\, dx.$$

904. *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Let n be a positive integer and let $\sigma(n)$ denote the sum of the positive integer divisors of n. If $A = \{n/\sigma(n);\ n \text{ is a positive integer}\}$, prove that A is dense in the interval $(0, 1)$.

905. *Proposed by the late Charles W. Trigg, San Diego, California.*

A permutation of the digits of the four-digit integer 1030 in the decimal system converts it to its equivalent 3001 in the septenary system. Find all four-digit base-ten numerals that can be converted to their base seven equivalents by permuting their digits.

906. *Proposed by Norman Schaumberger, Douglaston, New York.*

If a, b, and c are positive real numbers, prove that

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

907. *Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.*

For $k \geq 0$, evaluate the determinant of the $n \times n$ matrix $A_{n,k}$ whose (i,j)-entry is $(i + j + k - 2)!$. Denote by $n!!$ the product $\Pi_{m=1}^{n} m!$.

908. *Proposed by Andrew Cusumano, Great Neck, New York.*

Evaluate

$$\lim_{n\to\infty}\left[\frac{(n + 2)^{n+2}}{(n + 1)^{n+1}} - \frac{(n + 1)^{n+1}}{n^n}\right].$$

909. *Proposed by the late John M. Howell, Littlerock, California.*

For nonnegative integers n and k, let $a(n, 0) = n$, $a(0, k) = k$, and $a(n + 1, k + 1) = a(n + 1, k) + a(n, k + 1)$. Find a closed formula for $a(n, k)$.

910. *Proposed by William Chau, New York, New York.*

A triangle whose sides have lengths a, b, and c has area 1. Find the line segment of minimum length that joins two sides and separates the interior of the triangle into two parts of area α and $1 - \alpha$, where α is a given number between 0 and 1.

911. *Proposed by Norman Schaumberger, Douglaston, New York.*

If a, b, and c are the lengths of the sides of a triangle with semiperimeter s and area K, show that

$$\left(\frac{s}{s-a}\right)^{\frac{a}{s-a}} + \left(\frac{s}{s-b}\right)^{\frac{b}{s-b}} + \left(\frac{s}{s-c}\right)^{\frac{c}{s-c}} \geq \frac{s^4}{K^2}.$$

912. *Proposed by Paul S. Bruckman, Highwood, Illinois.*

Let p be a prime such that $p \equiv 1 \pmod{60}$. Show that there are positive integers r and s with $p = r^2 + s^2$ and 3 divides r or s and 5 divides r or s.

913. *Proposed by Kenneth B. Davenport, Pittsburgh, Pennsylvania.*

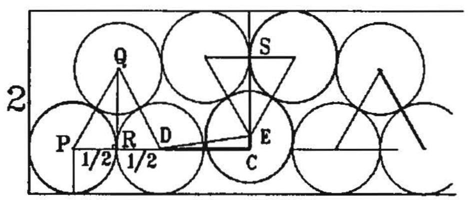
Find a closed formula for the sum

$$1^3 - 2^3 + 3^3 - 4^3 + \cdots + (-1)^{n+1} n^3.$$

Corrections

860. [Spring 1995, Spring 1996] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

This problem originally appeared in a column by the Japanese problems columnist Nob Yoshigahara. Find the minimal positive integer n so that $2n + 1$ circles of unit diameter can be packed inside a 2 by n rectangle.



II. *Comment and solution by Basil Rennie, Burnside, South Australia.*

The engineer looks at the figure on page 336, reproduced here, and sees a lot of slightly greasy roller bearings squeezed between two plates. They are obviously in unstable equilibrium, and so by giving them a little nudge they can be squeezed into a smaller length. Thus circle (Q) can move upward and to the left while circle (P) moves slightly upward, allowing all the other circles to move slightly leftward. By geometric methods similar to those in the solution we find that we gain 0.0077795 units by this shift. Furthermore, if now one more circle is added at the left end to the left of (P) and tangent to the bottom line, then there is room for another circle above this one. These two added circles require 0.9922204979 more width.

Now 107 sets of three circles plus 5 circles added on each end will fit in 159.53537 + 5.45987 = 164.99524, yielding 331 circles in a 2 by 165 rectangle.

When a "greasy shift" is applied to the "rhombi" packing of four circles "glued" together, then 469 circles fit into a 2 by 234 rectangle.

862. [Fall 1996], page 419. *Bram David Weiser, New York, New York, noted that the last sentence in the next to the last paragraph on page 419 should read:* Now $A = D + 1 + H = 7 + H \geq 8$, which is impossible. *An inequality, which is not incorrect but unnecessarily weak, appeared in place of the second equality.*

864. [Fall 1996], page 424. *H.-J. Seiffert pointed out that in the last line of proof I, the phrase* In base 10 *should be deleted. In a lapse of logic the editor added this unnecessary restriction.*

897. [Fall 1996], page 417. *Russell Euler noted that in the fourth line on page 418 the term bx_{x-1} should read bx_{i-1}.*

Solutions

875. [Spring 1996] *Proposed by Howe Ward Johnson, Iceboro, Maine.*

A certain restaurant chain used to advertise "28 flavors" of ice cream. In remembrance of many pleasant stops there, this problem is proposed. Replace each letter by a digit to reconstruct this base ten equation:

$$(ICE)^2 + 28 = ICONE.$$

Solution by Patty Revelant, student, Clarion University, Clarion, Pennsylvania.

Since $110^2 > 12000$, then $I = 1$ and $C = 0$. When 8 is added to E^2, the result must terminate in the digit E, which occurs only for $E = 2, 4, 7$, and 9. Since $107^2 > 11000$, then only 2 and 4 are candidates. Now $104^2 + 28 = 10844$, which requires that $N = E$. Hence, only 2 is permissible. The solution is $102^2 + 28 = 10432$.

Also solved by Avraham Adler, Charles Ashbacher, Frank P. Battles, Paul S. Bruckman, James Campbell, Crystal Casey, Christy Catania, Sandra Rena Chandler, William Chau, Mark Evans, Victor G. Feser, Jayanthi Ganapathy, Robert C. Gebhardt, S. Gendler, Brad Gilbert, Stephanie Hackett, Richard I. Hess, Peter A. Lindstrom, David E. Manes, Yoshinobu Murayoshi, William H. Peirce, Sheryle A. Westfall Proper, George W. Rainey, H.-J. Seiffert, Jesse Sharpe, Kenneth M. Wilke, Rex H. Wu, Chris Zaber, *and the* Proposer.

876. [Spring 1996] *Proposed by Peter A. Lindstrom, Irving, Texas.* Consider the portion of a typical calculator keyboard shown here.

