Topos-theoretic background*

Olivia Caramello
IHÉS

September 22, 2014

Contents

*chapter 1 of the book in preparation "Lattices of theories"

1 Introduction

In this chapter we provide the topos-theoretic preliminaries necessary for understanding the contents of the book, for the benefit of readers who are not familiar with topos theory. We shall only assume a familiarity with the basic notions of category theory.

Besides the original work [1], where the theory of toposes was first introduced, classical books on the subject, to which we shall systematically refer in the course of the chapter, are, in increasing order of technical sophistication, [23], [29] and [26].

The structure of this introductory chapter is as follows.

In section 2 we fix the notation and terminology used throughout the book.

In section 3 we introduce the notion of Grothendieck topos, review the basic properties of categories of sheaves and discuss the fundamental equivalence between geometric morphisms and flat functors.

In section 4 we introduce the concept of first-order theory and the various deductive systems for fragments of first-order logic that we shall consider in the course of the book.

In section 5 we review the way in which first-order theories can be soundly interpreted in categories having 'enough' structure; in particular, we explain the sense in which a Grothendieck topos (or, more generally, an elementary topos) can be regarded as a mathematical universe in which one can consider models of first-order theories.

In section 6 we present the fundamental concept of syntactic category of a first-order theory and illustrate the classification of models through structure-preserving functors defined on such categories. Lastly, we use syntactic categories for establishing the soundness and completeness for the categorical semantics defined in section 5.

In section 7 we review the fundamental notion of classifying topos of a geometric theory and discuss the appropriate kinds of interpretations between theories which induce morphisms between the associate classifying toposes. The theoretical presentation is accompanied by a few concrete examples of classifying toposes of theories naturally arising in mathematics.

In section 8 we explain the general unifying technique 'toposes as bridges' originally introduced in [9]. This technique, which allows to extract 'concrete' information from the existence of different representations for the classifying topos of a geometric theory, will be systematically exploited in the course of the book to establish theoretical results as well as applications.

2 Terminology and notation

The terminology and notation used in the book is essentially standard; some specific notations that we shall employ are:

- We shall denote by $\mathbf{Set}$ the category of sets and functions between sets.
- $\mathbf{Set}[\mathbb{T}]$ or $\mathcal{E}_{\mathbb{T}}$ will denote the classifying topos of a geometric theory $\mathbb{T}$.
- The (meta-)2-category of (large) categories, functors and natural transformations between them will be denoted by **CAT**.
- $\gamma_{\mathscr{E}} : \mathscr{E} \to \mathbf{Set}$ will denote the the unique (up to isomorphism) geometric morphism from a Grothendieck topos $\mathscr{E}$ to the category $\mathbf{Set}$ of sets.
- By a cartesian category (resp. cartesian functor) we mean a category with finite limits (resp. a finite-limit-preserving functor).
- Given a category $\mathcal{C}$, we shall often write $c \in \mathcal{C}$ to mean that c is an object of $\mathcal{C}$.
- The category of functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ will be denoted by $[\mathcal{C}, \mathcal{D}]$.
- The terminal object of a category $\mathcal{C}$, if it exists, will be denoted by $1_{\mathcal{C}}$, or simply by 1.
- The initial object of a category $\mathcal{C}$, if it exists, will be denoted by $0_{\mathcal{C}}$, or simply by 0.
- By a subterminal object in a category $\mathcal{C}$ having a terminal object 1 we mean an object c such that the unique arrow $c \to 1$ is a monomorphism.
- In a category with finite products, we shall denote by $< f, g >$ the arrow $c \to a \times b$ determined by the universal property of the product by a pair of arrows $f : c \to a$ and $g : c \to b$.
- The identity arrow on an object c in a category will be denoted by 1_c.

3 Grothendieck toposes

Before introducing the notion of Grothedieck topos, we need to talk about Grothendieck topologies on categories. Indeed, Grothendieck toposes will be defined in section 3.2 as categories of sheaves on a certain category with respect to a Grothendieck topology on it.

3.1 The notion of site

The notion of Grothendieck topology represents a categorical abstraction of the classical notion of covering in Topology of an open set of a topological space by a family of smaller open subsets.

Definition 3.1. (a) A *sieve* on an object c of a small category $\mathcal{C}$ is a set of arrows in $\mathcal{C}$ with codomain c such that for any $f \in S$, $f \circ g \in S$ for any arrow g composable with f in $\mathcal{C}$;

(b) A sieve S is said to be *generated* by a family $\mathcal{F}$ of arrows contained in it if every arrow in S factors through an arrow in $\mathcal{F}$;

(c) A *Grothendieck topology* on a category $\mathcal{C}$ is a function J assigning to every object c of $\mathcal{C}$ a set $J(c)$ of sieves on $\mathcal{C}$ in such a way that the following properties are satisfied:

- (Maximality axiom) For any object c of $\mathcal{C}$, the maximal sieve $M_c := \{f \mid \mathrm{cod}(f) = c\}$ on c belongs to $J(c)$;
- (Pullback stability) For any arrow $f : d \to c$ in $\mathcal{C}$ and any sieve $S \in J(c)$, the sieve $f^*(S) = \{g : e \to d \mid f \circ g \in S\}$ belongs to $J(d)$;
- (Transitivity) For any sieve S in $\mathcal{C}$ on c and any $T \in J(c)$, if $f^*(S) \in J(\mathrm{dom}(f))$ for all $f \in T$ then $S \in J(c)$.

(d) A *site* is a pair $(\mathcal{C}, J)$ consisting of a small category $\mathcal{C}$ and a Grothendieck topology J on $\mathcal{C}$.

(e) Given two cartesian sites (i.e., sites whose underlying categories are cartesian) $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a *morphism of sites* $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is a cartesian functor $F : \mathcal{C} \to \mathcal{D}$ which is cover-preserving (in the sense that the image under F of every J-covering sieve generates a K-covering sieve).

Remark 3.2. One can define morphisms of general, i.e. non-necessarily cartesian, sites; anyway, this general notion is more involved than that for cartesian sites and we shall not use it in the book, so we do not discuss it here.

A sieve on an object c is said to be J-covering, for a Grothendieck topology J on $\mathcal{C}$, if it belongs to $J(c)$.

The following notions will be important for illuminating the relationship between toposes and topological spaces.

Recall that a *lattice* is a partially ordered set with top and bottom element in which every pair of elements a and b has an infimum $a \wedge b$ and a supremum $a \vee b$.

Definition 3.3. (a) A *Heyting algebra* is a lattice H with bottom and top elements (denoted respectively 0 and 1) in which for any two elements $a, b \in H$ there exists an element $a \Rightarrow b$ satisfying the universal property that for any $c \in H$, $c \leq a \Rightarrow b$ if and only if $c \wedge a \leq b$. An Heyting algebra is said to be *complete* if it possess arbitrary suprema (equivalently, arbitrary infima).

(b) A *frame* F is a partially ordered set with arbitrary joins (and meets), in which the distributivity law of arbitrary joins with respect to finite meets holds:

$$(\bigvee_{i \in I} a_i) \wedge b = \bigvee_{i \in I} a_i \wedge b \ .$$

Since, by the Special Adjoint Functor Theorem, the functor $b \wedge - : F \to F$ has a right adjoint $b \Rightarrow - : F \to F$ if and only if it preserves arbitrary joins, a frame is the same thing as a complete Heyting algebra.

(c) A *locale* is a frame, regarded as an object of the opposite of the category **Frm** of frame and maps between them which preserve finite meets and arbitrary joins.

For any topological space X, the lattice $\mathcal{O}(X)$ of its open sets is a locale. Conversely, to any locale F one can associate a topological space X_F, whose points are the frame homomorphism $F \to \{0,1\}$ and whose open sets are the subsets of frame homomorphisms $F \to \{0,1\}$ which send a given element $f \in F$ to 1. In fact, the assignments $X \to \mathcal{O}(X)$ and $F \to X_f$ lift to a an adjunction between the category $\mathbf{Top}$ of topological spaces and continuous maps and the category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ of locales.

Examples 3.4. (a) The *trivial topology* T on a category $\mathcal{C}$ is the Grothendieck topology on $\mathcal{C}$ whose only covering sieves are the maximal ones.

(b) The *dense* or *double-negation topology* on a category $\mathcal{C}$ is the Grothendieck topology D on $\mathcal{C}$ whose covering sieves are exactly the stably non-empty ones (i.e., for any sieve S in $\mathcal{C}$ on an object c, $S \in D(c)$ if and only if $f^*(S) \neq \emptyset$ for all arrows f in $\mathcal{C}$ with codomain c).

(c) If $\mathcal{C}$ satisfies the *right Ore condition* (that is, the property that every pair of arrows with common codomain can be completed to a commutative square), the dense topology on $\mathcal{C}$ specializes to the *atomic topology* J_{at} on $\mathcal{C}$, whose covering sieves are exactly the non-empty ones.

(d) Given a topological space X, there is a natural topology $J_{\mathcal{O}(X)}$ on the category $\mathcal{O}(X)$ of its open sets (whose objects are the open sets U of X and whose arrows are the inclusions $V \subseteq U$ between them), whose covering sieves are precisely the ones generated by small covering families (i.e., families of inclusions $U_i \subseteq U$ $(i \in I)$ such that $\bigcup_{i\in I} U_i = U$).

(e) Given a locale L, regarded as a preorder category, the canonical topology J_L on L has as covering sieves the ones generated by families $\{l_i \leq l \mid i \in I\}$ such that $\bigvee_{i\in I} l_i = l$. Notice that the Grothendieck topology on $\mathcal{O}(X)$ defined at the previous point is just the canonical topology on the frame $\mathcal{O}(X)$.

3.2 Sheaves on a site

The introduction of the concept of Grothendieck topos stemmed from the observation that many important properties of topological spaces, such as compactness or connectedness, admit reformulations as categorically invariant properties of the associated categories of sheaves of sets on them; moreover, if the space X is sufficiently well-behaved (technically speaking, sober, cf. Remark 3.26 below), it can be recovered from the associated category $\mathbf{Sh}(X)$, as well as from the frame $\mathcal{O}(X)$, up to homeomorphism. Replacing topological spaces with the corresponding categories of sheaves on them presents the advantage that the latter are very rich in terms of categorical structure and can be studied by using a variety of different invariants naturally defined on them, notably including cohomology groups.

In studying the assignment $X \to \mathbf{Sh}(X)$ one immediately realizes that the definition of $\mathbf{Sh}(X)$ starting from X does not require the consideration of the points of X but only of the open sets of X and of the classical notion of covering for families of open subsets U_i of a given open set U (a given family $\{U_i \subseteq U \mid i \in I\}$ covers U if $\bigcup_{i\in I} U_i = U$). Now, the open sets of X can be organized in the preorder category $\mathcal{O}(X)$ (whose objects are the open sets of X and whose arrows are the inclusions between them), while the above-mentioned notion of covering can be formulated as an additional *datum* on this category.

Abstracting these two fundamental ingredients in the construction of categories of sheaves is precisely what led Grothendieck to introduce the notion of site reviewed in the last section. Sheaves on a general site $(\mathcal{C}, J)$ are defined in a formally analogous way to that of sheaves of sets on a topological space, as follows.

Definition 3.5. Let $(\mathcal{C}, J)$ be a site.

(a) A *presheaf* on a category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$.

(b) A *sheaf* on $(\mathcal{C}, J)$ is a presheaf $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ on $\mathcal{C}$ such that for every J-covering sieve $S \in J(c)$ and every family $\{x_f \in P(\text{dom}(f)) \mid f \in S\}$ such that $P(g)(x_f) = x_{f \circ g}$ for any $f \in S$ and any arrow g in $\mathcal{C}$ composable with f there exists a unique element $x \in P(c)$ such that $x_f = P(f)(x)$ for all $f \in S$.

(c) The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ has as objects the sheaves on $(\mathcal{C}, J)$ and as arrows the natural transformations between them, regarded as functors $\mathcal{C}^{\text{op}} \to \mathbf{Set}$.

(d) A *Grothendieck topos* is a category equivalent to a category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on a site.

(e) A *site of definition* of a Grothendieck topos $\mathcal{E}$ is a site $(\mathcal{C}, J)$ such that $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$.

(f) A *separating set* of objects for a category $\mathcal{E}$ is a set of objects $\mathcal{C}$ of $\mathcal{E}$ such that for any object e of $\mathcal{E}$, the arrows from objects of $\mathcal{C}$ to e are jointly epimorphic.

(g) A Grothendieck topology J on a small category $\mathcal{C}$ is said to be *subcanonical* if every representable functor $Hom_{\mathcal{C}}(-, c) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is a J-sheaf.

Examples 3.6. (a) The category $\mathbf{Set}$ of sets and functions between them is a Grothendieck topos. It can be represented as the category of sheaves on the one-point topological space.

(b) Given a small category $\mathcal{C}$ with the trivial topology T on it, the T-sheaves on $\mathcal{C}$ are clearly just the presheaves on $\mathcal{C}$; the category $\mathbf{Sh}(\mathcal{C}, T)$ thus coincides with the category $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ of functors $\mathcal{C}^{\text{op}} \to \mathbf{Set}$ and natural transformations between them.

(c) For any topological space X, the topos $\mathbf{Sh}(\mathcal{O}_X, J_{\mathcal{O}(X)})$, where $J_{\mathcal{O}(X)}$ is the Grothendieck topology defined in Example 3.4(d) coincides with the usual category $\mathbf{Sh}(X)$ of sheaves of sets on the space X.

(d) For any locale L, the topos $\mathbf{Sh}(L, J_L)$ is often simply denoted by $\mathbf{Sh}(L)$. The locale L can be recovered from it, up to isomorphism, as the frame $\text{Sub}_{\mathbf{Sh}(L,J_L)}(1)$ of subterminal objects in $\mathbf{Sh}(L, J_L)$.

More generally, for any *localic groupoid* $\mathcal{G}$ (i.e. groupoid internal to the category of locales, in the sense of section 5.3), there exists a Grothendieck topos $\mathbf{Sh}(\mathcal{G})$ classifying $\mathcal{G}$-principal bundles. By a theorem of Joyal and Tierney (cf. [27]), every Grothendieck topos can be represented in this form.

(e) Given a topological group G, the category $\mathbf{Cont}(G)$ whose objects are the continuous (left) actions of G on discrete sets and whose arrows are the equivariant maps between them is a Grothendieck topos. In fact, $\mathbf{Cont}(G) \simeq \mathbf{Sh}(\mathcal{C}, J_{at})$, where $\mathcal{C}$

is the full subcategory of $\mathbf{Cont}(G)$ on the non-empty transitive actions and J_{at} is the atomic topology on it.

More generally, for any *topological groupoid* $(d, c : G \to X)$ (i.e. groupoid internal to the category of topological spaces, in the sense of section 5.3), the category $\mathbf{Sh}_G(X)$ of G-equivariant sheaves on X is a Grothendieck topos. Butz and Moerdijk have shown in [3] that every Grothendieck topos with enough points can be represented in this form.

As Grothendieck observed himself, a site of definition for a given topos can be seen as a sort of *presentation* of it by generators and relations (one can think of the objects of the category underlying the site as defining the generators, and of the arrows and covering sieves as defining the relations). In fact, this intuition has been fruitfully exploited in [10] for building various kind of ordered structures presented by generators and relations.

As a group can have many different presentations, so a Grothendieck topos can have many different sites of definition, as for instance shown by the following theorem of [1]:

Theorem 3.7 (Comparison Lemma)**.** *Let $(\mathcal{C}, J)$ be a site and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ which is J-dense, in the sense that for every object c of $\mathcal{C}$, the sieve generated by the family of arrows to c from objects in $\mathcal{D}$ is J-covering. Then the toposes $\mathbf{Sh}(\mathcal{C}, J)$ and $\mathbf{Sh}(\mathcal{D}, J|_{\mathcal{D}})$ are equivalent, where $J|_{\mathcal{D}}$ is the Grothendieck topology on $\mathcal{D}$ induced by J and defined by: $S \in J|_{\mathcal{D}}(d)$ if and only if $\overline{S} \in J(d)$, where $\overline{S}$ is the sieve on $\mathcal{C}$ generated by the arrows in S.*

There are more refined versions of the Comparison Lemma in the literature; anyway, we shall not be concerned with them in this book. For a proof of the version reported above, we refer the reader to the Appendix of [29].

Another important source of different sites of definition for a given topos is provided by its separating sets of objects.

Definition 3.8. (a) A family of arrows $\{f_i : E_i \to E \mid i \in I\}$ with common codomain in a Grothendieck topos $\mathcal{E}$ is said to be *epimorphic* if for any two arrows $a, b : E \to F$ in $\mathcal{E}$, $a \circ f_i = b \circ f_i$ for all $i \in I$ implies $a = b$.

(b) The *canonical topology* $J_{\mathcal{E}}$ on a Grothendieck topos $\mathcal{E}$ is the Grothendieck topology on $\mathcal{E}$ whose covering sieves are exactly the sieves generated by small epimorphic families.

Every separating set of objects $\mathcal{C}$ of a Grothendieck topos $\mathcal{E}$, regarded as a full subcategory of $\mathcal{E}$, is $J_{\mathcal{E}}$-dense and yields a representation $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}|_{\mathcal{C}})$.

In spite of the existence of multiple representations for a given topos, there exists an intrinsic characterization, obtained by Giraud, of the notion of Grothendieck topos: a locally small category is a Grothendieck topos if and only if it is a cocomplete elementary topos with a separating set of objects (cf. section 5.4 for the definition of elementary topos). We shall not make use of this characterization in the book. Rather, we shall systematically exploit the existence of different sites of definition for a given topos (cf. section 8).

3.3 Basic properties of categories of sheaves

Grothendieck toposes are very rich in terms of categorical structure: they possess all small limits and colimits, exponentials and a subobject classifier (as defined below).

Recall that a *limit* (or *projective limit*, in the terminology of [1]) of a functor $D : \mathcal{I} \to \mathcal{E}$ is a universal cone over D, i.e. it is an object e of $\mathcal{E}$ together with arrows $\lambda_i : e \to D(i)$ indexed over the objects of $\mathcal{I}$ such that for any arrow $f : i \to j$ in $\mathcal{I}$, $D(f) \circ \lambda_i = \lambda_j$, and which is universal with such property. The notion of *colimit* of a functor (called *inductive limit* in [1]) is dual to that of limit. See pp. 10-23 of [29] for more details.

An important property of functors which admit a right (resp. a left) adjoint, which will be exploited in the sequel, is the fact that they preserve all the colimits (resp. all the limits) which exist in the domain category.

For any site $(\mathcal{C}, J)$, the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ has a left adjoint $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$, called the *associated sheaf functor*, which preserves finite limits.

Limits in $\mathbf{Sh}(\mathcal{C}, J)$ are computed pointwise as in $\mathbf{Set}$ (they are created by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$) while colimits in $\mathbf{Sh}(\mathcal{C}, J)$ can be calculated by taking the image of the colimit of the given diagram in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ under a_J.

To understand how exponentials are defined in general categories of sheaves, let us recall their construction in the base topos of sets.

For any two sets X and Y, we can always form the set Y^X of functions $X \to Y$. This set enjoys the following (universal) property in the category $\mathbf{Set}$ of sets: the familiar bijection

$$Hom_{\mathbf{Set}}(Z, Y^X) \cong Hom_{\mathbf{Set}}(Z \times X, Y)$$

is natural in both Y and Z and hence it gives rise to an adjunction between the functor $- \times X : \mathbf{Set} \to \mathbf{Set}$ (left adjoint) and the functor $(-)^X : \mathbf{Set} \to \mathbf{Set}$ (right adjoint).

In a general category $\mathcal{C}$ with finite products, one defines the *exponential* A^B of two objects A and B by the universal property that for any object C of $\mathcal{C}$ the arrows $C \to A^B$ are in natural bijective correspondence with the arrows $C \times B \to A$. In $\mathbf{Sh}(\mathcal{C}, J)$ exponentials are computed as in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$; specifically, an immediate application of Yoneda Lemma yields $P^Q(c) = Nat(yc \times Q, P)$ for all objects c of $\mathcal{C}$, where yc denotes the representable functor associated to the object c and Nat denotes the set of natural transformations.

Definition 3.9. A category $\mathcal{C}$ is said to be *cartesian closed* if it has finite products and exponentials for each object $c \in \mathcal{C}$.

Notice that a Heyting algebra can be described as a lattice H with 0 and 1 which is cartesian closed when regarded as a preorder category with products, i.e. such that for any pair of elements $x, y \in H$ there is an element $x \Rightarrow y$ satisfying the adjunction $z \leq (x \Rightarrow y)$ if and only if $z \wedge x \leq y$ (for any $z \in H$). For $x \in H$, we put $\neg x := x \Rightarrow 0$ and call it the pseudocomplement of x in H.

Let us now turn to the notion of *subobject classifier*. Recall that a *subobject* of an object c in a category $\mathcal{C}$ is an equivalence class of monomorphisms with codomain c modulo the equivalence relation which identifies two such monomorphisms when they factor one through one another. In the category $\mathbf{Set}$, subobjects of a set X correspond bijectively to subsets of X. The notion of subobject thus represents a natural categorical generalization of that of subset in Set Theory.

In the category $\mathbf{Set}$, subsets S of a given set X can be identified with their characteristic functions $\chi_S : X \to \{0, 1\}$; in fact, denoted by $\top : \{*\} = 1_{\mathbf{Set}} \to \{0, 1\}$ the function which sends $*$ to 1, we have a *pullback square*

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & \{*\} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[\chi_S]{} & \{0,1\} \end{array}$$

where $i : S \to X$ is the inclusion and $! : S \to \{*\}$ is the unique arrow in $\mathbf{Set}$ to the terminal object $1_{\mathbf{Set}} = \{*\}$.

