IDENTITIES FOR THE TOTAL NUMBER OF PARTS IN PARTITIONS OF INTEGERS

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Abstract. We consider the total number of parts in partitions of the natural number n, and derive identities relating this function to the number of partitions and to other familiar number theoretic functions. The total number of parts in partitions with distinct parts and partitions with other restrictions is also considered.

1. Introduction

Let $a_1 + a_2 + \cdots + a_k = n$, with $a_{i+1} \geq a_i$ for $i \geq 1$, $a_1 \geq 1$ be a partitition of n. It is well known that the ordinary generating function for partitions is

$$P(x) := \sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} \frac{1}{1-x^i}.$$

If we are interested to count the number of parts (or summands) in the partitions we can introduce the formal bivariate generating function

$$P(x,u) := \sum_{n\geq 0}\sum_{k\geq 1} p(n,k)x^n u^k = \prod_{i\geq 1} \frac{1}{1-ux^i},$$

where $p(n,k)$ counts partitions with exactly k parts.

Let $s(n)$ be the total number of parts in all partitions of the natural number n. Then the formula $s(n) = \sum_{k=1}^{n} kp(n,k)$ implies that

$$S(x) := \sum_{n\geq 0} s(n)x^n = \frac{\partial}{\partial u}P(x,u)|_{u=1} = P(x)\sum_{k\geq 1}\frac{z^k}{1-z^k} = P(x)\sum_{m\geq 1} d(m)x^m,$$

where $d(m)$ is the number of divisors of m.

By equating coefficients above we obtain the formula

$$s(n) = \sum_{i=1}^{n} d(i)p(n-i).$$

The $s(n)$ sequence begins as follows for $n \geq 0$,

$0, 1, 3, 6, 12, 20, 35, 54, 86, 128, 192, 275, 399, 556, 780, 1068, 1463, 1965, 2644, 3498, \cdots.$

In Sloane's online encyclopaedia of integer sequences [3] this is **A006128**.

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In addition, by transposing the Ferrers graph of each partition of n, it follows that $s(n)$ is also equal to the sum of the largest parts of all partitions of n. From this interpretation we can derive a further form for the generating function of $s(n)$:

Let $r_k(x)$ be the generating function for partitions with largest part equal to k. Then

$$r_k(x) = x^k \prod_{i=1}^{k} \frac{1}{1-x^i}.$$

Therefore

$$S(x) = \sum_{k\geq 1} k r_k(x) = \sum_{k\geq 1} k x^k \prod_{i=1}^{k} \frac{1}{1-x^i}.$$

The results given so far are known and appear without proof in Sloane's online encyclopaedia of integer sequences [3] for example. The earliest study of $s(n)$ is due to Erdös and Lehner [2].

Our aim in this section is to find some new identities involving $s(n)$. To do this we make use of a number well known partition identities which can be found for example in Andrews [1].

Theorem 1. *For $n \geq 1$,*

$$s(n) = \sum_{i\geq 1} (-1)^{i-1}(s(n-\omega(i)) + s(n-\omega(-i)) + d(n),$$

where $\omega(i) = \frac{i(3i-1)}{2}$, $i \geq 1$.

Proof. Firstly

$$S(x) \prod_{i\geq 1}(1-x^i) = \sum_{n\geq 1} d(m) x^m. \tag{1}$$

Using Euler's pentagonal number theorem

$$\prod_{i\geq 1}(1-x^i) = \left(1 + \sum_{i\geq 1}(-1)^i (x^{\omega(i)} + x^{\omega(-i)})\right),$$

we obtain

$$S(x)\left(1 + \sum_{i\geq 1}(-1)^i (x^{\omega(i)} + x^{\omega(-i)})\right) = \sum_{n\geq 1} d(m) x^m.$$

Equating coefficients of x^n on each side and rearranging gives the result. □

Recall that the ordinary generating function for partitions with distinct parts is

$$Q(x) := \sum_{n\geq 0} q(n) x^n = \prod_{i\geq 1}(1+x^i). \tag{2}$$

Theorem 2. *For $n \geq 1$,*

$$s(n) + \sum_{i\geq 1}(-1)^i(s(n-2\omega(i)) + s(n-2\omega(-i)) = \sum_{i=1}^{n} d(i)q(n-i).$$