7	8	9
4	5	6
1	2	3

a) Define a *small square number* to be a four-digit number formed by pressing in cyclic order four keys that form a small square, e.g. 1254 or 8569. Show that each small square number is divisible by 11.

b) Define a *large square number* to be a four-digit number formed by pressing in cyclic order the four keys that form the vertices of the large square, e.g. 9713 or 3179. Show that each large square number is divisible by 11.

c) Define a *diamond number* to be a four-digit number formed by pressing in cyclic order the four keys that form a diamond, e.g. 6842 or 2486. Show that each diamond number is divisible by 22.

d) Define a *big square number* to be an eight-digit number formed by pressing in cyclic order the eight keys that form the vertices and sides of the large square, e.g. 98741236 or 14789632. Show that each big square number is divisible by 11 and is divisible by neither 3 nor 5.

e) Define a *rectangular number* to be a six-digit number formed by pressing in cyclic order six keys that form the vertices and sides of a rectangle, e.g. 987456 or 478521. Show that each rectangular number is divisible by 111.

f) Define a *double triangle number* to be a six-digit number formed by pressing in any order the six keys that form the vertices of two right triangles with a common hypotenuse, e.g. 958956 or 421245. Show that each double triangle number is divisible by 3.

Solution by Rex H. Wu, Brooklyn, New York.

We use the following divisibility rules. A base ten number is divisible by 2 if and only if it ends with an even digit, by 3 if and only if the sum of its digits is divisible by 3, by 5 if and only if it ends with 0 or 5, by 11 if the sum of its digits in odd-numbered positions equals the sum of the digits in even-numbered positions (i.e. the number $abcdef$ is divisible by 11 if $a + c + e = b + d + f$), by 111 if the three sums of every third digit are all equal (i.e. the number $abcdefghi$ is divisible by 111 if $a + d + g = b + e + h = c + f + i$). Note that these last two tests are not "if and only if" conditions, but they will suffice here.

For parts (a) and (b), observe that the diagonals have the same sum. Thus $1 + 5 = 2 + 4$ and $9 + 1 = 7 + 3$. Hence any small or large square number is divisible by 11.

c) All the digits of any diamond number are even and their sums $4 + 6$ and $8 + 2$ are equal. Combining the divisibility rules for 2 and 11, it follows that any diamond number is divisible by 22.

d) Any big square number uses all the digits except 0 and 5. Since $1 + 7 + 9 + 3 = 2 + 4 + 8 + 6$, it is divisible by 11. The number cannot end in 0 or 5 and the sum of all its digits is 40, so it is divisible by neither 5 nor 3.

e) It is easy to see that a rectangular number satisfies our test for divisibility for 111. For example, $9 + 4 = 8 + 5 = 7 + 6$, so 987456, its reversal and their cyclic permutations are divisible by 111.

f) The digits of a double triangle number are the same as those of a small square number, whose diagonals have the same sum, as we have already noted. Since the sum of all its digits is equal to 3 times the diagonal sum, any double triangle number is divisible by 3.

Also solved by Paul S. Bruckman, James Campbell, Mark Evans, Victor G. Feser, S. Gendler, Richard I. Hess, David E. Manes, Kenneth M. Wilke, Monte J. Zerger, *and the* Proposer.

877. [Spring 1996] *Proposed by the late John M. Howell, Littlerock, California.*

For given constants a, b, c, d, let $a_0 = a$, $a_1 = b$, and, for $n > 1$, let $a_n = ca_{n-1} + da_{n-2}$.

a) Find a_n in terms of a, b, c, and d.

b) Find $\lim_{n\to\infty} (a_n / a_{n-1})$.

c) Find integers a, b, c, d so that the limit of part (b) is 3.

Solution by H.-J. Seiffert, Berlin, Germany.

a) Case 1: $c^2 + 4d = 0$. Here, an easy mathematical induction argument shows that

$$a_n = \left(\frac{c}{2}\right)^{n-1}\left[\left(b - \frac{ac}{2}\right)n + \frac{ac}{2}\right], \; n \geq 1. \tag{1}$$

If $c \neq 0$, this equation remains valid for $n = 0$.

Case 2: $c^2 + 4d \neq 0$. Assuming that a_n is of the form x^n, then the recursion formula becomes $x^{n+1} = cx^n + dx^{n-1}$, which reduces to the quadratic equation $x^2 - cx - d = 0$. The roots are

$$u = \frac{c + \sqrt{c^2 + 4d}}{2} \quad \text{and} \quad v = \frac{c - \sqrt{c^2 + 4d}}{2}$$

and we have

$$u^2 = cu + d, \; v^2 = cv + d, \; u + v = c,$$

$$u - v = \sqrt{c^2 + 4d}, \text{ and } uv = -d.$$

Since $x = u$ and $x = v$ each satisfy the recursion formula, it is easily proved by induction on n that

$$a_n = \frac{(b - av)u^n - (b - au)v^n}{u - v}, \; n \geq 0. \tag{2}$$

b) We must ensure that $a_n \neq 0$ for all sufficiently large n. This is not fulfilled if, for example, $a = b = 0$ or $c = d = 0$.

Case 1: $c^2 + 4d = 0$, $c \neq 0$, and $a \neq 0$ or $b \neq 0$. From (1) it follows that

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{c}{2}.$$

Case 2: $c^2 + 4d \neq 0$. Using (2) we find that

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \begin{cases} u & \text{if } b \neq av \text{ and } |v| < |u|, \\ v & \text{if } b \neq au \text{ and } |u| < |v|, \\ v & \text{if } b = av, \; b \neq au \text{ and } v \neq 0, \\ u & \text{if } b = au, \; b \neq av \text{ and } u \neq 0. \end{cases}$$

There are many nontrivial cases in which the limit does not exist. For example, if $abd \neq 0$ and $c = 0$, then $a_{2k} = ad^k$ and $a_{2k+1} = bd^k$. Here a_k/a_{k-1} is either ad/b or b/a, according as k is even or odd, and the limit does not exist if $a^2d \neq b^2$.

c) Let $r > 1$, take $c = 1$ and $d = r(r - 1)$. Then $u = r$ and $v = 1 - r$, so that we have $|v| < |u|$. From part (b) it follows that

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = r \text{ if } b \neq a(1 - r).$$

Therefore, a possible choice is $a = b = c = 1$ and $d = 6$.

Also solved by Paul S. Bruckman, Russell Euler, S. Gendler, Richard I. Hess, David E. Manes, Rex H. Wu, *and the* Proposer.

878. [Spring 1996] *Proposed by Andrew Cusumano, Great Neck, New York.*

If x is a solution to the equation $x^2 - ax + 1 = 0$, where a is an integer greater than 2, then show that x^3 can be written in the form $p + q\sqrt{r}$, where p, q, and r are integers.