This motivates the following

Definition 3.10. In a category $\mathcal{C}$ with finite limits, a *subobject classifier* is a monomorphism $\top : 1_{\mathcal{C}} \to \Omega$, such that for every monomorphism $m : a' \to a$ there is a unique arrow $\chi_m : a \to \Omega$, called the *classifying arrow* of m, such that we have a pullback square

$$\begin{array}{ccc} a' & \xrightarrow{\;!\;} & 1_{\mathcal{C}} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \top} \\ a & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

Equivalently, a subobject classifier is a representing object for the functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sending an object a to the poset $\mathrm{Sub}_{\mathcal{C}}(a)$ of subobjects of a and an arrow $f : a \to b$ to the pullback map $f^* : \mathrm{Sub}_{\mathcal{C}}(b) \to \mathrm{Sub}_{\mathcal{C}}(a)$.

Note that, for any object a of $\mathcal{C}$, we have an arrow $\in_a : a \times \Omega^a \to \Omega$ generalizing the belonging relation $\in$ of Set Theory, given by the transpose of the identity arrow on Ω^a.

Summarizing, we have the following theorem.

Theorem 3.11. *Let $(\mathcal{C}, J)$ be a site. Then*

(i) the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}]$ has a left adjoint $a_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$, called the associated sheaf functor, which preserves finite limits.

(ii) The category $\mathbf{Sh}(\mathcal{C}, J)$ has all (small) limits, which are preserved by the inclusion functor $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}]$; in particular, limits are computed pointwise and the terminal object $1_{\mathbf{Sh}(\mathcal{C},J)}$ of $\mathbf{Sh}(\mathcal{C}, J)$ is the functor $T : \mathcal{C}^{op} \to \mathbf{Set}$ sending each object $c \in \mathcal{C}$ to the singleton $\{*\}$.

(iii) The associated sheaf functor $a_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ preserves colimits; in particular, $\mathbf{Sh}(\mathcal{C}, J)$ has all (small) colimits.

(iv) The category $\mathbf{Sh}(\mathcal{C}, J)$ has exponentials, which are constructed as in the topos $[\mathcal{C}^{op}, \mathbf{Set}]$.

(v) The category $\mathbf{Sh}(\mathcal{C}, J)$ has a subobject classifier.

Subobject classifiers in Grothendieck toposes are constructed as follows.

- Given a site $(\mathcal{C}, J)$ and a sieve S in $\mathcal{C}$ on an object c, we say that S is *J-closed* if for any arrow $f : d \to c$, $f^*(S) \in J(d)$ implies that $f \in S$.

- Let us define $\Omega_J : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ by:

$$\Omega_J(c) = \{R \mid R \text{ is a } J\text{-closed sieve on } c\} \text{ (for an object } c \in \mathcal{C});$$
$$\Omega_J(f) = f^*(-) \text{ (for an arrow } f \text{ in } \mathcal{C}),$$

  where $f^*(-)$ denotes the operation of pullback of sieves in $\mathcal{C}$ along f.

  Then the arrow $\top : 1_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_J$ defined by:

  $\top(*)(c) = M_c$ (the maximal sieve on c) for each $c \in \mathcal{C}$

  is a subobject classifier for $\mathbf{Sh}(\mathcal{C}, J)$.

- The classifying arrow $\chi_{A'} : A \to \Omega_J$ of a subobject $A' \subseteq A$ in $\mathbf{Sh}(\mathcal{C}, J)$ is given by:

$$\chi_{A'}(c)(x) = \{f : d \to c \mid A(f)(x) \in A'(d)\}$$

  where $c \in \mathcal{C}$ and $x \in A(c)$.

For a detailed proof of this theorem the reader is referred to sections III.5-6-7 of [29].

The following theorem describes the main properties of subobject lattices in a Grothendieck topos.

Theorem 3.12. *(i) For any Grothendieck topos $\mathcal{E}$ and any object a of $\mathcal{E}$, the poset $\mathrm{Sub}_{\mathcal{E}}(a)$ of all subobjects of a in $\mathcal{E}$ is a complete Heyting algebra.*

(ii) For any arrow $f : a \to b$ in a Grothendieck topos $\mathcal{E}$, the pullback functor $f^* : \mathrm{Sub}_{\mathcal{E}}(b) \to \mathrm{Sub}_{\mathcal{E}}(a)$ has both a left adjoint $\exists_f : \mathrm{Sub}_{\mathcal{E}}(a) \to \mathrm{Sub}_{\mathcal{E}}(b)$ and a right adjoint $\forall_f : \mathrm{Sub}_{\mathcal{E}}(a) \to \mathrm{Sub}_{\mathcal{E}}(b)$.

A detailed proof of this result can be found in section III.8 of [29].

3.4 Geometric morphisms

There are two natural types of morphisms to consider between toposes: *geometric morphisms* and *logical functors*. The former preserve the 'geometric structure' of toposes, while the latter the 'elementary logical' one. This latter class is the natural class of morphisms to consider when one is interested in regarding an elementary topos as the syntactic category for a higher-order intuitionistic type theory (cf. section 5.4), while the former is the natural class of morphisms to consider between toposes regarded as classifying toposes of geometric theories (cf. section 7 below).

Accordingly, we will focus in this book on the former class:

Definition 3.13. (a) A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$ such that the left adjoint f^*, called the *inverse image* of f, preserves finite limits (notice that f^* always preserves colimits, it having a right adjoint, while f_* always preserves limits).

(b) A *geometric transformation* $\alpha : f \to g$ between two geometric morphisms $f : \mathcal{F} \to \mathcal{E}$ is a natural transformation $f^* \to g^*$ (equivalently, a natural transformation $g_* \to f_*$).

We shall denote by $\mathfrak{Btop}$ the 2-category of Grothendieck toposes, geometric morphisms and geometric transformations between them.

Recall that adjoints to a given functor are uniquely determined up to isomorphism. By the Special Adjoint Functor Theorem, giving a geometric morphism $\mathcal{F} \to \mathcal{E}$ is equivalent to giving a cocontinuous (i.e., colimit-preserving) finite-limit-preserving functor $\mathcal{E} \to \mathcal{F}$.

Detailed proofs of the facts stated in the following list of examples can be found in sections VII.1-2 and VII.10 of [29].

Examples 3.14. (a) Any continuous map $f : X \to Y$ of topological spaces induces a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$, whose direct image $\mathbf{Sh}(f)_*$ is given by: $\mathbf{Sh}(f)_*(P)(V) = P(f^{-1}(V))$.

(b) More generally, any arrow $f : L \to L'$ in $\mathbf{Loc}$ induces a geometric morphism $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$; conversely, any geometric morphism $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$ is, up to isomorphism, of this form. This defines a full and faithful embedding of the (2-)category $\mathbf{Loc}$ into the (2-)category $\mathfrak{Btop}$.

(c) Even more generally, any morphism of sites $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$, whose direct image is the functor $- \circ f^{\mathrm{op}} : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

(d) Any functor $f : \mathcal{C} \to \mathcal{D}$ induces a geometric morphism $[f, \mathbf{Set}] : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$, whose inverse image is given by $- \circ f : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$. This functor has both a left and a right adjoint, respectively given by the left and right Kan extensions along the functor f.

A geometric morphism whose inverse image has a left adjoint is said to be *essential*. If $\mathcal{C}$ and $\mathcal{D}$ are Cauchy-complete categories (i.e., categories in which all idempotents split) then every essential geometric morphism $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ is, up to isomorphism, of the form $[f, \mathbf{Set}]$ for some functor $f : \mathcal{C} \to \mathcal{D}$.

(e) For any site $(\mathcal{C}, J)$, the canonical inclusion functor $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the direct image of a geometric morphism $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ whose inverse image $a_J : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ is the associated sheaf functor.

(f) For any (Grothendieck) topos $\mathcal{E}$ and any arrow $f : F \to E$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}/E \to \mathcal{E}/F$ is the inverse image of a geometric morphism $\mathcal{E}/F \to \mathcal{E}/E$.

(g) Every Grothendieck topos $\mathcal{E}$ admits a unique (up to isomorphism) geometric morphism $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathbf{Set}$. The direct image of $\gamma_{\mathcal{E}}$ is the *global sections functor* $Hom_{\mathcal{E}}(1_{\mathcal{E}}, -) : \mathcal{E} \to \mathbf{Set}$, while the inverse image functor $\gamma_{\mathcal{E}}^*$ is given by $S \to \coprod_{s \in S} 1_{\mathcal{E}}$.

Definition 3.15. (a) A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be a *surjection* if its inverse image $f^* : \mathcal{E} \to \mathcal{F}$ is faithful (equivalently, conservative).

(b) A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be an *inclusion* if its direct image functor $f_* : \mathcal{F} \to \mathcal{E}$ is full and faithful.

Remark 3.16. By Theorem 5.34, (isomorphism classes of) geometric inclusions with codomain a topos $\mathcal{E}$ correspond precisely to the local operators on $\mathcal{E}$. We shall also call geometric inclusions into a topos $\mathcal{E}$ the *subtoposes* of $\mathcal{E}$.

Theorem 3.17 (Theorems VII.4.6 and VII.4.8 [29])**.** *Every geometric morphism of (Grothendieck) toposes can be factored, in a unique way up to commuting isomorphisms, as a surjection followed by an inclusion.*

Definition 3.18. By a *point* of a topos $\mathcal{E}$, we mean a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Examples 3.19. (a) For any site $(\mathcal{C}, J)$, the points of the topos $\mathbf{Sh}(\mathcal{C}, J)$ can be identified with the J-continuous flat functors $\mathcal{C} \to \mathbf{Set}$ (cf. Theorem 3.32 below).

(b) For any locale L, the points of the topos $\mathbf{Sh}(L)$ correspond precisely to the frame homomorphisms $L \to \{0, 1\}$.

(c) For any small category $\mathcal{C}$ and any object c of $\mathcal{C}$, we have a point $e_c : \mathbf{Set} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, whose inverse image is the evaluation functor at c.

Proposition 3.20. Any set of points P of a Grothendieck topos $\mathcal{E}$ indexed by a set X via a function $\xi : X \to P$ can be identified with a geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathcal{E}$.

Definition 3.21. (a) Let $\mathcal{E}$ be a topos and P be a collection of points of $\mathcal{E}$ indexed by a set X via a function $\xi : X \to P$. We say that P is *separating* for $\mathcal{E}$ if the points in P are jointly surjective, i.e. if the inverse image functors of the geometric morphisms in P jointly reflect isomorphisms (equivalently, if the geometric morphism $\tilde{\xi} : [X, \mathbf{Set}] \to \mathcal{E}$ is surjective).

(b) A topos is said to *have enough points* if the collection of all the points of $\mathcal{E}$ is separating for $\mathcal{E}$.

The following construction provides a way for endowing a given set of points of a topos with a natural topology.

Definition 3.22 (cf. [10])**.** Let $\xi : X \to \mathrm{P}$ be an indexing of a set P of points of a Grothendieck topos $\mathcal{E}$ by a set X. We define the *subterminal topology* $\tau_{\xi}^{\mathcal{E}}$ as the image of the function $\phi_{\mathcal{E}} : \mathrm{Sub}_{\mathcal{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathcal{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} \; .$$

We denote the space X endowed with the topology $\tau_{\xi}^{\mathcal{E}}$ by $X_{\tau_{\xi}^{\mathcal{E}}}$.

Proposition 3.23. If P is a *separating set* of points for $\mathcal{E}$ then the frame $\mathcal{O}(X_{\tau_{\xi}^{\mathcal{E}}})$ of open sets of $X_{\tau_{\xi}^{\mathcal{E}}}$ is isomorphic to $\mathrm{Sub}_{\mathcal{E}}(1)$ (via $\phi_{\mathcal{E}}$).

This result was used in [10] and [13] to build spectra for various kinds of partially ordered structures and, combined with other results, Stone-type and Priestley-type dualities for them.

Definition 3.24. A topological space is said to be *sober* if every irreducible closed subset of it is the closure of a unique point.

Example 3.25. Affine algebraic varieties with the Zariski topology are sober spaces.

Remark 3.26. The sober topological spaces are exactly the topological spaces X such that the canonical map $X \to X_{\tau_{\xi}^{\mathbf{Sh}(X)}}$ is an homeomorphisms. In other words, the sober topological spaces are exactly, up to homeomorphisms, the spaces of points of localic toposes.

3.5 Diaconescu's equivalence

The following result, already proved by Grothendieck in [1], is of fundamental importance in topos theory and is commonly referred to in the literature as Diaconescu's equivalence, in honour of R. Diaconescu who proved its relativization to an arbitrary base topos.

Before stating it, we need to introduce the equally fundamental notion of flat functor.

Recall that the *category of elements* $\int P$ of a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ has as objects the pairs (c, x) where c is an object of $\mathcal{C}$ and x is an element of $P(c)$ and as arrows $(c, x) \to (d, y)$ the arrows $f : c \to d$ in $\mathcal{C}$ such that $P(f)(y) = x$.

The following proposition exhibits a very general hom-tensor adjunction.

Proposition 3.27. Let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a locally small cocomplete category. Then, for any functor $A : \mathcal{C} \to \mathcal{E}$ the functor $R_A : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ defined for each $e \in \mathcal{E}$ and $c \in \mathcal{C}$ by:

$$R_A(e)(c) = Hom_{\mathcal{E}}(A(c), e)$$

has a left adjoint $- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$.

Sketch of proof. The left adjoint $- \otimes_{\mathcal{C}} A$ sends a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ to the 'generalized tensor product' $P \otimes_{\mathcal{C}} A = colim(A \circ \pi_P)$, where $\pi_P : \int P \to \mathcal{C}$ is the canonical projection to $\mathcal{C}$ from the category of elements $\int P$ of the presheaf P. For more details, see the proof of Theorem I.5.2 [29]. $\square$

Definition 3.28. (a) A functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a locally small topos $\mathcal{E}$ with small colimits is said to be *flat* if the functor $- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ preserves finite limits.

(b) The full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the flat functors will be denoted by **Flat**$(\mathcal{C}, \mathcal{E})$.

The following fundamental theorem allows to identify geometric morphisms to a topos of sheaves on a site with certain functors defined on the underlying category of the site.

Theorem 3.29. *Let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a Grothendieck topos. Then we have an equivalence of categories*

$$\textbf{\textit{Geom}}(\mathcal{E}, [\mathcal{C}^{op}, \mathbf{Set}]) \simeq \textbf{\textit{Flat}}(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$, which sends

- *a flat functor $A : \mathcal{C} \to \mathcal{E}$ to the geometric morphism $\mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ given by the functors R_A and $- \otimes_{\mathcal{C}} A$, and*
- *a geometric morphism $f : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ to the flat functor given by the composite $f^* \circ y$ of $f^* : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Sketch of proof. By definition of flat functor, the functor $- \otimes_{\mathcal{C}} A$ is the inverse image part of a geometric morphism $\mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Conversely, given a geometric morphism $f : \mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, the composite of f^* with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ yields a functor which is flat since $f^* = - \otimes_{\mathcal{C}} (f^* \circ y)$ (as the two functors take the same values on representables and they both preserve small colimits). The fact that $(- \otimes_{\mathcal{C}} A) \circ y \cong A$ follows immediately from the definition of $- \otimes_{\mathcal{C}} A$. For more details see the proof of Corollary I.5.4 [29]. $\square$

Recall that a small category $\mathcal{C}$ is said to be *filtered* if it is non-empty, for any objects $c, d \in \mathcal{C}$ there exists an object $e \in \mathcal{C}$ and two arrows $f : c \to e$ and $g : d \to e$ and for any parallel arrows $f, g : a \to b$ in $\mathcal{C}$ there exists an arrow $h : b \to c$ in $\mathcal{C}$ such that $h \circ f = h \circ g$.

For a detailed proof of the facts stated in the following proposition we refer the reader to sections VII.7 and VII.9 of [29].

Proposition 3.30. (a) A functor $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is flat if and only if it is a filtered colimit of representables, equivalently if and only if its category of elements $\int F$ is filtered. More generally, a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{E}$ with values in a Grothendieck topos $\mathcal{E}$ is flat if and only if it is *filtering*, that is if and only if the following conditions are satisfied:

(i) For any object E of $\mathcal{E}$ there exists some epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each index i an object b_i of $\mathcal{C}$ and a generalized element $E_i \to F(b_i)$ in $\mathcal{E}$;

(ii) For any two objects c and d in $\mathcal{C}$ and any generalized element $(x, y) : E \to F(c) \times F(d)$ in $\mathcal{E}$ there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each index i an object b_i of $\mathcal{C}$ with arrows $u_i : c \to b_i$ and $v_i : d \to b_i$ in $\mathcal{C}$ and a generalized element $z_i : E_i \to F(b_i)$ in $\mathcal{E}$ such that $< F(u_i), F(v_i) > \circ z_i =< x, y > \circ e_i$;

(iii) For any two parallel arrows $u, v : d \to c$ in $\mathcal{C}$ and any generalized element $x : E \to F(c)$ in $\mathcal{E}$ for which $F(u) \circ x = F(v) \circ x$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each index i an arrow $w_i : c \to b_i$ and a generalized element $y_i : E_i \to F(b_i)$ such that $w_i \circ u = w_i \circ v$ and $F(w_i) \circ y_i = x \circ e_i$.

(b) Let $\mathcal{C}$ be a category with finite limits and $\mathcal{E}$ be a Grothendieck topos. Then a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.

Definition 3.31. Let $\mathcal{E}$ be a Grothendieck topos. Given a site $(\mathcal{C}, J)$, a functor $F : \mathcal{C} \to \mathcal{E}$ is said to be J*-continuous* if it sends J-covering sieves to epimorphic families.

The full subcategory of **Flat**$(\mathcal{C}, \mathcal{E})$ on the J-continuous flat functors will be denoted by **Flat**$_J(\mathcal{C}, \mathcal{E})$.

Theorem 3.32. *For any site $(\mathcal{C}, J)$ and Grothendieck topos $\mathcal{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories*

$$\textbf{\textit{Geom}}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \textbf{\textit{Flat}}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

Sketch of proof. Appeal to Theorem 3.29

- identifying the geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ with the geometric morphisms $\mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, and

- using the characterization of such morphisms as the geometric morphisms $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ sends J-covering sieves to epimorphic families in $\mathcal{E}$.

For more details see section VII.7 of [29]. □

4 First-order logic

In Logic, *first-order languages* are a wide class of formal languages used for describing mathematical structures. The attribute 'first-order' means that all the universal and existential quantifications occurring in the axioms of the theory concern only individuals rather than collections of individuals. For instance, the property of a group to be Abelian can be expressed in a first-order way in the language of groups, by the formula $(\forall x)(\forall y)(x + y = y + x)$ whilst the property of the ordered set $\mathbb{R}$ of real numbers that every bounded subset of it admits a supremum is not expressible in a first-order way (in the language of ordered sets) since it involves a quantification over subsets of the given structure rather than over elements of it (the formula defining it is a second-order one).

A first-order language contains *sorts*, which are meant to represent different *kinds* of individuals, *terms*, which denote individuals, and *formulae*, which make assertions about the individuals. Compound terms and formulae are formed by using various logical operators, that is either connectives (such as $\wedge$, $\vee$, $\Rightarrow$ etc.) or quantifiers ($\exists$ and $\forall$). For example, as we shall see in Example 4.7, it is natural to axiomatize the notion of (small) category by using language with two sorts, one for objects and one for arrows.

It is well-known, at least since the work of A. Tarski, that first-order languages can always be interpreted in the context of (a given model of) set theory (sorts are interpreted as sets, function symbols as functions and relation symbols as subsets). We will show in this section that one can meaningfully interpret them also in a general category, provided that the latter possesses enough categorical structure. Sorts will be interpreted as *objects* of the given category, terms as *arrows* and formulae as *subobjects*, in a way that respects the logical structure of compound expressions.

4.1 First-order theories

Definition 4.1. A first-order *signature* Σ consists of the following data:

(a) A set Σ-Sort of *sorts*.

(b) A set Σ-Fun of *function symbols*, together with a map assigning to each $f \in \Sigma$-Fun its *type*, which consists of a finite non-empty list of sorts: we write

$$f : A_1 \cdots A_n \to B$$

to indicate that f has type $A_1, \ldots, A_n, B$ (if $n = 0$, f is called a *constant* of sort B). The number n is called the *arity* of f.

(c) A set Σ-Rel of *relation symbols*, together with a map assigning to each Σ-Rel its *type*, which consists of a finite list of sorts: we write

$$R \rightarrowtail A_1 \cdots A_n$$

to indicate that R has type $A_1, \ldots A_n$. The number n is called the *arity* of R.

For each sort A of a signature Σ we assume given a supply of variables of sort A, used to denote individuals of type A.

A variable is said to be *free* in a given formula if it appears non-quantified in it. For example, in the formula $(\exists x)(x + y = 0)$ the variable y is free while the variable x is not. Note that a formula makes assertions only about the non-quantified variables occurring in it. For this reason, formulae without free variables are called *sentences*, as their validity does not depend on the value of any variable.

Starting from variables, terms are built-up by repeated 'applications' of function symbols to them, as follows.