Proof. Multiplying (1) by (2) we obtain

$$S(x)\prod_{i\geq 1}(1-x^{2i}) = Q(x)\sum_{n\geq 1} d(m)x^m.$$

Equating coefficients of x^n on each side gives the result. □

More generally, let $b_r(n)$ denote the number of r-regular partitions of n, that is the number of partitions of n such that no part is divisible by r, or equivalently, the number of partitions of n such that no part occurs r or more times. It is known that the generating function of $b_r(n)$ is given by:

$$\sum_{n\geq 0} b_r(n)x^n = \prod_{n\geq 1}\frac{1-x^{rn}}{1-x^n}. \tag{3}$$

Then we have:

Theorem 3. *Let $m \geq 1$. Then*

$$s(n) + \sum_{j\geq 1}(-1)^j(s(n-2^m\omega(j)) + s(n-2^m\omega(-j))) = \sum_{k=1}^{n} b_{2^m}(n-k)d(k).$$

Proof. Identities (1) and (3) imply that

$$S(x)\prod_{n\geq 1}(1-x^{2^m}n) = (\sum_{n\geq 1} d(n)x^n)(\prod_{n\geq 1} b_{2^m}(n)x^n).$$

The conclusion now follows by matching coefficients of like powers of x.

□

Theorem 4. *For $n \geq 1$,*

$$\sum_{r\geq 0} s(n-\frac{r(r+1)}{2}) = \sum_{i+j+k=n} d(i)q(j)q(k)$$

Remarks: In the right member of the formula, we have $i, j, k \geq 0$.

Proof. Replacing x by x^2 in (1), we obtain:

$$S(x^2)\prod_{n\geq 1}(1-x^{2n}) = \sum_{n\geq 1} d(n)x^{2n},$$

hence

$$S(x^2)\prod_{n\ge1}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = (\sum_{n\ge1} d(n)x^{2n})\prod_{n\ge1}(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}).$$

By the Jacobi triple product identity, the left member of the preceding equation may be simplified, so that we get

$$S(x^2)\sum_{n=-\infty}^{\infty} x^{n^2}z^n = (\sum_{n\ge1} d(n)x^{2n})\prod_{n\ge1}(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}).$$

Letting $z = x$, we obtain:

$$S(x^2)\sum_{n=-\infty}^{\infty} x^{n^2+n} = (\sum_{n\ge1} d(n)x^{2n})\prod_{n\ge1}(1+x^{2n})\prod_{n\ge1}(1+x^{2n-2}),$$

that is,

$$S(x^2)\sum_{n=-\infty}^{\infty} x^{n^2+n} = 2(\sum_{n\ge1} d(n)x^{2n})\prod_{n\ge1}(1+x^{2n})^2.$$

Replacing x^2 by x, we have:

$$S(x)\sum_{n=-\infty}^{\infty} x^{\frac{n^2+n}{2}} = 2(\sum_{n\ge1} d(n)x^{n})\prod_{n\ge1}(1+x^{n})^2.$$

By symmetry, we have:

$$S(x)\sum_{n=0}^{\infty} x^{\frac{n^2+n}{2}} = (\sum_{n\ge1} d(n)x^{n})(\sum_{n\ge0} q(n)x^{n})^2,$$

that is,

$$(\sum_{n\ge0} s(n)x^n)(\sum_{n=0}^{\infty} x^{\frac{n^2+n}{2}}) = (\sum_{n\ge1} d(n)x^{n})(\sum_{n\ge0} q(n)x^{n})^2.$$

The conclusion now follows by matching coefficients of like powers of x. □

Theorem 5. *Let* $\prod_{n\ge1}(1-x^n) = \sum_{n\ge0} E(n)x^n$, *so that*

$$E(n) = \begin{cases} (-1)^r & \text{if } n = \omega(\pm r) \\ 0 & \text{otherwise} \end{cases}.$$

then

$$s(n) + \sum_{j\ge1}(-1)^j(2j+1)s(n-\frac{j(j+1)}{2}) = \sum_{i+j+k=n} d(i)E(j)E(k).$$