Solution by Heath Schulterman, student, Fort Smith Northside High School, Barling, Arkansas.

From the quadratic formula we know that the solutions to the above equation are

$$x = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

We cube these roots to see that

$$x^3 = \frac{a^3 - 3a}{2} \pm \frac{a^2 - 1}{2}\sqrt{a^2 - 4}.$$

If a is odd, then both $a^3 - 3a$ and $a^2 - 1$ are even numbers. Thus we may take $p = (a^3 - 3a)/2$, $q = (a^2 - 1)/2$, and $r = a^2 - 4$, and p, q, and r are all integers. If a is even, then $p = (a^3 - 3a)/2$ is still an integer. Now a^2 is divisible by 4, so we take $r = (a^2 - 4)/4$ and $q = a^2 - 1$, and both q and r are integers. In either case we have shown that x^3 can be written in the desired form.

Also solved by Avraham Adler, Miguel Amengual Covas, Frank P. Battles, Paul S. Bruckman, James Campbell, William Chau, Lisa M. Croft, Charles R. Diminnie, Russell Euler, George P. Evanovich, Mark Evans, Victor G. Feser, Jayanthi Ganapathy, Robert C. Gebhardt, S. Gendler, Richard I. Hess, Joe Howard, Thomas C. Leong, Peter A. Lindstrom, David E. Manes, Yoshinobu Murayoshi, William H. Peirce, Bob Prielipp, John F. Putz, H.-J. Seiffert, Skidmore College Problems Group, Kenneth M. Wilke, Rex H. Wu, Monte J. Zerger, *and the* Proposer.

879. [Spring 1996] *Proposed by Barton L. Willis, University of Nebraska at Kearney, Kearney, Nebraska.*

A Mystery Space. Let S be a set of ordered pairs of elements. Define binary operations $+$, $*$, and $\div$ on S by

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, b)*(c, d) = (ac, ad + bc),$$

and

$$(a, b) \div (c, d) = (a \div c, b \div c - ad \div c^2).$$

Although it might be fun to deduce properties of space S (commutativity, associativity, etc.), the problem is to find an application for S.

I. *Solution by Paul S. Bruckman, Highwood, Illinois.*

Let the ordered pair (a, b) denote the fraction b/a. A problem occurs if $a = 0$, which may be circumvented by including a point at infinity [$(0, b) = \infty$ if $b \neq 0$, and $(0, 0)$ is undefined] or by requiring that a be a positive integer.

One more relation needs to be defined:

$$(ka, kb) = (a, b) \text{ for all } k \neq 0.$$

The binary operation "+" is then defined as the operation whereby the numerators and denominators of the two fractions, respectively, are added. The result is called the *mediant* of the two given fractions, one application of which is found in baseball batting averages. A batter with 10 hits in 40 times at bat has a batting average of $1000 \times 10/40 = 250$. If in the next game that batter gets 2 hits in 4 times at bat, his or her average for the game is $1000 \times 2/4 = 500$ and the new accumulative batting average becomes the mediant $1000(10 + 2)/(40 + 4) = 273$ of the two previous fractions.

The binary operations "*" and "÷" represent respectively the sum and difference of the two fractions. That is,

$$(a, b) \text{"*"} (c, d) = b/a + d/c = (ad + bc)/ac = (ac, ad + bc)$$

and

$$(a, b) \text{"}\div\text{"} (c, d) = b/a - d/c = (bc - ad)/ac = (b/c - ad/c^2)/(a/c)$$

$$= (a/c, b/c - ad/c^2).$$

Of course, there is no unit under "+" (unless we allow $(0, 0)$), the unit under "*" is $(1, 0)$, and since we have (a, b) "÷" $(1, 0) = (a, b)$, then $(1, 0)$ is the right hand unit under "÷".

II. *Solution by David E. Manes, SUNY College at Oneonta, Oneonta, New York.*

Let $A(x) = \Sigma_{n\geq 0} a_n x^n$ be a formal power series in x with coefficients from the set of real numbers and with $a_0 \neq 0$. Let the elements of the ordered pair (a, b) be the first two coefficients of $A(x)$, that is, (a_0, a_1), the constant term and the coefficient of x. Then addition, multiplication, and division of ordered pairs correspond to the addition, multiplication, and division of these power series. Of course, we must ignore all terms of degree higher than 1.

III. *Solution by the Proposer.*

For a fixed value of x, say $x = r$, and functions f and g, both differentiable at r, let $(a, b) = (f(r), f'(r))$ and $(c, d) = (g(r), (g'(r))$. Then the formulas give the sum, difference, and quotient of functions and their derivatives. Thus the values of the function $H(x) = (1 + x)/(2 + x)$ and its derivative at $x = 4$ are given by

$$\frac{(1,0) + (4,1)}{(2,0) + (4,1)} = \frac{(5,1)}{(6,1)} = \left(\frac{5}{6}, \frac{1}{36}\right).$$

Thus $H(4) = 5/6$ and $H'(4) = 1/36$.

880. [Spring 1996] *Proposed by Rex H. Wu, Brooklyn, New York.*

Evaluate, where $i = \sqrt{-1}$,

$$\lim_{n\to\infty} \frac{n}{4i}(e^{2ai/n} - e^{-2ai/n}).$$

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Since $\sin x = (e^{ix} - e^{-ix})/2i$, we have

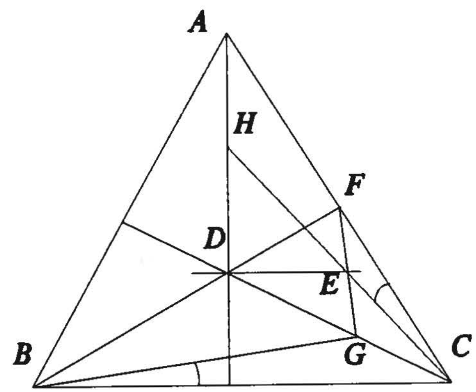
$$\lim_{n\to\infty} \frac{n}{4i}(e^{2ai/n} - e^{-2ai/n}) = \lim_{n\to\infty} \frac{n}{2} \sin\frac{2a}{n} = a.$$

Also solved by Avraham Adler, Miguel Amengual Covas, Frank P. Battles, James D. Brasher, Paul S. Bruckman, James Campbell, William Chau, Kenneth B. Davenport, Charles R. Diminnie, Russell Euler, George P. Evanovich, Jayanthi Ganapathy, Robert C. Gebhardt, Joe Howard, Thomas C. Leong, Peter A. Lindstrom, David E. Manes, Michael R. Pinter, Bob Prielipp, George W. Rainey, H.-J. Seiffert, Timothy Sipka, *and the* Proposer.