Definition 4.2. Let Σ be a signature. The collection of *terms* over Σ is defined recursively by the clauses below; simultaneously, we define the sort of each term and write $t : A$ to denote that t is a term of sort A.

a) $x : A$, if x is a variable of sort A.

b) $f(t_1, \ldots, t_n) : B$ if $f : A_1 \cdots A_n \to B$ is a function symbol and $t_1 : A_1, \ldots, t_n : A_n$.

Consider the following formation rules for recursively building classes of formulae F over Σ, together with, for each formula ϕ, the (finite) set FV(ϕ) of free variables of ϕ.

(i) *Relations*: $R(t_1, \ldots, t_n)$ is in F, if $R \rightarrowtail A_1 \cdots A_n$ is a relation symbol and $t_1 : A_1, \ldots, t_n : A_n$ are terms; the free variables of this formula are all the variables occurring in some t_i.

(ii) *Equality*: $(s = t)$ is in F if s and t are terms of the same sort; FV$(s = t)$ is the set of variables occurring in s or t (or both).

(iii) *Truth*: $\top$ is in F; FV$(\top) = \emptyset$.

(iv) *Binary conjunction*: $(\phi \wedge \psi)$ is in F, if ϕ and ψ are in F; FV$(\phi \wedge \psi) =$ FV$(\phi) \cup$ FV(ψ).

(v) *Falsity*: $\bot$ is in F; FV$(\top) = \emptyset$.

(vi) *Binary disjunction*: $(\phi \vee \psi)$ is in F, if ϕ and ψ are in F; FV$(\phi \vee \psi) =$ FV$(\phi) \cup$ FV(ψ).

(vii) *Implication*: $(\phi \Rightarrow \psi)$ is in F, if ϕ and ψ are in F; FV$(\phi \Rightarrow \psi) =$ FV$(\phi) \cup$FV(ψ).

(viii) *Negation*: $\neg\phi$ is in F, if ϕ is in F; FV$(\neg\phi) =$ FV(ϕ).

(ix) *Existential quantification*: $(\exists x)\phi$ is in F, if ϕ is in F and x is a variable; FV$((\exists x)\phi) =$ FV$(\phi) \setminus \{x\}$.

(x) *Universal quantification*: $(\forall x)\phi$ is in F, if ϕ is in F and x is a variable; FV$((\forall x)\phi) =$ FV$(\phi) \setminus \{x\}$.

(xi) *Infinitary disjunction*: $\bigvee\limits_{i \in I} \phi_i$ is in F, if I is a set, ϕ_i is in F for each $i \in I$ and FV$(\bigvee\limits_{i \in I} \phi_i) := \bigcup\limits_{i \in I}FV(\phi_i)$ is finite.

(xii) *Infinitary conjunction*: $\bigwedge\limits_{i \in I} \phi_i$ is in F, if I is a set, ϕ_i is in F for each $i \in I$ and FV$(\bigwedge\limits_{i \in I} \phi_i) := \bigcup\limits_{i \in I}FV(\phi_i)$ is finite.

A *context* is a finite list $\vec{x} = x_1, \dots, x_n$ of distinct variables (the empty context, for $n = 0$ is allowed and indicated by $[]$). We will often consider *formulae-in-context*, that is formulae ϕ equipped with a context $\vec{x}$ such that all the free variables of ϕ occur among $\vec{x}$; we will write either $\phi(\vec{x})$ or $\{\vec{x} \,.\, \phi\}$. The *canonical context* of a given formula is the context consisting of all the variables which appear freely in the formula.

Definition 4.3. In relation to the above-mentioned forming rules:

(a) The set of *atomic formulae* over Σ is the smallest set closed under *Relations* and *Equality*).

(b) The set of *Horn formulae* over Σ is the smallest set containing the class of atomic formulae and closed under *Truth* and *Binary conjunction*.

(c) The set of *regular formulae* over Σ is the smallest set containing the class of atomic formulae and closed under *Truth*, *Binary conjunction* and *Existential quantification*.

(d) The set of *coherent formulae* over Σ is the smallest set containing the set of regular formulae and closed under *False* and *Binary disjunction*.

(e) The set of *first-order formulae* over Σ is the smallest set closed under all the forming rules except for the infinitary ones.

(f) The *class* of *geometric formulae* over Σ is the smallest class containing the class of coherent formulae and closed under *Infinitary disjunction*.

(g) The *class* of *infinitary first-order formulae* over Σ is the smallest class closed under all the above-mentioned forming rules.

Definition 4.4. (a) By a *sequent* over a signature Σ we mean a formal expression of the form *($\phi \vdash_{\vec{x}} \psi$)*, where ϕ and ψ are formulae over Σ and $\vec{x}$ is a context suitable for both of them. The intended interpretation of this expression is that ψ is a logical consequence of ϕ in the context $\vec{x}$, i.e. that any assignment of individual values to the variables in $\vec{x}$ which makes ϕ true will also make ψ true.

(b) We say a sequent ($\phi \vdash_{\vec{x}} \psi$) is Horn (resp. regular, coherent, ...) if both ϕ and ψ are Horn (resp. regular, coherent, ...) formulae.

Notice that, in full first-order logic, the general notion of sequent is not really needed, since the sequent ($\phi \vdash_{\vec{x}} \psi$) expresses the same idea as $(\top \vdash (\forall \vec{x})(\phi \Rightarrow \psi))$.

Definition 4.5. (a) By a *theory* over a signature Σ, we mean a set $\mathbb{T}$ of sequents over Σ, whose elements are called the (non-logical) *axioms* of $\mathbb{T}$.

(b) We say that $\mathbb{T}$ is an *algebraic theory* if its signature Σ has a single sort and no relation symbols (apart from equality) and its axioms are all of the form $\top \vdash_{\vec{x}} \phi$ where ϕ is an atomic formula $(s = t)$ and $\vec{x}$ its canonical context.

(c) We say $\mathbb{T}$ is a *Horn* (resp. *regular*, *coherent*, *geometric*) theory if all the sequents in $\mathbb{T}$ are Horn (resp. regular, coherent, geometric).

(d) We say that $\mathbb{T}$ is a *universal* H*orn* theory if its axioms are all of the form $\phi \vdash_{\vec{x}} \psi$, where ϕ is a finite (possibly empty) conjunction of atomic formulae and ψ is an atomic formula or the formula $\bot$.

(e) We say that $\mathbb{T}$ is a *cartesian theory* if its axioms can be well-ordered in such a way that each axiom is cartesian relative to the sub-theory consisting of all the axioms preceding it in the ordering, in the sense that all the existential quantifications which appear in the given axiom are provably unique relative to that sub-theory.

(f) We say that $\mathbb{T}$ is a *propositional theory* if the signature of $\mathbb{T}$ has no sorts, i.e. it only consists of 0-ary relation symbols.

Remark 4.6. The study of the model theory of algebraic theories has been initiated by G. Birkhoff in the context of universal algebra and developed in a categorical setting starting from the pioneering work [28] by W. Lawvere introducing the functorial semantics of algebraic theories.

Example 4.7. An important example of cartesian theory is given by the first-order *theory of small categories* $\mathbb{C}$. The language of $\mathbb{C}$ consists of two sorts, O and A respectively for objects and arrows, two function symbols $\mathrm{dom}, \mathrm{cod} : A \to O$ formalizing domain and codomain, a function symbol $1 : O \to A$ formalizing the assignment to any object of the identity arrow on it and a ternary predicate C formalizing composition of arrows in the sense that $C(f, g, h)$ if and only if $h = f \circ g$ (notice that we have to use a relation symbol rather than a function symbol for formalizing composition since the latter is not everywhere defined). Over this signature, the axioms of $\mathbb{C}$ are the obvious ones. In particular, we have an axiom $(\mathrm{dom}(f) = cod(g) \vdash_{f,g} (\exists h)C(f, g, h))$ expressing the existence of the composite of two arrows such that the codomain of the first coincides with the domain of the second, which is cartesian relative to the sequent $(C(f, g, h) \wedge C(f, g, h') \vdash_{f,g,h,h'} h = h')$ expressing the functionality of the predicate C.

4.2 Deduction systems for first-order logic

To each of the fragments of first-order logic introduced above, we can naturally associate a *deduction system*, in the same spirit as in classical first-order logic. Such systems will be formulated as *sequent-calculi*, that is they will consist of inference rules enabling us to derive a sequent from a collection of others; we will write

$$\frac{\Gamma}{\sigma}$$

to mean that the sequent σ can be inferred by a collection of sequents Γ. A double line instead of the single line will mean that each of the sequents can be inferred from the other.

Given the axioms and inference rules below, the notion of *proof* (or *derivation*) is the usual one: a chain of inference rules whose premises are the axioms in the system and whose conclusion is the given sequent. Allowing the axioms of theory $\mathbb{T}$ to be taken as premises yields the notion of *proof relative to a theory* $\mathbb{T}$.

Consider the following rules.

- The *rules for finite conjunction* are the axioms

$$(\phi \vdash_{\vec{x}} \top) \quad ((\phi \wedge \psi) \vdash_{\vec{x}} \phi) \quad ((\phi \wedge \psi) \vdash_{\vec{x}} \psi)$$

  and the rule

$$\frac{(\phi \vdash_{\vec{x}} \psi)(\phi \vdash_{\vec{x}} \chi)}{(\phi \vdash_{\vec{x}} (\psi \wedge \chi))}$$

- The *rules for finite disjunction* are the axioms

$$(\bot \vdash_{\vec{x}} \phi) \quad (\phi \vdash_{\vec{x}} (\phi \vee \psi)) \quad (\psi \vdash_{\vec{x}} \phi \vee \psi)$$

  and the rule

$$\frac{(\phi \vdash_{\vec{x}} \chi)(\psi \vdash_{\vec{x}} \chi)}{((\phi \vee \psi) \vdash_{\vec{x}} \chi)}$$

- The *rules for infinitary conjunction (resp. disjunction)* are the infinitary analogues of the rules for finite conjunction (resp. disjunction).

- The *rules for implication* consist of the double rule

$$\frac{(\phi \wedge \psi \vdash_{\vec{x}} \chi)}{\overline{(\psi \vdash_{\vec{x}} (\phi \Rightarrow \chi))}}$$

- The *rules for existential quantification* consist of the double rule

$$\frac{(\phi \vdash_{\vec{x},y} \psi)}{\overline{((\exists y)\phi \vdash_{\vec{x}} \psi)}}$$

  provided that y is not free in ψ.

- The *rules for universal quantification* consist of the double rule

$$\frac{(\phi \vdash_{\vec{x},y} \psi)}{\overline{(\phi \vdash_{\vec{x}} (\forall y)\psi)}}$$

- The *distributive axiom* is

$$((\phi \wedge (\psi \vee \chi)) \vdash_{\vec{x}} ((\phi \wedge \psi) \vee (\phi \wedge \chi)))$$

- The *Frobenius axiom* is

$$((\phi \wedge (\exists y)\psi) \vdash_{\vec{x}} (\exists y)(\phi \wedge \psi))$$

  where y is a variable not in the context $\vec{x}$.

- The *Law of excluded middle* is

$$(\top \vdash_{\vec{x}} \phi \vee \neg\phi)$$

Remark 4.8. Note that these rules, although having being conceived with a particular semantics in mind (namely, the one that we shall describe in section 5), are completely formal; they do not have any meaning by themselves. Indeed, a fundamental principle of modern Logic is that of strictly separating syntax and semantics, distinguishing in particular the notion of *provability* in a given formal system (intended as the possibility of deriving an assertion from a given set of premises by repeatedly applying certain specified ‘inference rules’) from that of *validity* (or satisfaction) in a given structure. Of central importance is therefore the investigation of the connections between syntax and semantics, possibly leading to soundness and (in)completeness theorems relating the notion of provability in a deductive system and the notion of validity in an appropriately chosen class of structures. Results of this kind for the fragments of logic that we shall consider will be discussed in section 6.5.

4.3 Fragments of first-order logic

Definition 4.9. In addition to the usual structural rules of *sequent-calculi* (*Identity axiom*, *Equality rules*, *Substitution rule*, and *Cut rule*), our deduction systems consist of the following rules:

Algebraic logic no additional rules

Horn logic finite conjunction

Regular logic finite conjunction, existential quantification and Frobenius axiom

Coherent logic finite conjunction, finite disjunction, existential quantification, distributive axiom and Frobenius axiom

Geometric logic finite conjunction, infinitary disjunction, existential quantification, 'infinitary' distributive axiom, Frobenius axiom

Intuitionistic first-order logic all the finitary rules except for the law of excluded middle

Classical first-order logic all the finitary rules

Definition 4.10. We say a sequent σ is *provable* in an algebraic (regular, coherent, ...) theory $\mathbb{T}$ if there exists a derivation of σ relative to $\mathbb{T}$, in the appropriate fragment of first-order logic.

In geometric logic, intuitionistic and classical provability of geometric sequents coincide, provided that the ambient set theory satisfies the axiom of choice:

Theorem 4.11 (Barr's theorem - cf. Proposition D3.1.16 [26])**.** *Assuming that the ambient set-theory satisfies the axiom of choice, if a geometric sequent σ is derivable from the axioms of a geometric theory $\mathbb{T}$ using 'classical geometric logic' (i.e. the rules of geometric logic plus the Law of Excluded Middle), then there is also a constructive derivation of σ, not using the Law of Excluded Middle.*

Remark 4.12. Since a given theory can in general be regarded as belonging to more than one of the fragments of geometric logic that we considered above, it is natural to wonder whether the notion of provability of sequents over the signature of the theory does not depend on the fragment. A positive answer to this question will be obtained in section 8 an application of the 'bridge technique'.

5 Categorical semantics

Generalizing the classical Tarskian definition of satisfaction of first-order formulae in ordinary set-valued structures, one can naturally obtain, given a signature Σ, a notion of Σ-structure in a category with finite products, and define, according to the categorical structure present on the category, a notion of interpretation of an appropriate fragment of first-order logic in it.

Specifically, we will introduce various classes of 'logical' categories, each of them providing a semantics for a corresponding fragment of first-order logic:

Cartesian categories Cartesian logic

Regular categories Regular logic

Coherent categories Coherent logic

Geometric categories Geometric logic

Heyting categories First-order intuitionistic logic

Boolean coherent categories First-order classical logic

The process by which these classes of categories will be defined is quite canonical: first one looks at the set-theoretic structure needed to interpret the connectives and quantifiers occurring in the given fragment, then one proceeds to characterize this structure in categorical terms; the resulting categorical structure will be used for interpreting the given fragment of logic. For example, one immediately realizes that in set theory the conjunction $\phi \wedge \psi$ of two formulae is interpreted as the intersection of the interpretations of the two formulae ϕ and ψ; now, intersections of subsets can be characterized in categorical terms as pullbacks of subobjects. This ensures that, in any category $\mathcal{C}$ with finite limits, it is possible to give a meaning to the conjunction of two formulae which are interpretable in $\mathcal{C}$.

5.1 Structures in categories

Let us start defining the notion of Σ-structure in a category with finite products.

Definition 5.1. Let $\mathcal{C}$ be a category with finite products and Σ be a signature. A Σ-*structure* M in $\mathcal{C}$ is specified by the following data:

(i) A function assigning to each sort A in Σ-Sort, an *object* MA of $\mathcal{C}$. For finite strings of sorts, we define $M(A_1, \ldots, A_n) = MA_1 \times \cdots \times MA_n$ and set $M([])$ equal to the terminal object 1 of $\mathcal{C}$.

(ii) A function assigning to each function symbol $f : A_1 \cdots A_n \to B$ in Σ-Fun an *arrow* $Mf : M(A_1, \ldots, A_n) \to MB$ in $\mathcal{C}$.

(iii) A function assigning to each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ-Rel a *subobject* $MR \rightarrowtail M(A_1, \ldots, A_n)$ in $\mathcal{C}$.

Definition 5.2. A Σ-*structure homomorphism* $h : M \to N$ between two Σ-structures M and N in $\mathcal{C}$ is a collection of arrows $h_A : MA \to NA$ in $\mathcal{C}$ indexed by the sorts of Σ and satisfying the following two conditions:

(i) For each function symbol $f : A_1 \cdots A_n \to B$ in Σ-Fun, the diagram

$$\begin{array}{ccc} M(A_1, \ldots, A_n) & \xrightarrow{Mf} & MB \\ \Big\downarrow{\scriptstyle h_{A_1} \times \cdots \times h_{A_n}} & & \Big\downarrow{\scriptstyle h_B} \\ N(A_1, \ldots, A_n) & \xrightarrow[Nf]{} & NB \end{array}$$

commutes.

(ii) For each relation symbol $R \rightarrowtail A_1 \cdots A_n$ in Σ-Rel, there is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} MR & \longrightarrow & M(A_1, \ldots, A_n) \\ \downarrow & & \downarrow{\scriptstyle h_{A_1} \times \cdots \times h_{A_n}} \\ NR & \longrightarrow & M(A_1, \ldots, A_n) \ . \end{array}$$

Definition 5.3. Given a category $\mathcal{C}$ with finite products, Σ-structures in $\mathcal{C}$ and Σ-homomorphisms between them form a *category*, denoted by Σ-***Str***$(\mathcal{C})$. Identities and composition in Σ-**Str**$(\mathcal{C})$ are defined componentwise from those in $\mathcal{C}$.

Remark 5.4. If $\mathcal{C}$ and $\mathcal{D}$ are two categories with finite products then any functor $F : \mathcal{C} \to \mathcal{D}$ which preserves finite products and monomorphisms induces a functor Σ-**Str**(F) : Σ-**Str**$(\mathcal{C}) \to$ Σ-**Str**$(\mathcal{D})$ in the obvious way.

5.2 Classes of 'logical' categories

In this section we shall introduce classes of categories in which the fragments of first-order logic considered above can be naturally interpreted.

In any category $\mathcal{C}$ with pullbacks, pullbacks of monomorphisms are again monomorphisms; thus, for any arrow $f : a \to b$ in $\mathcal{C}$, we have a *pullback functor*

$$f^* : \mathrm{Sub}_{\mathcal{C}}(b) \to \mathrm{Sub}_{\mathcal{C}}(a) \ .$$

Recall that by a *finite limit* in a category $\mathcal{C}$ we mean a limit of a functor $F : \mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a *finite category* (i.e. a category with only a finite number of objects and arrows).

Definition 5.5. A *cartesian* category is any category with finite limits.

As we shall see below, in cartesian categories we can interpret atomic formulae as well as finite conjunctions of them; in fact, conjunctions will be interpreted as *pullbacks* (i.e. intersections) of subobjects.

Definition 5.6. (a) Given two monomorphisms $m_1 : a_1 \rightarrowtail c$ and $m_2 : a_2 \rightarrowtail c$ of an object c in a category $\mathcal{C}$, we say that m_1 factors through m_2 if there is a (necessarily unique) arrow $r : a_1 \to a_2$ in $\mathcal{C}$ such that $m_2 \circ r = m_1$. (Note that this defines a preorder relation $\leq$ on the collection $\mathrm{Sub}_{\mathcal{C}}(c)$ of subobjects of a given object c.)

(b) We say that a cartesian category $\mathcal{C}$ has *images* if we are given an operation assigning to each morphism f of $\mathcal{C}$ a subobject $Im(f)$ of its codomain, which is the least (in the sense of the preorder $\leq$) subobject of $cod(f)$ through which f factors.

(c) A *regular category* is a cartesian category $\mathcal{C}$ such that $\mathcal{C}$ has images and they are stable under pullback.

Proposition 5.7. Given an arrow $f : a \to b$ in a regular category $\mathcal{C}$, the pullback functor $f^* : \mathrm{Sub}_{\mathcal{C}}(b) \to \mathrm{Sub}_{\mathcal{C}}(a)$ has a left adjoint $\exists_f : \mathrm{Sub}_{\mathcal{C}}(a) \to \mathrm{Sub}_{\mathcal{C}}(b)$, which assigns to a subobject $m : c \rightarrowtail a$ the image of the composite arrow $f \circ m$.

In a regular category, every arrow $f : a \to b$ factors uniquely through its image $Im(f) \rightarrowtail b$ as the composite $a \to Im(f) \to b$ of $Im(f) \rightarrowtail b$ with an arrow $c(f) : a \to Im(f)$; arrows of the form $c(f)$ for some f are called *covers*. In fact, every arrow in a regular category can be factored uniquely as a cover followed by a monomorphism, and covers are precisely the arrows g such that $Im(g) = 1_{\mathrm{cod}(g)}$.

As we shall see below, in regular categories we can interpret formulae built-up from atomic formulae by using finite conjunctions and existential quantifications; in fact, the existential quantifiers will be interpreted as images of certain arrows.

Definition 5.8. A *coherent category* is a regular category $\mathcal{C}$ in which each $Sub_{\mathcal{C}}(c)$ has finite unions and each $f^* : \mathrm{Sub}_{\mathcal{C}}(b) \to \mathrm{Sub}_{\mathcal{C}}(a)$ preserves them.

As we shall see below, in coherent categories we can interpret formulae built-up from atomic formulae by using finite conjunctions, existential quantifications, and finite disjunctions; in fact, finite disjunctions will be interpreted as finite unions of subobjects.

Note in passing that, if coproducts exist, a union of subobjects of an object c may be constructed as the image of the induced arrow from the coproduct of such subobjects to c.

Definition 5.9. (a) A (large) category $\mathcal{C}$ is said to be *well-powered* if each of the preorders $Sub_{\mathcal{C}}(a)$, $a \in \mathcal{C}$, is equivalent to a small category.

(b) A *geometric category* is a well-powered regular category whose subobject lattices have arbitrary unions which are stable under pullback.

