PROBLEM DEPARTMENT

ASHLEY AHLIN* AND HAROLD REITER†

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2007. Solutions identified as by students are given preference.*

Problems for Solution.

1146. *Proposed by Douglas Shafer, University of North Carolina Charlotte*

Given six real constants a, b, c, d, e, and f, not all zero, a conic section C : $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is determined. Since rescaling the six coefficients by a nonzero number does not change C, we may view (a, b, c, d, e, f) as lying in $S^5 \subset \mathbf{R}^6$. If (a, b, c, d, e, f) is selected based on a uniform distribution on S^5, what is the probability that C is an ellipse?

No complete solutions to this problem were submitted. Therefore the problem will remain open until October 2007.

1147. *Proposed by Cecil Rousseau, University of Memphis*

The nth pentagonal number is $p(n) = n(3n - 1)/2$. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{p(n)}.$$

1148. *Proposed by Paul S. Bruckman, Sointula, BC*

Given a triangle with angle measures A, B, C, characterize those triangles such that $\cos 2A + \cos 2B + \cos 2C = -1$.

1149. *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

Let f and g be integer functions with $f(n) = 11n + 5$, $g(n + 1) = f(g(n))$ and $g(0) = 1$ and let $g(3)$ be nice and lucky and happy to you! Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the sixth of several such problems we plan for this column.

1150. *Alex Gordon, University of North Carolina Charlotte*

Peter tosses 25 fair coins and John tosses 20 fair coins. What is the probability that they toss the same number of heads? Non-computational solutions only please.

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1151. *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI*

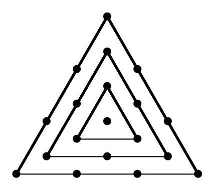
Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 \left\{\frac{1}{1-x}\right\}^2 dx,$$

where $\{a\}$ is *the fractional part* of a.

1152. *Russell Euler and Cathy George (student), Northwest Missouri State University*

Centered Triangular Numbers. The centered triangular numbers T_n are defined as follows. $T_1 = 1$, and for each integer $n > 1$, T_n is the number of dots in the $n-1$ triangles shown below. Note that the number of vertices on each edge of each triangle is one larger than the number on the edge of the next smaller triangle. Thus, the first four centered triangular numbers are $1, 4, 10, 19$. Express T_n in terms of the n^{th} triangular number t_n and n.



1153. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN*

Let $\phi = (1+\sqrt{5})/2$ denote the golden ratio. Establish the log-trig identity $\log^2(2\sin(2\pi/5)) + \log^2(2\sin(4\pi/5)) = (\log^2(\sqrt{5}) + \log^2(\phi))/2$.

1154. *Prithwijit De, University College Cork, Cork, Ireland*

Evaluate $\int_0^{\pi/4} \frac{\ln(\cot\theta)}{(\sin^n\theta+\cos^n\theta)^2} \cdot \sin^{n-1}(2\theta)\, d\theta$ for $n = 1, 2, 3, \ldots.$

1155. *Proposed by Stas Molchanov, University of North Carolina Charlotte, Charlotte, NC*

Prove that there is no solution to $x^5 - nx + 1 = 0, \quad n \geq 3$ obtainable with radicals.

1156. *Proposed by Paul S. Bruckman, Sointula, BC*

Given that a, b, c are real numbers such that a^4, b^4, c^4 satisfy the triangle inequality, with $\triangle(a^2, b^2, c^2)$ acute, let $F(x) = \sqrt{1-a^2x} + \sqrt{1-b^2x} + \sqrt{1-c^2x}$, defined over a suitable domain. Determine conditions for there to be a (real) solution of the equation $F(x) = 1$, and find any such solution(s). Solutions, if any, may be expressed in terms of triangle areas, using Heron's formula.

1157. *Proposed by S.C. Locke, Florida Atlantic University, Boca Raton, FL*

Over the course of a year, 137 people play chess games against each other, recording only the games which end in a win for one of the players. No player plays himself or herself, but players may have played against each other any number of times, including zero. For any subset A of the players, let A^+ denote the set of players who lost at least one game to at least one person in A. Prove that there is a set of players X such that $X \cap X^+ = \emptyset$ (no two players in X played a non-drawn game) and every player is in at least one of X, X^+ or $(X^+)^+$ (every player is in X, beaten by somebody in X, or beaten by somebody who was beaten by somebody in X).

1158. *Proposed by the Editors*

At ABC University, the football mascot does as many pushups after each ABCU score as the team has accumulated. The team always make extra points after touchdowns, so it scores only in increments of 3 and 7. For each sequence $a_1, a_2, \ldots, a_n$ where each a_k is 3 or 7, let $P(a_1, a_2, \ldots, a_n)$ denote the total number of pushups the mascot does for the scoring sequence $a_1, a_2, \ldots, a_n$. For example $P(3, 7, 3) = 3 + (3 + 7) + (3 + 7 + 3) = 26$. Call a positive integer k *accessible* if there is a scoring sequence $a_1, a_2, \ldots, a_n$ such that $P(a_1, a_2, \ldots, a_n) = k$. Is there a number K such that for all $t \geq K$, t is accessible? If not, prove it and if so, find K.

1159. *Mike Pinter*, Belmont University, Nashville, TN

In honor of the celebration this year of the 300$^{\text{th}}$ anniversary of the birth of Leonhard Euler and his significant contributions to mathematics in general and graph theory in particular, consider the following Base 12 alphametic:

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    G R A P H
+ T H E O R Y
  E U L E R S
```

Determine the maximum value possible for `E U L E R S`. Are there other values possible for `E U L E R S`?