881. [Spring 1996] *Proposed by Andrew Cusumano, Great Neck, New York.*

Let ABC be an equilateral triangle with center D. Let α be an arbitrary positive angle less than $30°$. Let BD meet CA at F. Let G be that point on segment CD such that angle $CBG = \alpha$, and let E be that point on FG such that angle $FCE = \alpha$. Prove that DE is parallel to BC.

Solution by William H. Peirce, Delray Beach, Florida.



We give an analytic solution in which the letter representing a point can be thought of as the complex number affix of that point, the ordered pair of Cartesian coordinates for the point, or the vector from the origin to the point. Two lemmas are used in this solution: (1) any point on the line through two distinct points can be expressed uniquely as a linear combination of the two points in which the two real coefficients add to 1, and (2) any point in the plane of three noncollinear (and therefore distinct) points can be expressed uniquely as a linear combination of the three points in which the three coefficients add to 1.

Thus $D = (A + B + C)/3$ and $F = (A + C)/2$. Let CE meet AD at H. Because triangles BGC and CHA are congruent, by Lemma 1 there is a real number λ that is a function of angle α such that both

$$G = (1 - \lambda)C + \lambda D \text{ and } H = (1 - \lambda)A + \lambda D.$$

Now E lies on both FG and CH, so there are real numbers μ and ν such that

$$E = (1 - \mu)F + \mu G \text{ and } E = (1 - \nu)C + \nu H.$$

In terms of A, B and C we therefore have, respectively,

$$E = \left(\frac{1}{2} - \frac{\mu}{2} + \frac{\lambda\mu}{3}\right)A + \frac{\lambda\mu}{3}B + \left(\frac{1}{2} + \frac{\mu}{2} - \frac{2\lambda\mu}{3}\right)C$$

and

$$E = \left(\nu - \frac{2\lambda\nu}{3}\right)A + \frac{\lambda\nu}{3}B + \left(1 - \nu + \frac{\lambda\nu}{3}\right)C$$

and they must be equal. Equating the corresponding coefficients of A, B, and C gives $\mu = \nu = 1/(3 - 2\lambda)$, and therefore

$$E = \frac{1}{3}A + \frac{\lambda}{3(3 - 2\lambda)}B + \frac{6 - 5\lambda}{3(3 - 2\lambda)}C.$$

From this expression we find that

$$D - E = \frac{1 - \lambda}{3 - 2\lambda}(B - C),$$

which shows that DE is parallel to BC.

Line DE is not defined and the solution breaks down when $\lambda = 0$, 1, or 3/2, which correspond respectively to $G = C$, D, or the midpoint of AB, that is, $\alpha = 0°$, $30°$, or $60°$. Otherwise there is no restriction on α or on the location of G on line CD, including the point at infinity.

Also solved by Miguel Amengual Covas, Paul S. Bruckman, Mark Evans, Yoshinobu Murayoshi, *and the* Proposer.

882. [Spring 1996] *Proposed by Rex Wu, Brooklyn, New York.*

Define, for any nonnegative integer m and any real number n,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}. \text{ Otherwise } \begin{bmatrix} n \\ m \end{bmatrix} = 0.$$

Find the values of

$$\text{(a)} \sum_{i=n}^{k} \begin{bmatrix} i \\ n \end{bmatrix} \quad \text{and} \quad \text{(b)} \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix}.$$

Solution by Thomas E. Leong, The City College of CUNY, New York, New York.

Note: we assume that $\binom{n}{0} = 1$ for all real n.

(a) A bit more generally we show that for any real number a and nonnegative integers n and k,

$$\binom{a}{n} + \binom{a+1}{n} + \binom{a+2}{n} + \cdots + \binom{a+k}{n} = \binom{a+k+1}{n+1} - \binom{a}{n+1}.$$

Recall Newton's binomial formula, that for any real number a and $|x| < 1$, we have

$$(1 + x)^a = 1 + \binom{a}{1}x + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \cdots.$$

Thus the desired sum is the coefficient of x^n in

$$(1 + x)^a + (1 + x)^{a+1} + \cdots + (1 + x)^{a+k}$$

$$= \frac{(1 + x)^{a+k+1} - (1 + x)^a}{(1 + x) - 1} = \frac{1}{x}[(1 + x)^{a+k+1} - (1 + x)^a],$$

which clearly is $\binom{a+k+1}{n+1} - \binom{a}{n+1}$. It follows that

$$\sum_{i=n}^{k} \binom{i}{n} = \binom{k+1}{n+1} - \binom{n}{n+1} = \binom{k+1}{n+1}.$$

(b) We have that

$$\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} = \sum_{i=0}^{k} \begin{bmatrix} k \\ k-i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix}$$

is the coefficient of x^k in

$$\left[x^k + \binom{k}{k-1}x^{k-1} + \cdots + \binom{k}{1}x + 1\right]\left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots\right]$$

$$= (1 + x)^k(1 + x)^n = (1 + x)^{n+k},$$

which indisputably is $\binom{n+k}{k}$. Thus we have

$$\sum_{i=1}^{k} \binom{k}{i} \binom{n}{i} = \binom{n+k}{k} - 1.$$

Also solved by Paul S. Bruckman, William Chau, Mark Evans, David E. Manes, H.-J. Seiffert, *and the* Proposer. *Bruckman, Manes, and Seiffert each proved Part (b) as an immediate consequence of the Vandermonde convolution formula, found on page 169 of [1]:*

$$\sum_{k} \binom{r}{m+r} \binom{s}{n-k} = \binom{r+s}{m+n}.$$

Reference

1. R. L. Graham, D. E. Knuth, O. Pasternak, *Concrete Mathematics*, 2nd ed, Addison-Wesley, New York, 1994.

883. [Spring 1996] *Proposed by Sammy Yu (student), University of South Dakota, Vermillion, South Dakota.*

M. N. Khatri [*Scripta Mathematica*, 1955, vol. 21, p. 94] found that from the identity $T(4) + T(9) = T(10)$, where $T(n) = n(n+1)/2$ is the nth triangular number, Pythagorean triples (5, 12, 13) and (8, 15, 17) produce the more general formulas $T(4 + 5k) + T(9 + 12k) = T(10 + 13k)$ and $T(4 + 8k) + T(9 + 15k) = T(10 + 17k)$, where k is a positive integer. Given p, q, r, so that $T(p) + T(q) = T(r)$, find Pythagorean triples (a, b, c) so that $a^2 + b^2 = c^2$ and $T(p + ak) + T(q + bk) = T(r + ck)$ for any positive integer k.