As we shall see below, in geometric categories we can interpret formulae built-up from atomic formulae by using finite conjunctions, existential quantifications, and infinitary disjunctions; in fact, disjunctions will be interpreted as unions of subobjects.

To understand how to categorically interpret quantifiers, let us analyse their interpretations in the category $\mathbf{Set}$.

Let X and Y be two sets. For any given subset $S \subseteq X \times Y$, we can consider the sets

$$\forall_p S := \{y \in Y \mid \text{for all } x \in X, (x, y) \in S\} \text{ and}$$

$$\exists_p S := \{y \in Y \mid \text{there exists } x \in X, (x, y) \in S\} .$$

The projection map $p : X \times Y \to Y$ induces a map at the level of powersets $p^* = p^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X \times Y)$. If we regard these powersets as poset categories (where the order is given by the inclusion relation) then this map becomes a functor. Also, the assignments $S \to \forall_p S$ and $S \to \exists_p S$ yield functors $\forall_p, \exists_p : \mathscr{P}(X \times Y) \to \mathscr{P}(Y)$.

Proposition 5.10. The functors $\exists_p$ and $\forall_p$ are respectively *left* and *right adjoints* to the functor $p^* : \mathscr{P}(Y) \to \mathscr{P}(X \times Y)$ which sends each subset $T \subseteq Y$ to its inverse image p^*T under p.

Of course, the proposition generalizes to the case of an arbitrary function in place of the projection p. This motivates the following

Definition 5.11. A *Heyting category* is a coherent category $\mathcal{C}$ such that for any arrow $f : a \to b$ in $\mathcal{C}$ the pullback functor $f^* : \mathrm{Sub}_{\mathcal{C}}(b) \to \mathrm{Sub}_{\mathcal{C}}(a)$ has a right adjoint $\forall_f : \mathrm{Sub}_{\mathcal{C}}(a) \to \mathrm{Sub}_{\mathcal{C}}(b)$ (as well as its left adjoint $\exists_f : \mathrm{Sub}_{\mathcal{C}}(a) \to \mathrm{Sub}_{\mathcal{C}}(b)$).

Proposition 5.12 (Lemma A1.4.13 [26])**.** Let $a_1 \rightarrowtail a$ and $a_2 \rightarrowtail a$ be subobjects in a Heyting category. Then there exists a largest subobject $(a_1 \Rightarrow a_2) \rightarrowtail a$ such that $(a_1 \Rightarrow a_2) \cap a_1 \leq a_2$. Moreover, the binary operation on subobjects thus defined is stable under pullback.

In particular, all the *subobject lattices* in a Heyting category are *Heyting algebras*.

The proposition ensures that in a Heyting category we may interpret full finitary first-order logic. The negation $\neg\phi(\vec{x})$ is interpreted as the Heyting pseudocomplement $\neg([[\vec{x} \ . \ \phi]])$; note that, whilst it is always the case that $a \wedge \neg a = 0$ in any Heyting algebra, $a \vee \neg a$ is in general different from 1, whence the law of excluded middle is not sound with respect to general Heyting categories. Indeed, the logic of Heyting categories is intuitionistic, not classical.

Definition 5.13. A coherent category $\mathcal{C}$ is said to be *Boolean* if every subobject $m : a \rightarrowtail c$ in $\mathcal{C}$ is complemented, in the sense that there exists a unique subobject n of c such that $m \cup n = 1_c$ and $m \cap n = 0_c$.

Since every Boolean algebra is a Heyting algebra, we obtain the following

Proposition 5.14. Any Boolean coherent category is a Heyting category.

Proposition 5.15 (cf. Lemma A1.4.18 [26])**.** Every geometric category is a Heyting category.

Proposition 5.16. Every Grothendieck topos is a geometric category.

Sketch of proof. Well-poweredness immediately follows from the fact that, by Giraud's theorem, every Grothendieck topos has a separating set of objects, while the other properties easily follow from Theorems 3.11 and 3.12. □

Thus every Grothendieck topos is a Heyting category. As we shall see in section 5.5, it is true more generally that every elementary topos is a Heyting category.

5.2.1 The internal language

Given a category $\mathcal{C}$ with finite products, one can define a first-order signature $\Sigma_{\mathcal{C}}$, called the *internal language* (or the *canonical signature*) of $\mathcal{C}$, for reasoning about $\mathcal{C}$ in a set-theoretic fashion.

Definition 5.17. The signature $\Sigma_{\mathcal{C}}$ has one sort $\ulcorner A \urcorner$ for each object A of $\mathcal{C}$, one function symbol $\ulcorner f \urcorner : \ulcorner A_1 \urcorner, \cdots, \ulcorner A_n \urcorner \to \ulcorner B \urcorner$ for each arrow $f : A_1 \times \cdots \times A_n \to B$ in $\mathcal{C}$, and one relation symbol $\ulcorner R \urcorner \rightarrowtail \ulcorner A_1 \urcorner \cdots \ulcorner A_n \urcorner$ for each subobject $R \rightarrowtail A_1 \times \cdots \times A_n$.

Note that there is a canonical $\Sigma_{\mathcal{C}}$-structure in $\mathcal{C}$, called the *tautological* $\Sigma_{\mathcal{C}}$*-structure*, which assigns A to $\ulcorner A \urcorner$, f to $\ulcorner f \urcorner$ and R to $\ulcorner R \urcorner$.

The usefulness of this notion lies in the fact that properties of $\mathcal{C}$ or constructions in it can often be formulated in terms of satisfaction of certain formulae over $\Sigma_{\mathcal{C}}$ in the canonical structure. The internal language can thus be used for proving things about $\mathcal{C}$.

Indeed, for any objects $A_1, \ldots, A_n$ of $\mathscr{E}$ and any first-order formula $\phi(\vec{x})$ over $\Sigma_{\mathscr{E}}$, where $\vec{x} = (x_1^{\ulcorner A_1 \urcorner}, \ldots, x_n^{\ulcorner A_n \urcorner})$, the set-theoretic expression $\{\vec{x} \in A_1 \times \cdots \times A_n \mid \phi(\vec{x})\}$ can be given a meaning, namely the interpretation of the formula $\phi(\vec{x})$ in the $\Sigma_{\mathscr{E}}$-structure $\mathcal{S}_{\mathscr{E}}$.

We shall see an example of the use of the internal language of a topos in section 5.5.

5.2.2 Completions of 'logical' categories

It is important for many purposes to be able to complete the 'logical' categories that we have considered in the last section with respect to certain kinds of colimits that they lack.

As far as it concerns regular categories, we can 'complete' them to regular categories in which quotients by equivalence relations always exist, by formally adding them. Such a construction is called *effectivization* and is characterized by the following universal property: for any regular category $\mathcal{C}$ there exists an effective regular category (i.e., regular category in which every equivalence relation occurs as the kernel pair of some morphism) $\mathbf{Eff}(\mathcal{C})$ with a full and faithful functor $i_{\text{eff}} : \mathcal{C} \to \mathbf{Eff}(\mathcal{C})$ such that for any effective regular category $\mathcal{D}$, the regular functors $\mathbf{Eff}(\mathcal{C}) \to \mathcal{D}$ correspond, naturally in $\mathcal{D}$, to the regular functors $\mathcal{C} \to \mathcal{D}$ (by composition with i_{eff}). For a detailed description of this construction the reader is referred to the proof of Corollary A3.3.10 [26].

Coherent (resp. geometric) categories do not possess in general finite (resp. arbitrary) coproducts, but it is possible to 'complete' them to coherent (resp. geometric) categories in which finite (resp. arbitrary) disjoint coproducts of objects exists. Such a construction is called *positivization* (resp. *infinitary positivization*) and is characterized by the following universal property:

(1) For any coherent category $\mathcal{C}$ there exists a positive coherent category (i.e., coherent category having disjoint finite coproducts) $\mathbf{Pos}(\mathcal{C})$ with a full and faithful functor $i_{\text{pos}} : \mathcal{C} \to \mathbf{Pos}(\mathcal{C})$ such that for any positive category $\mathcal{D}$, the coherent functors $\mathbf{Pos}(\mathcal{C}) \to \mathcal{D}$ correspond, naturally in $\mathcal{D}$, to the coherent functors $\mathcal{C} \to \mathcal{D}$ (by composition with i_{pos})

(2) For any geometric category $\mathcal{C}$ there exists a inf-positive coherent category (i.e., geometric category which has disjoint arbitrary set-indexed coproducts) $\infty\text{-}\mathbf{Pos}_{\infty}(\mathcal{C})$ with a full and faithful functor $i_{\infty\text{-pos}} : \mathcal{C} \to \infty\text{-}\mathbf{Pos}(\mathcal{C})$ such that for any ∞-positive category $\mathcal{D}$, the geometric functors $\infty\text{-}\mathbf{Pos}(\mathcal{C}) \to \mathcal{D}$ correspond, naturally in $\mathcal{D}$, to the geometric functors $\mathcal{C} \to \mathcal{D}$ (by composition with $i_{\infty\text{-pos}}$).

More details about these constructions can be found at p. 34-35 [26].

Definition 5.18. (a) A *pretopos* is a positive and effective coherent category.

(b) A ∞-*pretopos* is a ∞-positive and effective geometric category.

Remarks 5.19. (a) For any coherent category $\mathcal{C}$, the category $\mathbf{Eff}(\mathbf{Pos}(\mathcal{C}))$, with the canonical embedding $i_{\text{pr}} : \mathcal{C} \hookrightarrow \mathbf{Eff}(\mathbf{Pos}(\mathcal{C}))$, satisfies the universal property of the *pretopos completion* $\mathcal{P}_{\mathcal{C}}$ of $\mathcal{C}$, i.e. $\mathbf{Eff}(\mathbf{Pos}(\mathcal{C}))$ is a pretopos such that for any pretopos $\mathcal{D}$, the coherent functors $\mathcal{P}_{\mathcal{C}} \to \mathcal{D}$ correspond precisely to the coherent functors $\mathcal{C} \to \mathcal{D}$ (via composition with i_{pr}).

(b) For any geometric category $\mathcal{C}$, the category $\mathbf{Eff}(\infty\text{-}\mathbf{Pos}(\mathcal{C}))$, with the canonical embedding $i_{\infty\text{-pr}} : \mathcal{C} \hookrightarrow \mathbf{Eff}(\infty\text{-}\mathbf{Pos}(\mathcal{C}))$, satisfies the universal property of the ∞-*pretopos completion* $\mathcal{P}^{\infty}_{\mathcal{C}}$ of $\mathcal{C}$, i.e. $\mathbf{Eff}(\infty\text{-}\mathbf{Pos}(\mathcal{C}))$ is a ∞-pretopos such that for any ∞-pretopos $\mathcal{D}$, the geometric functors $\mathcal{P}^{\infty}_{\mathcal{C}} \to \mathcal{D}$ correspond precisely to the geometric functors $\mathcal{C} \to \mathcal{D}$ (via composition with $i_{\infty\text{-pr}}$).

5.3 Models of first-order theories in categories

First-order terms over a given signature can be interpreted in a category with finite products.

Definition 5.20. Let M be a Σ-structure in a category $\mathcal{C}$ with finite products. If $\{\vec{x} \,.\, t\}$ is a term-in-context over Σ (with $\vec{x} = x_1, \ldots, x_n$, $x_i : A_i$ $(i = 1, ..., n)$ and $t : B$, say), then an *arrow*

$$[[\vec{x} \,.\, t]]_M : M(A_1, \ldots, A_n) \to MB$$

in $\mathcal{C}$ is defined recursively by the following clauses:

a) If t is a variable, it is necessarily x_i for some unique $i \leq n$, and then $[[\vec{x} \,.\, t]]_M = \pi_i$, the ith product projection.

b) If t is $f(t_1, \ldots, t_m)$ (where $t_i : C_i$, say), then $[[\vec{x} \,.\, t]]_M$ is the composite

$$M(A_1, \ldots, A_n) \xrightarrow{([[\vec{x}.t_1]]_M, \ldots, [[\vec{x}.t_m]]_M)} M(C_1, \ldots, C_m) \xrightarrow{Mf} MB$$

In order to interpret first-order formulae in a given category $\mathcal{C}$, we need to have a certain amount of categorical structure present on $\mathcal{C}$ in order to meaningfully interpret the logical connectives and quantifiers which appear in the formulae. For example, to interpret finitary conjunctions, one needs to be able to form pullbacks, while to interpret disjunctions one needs to be able to take unions of subobjects, etc. In fact, the larger is the fragment of logic, the larger is the amount of categorical structure required to interpret it.

Let M be a Σ-structure in a category $\mathcal{C}$ with finite limits. A formula-in-context $\{\vec{x} \,.\, \phi\}$ over Σ (where $\vec{x} = x_1, \ldots, x_n$ and $x_i : A_i$, say) will be interpreted as a subobject $[[\vec{x} \,.\, \phi]]_M \rightarrowtail M(A_1, \ldots, A_n)$ according to the following recursive clauses:

- If $\phi(\vec{x})$ is $R(t_1, \ldots, t_m)$ where R is a relation symbol (of type $B_1, \ldots, B_m$, say), then $[[\vec{x} \,.\, \phi]]_M$ is the pullback

$$\begin{array}{ccc} [[\vec{x} \,.\, \phi]]_M & \longrightarrow & MR \\ \downarrow & & \downarrow \\ M(A_1, \ldots, A_n) & \xrightarrow{([[\vec{x}.t_1]]_M, \ldots, [[\vec{x}.t_m]]_M)} & M(B_1, \ldots, B_m) \end{array}$$

- If $\phi(\vec{x})$ is $(s = t)$, where s and t are terms of sort B, then $[[\vec{x} \,.\, \phi]]_M$ is the equalizer of $[[\vec{x} \,.\, s]]_M, [[\vec{x} \,.\, t]]_M : M(A_1, \ldots, A_n) \to MB$.
- If $\phi(\vec{x})$ is $\top$ then $[[\vec{x} \,.\, \phi]]_M$ is the top element of $\mathrm{Sub}_{\mathcal{C}}(M(A_1, \ldots, A_n))$.
- If ϕ is $\psi \wedge \chi$ then $[[\vec{x} \,.\, \phi]]_M$ is the intersection (= pullback)

$$\begin{array}{ccc} [[\vec{x} \,.\, \phi]]_M & \longrightarrow & [[\vec{x} \,.\, \chi]]_M \\ \downarrow & & \downarrow \\ [[\vec{x} \,.\, \psi]]_M & \longrightarrow & M(A_1, \ldots, A_n) \end{array}$$

- If $\phi(\vec{x})$ is $\bot$ and $\mathcal{C}$ is a coherent category then $[[\vec{x} \,.\, \phi]]_M$ is the bottom element of $\mathrm{Sub}_{\mathcal{C}}(M(A_1, \ldots, A_n))$.

- If ϕ is $\psi \vee \chi$ and $\mathcal{C}$ is a coherent category then $[[\vec{x} \,.\, \phi]]_M$ is the union of the subobjects $[[\vec{x} \,.\, \psi]]_M$ and $[[\vec{x} \,.\, \chi]]_M$.

- If ϕ is $\psi \Rightarrow \chi$ and $\mathcal{C}$ is a Heyting category, $[[\vec{x} \,.\, \phi]]_M$ is the implication $[[\vec{x} \,.\, \psi]]_M \Rightarrow [[\vec{x} \,.\, \chi]]_M$ in the Heyting algebra $\mathrm{Sub}_{\mathcal{C}}(M(A_1, \ldots, A_n))$ (similarly, the negation $\neg\psi$ is interpreted as the pseudocomplement of $[[\vec{x} \,.\, \psi]]_M$).

- If ϕ is $(\exists y)\psi$ where y is of sort B, and $\mathcal{C}$ is a regular category, then $[[\vec{x} \,.\, \phi]]_M$ is the image of the composite

$$[[\vec{x}, y \,.\, \psi]]_M \longrightarrow M(A_1, \ldots, A_n, B) \xrightarrow{\pi} M(A_1, \ldots, A_n)$$

  where π is the product projection on the first n factors.

- If ϕ is $(\forall y)\psi$ where y is of sort B, and $\mathcal{C}$ is a Heyting category, then $[[\vec{x} \,.\, \phi]]_M$ is $\forall_\pi([[\vec{x}, y \,.\, \psi]]_M)$, where π is the same projection as above.

- If ϕ is $\bigvee\limits_{i \in I} \phi_i$ and $\mathcal{C}$ is a geometric category then $[[\vec{x} \,.\, \phi]]_M$ is the union of the subobjects $[[\vec{x} \,.\, \phi_i]]_M$.

- If ϕ is $\bigwedge\limits_{i \in I} \phi_i$ and $\mathcal{C}$ has arbitrary intersections of subobjects then $[[\vec{x} \,.\, \phi]]_M$ is the intersection of the subobjects $[[\vec{x} \,.\, \phi_i]]_M$.

Definition 5.21. Let M be a Σ-structure in a category $\mathcal{C}$.

(a) If $\sigma = \phi \vdash_{\vec{x}} \psi$ is a sequent over Σ interpretable in $\mathcal{C}$, we say that σ is satisfied in M, and we write $M \vDash \sigma$, if $[[\vec{x} \,.\, \phi]]_M \leq [[\vec{x} \,.\, \psi]]_M$ in $\mathrm{Sub}_{\mathcal{C}}(M(A_1, \ldots, A_n))$.

(b) If $\mathbb{T}$ is a theory over Σ interpretable in $\mathcal{C}$, we say that M is a *model* of $\mathbb{T}$ if all the axioms of $\mathbb{T}$ are satisfied in M.

(c) We write $\mathbb{T}$-mod$(\mathcal{C})$ for the full subcategory of Σ-Str$(\mathcal{C})$ whose objects are the models of $\mathbb{T}$.

One can easily prove, by induction on the structure of ϕ, that for any geometric formula $\phi(\vec{x}) = \phi(x_1^{A_1}, \ldots, x_n^{A_n})$ over a signature Σ and any Σ-structure homomorphism $h : M \to N$, there is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} [[\vec{x} \,.\, \phi]]_M & \longrightarrow & M(A_1, \ldots, A_n) \\ \downarrow & & \downarrow {\scriptstyle h_{A_1} \times \cdots \times h_{A_n}} \\ [[\vec{x} \,.\, \phi]]_N & \longrightarrow & N(A_1, \ldots, A_n) \end{array} .$$

This is not in general true if ϕ is a first-order non-geometric formula. This motivates the following

Definition 5.22. A homomorphism of models of a first-order theory $\mathbb{T}$ in a category $\mathcal{C}$ is said to be an *elementary morphism* if there is a commutative diagram as above for all the first-order formulae $\phi(\vec{x})$ over the signature of $\mathbb{T}$. The category of $\mathbb{T}$-models in $\mathcal{C}$ and elementary embeddings between them will be denoted by $\mathbb{T}$-mod$_e(\mathcal{C})$.

We say that a *functor* $F : \mathcal{C} \to \mathcal{D}$ between two cartesian (resp. regular, coherent, geometric, Heyting) categories is *cartesian* (resp. *regular*, *coherent*, *geometric*, *Heyting*) if it preserves finite limits (resp. finite limits and images, finite limits and images and finite unions of subobjects, finite limits and images and arbitrary unions of subobjects, finite limits and images and Heyting implications between subobjects).

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *conservative* if it reflects isomorphism (i.e., for any arrow $f : c \to c'$ in $\mathcal{C}$, if $F(f)$ is an isomorphism in $\mathcal{D}$ then f is an isomorphism in $\mathcal{C}$).

Lemma 5.23 (Lemma D1.2.13 [26])**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a cartesian (resp. regular, coherent, Heyting, geometric) functor between categories of the appropriate kind; let M be a Σ-structure in $\mathcal{C}$, and let σ be a sequent over Σ interpretable in $\mathcal{C}$. If $M \vDash \sigma$ in $\mathcal{C}$ then $F(M) \vDash \sigma$ in $\mathcal{D}$. The converse implication holds if F is conservative.

Proof. An easy induction shows that F preserves the interpretations of all formulae-in-context interpretable in the appropriate class of categories; from this the first assertion immediately follows. To prove the second, it suffices to note that a sequent $\sigma = (\phi \vdash_{\vec{x}} \psi)$ is satisfied in M if and only if the inclusion $[[\vec{x} \, . \, \phi \wedge \psi]]_M \rightarrowtail [[\vec{x} \, . \, \phi]]_M$ is an isomorphism. □

The previous lemma immediately implies the following

Theorem 5.24. *If $\mathbb{T}$ is a regular (resp. coherent, ...) theory over Σ, then for any regular (resp. coherent, ...) functor $F : \mathcal{C} \to \mathcal{D}$ the functor Σ-**Str**$(F) : \Sigma$-**Str**$(\mathcal{C}) \to \Sigma$-**Str**$(\mathcal{D})$ defined above restricts to a functor $\mathbb{T}$-mod$(F) : \mathbb{T}$-mod$(\mathcal{C}) \to \mathbb{T}$-mod$(\mathcal{D})$.*

Remark 5.25. We shall see in section 6.5, by using the concept of syntactic category, that the categorical semantics defined above is *sound* and *complete*, in the sense that provability of a sequent in a given theory belonging to a fragment of first-order logic is equivalent to its validity in all the models of the theory in categories in which such a sequent is interpretable.

The following list of examples shows that the notion of model of a first-order theory in a category naturally captures several important notions of mathematical interest.

Examples 5.26. (a) A *topological group* can be seen as a model of the theory of groups in the category of topological spaces.