Proof. Identity (1) implies

$$S(x)\prod_{n\geq 1}(1-x^n)^3 = (\sum_{n\geq 1} d(n)x^n)\prod_{n\geq 1}(1-x^n)^2.$$

A well-known identity of Jacobi states that

$$\prod_{n\geq 1}(1-x^n)^3 = \sum_{j\geq 0}(-1)^j(2j+1)x^{\frac{j(j+1)}{2}}.$$

The conclusion now follows by matching coefficients of like powers of x. □

2. Partitions with distinct parts

Let $a_1 + a_2 + \cdots + a_k = n$, with $a_{i+1} > a_i$ for $i \geq 1$, $a_1 \geq 1$ be a partitition of n with distinct parts, called *distinct partitions*. As mentioned above, the ordinary generating function for distinct partitions is

$$Q(x) := \sum_{n\geq 0} q(n)x^n = \prod_{i\geq 1}(1+x^i).$$

To count the number of parts in distinct partitions we can introduce the bivariate generating function

$$Q(x,u) := \sum_{n\geq 0}\sum_{k\geq 1} q(n,k)x^n u^k = \prod_{i\geq 1}(1+ux^i),$$

where $q(n,k)$ counts distinct partitions with exactly k parts.

Let $s_d(n)$ be the total number of parts in all distinct partitions of the natural number n. Then the formula $s_d(n) = \sum_{k=1}^{n} kq(n,k)$ implies that

$$S_d(x) := \sum_{n\geq 0} s_d(n)x^n = \frac{\partial}{\partial u}Q(x,u)|_{u=1} = Q(x)\sum_{k\geq 1}\frac{z^k}{1+z^k} = Q(x)\sum_{m\geq 1} e(m)x^m, \quad (4)$$

where $e(m)$ is the number of odd divisors of m minus the number of even divisors of m.

Note that if $n = 2^k m$, where $k \geq 0$, $2 \not| m$, then $e(n) = -(k-1)d(m)$.

By equating coefficients above we obtain the formula

$$s_d(n) = \sum_{i=1}^{n} e(i)q(n-i).$$

The $s_d(n)$ sequence begins as follows for $n \geq 0$,

$$0, 1, 1, 3, 3, 5, 8, 10, 13, 18, 25, 30, 40, 49, 63, 80, 98, 119, 179, 218, \cdots.$$

In Sloane's online encyclopaedia of integer sequences [3] this is **A015723**.

Theorem 6. *For* $n \geq 1$,

$$s_d(n)+\sum_{i\geq 1}(-1)^i(s_d(n-\omega(i))+s_d(n-\omega(-i)) = e(n)+\sum_{i\geq 1}(-1)^{i-1}(e(n-2\omega(i))+e(n-2\omega(-i)).$$

Proof. We have from (4)

$$S_d(x)\prod_{i\geq 1}(1-x^i) = \prod_{i\geq 1}(1-x^{2i})\sum_{m\geq 1}e(m)x^m.$$

Using the pentagonal number theorem

$$S_d(x)\left(1+\sum_{i\geq 1}(-1)^i(x^{\omega(i)}+x^{\omega(-i)})\right) = \left(1+\sum_{i\geq 1}(-1)^i(x^{2\omega(i)}+x^{2\omega(-i)})\right)\sum_{m\geq 1}e(m)x^m.$$

Equating coefficients of x^n on each side gives the result. □

Theorem 7. *For* $n \geq 1$,

$$\sum_{j=1}^{n/2} s_d(n-2j)p(j) = \sum_{i=1}^{n} e(i)p(n-i).$$

Proof. Again from (4),

$$S_d(x)P(x^2) = P(x)\sum_{m\geq 1}e(m)x^m.$$

Equating coefficients of x^n on each side gives the result. □

Theorem 8. *For* $n \geq 1$,

$$\sum_{i=1}^{n}(-1)^i s_d(i)q(n-i) = \sum_{j=1}^{n/2} e(n-2j)q(j).$$

Proof. If we replace x by $-x$ in (4),

$$S_d(-x) = \sum_{m\geq 1}(-1)^m e(m)x^m \prod_{i\geq 1}(1+x^{2i})\prod_{i\geq 1}(1-x^{2i-1}) \tag{5}$$