Solution by William H. Peirce, Delray Beach, Florida.

Given a Pythagorean triple (a, b, c) and positive integers p, q, and r such that $T(p) + T(q) = T(r)$, the equation $T(p + ak) + T(q + bk) = T(r + ck)$ reduces to

$$a(2p + 1) + b(2q + 1) = c(2r + 1). \qquad (1)$$

Pythagorean triples can be written as

$$a = u^2 - v^2,\ b = 2uv, \text{ and } c = u^2 + v^2. \qquad (2)$$

Now substitute (2) into (1) and simplify to obtain

$$(r - p)u^2 - (2q + 1)uv + (p + r + 1)v^2 = 0,$$

whose two solutions are

$$(r - p)u = qv \text{ and } (r - p)u = (q + 1)v.$$

From these equations we take $(u, v) = (q, r - p)$ and $(q + 1, r - p)$, which give

$$(a, b, c) = (q^2 - [r - p]^2, 2q[r - p], q^2 + [r - p]^2)$$

and

$$(a, b, c) = ([q + 1]^2 - [r - p]^2, 2[q + 1][r - p], [q + 1]^2 + [r - p]^2).$$

Finally, any factors common to a, b, and c are removed to give the final two primitive solution triples (a, b, c).

For example, $(p, q, r) = (9, 13, 16)$ satisfy $T(p) + T(q) = T(r)$. Then $(u, v) = (13, 7)$ or $(14, 7)$, which produce the triples $(a, b, c) = (120, 182, 218)$ and $(147, 196, 245)$. Removing the common factor 2 from the first triple and 49 from the second gives us the two primitive solution triples $(60, 91, 109)$ and $(3, 4, 5)$.

Also solved by Paul S. Bruckman, Richard I. Hess, Joe Howard, David E. Manes, Bob Prielipp, *and the* Proposers.

884. [Spring 1996] *Proposed by Seema Chauhan, Lucknow, India.*

a) Held every day is a tutorial class in which $2m$ students are enrolled. Exactly m of these students, selected at random, attend class on any given day. If the class meets for exactly $2r$ days, find the probability that in the end each student has attended exactly r classes.

*b) The class of part (a) contains m boys and m girls. For each p, $0 \leq p \leq r$, find the probability that each girl attends exactly $r + p$ classes and each boy attends just $r - p$ classes.

Solution by Paul S. Bruckman, Highwood, Illinois.

We solve Part (b) first, and show that it is a generalization of Part (a). For brevity we write

$$M = \binom{2m}{m} \text{ and } a = M/2 = \binom{2m-1}{m}.$$

Let $N_{r,p,a}$ and $T_{r,a}$ denote the number of ways to form $2r$ classes out of the $2m$ students with the restrictions as described and without restriction, respectively. Each class consists of k males and $m-k$ females, say, where $0 \le k \le m$, and the total number of ways in which such a class is made up is equal to

$$\sum_{k=0}^{m}\binom{m}{k}\binom{m}{m-k} = \binom{2m}{m} = M.$$

In counting $T_{r,a}$, each class is made up independently of the others, and so

$$T_{r,a} = M^{2r}. \tag{1}$$

In counting $N_{r,p,a}$, if a particular one of the M possible selections of m students in a class occurs a total of n times, say, then the classes consisting of the remaining m students must also occur n times, with one exception if $p > 0$. That exception occurs if the class is made up entirely of boys; if such classes occur n times, say, then the classes consisting entirely of girls must occur $n + 2p$ times, in order to fulfill the conditions of the problem.

Note that we may express $T_{r,a}$ in terms of multinomial coefficients, as

$$T_{r,a} = \sum_{n_1+n_2+\cdots+n_M=2r}\binom{2r}{n_1,n_2,\ldots,n_M}. \tag{2}$$

Similarly, based on the foregoing comments, we may express $N_{r,p,a}$ as

$$N_{r,p,a} = \sum_{n_1+n_2+\ldots+n_a=r-p}\binom{2r}{n_1,n_1+2p,n_2,n_2,\ldots,n_a,n_a}. \tag{3}$$

We also have

$$N_{r,p,a} = \binom{2r}{r-p}\sum_{n_1+n_2+\ldots+n_a=r-p}\binom{r-p}{n_1,n_2,\ldots,n_a}\binom{r+p}{n_1+2p,n_2,\ldots,n_a}. \tag{4}$$

We make the substitution

$$S_{r,p,a} = N_{r,p,a} \div \binom{2r}{r-p}. \tag{5}$$

Note that

$$S_{r,p,a} = \sum_{n=0}^{r-p}\binom{r-p}{n}\binom{r+p}{n}\sum_{n_2+n_3+\ldots+n_a=n}\binom{n}{n_1,n_2,\ldots,n_a}^2.$$

Temporarily we detach a from its dependence on m, and allow a to be any positive integer. With this proviso, we then see that

$$S_{r,p,a} = \sum_{n=0}^{r-p}\binom{r-p}{n}\binom{r+p}{n}S_{n,0,a-1}. \tag{6}$$

Next we form the formal generating function $F_{p,a}(x)$ by

$$F_{p,a}(x) = \sum_{r=p}^{\infty}\frac{S_{r,p,a}x^{r-p}}{(r-p)!(r+p)!}. \tag{7}$$

Substitute the recurrence expression of (6) into (7) to obtain

$$F_{p,a}(x) = \sum_{s=0}^{\infty}\frac{x^s}{s!(s+2p)!}\sum_{n=0}^{\infty}\frac{S_{n,0,a-1}x^n}{n!n!} = I_p(x)\cdot F_{0,a-1}(x), \tag{8}$$

where

$$I_p(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!(n+2p)!}. \tag{9}$$

Define $S_{0,0,0} = 1$ and $S_{r,0,0} = 0$ if $r > 0$. We see from (7) and (8) that $F_{0,a}(x) = I_0(x)F_{0,a-1}(x)$, and that $F_{0,0}(x) = 1$. It follows that

$$F_{0,a}(x) = (I_0(x))^a. \tag{10}$$

Returning to (8), we then see that

$$F_{p,a}(x) = I_p(x)(I_0(x))^{a-1}. \tag{11}$$

It then follows that $N_{r,p,a}$ is $(2r)!$ times the coefficient of x^{r-p} in the expansion of $F_{p,a}(x)$ as given by (11). The first few terms of this expansion (in terms of $S_{r,p,a}$) are

$$S_{0,0,a} = 1,\ S_{1,0,a} = a,\ S_{1,1,a} = 1;$$

$$S_{2,0,a} = 2a^2 - a,\ S_{2,1,a} = 3a - 2,\ S_{2,2,a} = 1;$$

$$S_{3,0,a} = 6a^3 - 9a^2 + 4a;\ S_{3,1,a} = 12a^2 - 22a + 11;\ S_{3,2,a} = 5a - 4;\ S_{3,3,a} = 1.$$