More generally, a *topological groupoid* can be seen as a model of the theory $\mathbb{G}$ of groupoids (obtained from the theory $\mathbb{C}$ of small categories considered in Example 4.7 by adding a unary function symbol Inv of sort A for formalizing the operation of inverse of an arrow and the obvious axioms for it) in the category of topological spaces.

(b) Similarly, an *algebraic* (resp. *Lie*) *group* is a model of the algebraic theory of groups in the category of algebraic varieties (resp. the category of smooth manifolds).

(c) A *sheaf of rings* (more generally, a sheaf of models of a Horn theory $\mathbb{T}$) on a topological space X can be seen as a model of the theory of rings (resp. of the theory $\mathbb{T}$) in the topos $\mathbf{Sh}(X)$ of sheaves on X.

(d) A *sheaf of local rings* on a topological space X (in the sense that all its stalks are local rings) is precisely a model of the theory of local rings in the topos $\mathbf{Sh}(X)$.

(e) A *family of set-based models* of a theory $\mathbb{T}$ indexed over a set I can be seen as a model of $\mathbb{T}$ in the functor category $[I, \mathbf{Set}]$.

Remark 5.27. One advantage of categorical semantics is that a given piece of syntax can be soundly interpreted in a variety of independent contexts. The concrete results obtained in this way cannot in general be proved one from the other, but arise as different instances of a unique general result lying at the syntactic level. This can be useful in practice to avoid reproving the same results in different contexts, when it is possible to lift them to a syntactic level. For example, the fact that the category of algebraic groups and that of Lie groups have finite products are both immediate consequences of the syntactic property of the theory of groups to be algebraic.

5.4 Elementary toposes

The notion of elementary topos was introduced by W. Lawvere and M. Tierney in the late sixties as a generalization of the concept of Grothendieck topos in which it would still be possible to do some sort of abstract sheaf theory and consider models of arbitrary finitary first-order theories.

Definition 5.28. An *elementary topos* is a cartesian closed category with finite limits and a subobject classifier.

The notion of elementary topos can be formalized elementarily in the first-order language of categories (cf. Example 4.7), which explains the attribute ‘elementary’ to this kind of toposes.

By Theorem 3.11, every Grothendieck topos is an elementary topos. There are examples of elementary toposes which are not Grothendieck toposes, and which are useful in connection with the logical study of higher-order intuitionistic type theories and realizability. In fact, elementary toposes are exactly the syntactic categories, with respect to logical functors, of higher-order intuitionistic type theories.

As shown by Giraud’s theorem, the essential feature which distinguishes Grothendieck toposes among general elementary toposes is the fact that they admit sites of definition. The presence of sites allows one to effectively use a geometric intuition when dealing with toposes, and to study first-order mathematical theories of a general specified form (technically speaking, geometric theories, cf. section 7.1 below) through their classifying toposes. On the other hand, when studying aspects of toposes having to do with their elementary categorical structure, it is often possible, and even natural, to generalize such results to the level of elementary toposes. We shall see various instances of this phenomenon in the book.

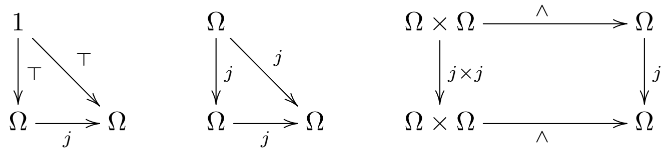
The name ‘topos’ for this kind of categories is justified by the fact that it is possible to lift to this level of generality many natural notions and constructions which apply to Grothendieck toposes. For instance, the notion of Grothendieck topology J on a category $\mathcal{C}$ corresponds to an invariant-level notion defined at the level of the presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, namely to a *local operator* (also called *Lawvere-Tierney topology*) on it. Before introducing this notion, we need to recall the following result. By an internal Heyting algebra in a topos $\mathcal{E}$ we mean a model of the algebraic theory of Heyting algebras in $\mathcal{E}$, in the sense of section 5.3: in other words, it is an object L of the topos $\mathcal{E}$ with arrows $\wedge, \vee, \Rightarrow : L \times L \to L$ and $0, 1 : 1_{\mathcal{E}} \to L$ which make commutative the diagrams which express the identities used in the equational definition of a Heyting algebra.

Theorem 5.29. *Any elementary topos $\mathcal{E}$ is an Heyting category, and the subobject classifier Ω of $\mathcal{E}$ has the structure of an internal Heyting algebra, inducing by the Yoneda Lemma a natural structure of Heyting algebra on each subobject lattice in the topos.*

Sketch of proof. Let $\mathcal{E}$ be an elementary topos. One can prove that $\mathcal{E}$ has all finite colimits (cf. section IV.5 of [29]). Using this, one can construct cover-mono factorizations for arrows which are stable under pullback, thereby showing that $\mathcal{E}$ is a regular category. The pullback functors between subobject lattices in $\mathcal{E}$ thus admit left adjoints. The existence of right adjoints to them follows from that of the cartesian closed structure (cf. the proof of Theorem I.9.4 [29]). The existence of such right adjoints ensures that finite unions of subobjects in $\mathcal{E}$, which exist in $\mathcal{E}$ since $\mathcal{E}$ has finite colimits as well as images, are stable under pullback. Given this, it is not hard to prove that Ω has the structure of an internal Heyting algebra. Specifically, one defines $0 : 1_{\mathcal{E}} \to \Omega$ as the classifying arrow of the zero subobject $0 \rightarrowtail 1$, $1 : 1_{\mathcal{E}} \to \Omega$ to be the arrow $\top$, $\wedge : \Omega \times \Omega \to \Omega$ to be the classifying arrow of the monomorphism $< \top, \top >: 1_{\mathcal{E}} \to \Omega \times \Omega, \Rightarrow: \Omega \times \Omega \to \Omega$ to be the classifying arrow of the equalizer of the arrows $\wedge$ and π_1 and $\vee : \Omega \times \Omega \to \Omega$ is the classifying arrow of the union of the two subobjects $\pi_1^*(\top)$ and $\pi_2^*(\top)$, where π_1 and π_2 are the two canonical projections $\Omega \times \Omega \to \Omega$ (for more details, see the proof of Lemma A1.6.3 [26]).

It is also immediate to see that the Heyting algebra structure of the subobject lattices in $\mathcal{E}$ is induced by this internal structure via the Yoneda Lemma. □

Definition 5.30. Let $\mathcal{E}$ be an elementary topos, with subobject classifier $\top : 1 \to \Omega$. A *local operator* (or *Lawvere-Tierney topology*) on $\mathcal{E}$ is an arrow $j : \Omega \to \Omega$ in $\mathcal{E}$ such that the diagrams



commute (where $\wedge : \Omega \times \Omega \to \Omega$ is the meet operation of the internal Heyting algebra Ω).

Interestingly, the notion of local operator admits several equivalent characterizations.

Definition 5.31. (a) A *closure operation* on subobjects in an elementary topos $\mathcal{E}$ is an operation c sending any subobject m in $\mathrm{Sub}_{\mathcal{E}}(A)$ to a subobject $c(m)$ in $\mathrm{Sub}_{\mathcal{E}}(A)$ in such a way that $m \leq c(m)$ and $c(c(m)) = c(m)$ for all m, where $\leq$ is the natural ordering between subobjects.

(b) A *universal closure operation* on an elementary topos $\mathcal{E}$ is a closure operation c on subobjects which commutes with pullback (= intersection) of subobjects.

Theorem 5.32. *For any elementary topos $\mathcal{E}$, there is a bijection between universal closure operations on $\mathcal{E}$ and local operators on $\mathcal{E}$.*

Sketch of proof. The bijection sends a universal closure operation c on $\mathcal{E}$ to the local operator $j_c : \Omega \to \Omega$ given by classifying map of the subobject $c(1 \overset{\top}{\rightarrowtail} \top)$, and a local operator j to the closure operation c_j induced by composing classifying arrows with j. For more details, see the proof of Lemma A4.4.2 [26]. □

One can define an abstract notion of c-sheaf, for a universal closure operation c on an elementary topos $\mathcal{E}$, as follows.

Definition 5.33. Let $\mathcal{E}$ be an elementary topos and c a universal closure operation on $\mathcal{E}$. An object A of $\mathcal{E}$ is said to be a *c-sheaf* if for any monomorphism $m : B' \rightarrowtail B$ in $\mathcal{E}$ which is c-dense (i.e. such that $c(m) = 1_B$) and any arrow $f' : B' \to A$ in $\mathcal{E}$, there exists exactly one arrow $f : B \to A$ such that $f \circ m = f'$.

The full subcategory of $\mathcal{E}$ on the c-sheaves will be denoted by $\mathbf{sh}_c(\mathcal{E})$.

For a proof of the following theorem the reader is referred to sections A4.3 and A4.4 of [26].

Theorem 5.34. *For any local operator j on an elementary (resp. Grothendieck) topos $\mathcal{E}$, $\mathbf{sh}_{c_j}(\mathcal{E})$ is an elementary (resp. Grothendieck) topos, and the inclusion $\mathbf{sh}_{c_j}(\mathcal{E}) \hookrightarrow \mathcal{E}$ has a left adjoint $a_j : \mathcal{E} \to \mathbf{sh}_{c_j}(\mathcal{E})$ which preserves finite limits. In fact, local operators on $\mathcal{E}$ also correspond bijectively to (equivalence classes of) geometric inclusions to $\mathcal{E}$.*

Theorem 5.35. *Let $\mathcal{C}$ be a small category. Then the Grothendieck topologies J on $\mathcal{C}$ correspond exactly to the local operators on the presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. In fact, if J is the Grothendieck topology corresponding to a local operator j, an object of $[\mathcal{C}^{op}, \mathbf{Set}]$ is a J-sheaf (in the sense of Grothendieck toposes) if and only if it is a c_j-sheaf (in the sense of universal closure operations).*

Sketch of proof. The correspondence sends a local operator $j : \Omega \to \Omega$ to the subobject $J \rightarrowtail \Omega$ which it classifies, that is to the Grothendieck topology J on $\mathcal{C}$ defined by:

$$S \in J(c) \text{ if and only if } j(c)(S) = M_c$$

Conversely, it sends a Grothendieck topology J, regarded as a subobject $J \rightarrowtail \Omega$, to the arrow $j : \Omega \to \Omega$ that classifies it.

For more details, see section V.4 of [29]. □

5.5 Toposes as mathematical universes

We have seen in section 5.2.1 that, given a category $\mathcal{C}$ with finite products, there is a first-order signature $\Sigma_{\mathcal{C}}$, called the internal language of $\mathcal{C}$, which can be profitably used for reasoning about $\mathcal{C}$ in a set-theoretic fashion, that is by using 'elements'.

If $\mathcal{C}$ is an elementary topos, we can extend the internal language by allowing the formation of formulae of the kind $\tau \in \Gamma$, where τ is a term of sort A and Γ is a term of sort Ω^A. Indeed, we may interpret this formula as the subobject whose classifying arrow is the composite

$$W \xrightarrow{\quad <\tau,\Gamma> \quad} A \times \Omega^A \xrightarrow{\in_A} \Omega$$

where W denotes the product of (the objects representing the) sorts of the variables occurring either in τ or in Γ (considered without repetitions) and $< \tau, \Gamma >$ denotes the induced map to the product. Note that an object A of $\mathcal{C}$ gives rise to a constant term of type Ω^A. In fact, Ω^A behaves like the power set of A in $\mathcal{C}$.

Thus in a topos we can also interpret all the common formulae that we use in Set Theory.

Note that, the logic of a topos being at least intuitionistic (cf. Theorems 5.29 and 6.5), we are assured that any formal proof involving first-order sequents over the signature $\Sigma_{\mathscr{E}}$ will be automatically valid in the structure $\mathcal{S}_{\mathscr{E}}$ provided that the law of excluded middle or any other non-constructive (i.e., non-intuitionistic) principles are not employed in it. In other words, the existence of the internal language of a topos and the corresponding tautological structure justifies the use of the standard set-theoretic intuition when working 'inside' the given topos provided that only constructive principles are used in the proofs.

An example of a reformulation of basic properties of sets in the internal language of a topos is provided by the following proposition:

Proposition 5.36 (cf. Lemma D1.3.11 [26])**.** Let $\mathscr{E}$ be a topos. The following statements hold

(i) $f : A \to A$ is the identity arrow if and only if $(\top \vdash_x f(x) = x)$ holds in $\mathcal{S}_{\mathscr{E}}$.

(ii) $f : A \to C$ in the composite of $g : A \to B$ and $h : B \to C$ if and only if $(\top \vdash_x f(x) = h(g(x))$ holds in $\mathcal{S}_{\mathscr{E}}$.

(iii) $f : A \to B$ is a monomorphism if and only if $(f(x) = f(x') \vdash_x x = x')$ holds in $\mathcal{S}_{\mathscr{E}}$.

(iv) $f : A \to B$ is an epimorphism if and only if $(\top \vdash_x (\exists x)(f(x) = y))$ holds in $\mathcal{S}_{\mathscr{E}}$.

(v) A is a terminal object if and only if $(\top \vdash (\exists x)\top)$ and $(\top \vdash_{x,x'} (x = x'))$ hold in $\mathcal{S}_{\mathscr{E}}$.

5.5.1 Validity in an elementary topos

Definition 5.37. We say that a first-order formula $\phi(\vec{x})$ over a signature Σ is *valid* in an elementary topos $\mathcal{E}$ if for every Σ-structure M in $\mathcal{E}$ the sequent $\top \vdash_{\vec{x}} \phi$ is satisfied in M.

Theorem 5.38. *Let Σ be a signature and $\phi(\vec{x})$ a first-order formula over Σ. Then $\phi(\vec{x})$ is provable in intuitionistic (finitary) first-order logic if and only if it is valid in every elementary topos.*

Sketch of proof. The soundness result follows from Theorems 5.29 and 6.5. The completeness part follows from the existence of canonical Kripke models and the fact that, given a poset P and a Kripke model $\mathcal{U}$ on P there is a model $\mathcal{U}^*$ in the topos $[P, \mathbf{Set}]$ such that the first-order sequents valid in $\mathcal{U}$ are exactly those valid in $\mathcal{U}^*$. For more details, see section 8.4 of [23]. □

Hence an elementary topos can be considered as a *mathematical universe* in which one can do mathematics similarly to how one does it in the classical context of sets, with the only exception that one must in general argue constructively.

5.5.2 Kripke-Joyal semantics

Kripke-Joyal semantics represents the analogue for toposes of the usual Tarskian notion of satisfaction of a first-order formula by a tuple of elements of a structure in which the given formula is interpretable. In the context of toposes, it makes no sense to speak of elements of a structure in a topos, but we can replace the classical notion of element of a set with that of *generalized element* of an object: a generalized element of an object B of a topos $\mathcal{E}$ is simply an arrow $\alpha : E \to B$ with codomain B. Note that, by the Yoneda lemma, an object is determined, up to isomorphism, by the collection of all its generalized elements (represented by the functor $Hom_{\mathcal{E}}(-, B)$).

Definition 5.39. Let $\mathcal{E}$ be an elementary topos and M be a Σ-structure in $\mathcal{E}$. Given a first-order formula $\phi(x)$ over Σ in a variable x of sort A and a generalized element $\alpha : E \to MA$ of MA, we define

$$E \models_M \phi(\alpha) \quad \text{if and only if} \quad \alpha \text{ factors through } [[x \,.\, \phi]]_M \rightarrowtail MA$$

Of course, the definition can be extended to formulae with an arbitrary (finite) number of free variables.

Theorem 5.40. *If $\alpha : E \to MA$ is a generalized element of MA while $\phi(x)$ and $\psi(x)$ are formulae with a free variable x of sort A then*

- $E \models (\phi \wedge \psi)(\alpha)$ *if and only if* $U \models \phi(\alpha)$ *and* $U \models \psi(\alpha)$.
- $E \models (\phi \vee \psi)(\alpha)$ *if and only if there are jointly epimorphic arrows* $p : E' \to E$ *and* $q : E'' \to E$ *such that both* $E' \models \phi(\alpha \circ p)$ *and* $E'' \models \psi(\alpha \circ q)$.
- $E \models (\phi \Rightarrow \psi)(\alpha)$ *if and only if for any arrow* $p : E' \to E$ *such that* $E' \models \phi(\alpha \circ p)$*, then* $E' \models \psi(\alpha \circ p)$.
- $E \models (\neg\phi)(\alpha)$ *if and only if whenever* $p : E' \to E$ *is such that* $E' \models \phi(\alpha \circ p)$*, then* $E' \cong 0_{\mathcal{E}}$.

If $\phi(x, y)$ has an additional free variable y of sort B then

- $E \models (\exists y)\phi(\alpha, y)$ *if and only if there exist an epi* $p : E' \to E$ *and a generalized element* $\beta : E' \to B$ *such that* $E' \models \phi(\alpha \circ p, \beta)$.
- $E \models (\forall y)\phi(\alpha, y)$ *if and only if for every object* E'*, for every arrow* $p : E' \to E$ *and every generalized element* $c : E' \to B$ *one has* $E' \models \phi(\alpha \circ p, \beta)$.

Remark 5.41. If $\mathcal{E}$ is a Grothendieck topos then also infinitary formulae can be interpreted in it. The infinitary version of the second point of the theorem reads as follows:

$E \models (\bigvee_{i \in I} \phi_i)(\alpha)$ if and only if there is an epimorphic family of arrows $\{p_i : E_i \to E \mid i \in I\}$ such that $E_i \models \phi(\alpha \circ p_i)$ for all $i \in I$.

6 Syntactic categories

In this section we shall introduce the fundamental notion of *syntactic category* of a theory within a given fragment of first-order logic.

6.1 Definition

Let $\phi(\vec{x})$ and $\psi(\vec{y})$ be two formulae over a first-order signature, where $\vec{x}$ and $\vec{y}$ are contexts of the same type and length. We say that $\phi(\vec{x})$ and $\psi(\vec{y})$ are α*-equivalent* if $\psi(\vec{y})$ is obtained from $\phi(\vec{x})$ by an *acceptable renaming*, i.e. every free occurrence of x_i is replaced by y_i in ϕ and each x_i is free for y_i in ϕ. We write $\{\vec{x} \,.\, \phi\}$ for the α-equivalence class of the formula $\phi(\vec{x})$.

Definition 6.1. (a) Let $\mathbb{T}$ be a geometric theory over a signature Σ. The *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as *objects* the α-equivalence classes of geometric formulae-in-context $\{\vec{x} . \phi\}$ over Σ and as *arrows* $\{\vec{x} . \phi\} \to \{\vec{y} . \psi\}$ (where the contexts $\vec{x}$ and $\vec{y}$ are supposed to be disjoint without loss of generality) the $\mathbb{T}$-provable-equivalence classes $[\theta]$ of geometric formulae $\theta(\vec{x}, \vec{y})$ which are $\mathbb{T}$-provably functional i.e. such that the sequents

$$(\phi \vdash_{\vec{x}} (\exists y)\theta),$$
$$(\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi), \text{ and}$$
$$((\theta \wedge \theta[\vec{z}/\vec{y}]) \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{y} = \vec{z}))$$

are provable in $\mathbb{T}$.

The *composite* of two arrows

$$\{\vec{x} . \phi\} \xrightarrow{[\theta]} \{\vec{y} . \psi\} \xrightarrow{[\gamma]} \{\vec{z} . \chi\}$$

is defined as the $\mathbb{T}$-provable-equivalence class of the formula $(\exists \vec{y})\theta \wedge \gamma$.

The *identity* arrow on an object $\{\vec{x} . \phi\}$ is the arrow

$$\{\vec{x} . \phi\} \xrightarrow{[\phi \wedge \vec{x'} = \vec{x}]} \{\vec{x'} . \phi[\vec{x'}/\vec{x}]\}$$

(b) We define the *cartesian* (resp. *regular*, *coherent*, *first-order*) *syntactic category* $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$ (resp. $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$, $\mathcal{C}_{\mathbb{T}}^{\text{fo}}$) of a cartesian (resp. regular, coherent, first-order) theory $\mathbb{T}$ by replacing the word 'geometric' with 'cartesian' (resp. 'regular', 'coherent', 'first-order') in the above definition.

(c) We define the *classical syntactic category* $\mathcal{C}_{\mathbb{T}}^{\text{fo-cl}}$ of a first-order theory $\mathbb{T}$ by replacing, in the above definition, the word 'geometric' with 'first-order' and the notion of (intuitionistic) provability with that of provability in classical first-order logic.

(d) Given a universal Horn theory $\mathbb{T}$, we define the *algebraic syntactic category* $\mathscr{C}_{\mathbb{T}}^{\text{alg}}$ of $\mathbb{T}$ as the category whose objects are the finite conjunctions of atomic formulae-in-context (up to α-equivalence) over the signature of $\mathbb{T}$ and whose arrows $\{\vec{x} . \phi\} \to \{\vec{y} . \psi\}$ (where the contexts $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$ are supposed to be disjoint, without loss of generality) are sequences of terms $t_1(\vec{x}), \ldots, t_m(\vec{x})$ such that the sequent $(\phi \vdash_{\vec{x}} \psi(t_1(\vec{x}), \ldots, t_m(\vec{x})))$ is provable in $\mathbb{T}$, modulo the equivalence relation which identifies two such sequences $\vec{t}$ and $\vec{t'}$ precisely when the sequent $(\phi \vdash_x \vec{t}(\vec{x}) = \vec{t'}(\vec{x}))$ is provable in $\mathbb{T}$.