or

$$S_d(-x)Q(x) = \sum_{m\geq 1}(-1)^m e(m)x^m Q(x^2).$$

Equating coefficients of x^n on each side gives the result. □

Theorem 9. *For $n \geq 1$,*

$$\sum_{i\geq 0}(-1)^{\frac{i(i+1)}{2}} s_d(n - \frac{i(i+1)}{2}) = e(n) + \sum_{j\geq 1}(-1)^j(e(n-4\omega(i)) + e(n-4\omega(-i))).$$

Proof. In (5), multiply both sides to get

$$S_d(-x)\frac{\prod_{i\geq 1}(1-x^{2i})}{\prod_{i\geq 1}(1-x^{2i-1})} = \sum_{m\geq 1}(-1)^m e(m)x^m \prod_{i\geq 1}(1+x^{2i})\prod_{i\geq 1}(1-x^{2i}).$$

Then

$$S_d(-x)\sum_{i\geq 0}x^{i(i+1)/2} = \sum_{m\geq 1}(-1)^m e(m)x^m \prod_{i\geq 1}(1-x^{4i}).$$

Replace x by $-x$,

$$S_d(x)\sum_{i\geq 0}(-x)^{i(i+1)/2} = \sum_{m\geq 1} e(m)x^m \prod_{i\geq 1}(1-x^{4i}).$$

Equating coefficients of x^n on each side gives the result. □

3. Partitions into distinct odd parts

Let $s_o(n)$ denote the total number of parts in all partitions of the natural number n into distinct odd parts. Let the corresponding generating function be given by:

$$S_o(x) := \sum_{n\geq 0} s_o(n)x^n.$$

Let the generating function for partitions into distinct odd parts be given by:

$$Q_0(x) = \sum_{n\geq 0} q_0(n)x^n = \prod_{i\geq 1}(1+x^{2i-1}).$$

Then, reasoning as in prior sections, we obtain:

$$S_o(x) = Q_0(x)\sum_{k\geq 1}\frac{x^{2k-1}}{1+x^{2k-1}}.$$

Now let

$$T(x) = \sum_{k\geq 1}\frac{x^{2k-1}}{1+x^{2k-1}} = \sum_{k\geq 1}x^{2k-1}\sum_{j\geq 0}(-1)^j(x^{2k-1})^j =$$

$$\sum_{k\geq 1}\sum_{j\geq 0}(-1)^j(x^{2k-1})^{j+1} = \sum_{k\geq 1}\sum_{i\geq 1}(-1)^{i-1}(x^{2k-1})^i.$$

Let $n = (2k-1)i$. Then, reversing the order of summation in the double sum, we have:

$$T(x) = \sum_{r\geq 1}(\sum_{n/i\,odd}(-1)^i)x^n.$$

Let $f(n) = \sum_{n/i\,odd}(-1)^i$. If n is odd, then $f(n) = d(n)$; If $n = 2^k t$ where $k \geq 1$ and t is odd, then $f(n) = -d(t)$. Therefore

$$f(n) = (-1)^{n-1}d(t).$$

Now we have

$$S_o(x) = Q_0(x)T(x), \quad T(x) = \sum_{n\geq 1} f(n)x^n.$$

Matching coefficients of like powers of x, we obtain the identity:

$$s_o(n) = \sum_{j=1}^{n} q_0(n-j)f(j).$$

The first few terms of the $s_o(n)$ sequence for $n \geq 1$ are given below:
$1, 0, 1, 2, 1, 2, 1, 4, 4, 4, 4, 6, 7, 6, 10, 12, 13, 12, 16, 18, 22, 22, 25, 32, 36, 36, 42, 50, 53, 58, \cdots$.
This sequence is not presently in Sloane [3].