Then the required probability is given by

$$P_{r,p,a} \equiv \frac{N_{r,p,a}}{T_{r,a}} = \binom{2r}{r-p} M^{-2r} S_{r,p,a}.$$

In terms of $M = 2a$ the first few results are

$$P_{1,0,a} = 1/M,\ P_{1,1,a} = 1/M^2;\ P_{2,0,a} = 3(M-1)/M^3,$$

$$P_{2,1,a} = (6M-8)/M^4,\ P_{2,2,a} = 1/M^4;\ P_{3,0,a} = 5(3M^2 - 9M + 8)/M^5,$$

$$P_{3,1,a} = 15(3M^2 - 11M + 11)/M^6,\ P_{3,2,a} = 3(5M-8)/M^6,\ P_{3,3,a} = 1/M^6.$$

As mentioned, Part (a) is the special case of Part (b) for which $p = 0$.

Part (a) also solved by the Proposer.

885. [Spring 1996] *Proposed by Arthur Marshall, Madison, Wisconsin.*
Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{3^{(2n-1)/2}(2n-1)}.$$

Solution by Kenneth P. Davenport, Pittsburgh, Pennsylvania.
Since

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1},$$

then we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{3^{(2n-1)/2}(2n-1)} = 6\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1/\sqrt{3})^{2n-1}}{2n-1} = 6\tan^{-1}\frac{1}{\sqrt{3}} = \pi.$$

Also solved by Frank P. Battles, Paul S. Bruckman, James Campbell, Charles R. Diminnie, Russell Euler, George P. Evanovich, Jayanthi Ganapathy, Robert C. Gebhardt, Richard I. Hess, Joe Howard, Thomas C. Leong, Peter A. Lindstrom, David E. Manes, Bob Prielipp, H.-J. Seiffert, Rex H. Wu, *and the* Proposer.

886. [Spring 1996] *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Find the general solution in integers to the equation $x^2 - 8y + 7 = 0$.

I. *Solution by David Tascione, student, St. Bonaventure University, St. Bonaventure, New York.*

Suppose that (x, y) is a solution of $x^2 = 8y - 7$. Since the right side is odd for any integer y, then x must be odd. We therefore take $x = 2n + 1$ for n an integer. Then

$$y = [(2n+1)^2 + 7]/8 = 1 + n(n+1)/2$$

which is an integer. It follows that $(x, y) = (2n + 1, 1 + n(n + 1)/2)$ is the general solution.

II. *Solution by Skidmore College Problem Group, Saratoga Springs, New York.*

It is clear that $y = (x^2 + 7)/8$, so if y is an integer, then

$$x^2 \equiv 1 \pmod 8.$$

This congruence is true for all x where x is an odd positive integer. It can be shown by induction that all integer solutions may be written as a sequence of ordered pairs (x_i, y_i) where $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (3, 2)$, and for $i > 1$, $(x_i, y_i) = (x_{i-1} + 2,\ x_{i-1} + y_{i-2})$.

Also solved by Miguel Amengual Covas, Charles Ashbacher, Frank P. Battles, Paul S. Bruckman, James Campbell, Sandra Rena Chandler, William Chau, Kenneth B. Davenport, George P. Evanovich, Mark Evans, Victor G. Feser, Jayanthi Ganapathy, Robert C. Gebhardt, Richard I. Hess, Thomas C. Leong, Peter A. Lindstrom, David E. Manes, Yoshinobu Murayoshi, William H. Peirce, Bob Prielipp, George W. Rainey, H.-J. Seiffert, Timothy Sipka, Kenneth M. Wilke, Rex H. Wu, *and the* Proposer.

887. [Spring 1996] *Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.*

The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$. Compute the following sums involving Fibonacci numbers:

$$S_{1,n} = \sum_{k=1}^{n} \frac{1}{F_{2k-1}F_{2k+1}} \quad \text{and} \quad S_{2,n} = \sum_{k=1}^{n} \frac{1}{F_{2k}F_{2k+2}}.$$

Also find their limits S_1 and S_2 as $n \to \infty$. Express the finite sums as rational numbers in lowest terms. Finally, simplify each of the following expressions:

$$a = \frac{S_1^2}{S_2}, \quad b = \frac{1}{S_{2,n}} - \frac{1}{S_{1,n}}, \quad c = S_1 - S_2,$$

$$d = \frac{S_{1,n}}{S_{2,n}} - S_{1,n}, \quad \text{and} \quad e = \frac{1}{S_{1,n}^2} - \frac{1}{S_{2,n}}.$$

Solution by the Proposer.

The Fibonacci numbers satisfy the determinantal relations

$$\begin{vmatrix} F_{j+2} & F_j \\ F_{j+1} & F_{j-1} \end{vmatrix} = \begin{vmatrix} F_{j+1} & F_j \\ F_j & F_{j-1} \end{vmatrix} = -\begin{vmatrix} F_j & F_{j-1} \\ F_{j-1} & F_{j-2} \end{vmatrix} = (-1)^j \begin{vmatrix} F_3 & F_2 \\ F_2 & F_1 \end{vmatrix} = (-1)^j.$$

Hence we can compute $S_{1,n}$ and $S_{2,n}$ as telescoping series

$$S_{1,n} = \sum_{k=1}^{n} \left(\frac{F_{2k}}{F_{2k+1}} - \frac{F_{2k-2}}{F_{2k-1}} \right) = \frac{F_{2n}}{F_{2n+1}} - \frac{0}{1} = \frac{F_{2n}}{F_{2n+1}}$$

and

$$S_{2,n} = \sum_{k=1}^{n} \left(-\frac{F_{2k+1}}{F_{2k+2}} + \frac{F_{2k-1}}{F_{2k}} \right) = -\frac{F_{2n+1}}{F_{2n+2}} + 1.$$

Since successive Fibonacci numbers are relatively prime, then $S_{1,n} = F_{2n}/F_{2n+1}$ and $S_{2,n} = F_{2n}/F_{2n+2}$ are rational numbers in lowest terms. Also $S_{1,n}$ and $S_{1,n}/S_{2,n} = F_{2n+1}/F_{2n+2}$ have a common limit S_1 as $n \to \infty$. Hence $a = S_1^2/S_2 = 1$. Since

$$b = 1/S_{2n} - 1/S_{1,n} = (F_{2k+2} - F_{2k+1})/F_{2k} = 1,$$

we have $1/S_1^2 - 1/S_1 = 1$, so that $S_1^2 + S_1 - 1 = 0$. Then

$S_1 = (\sqrt{5} - 1)/2$, $S_2 = S_1^2 = (3 - \sqrt{5})/2$, $c = S_1 - S_1^2 = S_1^3 = \sqrt{5} - 2$,

$d = (F_{2n+2} - F_{2n})/F_{2n+1} = 1$, and $e = (F_{2n+1}^2 - F_{2n}F_{2n+2})F_{2n}^2 = 1/F_{2n}^2$.