(e) We shall say that two geometric formulae-in-context $\{\vec{x} . \phi\}$ and $\{\vec{y} . \psi\}$, where $\vec{x}$ and $\vec{y}$ are disjoint, are $\mathbb{T}$-*equivalent* if they are isomorphic objects in the syntactic category $\mathscr{C}_{\mathbb{T}}$, that is, if there exists a geometric formula $\theta(\vec{x}, \vec{y})$ which is $\mathbb{T}$-provably functional from $\{\vec{x} . \phi\}$ to $\{\vec{y} . \psi\}$ and which moreover satisfies the property that the sequent $(\theta \wedge \theta[\vec{x'}/\vec{x}] \vdash_{\vec{x},\vec{x'},\vec{y}} \vec{x} = \vec{x'})$ is provable in $\mathbb{T}$.

Lemma 6.2 (Lemma 1.4.4(iv) [26])**.** Any subobject of $\{\vec{x} . \phi\}$ in $\mathcal{C}_{\mathbb{T}}$ is isomorphic to one of the form

$$\{\vec{x'} . \psi[\vec{x'}/\vec{x}]\} \xrightarrow{[\psi \wedge \vec{x'} = \vec{x}]} \{\vec{x} . \phi\}$$

where ψ is a formula such that the sequent $\psi \vdash_{\vec{x}} \phi$ is provable in $\mathbb{T}$. We will denote this subobject simply by $[\psi]$.

Moreover, for two such subobjects $[\psi]$ and $[\chi]$, we have $[\psi] \leq [\chi]$ in $\text{Sub}_{\mathcal{C}_{\mathbb{T}}}(\{\vec{x} . \phi\})$ if and only if the sequent $\psi \vdash_{\vec{x}} \chi$ is provable in $\mathbb{T}$.

Theorem 6.3. *(i) For any cartesian theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}^{cart}$ is a cartesian category.*

(ii) For any regular theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}^{reg}$ is a regular category.

(iii) For any coherent theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}^{coh}$ is a coherent category.

(iv) For any first-order theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}^{fo}$ is a Heyting category.

(v) For any geometric theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}$ is a geometric category.

(vi) For any first-order theory $\mathbb{T}$, $\mathcal{C}_{\mathbb{T}}^{fo\text{-}cl}$ is a Boolean coherent category.

Sketch of proof. The structure of subobject lattices in syntactic categories directly reflects the structure of the formulae involved in the fragment of logic in question, via the identification provided by Lemma 6.2; for instance, unions of subobjects correspond to disjunctions, intersections to conjunctions, top elements of lattices $\mathrm{Sub}(\{\vec{x} \,.\, \phi\})$ to the associated formulae $\{\vec{x} \,.\, \phi\}$ (regarded as identical subobjects $[\phi]$) and bottom elements, if they exist, to the formulae $\{\vec{x} \,.\, \bot\}$. Pullbacks correspond in a natural way to substitutions, and the cover-mono factorizations of a morphism correspond to existential quantifications (the factorization of a morphism $[\theta(\vec{x}, \vec{y})] : \{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, \psi\}$ is given by the canonical arrows $\{\vec{x} \,.\, \phi\} \to \{\vec{y} \,.\, (\exists \vec{x})\theta(\vec{x}, \vec{y})\} \rightarrowtail \{\vec{y} \,.\, \psi\}$). For more details, we refer the reader to the proof of Lemma D1.4.10 [26]. □

Definition 6.4. Let $\mathbb{T}$ be a geometric theory over a signature Σ. The *universal model* of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{T}}$ is defined as the structure $M_{\mathbb{T}}$ which assigns

- to a sort A the object $\{x^A \,.\, \top\}$ where x^A is a variable of sort A,
- to a function symbol $f : A_1 \cdots A_n \to B$ the morphism

$$\{x_1^{A_1}, \ldots, x_n^{A_n} \,.\, \top\} \xrightarrow{[f(x_1^{A_1}, \ldots, x_n^{A_n}) = y^B]} \{y^B \,.\, \top\}$$

  and

- to a relation symbol $R \rightarrowtail A_1 \cdots A_n$ the subobject

$$\{x_1^{A_1}, \ldots, x_n^{A_n} \,.\, R(x_1^{A_1}, \ldots, x_n^{A_n})\} \xrightarrow{[R(x_1^{A_1}, \ldots, x_n^{A_n})]} \{x_1^{A_1}, \ldots, x_n^{A_n} \,.\, \top\}$$

Theorem 6.5. *Let $\mathbb{T}$ be a geometric theory. Then*

(i) For any geometric formula-in-context $\{\vec{x} \,.\, \phi\}$ over Σ, the interpretation $[[\vec{x} \,.\, \phi]]_{M_{\mathbb{T}}}$ in $M_{\mathbb{T}}$ is the subobject $[\phi] : \{\vec{x} \,.\, \phi\} \rightarrowtail \{\vec{x} \,.\, \top\}$.

(ii) A geometric sequent $\phi \vdash_{\vec{x}} \psi$ is satisfied in $M_{\mathbb{T}}$ if and only if it is provable in $\mathbb{T}$.

Sketch of proof. The first part of the theorem can be easily proved by induction on the structure of ϕ, while the second follows from the first by appealing to Lemma 6.2. □

Remark 6.6. Of course, the definition and theorem above admit obvious variants for the other fragments of first-order logic, namely cartesian, regular, coherent, intuitionistic and classical first-order logic.

6.2 Syntactic sites

The idea behind the notion of syntactic site is that the property of a functor to preserve the logical structure on a given syntactic category can be interpreted as a form of continuity with respect to a well-defined Grothendieck topology defined on it.

Before introducing such topologies, a few remarks are in order.

Recall that every arrow in a regular category can be factored uniquely as a cover followed by a monomorphism, and covers are precisely the arrows g such that $Im(g) = 1_{\mathrm{cod}(g)}$.

In a coherent (resp. geometric) category, a finite (resp. small) *covering family* is a family of arrows such that the union of their images is the maximal subobject.

Definition 6.7. (a) For a regular category $\mathcal{C}$, the *regular topology* is the Grothendieck topology $J_{\mathcal{C}}^{\mathrm{reg}}$ on $\mathcal{C}$ whose covering sieves are those which contain a cover.

(b) For a coherent category $\mathcal{C}$, the *coherent topology* is the Grothendieck topology $J_{\mathcal{C}}^{\mathrm{coh}}$ on $\mathcal{C}$ whose covering sieves are those which contain finite covering families.

(c) For a geometric category $\mathcal{C}$, the *geometric topology* is the Grothendieck topology $J_{\mathcal{C}}^{\mathrm{geom}}$ on $\mathcal{C}$ whose covering sieves are those which contain small covering families.

Proposition 6.8. Given regular (resp. coherent, geometric) categories $\mathcal{C}$ and $\mathcal{D}$, a cartesian functor is regular (resp. coherent, geometric) if and only it sends $J_{\mathcal{C}}^{\mathrm{reg}}$-covering (resp. $J_{\mathcal{C}}^{\mathrm{coh}}$-covering, $J_{\mathcal{C}}^{\mathrm{geom}}$-covering) sieves to covering families.

Remark 6.9. The Grothendieck topologies $J_{\mathcal{C}}^{\mathrm{reg}}$,$J_{\mathcal{C}}^{\mathrm{coh}}$ and $J_{\mathcal{C}}^{\mathrm{geom}}$ are all subcanonical, whence for any regular (resp. coherent, geometric) category we have a Yoneda embedding $y^{\mathrm{reg}} : \mathcal{C} \hookrightarrow \mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}}^{\mathrm{reg}})$ (resp. $y^{\mathrm{coh}} : \mathcal{C} \hookrightarrow \mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}}^{\mathrm{coh}})$, $y^{\mathrm{geom}} : \mathcal{C} \hookrightarrow \mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}}^{\mathrm{geom}})$).

We shall denote by $J_{\mathbb{T}}^{\mathrm{reg}}$ (resp. by $J_{\mathbb{T}}^{\mathrm{coh}}$, by $J_{\mathbb{T}}$) the regular (resp. coherent, geometric) topology on the regular (resp. coherent, geometric) category $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$ (resp. $\mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}$, $\mathcal{C}_{\mathbb{T}}$) and refer to them as to the *syntactic topologies* on the syntactic categories in question.

We shall denote by $\mathbf{Cart}(\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}, \mathcal{D})$ (resp. $\mathbf{Reg}(\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}, \mathcal{D})$, $\mathbf{Coh}(\mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}, \mathcal{D})$, $\mathbf{Geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$, $\mathbf{Heyt}(\mathcal{C}_{\mathbb{T}}^{\mathrm{fo}}, \mathcal{D})$, $\mathbf{Bool}(\mathcal{C}_{\mathbb{T}}^{\mathrm{fo\text{-}cl}}, \mathcal{D})$) the categories of (resp. regular, coherent, geometric, Heyting, coherent) functors from $\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$ (resp. $\mathcal{C}_{\mathbb{T}}^{\mathrm{reg}}$, $\mathcal{C}_{\mathbb{T}}^{\mathrm{coh}}$, $\mathcal{C}_{\mathbb{T}}$, $\mathcal{C}_{\mathbb{T}}^{\mathrm{fo}}$, $\mathcal{C}_{\mathbb{T}}^{\mathrm{fo\text{-}cl}}$) to a cartesian (resp. regular, coherent, geometric, Heyting, Boolean coherent) category $\mathcal{D}$.

6.3 Models as functors

The importance of syntactic categories consists in the fact that they allow to associate to a theory (in the sense of axiomatic presentation), which is a 'linguistic', un-structured kind of entity, a well-structured mathematical object whose 'geometry' incarnates the syntactic aspects of the theory. A most notable fact is that the models of the theory can be recovered as functors defined on the syntactic category respecting the 'logical' structure on it. More specifically, we have the following

Theorem 6.10. *(i) For any cartesian theory $\mathbb{T}$ and cartesian category $\mathcal{D}$, we have an equivalence of categories **Cart**$(\mathcal{C}_{\mathbb{T}}^{cart}, \mathcal{D}) \simeq \mathbb{T}$-mod$(\mathcal{D})$ natural in $\mathcal{D}$.*

*(ii) For any regular theory $\mathbb{T}$ and regular category $\mathcal{D}$, we have an equivalence of categories **Reg**$(\mathcal{C}_{\mathbb{T}}^{reg}, \mathcal{D}) \simeq \mathbb{T}$-mod$(\mathcal{D})$ natural in $\mathcal{D}$.*

(iii) *For any coherent theory $\mathbb{T}$ and coherent category $\mathcal{D}$, we have an equivalence of categories* ***Coh***$(\mathcal{C}_{\mathbb{T}}^{coh}, \mathcal{D}) \simeq \mathbb{T}$*-mod*$(\mathcal{D})$ *natural in* $\mathcal{D}$.

(iv) *For any geometric theory $\mathbb{T}$ and geometric category $\mathcal{D}$, we have an equivalence of categories* ***Geom***$(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \simeq \mathbb{T}$*-mod*$(\mathcal{D})$ *natural in* $\mathcal{D}$.

(v) *For any finitary first-order theory $\mathbb{T}$ and Heyting category $\mathcal{D}$, we have an equivalence of categories* ***Heyt***$(\mathcal{C}_{\mathbb{T}}^{fo\text{-}cl}, \mathcal{D}) \simeq \mathbb{T}$*-mod*$_e(\mathcal{D})$ *natural in* $\mathcal{D}$.

(vi) *For any finitary first-order theory $\mathbb{T}$ and Booelan category $\mathcal{D}$, we have an equivalence of categories* ***Bool***$(\mathcal{C}_{\mathbb{T}}^{fo\text{-}cl}, \mathcal{D}) \simeq \mathbb{T}$*-mod*$_e(\mathcal{D})$ *natural in* $\mathcal{D}$.

Sketch of proof. One half of the equivalence sends a model $M \in \mathbb{T}$-mod$(\mathcal{D})$ to the functor $F_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ assigning to a formula $\{\vec{x} \,.\, \phi\}$ (the domain of) its interpretation $[[\vec{x} \,.\, \phi]]_M$ in M.

The other half of the equivalence sends a functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ to the image $F(M_{\mathbb{T}})$ of the universal model $M_{\mathbb{T}}$ of $\mathbb{T}$ under F.

For more details, see the proof of Theorem D1.4.7 [26]. □

Remark 6.11. By Theorem 6.3, the properties of Theorem 6.10 characterize syntactic categories up to equivalence, as representing objects of the (2-)functors $\mathcal{D} \to \mathbb{T}$-mod$(\mathcal{D})$ (resp. $\mathcal{D} \to \mathbb{T}$-mod$_e(\mathcal{D})$ in the case of intuitionistic and classical first-order syntactic categories).

The concept of syntactic category of a first-order theory also allows to formalize the idea of a 'dictionary' between two theories allowing to translate formulae in the language of the former into formulae in the language of the latter in such a way as to induce an equivalence between their categories of models (inside categories possessing the required categorical structure). This is realised by the notion of bi-interpretation or, more generally, of interpretation of a theory into another.

Definition 6.12. Within a given fragment of first-order logic, an *interpretation* (resp. a *bi-interpretation*) of a theory $\mathbb{T}$ into a theory $\mathbb{S}$ is a functor (resp. an equivalence) between their respective syntactic categories which respects the logical structure on them.

Notice that any interpretation $I : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{S}}$ induces a functor $s_I : \mathbb{S}$-mod$(\mathcal{D}) \to \mathbb{T}$-mod$(\mathcal{D})$, for any 'logical' category $\mathcal{D}$ of the appropriate kind. However, it is not true in general that any functor $\mathbb{S}$-mod$(\mathcal{D}) \to \mathbb{T}$-mod$(\mathcal{D})$ is induced by an interpretation.

6.4 Categories with 'logical structure' as syntactic categories

Theorem 6.13. *Any cartesian (resp. regular, coherent, geometric) category is, up to categorical equivalence, the regular (resp. coherent, geometric) syntactic category of some cartesian (resp. regular, coherent, geometric) theory.*

Proof. By Theorem 6.3, the cartesian (resp. regular, coherent, geometric) syntactic category of a cartesian (resp. regular, coherent, geometric) theory $\mathbb{T}$ is cartesian (resp. regular, coherent, geometric) and satisfies the property that the category of cartesian (resp. regular, coherent, geometric) functors from it to any cartesian (resp. regular, coherent, geometric) category $\mathcal{D}$ is naturally equivalent to the category of models of $\mathbb{T}$ in $\mathcal{D}$ (cf. Theorem 6.10). Let us now show that, conversely, any cartesian (resp. regular, coherent, geometric) category $\mathcal{C}$ is equivalent to the syntactic category of a cartesian (resp. regular, coherent, geometric) theory.

Let $\mathbb{T}^{\mathcal{C}}$ be the theory over the internal language $\Sigma_{\mathcal{C}}$ of $\mathcal{C}$ formed by the following cartesian sequents over $\Sigma_{\mathcal{C}}$:

$$(\top \vdash_x (\ulcorner f \urcorner(x) = x))$$

for any identity arrow f in $\mathcal{C}$;

$$(\top \vdash_x (\ulcorner f \urcorner(x) = \ulcorner h \urcorner(\ulcorner g \urcorner(x))))$$

for any triple of arrows f, g, h of $\mathcal{C}$ such that f is equal to the composite $h \circ g$;

$$(\top \vdash_{[]} (\exists x)\top) \text{ and } (\top \vdash_{x,x'} (x = x'))$$

where x and x' are of sort $\ulcorner 1 \urcorner$, 1 being the terminal object of $\mathcal{C}$;

$$\begin{gathered}(\top \vdash_x (\ulcorner h \urcorner(\ulcorner f \urcorner(x)) = \ulcorner k \urcorner(\ulcorner g \urcorner(x)))),\\ (((\ulcorner f \urcorner(x) = \ulcorner f \urcorner(x')) \wedge (\ulcorner g \urcorner(x) = \ulcorner g \urcorner(x'))) \vdash_{x,x'} (x = x')), \text{ and}\\ ((\ulcorner h \urcorner(y) = \ulcorner k \urcorner(z)) \vdash_{y,z} (\exists x)((\ulcorner f \urcorner(x) = y) \wedge (\ulcorner g \urcorner(x) = z)))\end{gathered}$$

for any pullback square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ c & \xrightarrow{k} & d \end{array}$$

in $\mathcal{C}$.

It is an immediate to see that for any cartesian category $\mathcal{D}$, the $\mathbb{T}^{\mathcal{C}}$-models are 'the same thing' as cartesian functors $\mathcal{C} \to \mathcal{D}$. So we have an equivalence of categories $\mathbb{T}^{\mathcal{C}}\text{-mod}(\mathcal{D}) \simeq \mathbf{Cart}(\mathcal{C}, \mathcal{D})$ natural in $\mathcal{D} \in \mathbf{Cart}$. Since we also have an equivalence $\mathbf{Cart}(\mathcal{C}^{\text{cart}}_{\mathbb{T}^{\mathcal{C}}}, \mathcal{D}) \simeq \mathbb{T}^{\mathcal{C}}\text{-mod}(\mathcal{D})$ natural in $\mathcal{D} \in \mathbf{Cart}$ (by definition of syntactic category), by composing the two we find an equivalence $\mathbf{Cart}(\mathcal{C}, \mathcal{D}) \simeq \mathbf{Cart}(\mathcal{C}^{\text{cart}}_{\mathbb{T}^{\mathcal{C}}}, \mathcal{D})$ natural in $\mathcal{D} \in \mathbf{Cart}$ and hence, by the (2-dimensional) Yoneda Lemma, a natural equivalence of categories $\mathcal{C}^{\text{cart}}_{\mathbb{T}^{\mathcal{C}}} \simeq \mathcal{C}$, one half of which sends a formula $\phi(\vec{x})$ to (the domain of) its interpretation $[[\vec{x} \,.\, \phi]]$ in the tautological $\Sigma_{\mathcal{C}}$-structure in $\mathcal{C}$.

One can easily extend this result to more general fragments of geometric logic. Given a Grothendieck topology J on a cartesian category $\mathcal{C}$, let us denote by $\mathbb{T}^{\mathcal{C}}_J$ the theory obtained from $\mathbb{T}^{\mathcal{C}}$ by adding the axioms

$$(\top \vdash_x \bigvee_{i \in I} (\exists y_i)(\ulcorner f_i \urcorner(y_i) = x))$$

for each J-covering family $\{f_i : B_i \to A \mid i \in I\}$. It is immediate to see, recalling Proposition 6.8, that if $\mathcal{C}$ is a regular (resp. coherent, geometric) category then for any regular (resp. coherent, geometric) category $\mathcal{D}$, the regular (resp. coherent, geometric) functors $\mathcal{C} \to \mathcal{D}$ are exactly the cartesian functors on $\mathcal{C}$ which are J-continuous, where J is the regular (resp. coherent, geometric) coverage on $\mathcal{C}$. So we conclude as above that if $\mathcal{C}$ is a regular (resp. coherent, geometric) category then there is an equivalence of categories $\mathcal{C}^{\text{reg}}_{\mathbb{T}^{\mathcal{C}}_J} \simeq \mathcal{C}$ (resp. $\mathcal{C}^{\text{coh}}_{\mathbb{T}^{\mathcal{C}}_J} \simeq \mathcal{C}$, $\mathcal{C}^{\text{geom}}_{\mathbb{T}^{\mathcal{C}}_J} \simeq \mathcal{C}$) one half of which sends a formula $\phi(\vec{x})$ to (the domain of) its interpretation $[[\vec{x} \,.\, \phi]]$ in the canonical $\Sigma_{\mathcal{C}}$-structure in $\mathcal{C}$. □

Remark 6.14. The fact that every cartesian (resp. regular, coherent, geometric) category $\mathcal{C}$ is naturally equivalent to the syntactic category of a theory $\mathbb{T}$ enables us to interpret categorical constructions on $\mathcal{C}$ as logical operations in $\mathbb{T}$.

6.5 Soundness and completeness

Theorem 6.15 (Soundness)**.** *Let $\mathbb{T}$ be a cartesian (resp. regular, coherent, first-order, geometric) theory over a signature $\mathbb{T}$, and let M be a model of $\mathbb{T}$ in a cartesian (resp. regular, coherent, geometric, Heyting) category $\mathcal{C}$. If σ is a sequent (in the appropriate fragment of first-order logic over Σ) which is provable in $\mathbb{T}$, then σ is satisfied in M.*

Proof. The thesis immediately follows from Theorems 6.5 and 6.10 in view of Remark 6.6. □

Theorem 6.16 (Strong completeness)**.** *Let $\mathbb{T}$ be a Horn (resp. regular, coherent, geometric, first-order,) theory. If a Horn (resp. regular, coherent, geometric, Heyting,) sequent σ is satisfied in all models of $\mathbb{T}$ in cartesian (resp. regular, coherent, geometric, Heyting) categories, then it is provable in $\mathbb{T}$.*

Proof. Thanks to the construction of syntactic categories, the proof of completeness with respect to all the models in categories of the appropriate kind becomes a tautology (cf. Theorem 6.5). □

Let us now discuss classical completeness, that is completeness with respect to the class of set-based models of the theory.

Definition 6.17. Within a given fragment of geometric logic, a theory $\mathbb{T}$ is said to have *enough models* (in $\mathbf{Set}$) if for any geometric sequent σ over its signature, if σ is valid in all the set-based models of $\mathbb{T}$ then σ is provable in $\mathbb{T}$ (regarded as a theory in the given fragment).

