# Problem Department

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*This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

*All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to* `hbreiter@email.uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2007. Solutions identified as by students are given preference.*

## Problems for Solution

### 1138

*Proposed by Leo Schneider, John Carroll University, Cleveland, OH*

Consider the graphs of $x^y = y^x$ and $y = mx$ in the first quadrant.

(a) Find functions $f$ and $g$ so that $(x, y) = (f(m), g(m))$ gives the coordinates of the point of intersection between the two graphs for each positive $m \neq 1$.

(b) Evaluate $\lim_{m\to 1} f(m)$ and $\lim_{m\to 1} g(m)$.

(c) Prove that the graph of $x^y = y^x$, $y \neq x$, yields a decreasing function of $y$ in terms of $x$ with a missing point discontinuity at $y = x$.

(d) Use the above to decide which is larger, $e^\pi$ or $\pi^e$.

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## 1139

*Proposed by Brian Bradie, Christopher Newport University, Newport News, VA*

Simplify
$$\arctan\left(\frac{1+t}{1-t}\right) - \arctan t.$$

## 1140

*Proposed by Cecil Rousseau, University of Memphis, Memphis TN*

For each $n \geq 0$ determine
$$\min \int_0^\infty p^2(x)\, e^{-x}\, dx,$$
when the minimum is taken over all $p \in \mathbf{R}[x]$ satisfying $\deg p \leq n$ and $p(0) = 1$. Show that the minimizing polynomial is unique and express it as a single member of a classical family of orthogonal polynomials.

## 1141

*Proposed by Arthur Holshouser, Charlotte, NC*

We call $((S,\cdot),(S,*),f)$ a standard structure if (1) $(S,\cdot),(S,*)$ are two district binary operators on $S$, (2) if $f: S \to S$ is a bijection on $S$, (3) $f: (S,\cdot) \to (S,*)$ is an isomorphism and (4) $f: (S,*) \to (S,\cdot)$ is an isomorphism.

(a) Suppose $((S,\cdot),(S,*),f)$ is a standard structure.

   (1) show that neither of $f: (S,\cdot) \to (S,\cdot), f: (S,*) \to (S,*)$ are automorphisms.

   (2) Show that both of $ff: (S,\cdot) \to (S,\cdot), ff: (S,*) \to (S,*)$ are automorphisms where $ff$ denotes the composition of the two functions $f$ and $f$.

(b) Suppose $(S,\cdot)$ is a binary operator on $S, f: S \to S$ is a bijection on $S, f: (S,\cdot) \to (S,\cdot)$ is not an automorphism, and $ff: (S,\cdot) \to (S,\cdot)$ is an automorphism. In particular note that if $f: (S,\cdot) \to (S,\cdot)$ is not an automorphism and $ff = I$, the identity function, the (b) is satisfied.

Define $(S,*)$ by $\forall a,b \in S, a*b = f\left(f^{-1}(a) \cdot f^{-1}(b)\right)$. Show that $((S,\cdot),(S,*),f)$ is a standard structure.

## 1142

*Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

From among all $K$-element subsets of an $N$-element set choose $M$ sets. Let $M = f(N, K)$ be the least $M$ which guarantees that some two of the chosen sets are disjoint – and let $f(15, 6)$ be nice and lucky and happy for you! Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the fifth of several such problems we plan for this column.

## 1143

*Proposed by Leo Schneider, John Carroll University, Cleveland, OH*

Compute the number of ordered pairs of positive integers $(m, n)$ that solve

$$2m + 6n = 2006$$

and for which $m + n$ is a multiple of 13.

## 1144

*Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let $\alpha, \beta, \gamma$ be the angles of an acute triangle $ABC$ with semiperimeter $s$ and circumradius $R$. Prove that

$$\csc\alpha^{\sin\alpha} + \csc\beta^{\sin\beta} + \csc\gamma^{\sin\gamma} < 6 - \frac{s}{R}$$

## 1145

*Bathoo U. Kyaw Htoo, Myint Nge Qtr, Myeik, Myanmar*

Given a positive integer $n$, an *associate* of $n$ is any integer than can be obtained by permuting digits of $n$ (and dropping any initial zeros). Thus the associates of 1022 are 122, 212, 221, 1022, 1202, 1220, 2012, 2102, 2120, 2021, 2201, and 2210. Find a formula for $\Sigma(n)$, the sum of all associates of $n$. To express the formula for $\Sigma(n)$ in compact form, let $m_0, m_1, \ldots, m_9$ denote the multiplicities of the decimal digits $0, 1, \ldots, 9$, respectively, in the decimal representation of $n$, and let $m = m(n) = \sum_{k=0}^{9} m_k$ and $s(n) = \sum_{k=1}^{n} km_k$. Thus $m$ is the number of decimal digits of $n$ and $s$ is their sum. Finally, let $R_m = (10^m - 1)/9$ denote the $m$-digit repunit.

## 1146

*Douglas Shafer, University of North Carolina Charlotte*

Given six real constants $a$, $b$, $c$, $d$, $e$, and $f$, not all zero, a conic section $C : ax^2 + bxy + cy^2 + dx + ey + f = 0$ is determined. Since rescaling the six coefficients by a nonzero number does not change $C$, we may view $(a, b, c, d, e, f)$ as lying in $S^5 \subset \mathbf{R}^6$. If $(a, b, c, d, e, f)$ is selected based on a uniform distribution on $S^5$, what is the probability that $C$ is an ellipse?

## 1147

*Proposed by Alexander Povolotsky, Goodrich Optical Systems, Chelmsford, MA*

It is known that

$$n(n+1)[n+(n+1)] = 6(1+4+9+16+..+n^2)$$

and that

$$n(n+1)(n+2)[n+(n+1)+(n+2)] = 36[1+(1+4)+(1+4+9)+..+(1+4+9+16+..+n^2)].$$

Find

$$n(n+1)(n+2)(n+3)...(n+k)[n+(n+1)+(n+2)+(n+3)+..+(n+k)].$$

## 1148

*Walter Whiteley, York University, Toronto, Canada*

Given a tetrahedron with all its four faces of area 1, does it follow that the tetrahedron is regular? If your answer is yes, give a proof. If your answer is no, describe the kinds of tetrahedra that you can obtain under the given face area constraint. Note: there is a $500 prize for the best undergraduate student solution to this problem donated by Professor Ali Amir-Moez.