In summary,

$S_{1,n} = F_{2n}/F_{2n+1}$, $S_{2,n} = F_{2n}/F_{2n+2}$, $S_1^2 = S_2$, $a = b = d = S_1 + S_2 = 1$,

$$c = S_1 - S_2 = \sqrt{5} - 2, \text{ and } e = 1/F_{2n}^2.$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, George P. Evanovich, Mark Evans, Richard I. Hess, David E. Manes, William H. Peirce, Bob Prielipp, *and* H.-J. Seiffert.

Notice

Starting in the Fall of 1995, to save space in this JOURNAL for articles, we stopped listing the affiliations of also-solvers. It has been made clear to us that this was not a popular decision. So we have reconsidered and shall again print your affiliations. Since this decision was made just as this issue was being sent to press, that listing will return in the Fall of 1997. We try our best to make this JOURNAL interesting and appropriate for its audience and we welcome your comments, criticisms, and suggestions.

THE RICHARD V. ANDREE AWARDS

The Richard V. Andree Awards are given annually to the authors of the three papers written by students that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the **Pi Mu Epsilon Journal** in the past year.

Until his death in 1987, Richard V. Andree was Professor Emeritus of Mathematics at the University of Oklahoma. He had served **Pi Mu Epsilon** for many years and in a variety of capacities: as President, as Secretary-Treasurer, and as Editor of the **Journal**.

The winner of the first prize for 1996 is **Rick Mohr** for his paper "Nearness of Normals", this **Journal** 10(1994-99)#4, 257-264.

The winner of the second prize is **Ryan Bennink** for his paper "Red Light, Green Light: A Model of Traffic Signal Systems", this **Journal** 10(1994-99)#5, 353-363.

Since there was a two-way tie for third place, there will be four awards this year. The winners are **Carolyn Farruggia, Michael Lawrence,** and **Brian Waterhouse** for their paper "The Elimination of a Family of Periodic Parity Vectors in the 3x + 1 Problem", this **Journal** 10(1994-99)#4, 275-280, and **Mark Tomforde** for his paper "Self-Similarity and Fractal Dimension of Certain Generalized Arithmetical Triangles", this **Journal** 10(1994-99)#5, 379-389.

At the time the papers written **Mr. Mohr** was at Rose-Hulman Institute of Technology; **Mr. Bennink** was at Hope College; **Ms. Farruggia, Mr. Lawrence** and **Mr. Waterhouse** were at the University of Scranton; and **Mr. Tomforde** was at Gustavus Adolphus College.

The officers and councilors of the Society congratulate the winners on their achievements and wish them well for their futures.

1996 NATIONAL PI MU EPSILON MEETING

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held at the University of Washington in Seattle from August 9 through August 11. As in the past, the meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections.

The J. Sutherland Frame Lecturer was **J. Kevin Colligan**, National Security Agency. His presentation was on "Webs, Sieves, and Money: Number Theory's Rubber Hits the I-Way Road."

The following thirty student papers were presented at the meeting. An asterisk (*) before the name of the presenter indicates that the speaker received a best paper award.

Program - Student Paper Pi Mu Epsilon Sessions

The Math Behind the Schrodinger Equation	**Janet Bernard** Virginia Epsilon Longwood College
Extensions of the Tower of Hanoi	* **Scott Clark** Ohio Xi Youngstown State University
Graphs of Groups and Sub-Groups	**James A. Cole** Arkansas Beta Hendrix College
Reflection Groups in Two and Three Dimensions	**Katherine Crow** Massachusetts Beta Holy Cross College
Qualitative Analysis of Dynamic Systems	* **Philip Darcy** New York Omega St. Bonaventure University

A Comparative Study on the Variance of Thickness, Gogan Hardness, and Specific Gravity in Brakes — **Scott Delaney**, Virginia Zeta, Mary Washington College

Genetic Algorithms — **Brian Drobet**, Michigan Alpha, Michigan State University

The Creation of the National PME Home Page — **Nathan Gibson**, **Anu Karna**, Massachusetts Alpha, Worcester Polytechnic Institute

Effects of the Tide — **Jacqueline Gosz**, Wisconsin Delta, St. Norbert College

Computing Integrals for the Invariant Measure of Elementary Fractals — *** Stephen Haptonstahl**, Louisiana Alpha, Louisiana State University

Selection Schemes for Judging Essay Contests — **Andrew Hetzel**, Ohio Zeta, University of Dayton

Density Functional Theory in Chemistry — **John Hybl**, Michigan Alpha, Michigan State University

Graceful Creatures of the Sea — *** Kim Jordan**, Ohio Xi, Youngstown State University

Delgarno-Lewis Methods for Second Order Energies of Neon — **Robert Komara**, Ohio Xi, Youngstown State University

Two-Color Rado Numbers for Some Inequalities — *** Pi-Yeh Liu**, Pennsylvania Lambda, Clarion University

Matrix Integrals and the Topology of the Moduli Spaces of Riemann Surfaces — **Laura Loos**, California Lambda, University of California, Davis

The KGB, Espionage, and RSA Encryption — *** Vincent Lucarelli**, Ohio Xi, Youngstown State University

Student Tutorials in Maple-Mathematical Software — **Cali Manning**, Alabama Alpha, University of Alabama

Minimum Defining Sets in Vertex Colorings of Networks — **Thayer Morrill**, Ohio Delta, Miami University

Sliding Piece Puzzles with Permutations on the Rows and Columns — **David Murphy**, Michigan Epsilon, Western Michigan University

Applications of Advanced Mathematics in Gardening — **Clinton David Nolan III**, Georgia Epsilon, Valdosta State University

The Studentization of Individual Multivariate Observations — **Sherry Parker**, Louisiana Epsilon, McNeese State University

Nim's Sum — **Wendy Rigterink**, Ohio Delta, Miami University

Ovals in Kaleidoscopic Tilings on Closed Surfaces — **Dennis A. Schmidt**, Wisconsin Delta, St. Norbert College

A Paradox of Function Approximations — **Mark Short**, Wisconsin Delta, Carthage College

Pipeflow in the Region of a Bifurcation — * **Eugene Sy**, California Lambda, University of California, Davis

A Discrete Time Martingale Discussion of the Free Group with Two Generators a and b — **Michael Everett Theriot, Jr.**, Louisiana Alpha, Louisiana State University

Days of the Week from Dates — **Bryan Treusch**, Ohio Xi, Youngstown State University

Shape: The Unsolved Mystery — **Rebecca Weingard**, Virginia Alpha, University of Richmond

To Be or Not To Be Written by Shakespeare — **Hande Yurttan**, D.C. Beta, Georgetown University

List of Referees

The editor wishes to acknowledge the substantial contributions made by the following mathematicians who reviewed manuscripts for the **Pi Mu Epsilon Journal** during the past year.