The following theorem is essentially equivalent, via the notion of Morleyization of a first-order theory discussed in section 7.1, to Gödel's classical completeness theorem for classical first-order logic.

Theorem 6.18 (Classical completeness for coherent logic, cf. Corollary D1.5.10(ii) [26])**.** *Assuming the axiom of choice, every coherent theory has enough models.*

Remark 6.19. Note that, having extended the notion of model from $\mathbf{Set}$ to an arbitrary 'logical' category, it is no longer necessary, as in classical finitary first-order logic, to appeal to non-constructive principles such as the axiom of choice to ensure the existence of enough models of the theory for ensuring provability in it. The universal model $M_{\mathbb{T}}$ in the syntactic category of $\mathbb{T}$ exists constructively, and all the other models of $\mathbb{T}$ are images of it under structure-preserving functors. This represents a very strong form of completeness: the universal model is literally made of formulae, and represents a first-order analogue of the Lindenbaum-Tarski algebra of a propositional theory.

7 Geometric logic and classifying toposes

7.1 Geometric theories

In this book we shall be mostly concerned with geometric theories, because of their connection with the theory of Grothendieck toposes.

Geometric theories are linked to Grothendieck toposes via the notion of classifying topos. Indeed, as we shall see in section 7.2, every geometric theory has a classifying topos and conversely any Grothendieck topos can be seen as the classifying topos of some geometric theory.

Recall that a *geometric theory* over a first-order signature Σ is a theory whose axioms are sequents of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are *geometric formulae*, i.e. formulae with finite number of free variables in the context $\vec{x}$ built up from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.

Whilst the notion of geometric theory might seem quite restrictive at first sight, it turns out that most of the (first-order) theories naturally arising in Mathematics have a geometric axiomatization (over their signature). In fact, the possibility of employing infinitary disjunctions (of an arbitrary cardinality) in the construction of geometric formulae makes geometric logic particularly expressive and suitable for axiomatizing theories which do not belong to the realm of classical first-order logic (think for example of the property of an element of a ring to be nilpotent, or to the notion of algebraic extension of a given field, or of that of ℓ-group with strong unit). Anyway, if a finitary first-order theory $\mathbb{T}$ is not geometric, one can canonically construct a coherent theory $\mathbb{T}'$ over a larger signature, called the Morleyization of $\mathbb{T}$, whose models in the category $\mathbf{Set}$ of sets (and, more generally, in any Boolean coherent category) can be identified with those of $\mathbb{T}$:

Proposition 7.1 (Lemma D1.5.13 [26])**.** Let $\mathbb{T}$ be a first-order theory over a signature Σ. Then there is a signature Σ' containing Σ, and a coherent theory $\mathbb{T}'$ over Σ', called the *Morleyization* of $\mathbb{T}$, such that we have

$$\mathbb{T}\text{-mod}_e(\mathcal{C}) \simeq \mathbb{T}'\text{-mod}(\mathcal{C})$$

for any Boolean coherent category $\mathcal{C}$.

Sketch of proof. The signature Σ' of $\mathbb{T}'$ has, in addition to all the sorts, function symbols and relation symbols of the signature Σ of $\mathbb{T}$, two relation symbols $C_\phi \rightarrowtail A_1 \cdots A_n$ and $D_\phi \rightarrowtail A_1 \cdots A_n$ for each first-order formula ϕ over Σ (where $A_1 \cdots A_n$ is the string of sorts corresponding to the canonical context of ϕ), while the axioms of $\mathbb{T}'$ are given by the sequents of the form $(C_\phi \vdash_{\vec{x}} C_\psi)$ for any axiom $(\phi \vdash_{\vec{x}} \psi)$ of $\mathbb{T}$, plus a set of coherent sequents involving the new relation symbols C_ϕ and D_ϕ which ensure that in any model M of $\mathbb{T}'$ in a Boolean coherent category $\mathcal{C}$, the interpretation of C_ϕ coincides with the interpretation of ϕ and the interpretation of D_ϕ coincides with the complement of the interpretation of ϕ (cf. p. 859-860 [26] for the details). □

Proposition 7.2. Let $\mathbb{T}$ be a finitary first-order theory over a signature Σ and $\mathbb{T}'$ its Morleyization. Then

(i) For any finitary first-order sequent $\sigma := (\phi \vdash_{\vec{x}} \psi)$ over Σ, the sequent is provable in $\mathbb{T}$ using classical first-order logic if and only if the sequent $(C_\phi \vdash_{\vec{x}} C_\psi)$ is provable in $\mathbb{T}'$ using coherent logic.

(ii) The classical first-order syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{fo-cl}}$ of $\mathbb{T}$ is isomorphic to the coherent syntactic category of $\mathbb{T}'$, and to the classical first-order syntactic category of $\mathbb{T}'$.

Proof. (i) This follows immediately from the axioms defining $\mathbb{T}'$.

(ii) It is immediately verified that every finitary first-order (resp. coherent) formula over the signature of $\mathbb{T}'$ is classically provably equivalent (resp. provably equivalent in coherent logic) to a coherent formula over the signature of $\mathbb{T}'$. It follows that the coherent syntactic category of $\mathbb{T}'$ is Boolean and that every morphism between models of $\mathbb{T}'$ in Boolean coherent categories is an elementary morphism. Hence, by Proposition 7.1, both the coherent syntactic category of $\mathbb{T}'$ and the classical first-order syntactic category of $\mathbb{T}'$ satisfy the universal property of the category $\mathcal{C}_{\mathbb{T}}^{\text{fo-cl}}$ with respect to models of $\mathbb{T}$ in Boolean coherent categories whence, by universality, these two categories are naturally equivalent to $\mathcal{C}_{\mathbb{T}}^{\text{fo-cl}}$ (cf. Remark 6.11), as required. □

Remark 7.3. It follows immediately from the proof of part (ii) of the proposition that a first-order theory is complete in the sense of classical Model Theory (i.e., any first-order sentence over the signature of the theory is either provably false or provably true, but not both) if and only if its Morleyization is complete in the sense of geometric logic (i.e., any geometric sentence over its signature is either provably false or provably true, but not both).

The notion of Morleyization is important because it enables us to study any kind of first-order theory by using the methods of Topos Theory. In fact, we can expect many important properties of first-order theories to be naturally expressible as properties of their Morleyizations, and these latter properties to be in turn expressible in terms of invariant properties of their classifying toposes (cf. for instance Remark 7.3).

Still, one can 'turn' a finitary first-order theory $\mathbb{T}$ into a geometric one in alternative ways, i.e. by simply adding some sorts to the signature of $\mathbb{T}$ and axioms over the extended signature so to ensure that each of the first-order formulae which appear in the axioms of $\mathbb{T}$ becomes equivalent to a geometric formula in the new theory and the set-based models of the latter can be identified with those of $\mathbb{T}$. The Morleyization just represents a canonical, generally 'non-economical' way of doing this which works uniformly for any finitary first-order theory.

7.2 The notion of classifying topos

We have seen in section 5.2 (cf. Proposition 5.16) that every *Grothendieck topos* $\mathcal{E}$ is a geometric category. Thus we can consider models of *geometric theories* in $\mathcal{E}$. Inverse image functors of geometric morphisms of toposes preserve finite limits (by definition) and arbitrary colimits (having a right adjoint); so they are *geometric functors* and hence they preserve the interpretation of (arbitrary) geometric formulae (cf. the proof of Lemma 5.23). In general, such functors are *not* Heyting functors, which explains why the following definition only makes sense for geometric theories.

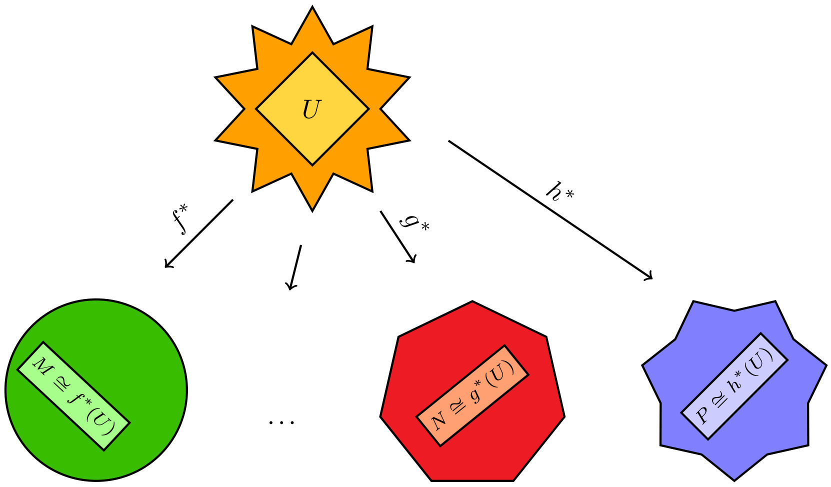
Definition 7.4. Let $\mathbb{T}$ be a geometric theory over a given signature. A *classifying topos* of $\mathbb{T}$ is a Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ (also denoted $\mathcal{E}_{\mathbb{T}}$) such that for any Grothendieck topos $\mathcal{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$$

natural in $\mathcal{E}$.

Naturality means that for any geometric morphism $f : \mathcal{E} \to \mathcal{F}$, we have a commutative square (up to natural isomorphism)

$$\begin{array}{ccc} \mathbf{Geom}(\mathcal{F}, \mathbf{Set}[\mathbb{T}]) & \xrightarrow{\simeq} & \mathbb{T}\text{-mod}(\mathcal{F}) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle \mathbb{T}\text{-mod}(f^*)} \\ \mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]) & \xrightarrow{\simeq} & \mathbb{T}\text{-mod}(\mathcal{E}) \end{array}$$



Classifying topos

in the (meta-)category **CAT** of categories.

In other words, there is a model U of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$, called 'the' *universal model* of $\mathbb{T}$, characterized by the universal property that any model M in a Grothendieck topos can be obtained, up to isomorphism, as a pullback $f^*(U)$ of the model U along the inverse image f^* of a unique (up to isomorphism) geometric morphism from $\mathcal{E}$ to $\mathbf{Set}[\mathbb{T}]$.

Remark 7.5. The classifying topos of a geometric theory $\mathbb{T}$ can be seen as a *representing object* for the (pseudo-)functor

$$\mathbb{T}\text{-mod} : \mathfrak{BTop}^{\text{op}} \to \mathbf{CAT}$$

which assigns

- to a topos $\mathcal{E}$ the category $\mathbb{T}\text{-mod}(\mathcal{E})$ of models of $\mathbb{T}$ in $\mathcal{E}$ and
- to a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ the functor $\mathbb{T}\text{-mod}(f^*) : \mathbb{T}\text{-mod}(\mathcal{F}) \to \mathbb{T}\text{-mod}(\mathcal{E})$ sending a model $M \in \mathbb{T}\text{-mod}(\mathcal{F})$ to its image $f^*(M)$ under the functor f^*.

In particular, classifying toposes are *unique up to categorical equivalence*.

In the above picture, the big shapes represent different toposes while the inner lighter shapes represent models of a given theory inside them; in particular, the dark yellow star represent the classifying topos of a given theory and the light yellow diamond represent the universal model of the theory inside it.

As the picture illustrates, all the models of the theory, including all the classical set-based models, are sorts of 'shadows' of the universal model lying in the classifying topos. This indicates that the symmetries of the theory are best understood by adopting the point of view of its classifying topos, since all the other models are images of this inner 'core' under 'deformations' realized by structure-preserving functors.

Contrary to what happens in classical logic, where one is forced to appeal to non-constructive principles such as the axiom of choice in order to ensure the existence of 'enough' set-based models for faithfully representing a finitary first-order theory, here we dispose of a constructively defined universe in which the syntax and semantics of the theory meet yielding a strong form of completeness (cf. Theorem 6.5) and definability (cf. [7]). Moreover, classifying toposes not only exist for finitary geometric theories, but also for arbitrary infinitary ones, theories for which a classical completeness theorem no longer holds in general (one can easily exhibit non-contradictory infinitary geometric theories without any set-based models).

Definition 7.6. Let $\mathbb{T}$ be a geometric theory. A *universal model* of a geometric theory $\mathbb{T}$ is a model $U_{\mathbb{T}}$ of $\mathbb{T}$ in a Grothendieck topos $\mathcal{G}$ such that for any $\mathbb{T}$-model M in a Grothendieck topos $\mathcal{F}$ there exists a unique (up to isomorphism) geometric morphism $f_M : \mathcal{F} \to \mathcal{G}$ such that $f_M^*(U_{\mathbb{T}}) \cong M$.

Remarks 7.7. (a) By the (2-dimensional) Yoneda lemma, if a topos $\mathcal{G}$ contains a universal model of a geometric theory $\mathbb{T}$ then $\mathcal{G}$ satisfies the universal property of the classifying topos of $\mathbb{T}$. Conversely, if a topos $\mathcal{E}$ classifies a geometric theory $\mathbb{T}$ then $\mathcal{E}$ contains a universal model of $\mathbb{T}$.

(b) If M and N are universal models of a geometric theory $\mathbb{T}$ lying respectively in toposes $\mathcal{F}$ and $\mathcal{G}$ then there exists a unique up to isomorphism (geometric) equivalence between $\mathcal{F}$ and $\mathcal{G}$ such that its inverse image functors send M and N to each other (up to isomorphism).

The classifying topos of a geometric theory can be canonically built as the category of sheaves on the geometric syntactic category of the theory with respect to the geometric topology on it. For smaller fragments of geometric logic, such as for example cartesian (resp. regular, coherent) logic, there exist variations of this syntactic construction, consisting in replacing the geometric syntatic site of the theory with its cartesian (resp. regular, coherent) syntactic site. In particular, any finitary algebraic theory can be realized as the topos of presheaves on its cartesian syntactic category.

More specifically, the following result holds.

Theorem 7.8. *(i) For any universal Horn theory $\mathbb{T}$, the topos $[(\mathcal{C}_{\mathbb{T}}^{alg})^{op}, \mathbf{Set}]$ classifies $\mathbb{T}$.*

(ii) For any cartesian theory $\mathbb{T}$, the topos $[(\mathcal{C}_{\mathbb{T}}^{cart})^{op}, \mathbf{Set}]$ classifies $\mathbb{T}$.

(iii) For any regular theory $\mathbb{T}$, the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{reg}, J_{\mathbb{T}}^{reg})$ classifies $\mathbb{T}$.

(iv) For any coherent theory $\mathbb{T}$, the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{coh}, J_{\mathbb{T}}^{coh})$ classifies $\mathbb{T}$.

(v) For any geometric theory $\mathbb{T}$, the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ classifies $\mathbb{T}$.

Sketch of proof. The proof of (i) requires a slightly different argument from that of the other points and can be found in [2].

By Diaconescu's equivalence (cf. Theorem 3.32) and Proposition 3.30, the geometric morphisms $\mathcal{E} \to [(\mathcal{C}_{\mathbb{T}}^{\text{cart}})^{\text{op}}, \mathbf{Set}]$ (resp. $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{reg}}, J_{\mathbb{T}}^{\text{reg}})$, $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathbb{T}}^{\text{coh}})$, $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}), J_{\mathbb{T}})$) correspond, naturally in $\mathcal{E}$, to the cartesian functors $\mathcal{C}_{\mathbb{T}}^{\text{cart}} \to \mathcal{E}$ (resp. to the cartesian $J_{\mathbb{T}}^{\text{reg}}$-continuous functors $\mathcal{C}_{\mathbb{T}}^{\text{reg}} \to \mathcal{E}$, to the cartesian $J_{\mathbb{T}}^{\text{coh}}$-continuous functors $\mathcal{C}_{\mathbb{T}}^{\text{coh}} \to \mathcal{E}$, to the cartesian $J_{\mathbb{T}}$-continuous functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$). The thesis thus follows from Theorem 6.10 and Proposition 6.8. □

Remark 7.9. The universal model of the theory $\mathbb{T}$ in its classifying topos is given in each case by the image of the universal model $M_{\mathbb{T}}$ of $\mathbb{T}$ in its syntactic category (as described in Definition 6.4) under the Yoneda embedding from it to the classifying topos of $\mathbb{T}$ (recall from Remark 6.9 that all the syntactic topologies are subcanonical).

As a corollary, one obtains the following fundamental result, due to A. Joyal, G. Reyes and M. Makkai.

Theorem 7.10. *Every geometric theory has a classifying topos.*

□

The following result represents a converse to this theorem:

Theorem 7.11. *Every Grothendieck topos is the classifying topos of some geometric theory.*

Proof. Let $\mathbf{Sh}(\mathcal{C}, J)$ be a Grothendieck topos. Diaconescu's equivalence provides, for any Grothendieck topos $\mathcal{E}$, an equivalence between the category $\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J))$ of geometric morphisms from $\mathcal{E}$ to $\mathbf{Sh}(\mathcal{C}, J)$ and the category $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$ of J-continuous flat functors from $\mathcal{C}$ to $\mathcal{E}$, naturally in $\mathcal{E}$. Now, we can construct a geometric theory $\mathbb{T}^{\mathcal{C}}_{J}$ such that its models in any Grothendieck topos $\mathcal{E}$ can be identified precisely with the J-continuous flat functors from $\mathcal{C}$ to $\mathcal{E}$ (and the homomorphisms of $\mathbb{T}^{\mathcal{C}}_{J}$-models can be identified with natural transformations between the corresponding flat functors); clearly, $\mathbb{T}^{\mathcal{C}}_{J}$ will be classified by the topos $\mathbf{Sh}(\mathcal{C}, J)$. We shall call $\mathbb{T}^{\mathcal{C}}_{J}$ the *theory of J-continuous flat functors on $\mathcal{C}$*. It is instructive to write down explicitly an axiomatization of $\mathbb{T}^{\mathcal{C}}_{J}$.

The signature of $\mathbb{T}^{\mathcal{C}}_{J}$ has one sort $\ulcorner A \urcorner$ for each object A of $\mathcal{C}$, and one function symbol $\ulcorner f \urcorner : \ulcorner A \urcorner \to \ulcorner B \urcorner$ for each arrow $f : A \to B$ in $\mathcal{C}$. The axioms of $\mathbb{T}^{\mathcal{C}}_{J}$ are the following (to indicate that a variable x has sort $\ulcorner A \urcorner$ we write x^A):

$$(\top \vdash_x (\ulcorner f \urcorner(x) = x)) \tag{1}$$

for any identity arrow f in $\mathcal{C}$;

$$(\top \vdash_x (\ulcorner f \urcorner(x) = \ulcorner h \urcorner(\ulcorner g \urcorner(x)))) \tag{2}$$

for any triple of arrows f, g, h of $\mathcal{C}$ such that f is equal to the composite $h \circ g$;

$$\top \vdash_{[]} \bigvee_{A \in Ob(\mathcal{C})} (\exists x^A)\top \tag{3}$$

(where the disjunction ranges over all the objects of $\mathcal{C}$);

$$(\top \vdash_{x^A, y^B} \bigvee_{A \xleftarrow{f} C \xrightarrow{g} B} (\exists z^C)(\ulcorner f \urcorner(z^C) = x^A \wedge \ulcorner g \urcorner(z^C) = y^B)) \tag{4}$$

for any objects A, B of $\mathcal{C}$ (where the disjunction ranges over all the cones $A \xleftarrow{f} C \xrightarrow{g} B$ on the discrete diagram given by the pair of objects A and B);

$$(\ulcorner f \urcorner(x^A) = \ulcorner g \urcorner(x^A) \vdash_{x^A} \bigvee_{h: C \to A \in Eq(f,g)} (\exists z^C)(\ulcorner h \urcorner(z^C) = x^A)) \tag{5}$$

for any pair of arrows $f, g : A \to B$ in $\mathcal{C}$ with common domain and codomain (where the disjunction ranges over all the arrows h which equalize f and g);

$$(\top \vdash_{x^A} \bigvee_{i \in I} (\exists y_i^{B_i})(\ulcorner f_i \urcorner(y_i^{B_i}) = x^A)) \tag{6}$$

for each J-covering family $\{f_i : B_i \to A \mid i \in I\}$.

Notice that the first two groups of axioms express functoriality, the third, fourth and fifth together express flatness (in terms of filteredness of the corresponding category of elements), while the sixth group of axioms expresses J-continuity. □

The following notion will be systematically studied in section 8.2.

Definition 7.12. Two geometric theories are said to be *Morita-equivalent* if they have equivalent classifying toposes.

From the above discussion it follows that Grothendieck toposes can be thought of as canonical representatives for Morita-equivalence classes of geometric theories. We shall extensively build on this remark in section 8.

Remark 7.13. It is important to note that the above-described method for constructing classifying toposes via syntactic sites is by no means the only one for 'calculating' the classifying topos of a geometric theory. Alternative techniques, of more 'semantic' or 'geometric' nature have been developed. For instance, as we shall see below, every representation of a geometric theory as an extension of a given geometric theory over its signature leads to a representation of its classifying topos as a subtopos of the classifying topos of the latter theory; applied to extensions $\mathbb{S}$ of theories $\mathbb{T}$ classified by a presheaf topos, this leads to a 'semantic' representation for the classifying topos of $\mathbb{S}$ as a topos of sheaves on the opposite of the category of finitely presentable $\mathbb{T}$-models. More generally, as we shall argue in section 8, it is reasonable to expect 'different ways of looking at a certain theory' to materialize into different representations of its classifying topos.

Remark 7.14. The notions of geometric theory and classifying topos can be 'relativized' to an arbitrary base topos. For more details the reader is referred to sections 6.5 of [25] and B4.2 of [26].