Theorem 10. *If $n = 2^k t$ where $k \geq 0$ and t is odd, then*

$$\sum_{j=1}^{n}(-1)^{j-1}q(n-j)s_o(j) = d(t).$$

Proof. Recall that if $n = 2^k t$ where $k \geq 0$ and t is odd, then

$$S_o(x) = Q_0(x)\sum_{n\geq 1} f(n)x^n,$$

where $f(n) = (-1)^{n-1}d(t)$. Therefore

$$S_o(-x) = Q_0(-x)\sum_{n\geq 1}(-1)^n f(n)x^n,$$

that is,

$$S_o(-x) = \prod_{i\geq 1}(1-x^{2i-1})\sum_{n\geq 1} -d(t)x^n.$$

This implies

$$S_o(-x)\prod_{i\geq 1}(1+x^i) = -\sum_{n\geq 1} d(t)x^n,$$

that is

$$\sum_{n\geq 1}(-1)^n s_o(n)x^n \sum_{n\geq 0} q(n)x^n = -\sum_{n\geq 1} d(t)x^n.$$

Multiplying this last equation by -1 and then matching coefficients of like powers of x, we arrive at our conclusion. □

A well known bijection shows that there are the same number of self conjugate partitions of n as there are partitions of n into distinct odd parts. However, the number of parts in a self conjugate partition and its corresponding partitions into distinct odd parts need not be the same. In fact the bijection shows that if n_1 is the largest part in the distinct odd partition then its corresponding self conjugate partition has number of parts equal to $\frac{n_1+1}{2}$.

Thus if $s_c(n)$ denotes the sum of the number of parts in all self conjugate partitions of n, then

$$s_c(n) = \sum_{\lambda\vdash n}\frac{n_1+1}{2},$$

where the sum is over all partitions λ of n into distinct odd parts, and n_1 is the largest part of λ.

We use this to show

Theorem 11.

$$S_c(x) := \sum_{n\geq 1} s_c(n)x^n = \sum_{k\geq 1} kx^{2k-1}\prod_{i=1}^{k-1}(1+x^{2i-1}).$$

Proof. Let $t_{2k-1}(x)$ be the generating function for partitions λ of n into distinct odd parts with largest part $2k-1$. Then

$$t_{2k-1}(x) = x^{2k-1}\prod_{i=1}^{k-1}(1+x^{2i-1}).$$

Thus the generating function for the sum of the largest parts in partitions of n into distinct odd parts is

$$l(x) := \sum_{n\geq 1} l(n)x^n = (\sum_{k\geq 1}(2k-1)t_{2k-1}(x) = \sum_{k\geq 1}(2k-1)x^{2k-1}\prod_{i=1}^{k-1}(1+x^{2i-1}).$$

Now

$$\begin{aligned} s_c(n) &= \sum_{\lambda\vdash n}\frac{n_1+1}{2} = \frac{1}{2}(\sum_{\lambda\vdash n} n_1 + \sum_{\lambda\vdash n} 1) \\ &= \frac{l(n)+q_0(n)}{2}. \end{aligned}$$

Using the fact that $Q_0(x) = \sum_{k\geq 1} t_{2k-1}(x)$ we find that

$$S_c(x) := \frac{1}{2}(l(x) + Q_0(x)) = \sum_{k\geq 1} t_{2k-1}(x) = \sum_{k\geq 1} kx^{2k-1} \prod_{i=1}^{k-1}(1 + x^{2i-1}),$$

as claimed.

$\square$

The $s_c(n)$ sequence begins as follows for $n \geq 0$,

$$0, 1, 0, 2, 2, 3, 3, 4, 7, 8, 9, 10, 15, 16, 18, 23, 30, 32, 35, 42, 51, 59, 63, \cdots .$$

In Sloane's online encyclopaedia of integer sequences [3] this is **A067619**.

By analogy with partitions into distinct odd parts, it is natural to try express

$$S_c(x) = Q_0(x)R(x), \quad R(x) := \sum_{n\geq 1} r(n)x^n.$$

However the integer coefficients $r(n)$ that appear do not seem to follow any discernable pattern. The sequence of $r(n)$ values for $n \geq 1$ begins as follows:

$$1, -1, 3, -2, 5, -5, 8, -7, 13, -13, 18, -19, 26, -29, 39, -40, 52, -60, 72, -81, 101, \cdots .$$

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