Phil Barker, University of Missouri-Kansas City
Philip J. Byrne, College of St. Benedict
Gerard Buskes, University of Mississippi
Hang Chen, Central Missouri State University
Curtis Cooper, Central Missouri State University
Richard Delaware, University of Missouri - Kansas City
Underwood Dudley, DePauw University
Larry Eifler, University of Missouri - Kansas City
J. Douglas Faires, Youngstown State University
James Foran, University of Missouri - Kansas City
Ramesh Garimella, Tennessee Technological University
Betty Jean Harmsen, Northwest Missouri State University
Terry King, Northwest Missouri State University
Michael Kinyon, Indiana University South Bend
Dennis Malm, Northwest Missouri State University
Wayne McDaniel, University of Missouri - St. Louis
Allan D. Mills, Tennessee Technological University
Michael Motto, Northwest Missouri State University
Charles B. Pierre, Clark Atlantic University
Jawad Sadek, Northwest Missouri State University
Norman Schaumberger, Bronx Community College
Lawrence Somer, The Catholic University of America
Michael Steiner, University of Texas - Austin
Songlin Tian, Central Missouri State University
Jingcheng Tong, University of North Florida

MISCELLANY

Letter to the Editor

Basil Rennie of South Australia sent the following comment on the note "Approximating $e^n/2$ with nearly n + 1/3 Terms" by **Frame** and **Frenzen** (Volume 10, Number 5, Fall 1996).

> These authors are treading in the footsteps of Ramanujan, who in a partial solution of a problem in The Journal of the Indian Math Soc. commented that if

$S = 1 + n + n^2/2 + \cdots + n^{n-1}/(n-1)!$ and $M = n^n/n!$, then

$$e^n \quad 2S + \frac{2}{3} M\left(1 + \frac{4}{45} n^{-1} - \frac{8}{945} n^{-2} - \frac{16}{2835} n^{-3} + \cdots\right).$$

In Memoriam

Leon Bankoff practiced dentistry for sixty years in Beverly Hills, California. Among his other interests, such as piano and guitar, he lectured and wrote papers both on dentistry and mathematics. His specialty was geometry, and the figure he loved best was the arbelos, or shoemaker's knife. It is said that the test of a mathematician is not what he himself has discovered, but what he inspired others to do. Leon's discovery of a third circle congruent to each of the twin circles of Archimedes motivated the discovery of several other members of that family of circles. An article on those circles is in progress.

Dr. Bankoff edited this Problem Department from 1968 to 1981, setting and maintaining a high standard of excellence in the more than 300 problems he included in these pages. His influence continues today.

Leon and I became good friends, first through correspondence and later through many personal meetings. On February 16, 1997, he died of cancer at age 88. Right up to his last few weeks he worked on the manuscript for a proposed book on the properties of the arbelos, carrying on work started by his collaboration with the late Victor Thébault.

He was a gentleman and a scholar, a true friend.

Clayton W. Dodge

Francis Regan, professor and chair emeritus of St. Louis University, died on February 18, 1996. Born on January 10, 1903, he was a member of the American Mathematical Society for sixty-three years. Francis was editor of the **Pi Mu Epsilon Journal** from 1957 to 1963.

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ATLANTA OLYMPICS!

The 1997 Meeting of the Pi Mu Epsilon National Honorary Mathematics Society will be held in Atlanta, GA, from August 1-3. The meeting will be held in conjunction with the MAA Mathfest, which will run from August 1-4. Pi Mu Epsilon will again coordinate its national meeting with that of the MAA student chapters.

The Pi Mu Epsilon meeting will begin with a reception on the evening of Friday, August 1. On Saturday morning, August 2, the Pi Mu Epsilon Council will have its annual meeting. The student presentations will begin later that same day. The presentations will continue on Sunday, August 3. The Pi Mu Epsilon Banquet will take place that evening, followed by the J. Sutherland Frame Lecture. This year's Frame lecture will be given by Philip Straffin, of Beloit College. Pi Mu Epsilon members are encouraged to participate in the MAA Student Chapter Workshop and Student Lecture, both of which will take place on Monday, August 4.

*TRAVEL SUPPORT FOR STUDENT SPEAKERS

Pi Mu Epsilon will provide travel support for student speakers at the national meeting. The first speaker is eligible for 25 cents per mile, up to a maximum of $600. If a student chooses to use public transportation, PME will reimburse for the actual cost of transportation, up to a maximum of $600. In case this request exceeds 25 cents per mile, receipts should be presented. The first four additional speakers are eligible for 20% of whatever amount the first speaker receives. In the case of more than one speaker from one chapter, the speakers may share the allowance in any way that they see fit. If a chapter is not represented by a student speaker, Pi Mu Epsilon will provide one-half support for a student delegate. For further information about the meeting and the travel support:

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PI MU EPSILON

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Saint Olaf College

Sponsored by: St. Norbert College Chapter of ΠΜΕ
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The conference will begin on Friday evening and continue through Saturday noon. Highlights of the conference will include sessions for student papers and two presentations by Professor Humke, one on Friday evening and one on Saturday morning. Anyone interested in undergraduate mathematics is welcome to attend. All students (who have not yet received a master's degree) are encouraged to present papers. The conference is free and open to the public.

For information, contact:

Rick Poss, St. Norbert College
De Pere, WI 54115
(414) 403-3198
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Σ Ν Δ

PI MU EPSILON

T-SHIRTS

The shirts are white, Hanes® BEEFY-T®, pre-shrunk, 100% cotton. The front has a large Pi Mu Epsilon shield (in black), with the line "1914 - ∞" below it. The back of the shirt has a "Π Μ Ε" tiling, designed by **Doris Schattschneider,** in the PME colors of gold, lavender, and violet. The shirts are available in sizes **large** and **X-large**. The price is only $10 per shirt, which includes postage and handling. To obtain a shirt, send your check or money order, payable to **Pi Mu Epsilon**, to:

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