7.3 Interpretations and geometric morphisms

The mapping from the collection of geometric theories to the collection of Grothendieck toposes sending a theory to its classifying topos can be made functorial, as follows.

A natural notion of morphism between theories is given by the notion of interpretation of one theory into another. In fact, there are many possible variants of this notion, all of which inducing geometric morphisms between the corresponding classifying toposes.

Definition 7.15. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two cartesian (resp. regular, coherent, geometric) theories.

(a) A cartesian (resp. regular, coherent, geometric) interpretation of $\mathbb{T}_1$ in $\mathbb{T}_2$ is a cartesian (resp. regular, coherent, geometric) functor from $\mathcal{C}^{\text{cart}}_{\mathbb{T}_1}$ (resp. $\mathcal{C}^{\text{reg}}_{\mathbb{T}_1}$, $\mathcal{C}^{\text{coh}}_{\mathbb{T}_1}$, $\mathcal{C}_{\mathbb{T}_1}$) to $\mathcal{C}^{\text{cart}}_{\mathbb{T}_2}$ (resp. $\mathcal{C}^{\text{reg}}_{\mathbb{T}_1}$, $\mathcal{C}^{\text{coh}}_{\mathbb{T}_1}$, $\mathcal{C}_{\mathbb{T}_2}$).

(b) The cartesian (resp. regular, coherent, geometric) theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are said to be *cartesianly bi-interpretable* (resp. *regularly bi-interpretable*, *coherently bi-interpretable*, *geometrically bi-interpretable*) if their cartesian (resp. regular, coherent, geometric) syntactic categories are equivalent.

Any cartesian functor $\mathcal{C}_{\mathbb{T}_1}^{\text{cart}} \to \mathcal{C}_{\mathbb{T}_2}^{\text{cart}}$ induces a geometric morphism

$$[\mathcal{C}_{\mathbb{T}_2}^{\text{cart}^{\text{op}}}, \mathbf{Set}] \to [\mathcal{C}_{\mathbb{T}_1}^{\text{cart}^{\text{op}}}, \mathbf{Set}]$$

(cf. Example 3.14(c)).

Since the regular (resp. coherent, geometric) functors $\mathcal{C}_{\mathbb{T}_1}^{\text{reg}} \to \mathcal{C}_{\mathbb{T}_2}^{\text{reg}}$ (resp. $\mathcal{C}_{\mathbb{T}_1}^{\text{coh}} \to \mathcal{C}_{\mathbb{T}_2}^{\text{coh}}$, $\mathcal{C}_{\mathbb{T}_1} \to \mathcal{C}_{\mathbb{T}_2}$) are exactly the cartesian functors which send $J_{\mathbb{T}_1}^{\text{reg}}$-covering sieves (resp. $J_{\mathbb{T}_1}^{\text{coh}}$-covering sieves, $J_{\mathbb{T}_1}$-covering sieves) on $\mathcal{C}_{\mathbb{T}_1}^{\text{reg}}$ (resp. $\mathcal{C}_{\mathbb{T}_1}^{\text{coh}}$, $\mathcal{C}_{\mathbb{T}_1}$) to $J_{\mathbb{T}_2}^{\text{reg}}$-covering sieves (resp. $J_{\mathbb{T}_2}^{\text{coh}}$-covering sieves, $J_{\mathbb{T}_2}$) on $\mathcal{C}_{\mathbb{T}_2}^{\text{reg}}$ (resp. $\mathcal{C}_{\mathbb{T}_2}^{\text{coh}}$, $\mathcal{C}_{\mathbb{T}_2}$), they induce geometric morphisms $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}_2}^{\text{reg}}, J_{\mathbb{T}_2}^{\text{reg}}) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_1}^{\text{reg}}, J_{\mathbb{T}_1}^{\text{reg}})$ (resp. $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}_2}^{\text{coh}}, J_{\mathbb{T}_2}^{\text{coh}}) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_1}^{\text{coh}}, J_{\mathbb{T}_1}^{\text{coh}})$, $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}_2}, J_{\mathbb{T}_2}) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_1}, J_{\mathbb{T}_1})$), again by Example 3.14(c).

Let us now discuss the relationship between syntactic categories and classifying toposes. Whilst if is always possible to recover the cartesian syntactic category of a cartesian theory from its classifying topos up to equivalence, for regular (resp. coherent) theories it is only possible in general to recover the effectivization of the regular syntactic category (resp. the effective positivization of the coherent syntactic category) of the theory.

The topos-theoretic invariants which are useful in this respect are the following:

Definition 7.16. (a) An object A of a Grothendieck topos is said to be *irreducbile* if every epimorphic family in the topos with codomain A contains a split epimorphism.

(b) An object A of a Grothendieck topos is said to be *compact* if every epimorphic family with codomain A contains a finite epimorphic sub-family.

(c) An object A of a Grothendieck topos is said to be *supercompact* if every epimorphic family with codomain A contains an epimorphism.

(d) An object A of a Grothendieck topos $\mathcal{E}$ is said to be *coherent* if it is compact and for any arrow $u : B \to A$ in $\mathcal{E}$ such that B is compact, the domain of the kernel pair of u is compact.

(e) An object A of a Grothendieck topos $\mathcal{E}$ is said to be *supercoherent* if it is supercompact and for any arrow $u : B \to A$ in $\mathcal{E}$ such that B is supercompact, the domain of the kernel pair of u is supercompact.

For a proof of the following proposition we refer the reader to section D3.3 of [26] for points (ii) and (iii) and to [8] for point (i).

Proposition 7.17. (i) The cartesian syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{cart}}$ of a cartesian theory $\mathbb{T}$ can be recovered, up to equivalence, from its classifying topos $[\mathcal{C}_{\mathbb{T}}^{\text{cart}^{\text{op}}}, \mathbf{Set}]$ as the full subcategory of it on its irreducible objects.

(ii) The effectivization $\mathbf{Eff}(\mathcal{C}_{\mathbb{T}}^{\text{reg}})$ of the regular syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{cart}}$ of a regular theory $\mathbb{T}$ can be recovered, up to equivalence, from its classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{reg}}, J_{\mathbb{T}}^{\text{reg}})$ as the full subcategory of it on its regular objects.

(iii) The positive effectivization $\mathbf{Pos}(\mathbf{Eff}(\mathcal{C}_{\mathbb{T}}^{\text{reg}}))$ of the coherent syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{cart}}$ of a coherent theory $\mathbb{T}$ can be recovered, up to equivalence, from its classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathbb{T}}^{\text{coh}})$ as the full subcategory of it on its coherent objects.

Notice that every object of $\mathbf{Eff}(\mathcal{C}_{\mathbb{T}}^{\text{reg}})$ has a syntactic description as a 'formal quotient' of an object of $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$ by an equivalence relation on $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$, and similarly any object of $\mathbf{Eff}(\mathbf{Pos}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}))$ is a 'formal quotient' in $\mathbf{Pos}(\mathcal{C}_{\mathbb{T}}^{\text{coh}})$ of a 'formal finite coproduct' of objects of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ by an equivalence relation on $\mathbf{Pos}(\mathcal{C}_{\mathbb{T}}^{\text{coh}})$.

This motivates the following

Definition 7.18. (a) A *generalized regular interpretation* of a regular theory $\mathbb{T}_1$ in another regular theory $\mathbb{T}_2$ is a regular functor $\mathbf{Eff}(\mathcal{C}_{\mathbb{T}_1}^{\text{reg}}) \to \mathbf{Eff}(\mathcal{C}_{\mathbb{T}_2}^{\text{reg}})$.

(b) A *generalized coherent interpretation* of a coherent theory $\mathbb{T}_1$ in another coherent theory $\mathbb{T}_2$ is a coherent functor $\mathbf{Eff}(\mathbf{Pos}(\mathcal{C}_{\mathbb{T}_1}^{\text{coh}})) \to \mathbf{Eff}(\mathbf{Pos}(\mathcal{C}_{\mathbb{T}_2}^{\text{coh}}))$.

Of course, as their classical counterparts, generalized interpretations also induce geometric morphisms between the associated classifying toposes.

Concerning geometric theories, the universal properties of the classifying topos and of the geometric syntactic category of a geometric theory, in combination with Giraud's theorem, ensure that the ∞-pretopos completion of the geometric syntactic category of a geometric theory is equivalent to its classifying topos.

The following proposition represents the 'functorialization' of Proposition 7.17. Before stating it, we need to introduce some definitions.

A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is said to be *coherent* if the inverse image functor $f^* : \mathcal{F} \to \mathcal{E}$ of f sends coherent objects of $\mathcal{F}$ to coherent objects of $\mathcal{E}$. Similarly, a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is said to be *regular* if $f^* : \mathcal{F} \to \mathcal{E}$ sends supercoherent objects of $\mathcal{F}$ to supercoherent objects of $\mathcal{E}$.

Proposition 7.19. (i) Every essential geometric morphism $\mathcal{E}_{\mathbb{T}_2} \to \mathcal{E}_{\mathbb{T}_1}$ between the classifying toposes of two cartesian theories $\mathbb{T}_1$ and $\mathbb{T}_2$ is induced by a (unique up to isomorphism) cartesian interpretation of $\mathbb{T}_1$ in $\mathbb{T}_2$.

(ii) Every regular geometric morphism $\mathcal{E}_{\mathbb{T}_2} \to \mathcal{E}_{\mathbb{T}_1}$ between the classifying toposes of two regular theories $\mathbb{T}_1$ and $\mathbb{T}_2$ is induced by a (unique up to isomorphism) generalized regular interpretation of $\mathbb{T}_1$ in $\mathbb{T}_2$.

(iii) Every coherent geometric morphism $\mathcal{E}_{\mathbb{T}_2} \to \mathcal{E}_{\mathbb{T}_1}$ between the classifying toposes of two coherent theories $\mathbb{T}_1$ and $\mathbb{T}_2$ is induced by a (unique up to isomorphism) coherent interpretation of $\mathbb{T}_1$ in $\mathbb{T}_2$.

Concerning geometric theories, we note that, since the classifying topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$ is the ∞-pretopos completion of $\mathcal{C}_{\mathbb{T}}$, we can regard any geometric morphism $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}_1}, J_{\mathbb{T}_1}) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_2}, J_{\mathbb{T}_2})$ as a generalized geometric interpretation of $\mathbb{T}_1$ in $\mathbb{T}_2$.

Finally, we note that we can give an alternative description of the notions of interpretations defined above, as follows. If $\mathbb{T}$ is a cartesian (resp. regular, coherent, geometric) theory over a signature Σ then, by Theorem 6.10, the category of cartesian (resp. regular, coherent, geometric) functors from $\mathcal{C}_{\mathbb{T}}^{\text{cart}}$ (resp. $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$, $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$, $\mathcal{C}_{\mathbb{T}}$) to any cartesian (resp. regular, coherent, geometric) category $\mathcal{D}$ is naturally equivalent to the category of models of $\mathbb{T}$ in $\mathcal{D}$, where the equivalence sends each model $M \in \mathbb{T}\text{-mod}(\mathcal{E})$ to the functor $F_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ assigning to a formula $\phi(\vec{x})$ its interpretation $[[\vec{x} \,.\, \phi]]_M$ in M. Thus, any geometric interpretation of a geometric theory $\mathbb{T}_1$ in a geometric theory $\mathbb{T}_2$ corresponds to an internal $\mathbb{T}_1$-model in the syntactic category $\mathcal{C}_{\mathbb{T}_2}$ of $\mathbb{T}_2$, and similarly for cartesian, regular and coherent theories (and their generalized interpretations). Note that if $\mathbb{T}_2$ has enough models (for example, if $\mathbb{T}_2$ is coherent assuming the axiom of choice) then the condition that a certain Σ-structure U, where Σ is the signature of

$\mathbb{T}_1$, in the (effectivization or pretopos completion of the) syntactic category of $\mathbb{T}_2$ be a $\mathbb{T}_1$-model is equivalent to the requirement that for any $\mathbb{T}_2$-model M in $\mathbf{Set}$ the functor F_M send U to a $\mathbb{T}_1$-model.

Remarks 7.20. (a) Since a given theory can in general belong to more than one fragment of geometric logic, it is natural to wonder if the notions of bi-interpretability of theories given above for different fragments of geometric logic are compatible with each other. We shall answer this question in the affirmative in section 8.4 below.

(b) Since the classification of models of geometric theories already takes place at the level of geometric categories (cf. Theorem 6.10), one may naturally wonder why we should consider more complicated structures such as classifying toposes in place of them. In fact, there are many reasons for doing this.

First, a Grothendieck topos is, in terms of internal categorical structure, much richer than a geometric category, and hence constitutes a mathematical environment which is naturally more amenable to computations and which enjoys higher degrees of symmetry. In fact, it is a standard process in mathematics to complete mathematical entities with respect to natural operations or properties through the formal addition of 'imaginaries' and work in the extended, more 'symmetric' environment, to solve problems posed in the original context (think for instance of the real line and the complex plane).

Second, a very important aspect of Grothendieck toposes, on which the theory of topos-theoretic 'bridges' described in section 8 is based, is the fact that they admit a very well-behaved, although highly non-trivial, representation theory. The abstract relationship between an object and its different 'presentations' is geometrically incarnated in the context of Grothendieck toposes in the form of relationships between a given topos and its different sites of definition (or, more generally, its different representations); the possibility of working at two levels, rather than just one, makes the theory extremely fruitful and technically flexible.

Third, the 2-category $\mathfrak{Btop}$ of Grothendieck toposes is itself quite rich in terms of (2-)categorical structure (cf. section B4 of [26]).

7.4 Classifying toposes for propositional theories

Definition 7.21. (a) A *propositional theory* is a geometric theory over a signature Σ which has no sorts.

(b) A *localic topos* is any topos of the form $\mathbf{Sh}(L)$ for a locale L.

Remark 7.22. The signature of a propositional theory merely consists of a set of 0-ary relation symbols, which define the atomic propositions; all the other formulae (i.e., sentences) are built from them by using finitary conjunctions and infinitary disjunctions.

Propositional theories are useful for describing subsets of a given set with particular properties, such as filters on a frame or prime ideals on a commutative ring. We shall see specific examples below.

Proposition 7.23 (cf. Remark D3.1.14 [26])**.** Localic toposes are precisely the classifying toposes of propositional theories.

Specifically, given a locale L, we can consider the propositional theory $\mathbb{P}_L$ of *completely prime filters* in L, defined as follows. We take one atomic proposition F_a (to be thought of as the assertion that a is in the filter) for each $a \in L$; the axioms are

$$(\top \vdash F_1),$$

all the sequents of the form

$$(F_a \wedge F_b \vdash F_{a \wedge b}),$$

for any $a, b \in L$, and all the sequents of the form

$$F_a \vdash \bigvee_{i \in I} F_{a_i}$$

whenever $\bigvee_{i \in I} a_i = a$ in L.

In fact, for any locale L, the topos $\mathbf{Sh}(L)$ classifies the theory $\mathbb{P}_L$.

Notice that the set-based models of the theory $\mathbb{P}_L$ are the completely prime filters on L, that is the subsets $F \subseteq L$ such that $1 \in F, a \wedge b \in F$ whenever $a, b \in F$ and for any family of elements $\{a_i \in L \mid i \in I\}$ such that $\bigvee_{i \in I} a_i \in F$, there exists $i \in I$ such that $a_i \in F$.

Remark 7.24. Recall that the *Lindenbaum-Tarski algebra* $\mathcal{L}_{\mathbb{T}}$ of a propositional theory $\mathbb{T}$ is the algebra consisting of the provable-equivalence classes of sentences over its signature. In the case of a geometric propositional theory $\mathbb{T}$, $\mathcal{L}_{\mathbb{T}}$ is clearly a frame, equivalent to the geometric syntactic category of $\mathbb{T}$. The classifying topos of $\mathbb{T}$ is simply given by the category of sheaves $\mathbf{Sh}(\mathcal{L}_{\mathbb{T}}, J_{\mathcal{L}_{\mathbb{T}}})$ on this locale. This indicates that the classifying topos of a geometric theory represents a natural first-order analogue of the Lindenbaum-Tarski algebra of a propositional theory.

We shall encounter again propositional theories in section 7.6.

7.5 Classifying toposes for cartesian theories

In order to describe classifying toposes for cartesian theories, we have to recall the notion of finitely presented model of such a theory.

Definition 7.25. Let $\mathbb{T}$ be a cartesian theory over a signature Σ. We say that a $\mathbb{T}$-model M in $\mathbf{Set}$ is *finitely presented* by a cartesian formula-in-context $\phi(\vec{x})$, where $A_1 \cdots A_n$ is the string of sorts associated to $\vec{x}$ if there exists a string of elements $(\xi_1, \ldots, \xi_n) \in MA_1 \times \cdots \times MA_n$, called the *generators* of M, such that for any $\mathbb{T}$-model N in $\mathbf{Set}$ and string of elements $\vec{b} = (b_1, \ldots, b_n) \in MA_1 \times \cdots \times MA_n$ such that $(b_1, \ldots, b_n) \in [[\vec{x} \,.\, \phi]]_N$, there exists a unique arrow $f^{\vec{b}} : M \to N$ in $\mathbb{T}$-mod$(\mathbf{Set})$ such that $(f^{\vec{b}} A_1 \times \cdots \times f^{\vec{b}} A_n)((\xi_1, \ldots, \xi_n)) = (b_1, \ldots, b_n)$.

We denote by f.p.$\mathbb{T}$-mod$(\mathbf{Set})$ the full subcategory of $\mathbb{T}$-mod$(\mathbf{Set})$ on the finitely presented models.

Theorem 7.26 (Corollary D3.1.2 [26]). *For any cartesian theory $\mathbb{T}$, we have an equivalence of categories*

$$\textit{f.p.}\mathbb{T}\textit{-mod}(\mathbf{Set}) \simeq (\mathcal{C}_{\mathbb{T}}^{cart})^{op}$$

In particular, $\mathbb{T}$ is classified by the topos $[\textit{f.p.}\mathbb{T}\textit{-mod}(\mathbf{Set}), \mathbf{Set}]$.

Remark 7.27. If $\mathbb{T}$ is a universal Horn theory (in particular, a finitary algebraic theory) then the category $\mathscr{C}_{\mathbb{T}}^{\text{alg}}$ is dual to the category of finitely presented $\mathbb{T}$-algebras, and the presheaf topos $[\mathscr{C}_{\mathbb{T}}^{\text{alg}^{\text{op}}}, \mathbf{Set}]$ satisfies the universal property of the classifying topos for $\mathbb{T}$ (cf. [2]).

Examples 7.28. (a) The (algebraic) theory of Boolean algebras is classified by the topos $[\mathbf{Bool}_{\text{fin}}, \mathbf{Set}]$, where $\mathbf{Bool}_{\text{fin}}$ is the category of finite Boolean algebras and Boolean algebra homomorphisms between them.

(b) The (algebraic) theory of commutative rings with unit is classified by the topos $[\mathbf{Rng}_{\text{f.g.}}, \mathbf{Set}]$, where $\mathbf{Rng}_{\text{f.g.}}$ is the category of finitely generated rings and ring homomorphisms between them.

7.6 Further examples

- **The Zariski topos**

Let Σ be the one-sorted signature for the theory $\mathbb{T}$ of commutative rings with unit i.e. the signature consisting of two binary function symbols $+$ and $\cdot$, one unary function symbol $-$ and two constants 0 and 1.

The *coherent theory of local rings* is obtained from $\mathbb{T}$ by adding the sequents

$$(0 = 1 \vdash_{[]} \bot)$$

and

$$((\exists z)((x + y) \cdot z = 1) \vdash_{x,y} ((\exists z)(x \cdot z = 1) \vee (\exists z)(y \cdot z = 1))),$$

Definition 7.29. The *Zariski topos* is the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{op}, J)$ of sheaves on the opposite of the category $\mathbf{Rng}_{f.g.}$ of finitely generated rings with respect to the topology J on $\mathbf{Rng}_{f.g.}^{\text{op}}$ defined by: given a cosieve S in $\mathbf{Rng}_{f.g.}$ on an object A, $S \in J(A)$ if and only if S contains a finite family $\{\xi_i : A \to A[{s_i}^{-1}] \mid 1 \leq i \leq n\}$ of canonical inclusions $\xi_i : A \to A[{s_i}^{-1}]$ in $\mathbf{Rng}_{f.g.}$ where $\{s_1, \ldots, s_n\}$ is any set of elements of A which is not contained in any proper ideal of A.

Proposition 7.30 (Theorem VII.6.3 [29])**.** The (coherent) theory of local rings is classified by the Zariski topos.

More generally, one can prove that the big Zariski topos of an affine scheme $Spec(A)$ classifies the theory of local A-algebras. On the other hand, the small Zariski topos $Spec(A)$ classifies the theory of localizations of the ring A, equivalently the propositional theory $\mathbb{P}$ of *prime filters* on A defined as follows. The signature of $\mathbb{P}$ consists of a propositional symbol $P(a)$ for each element $a \in A$, and the axioms of $\mathbb{P}$ are the following:

$$(\top \vdash P_{1_A});$$

$$(P_{0_A} \vdash \bot);$$

$$(P_{a \cdot b} \dashv\vdash P_a \wedge P_b)$$

for any a, b in A;

$$(P_{a+b} \vdash P_a \vee P_b)$$

for any $a, b \in A$.

The models of $\mathbb{P}$ in $\mathbf{Set}$ are precisely the prime filters on A, that is the subsets S of A such that the complement $A \setminus S$ is a prime ideal. In fact, the Zariski topology is homeomorphic to the subterminal topology (in the sense of Definition 3.22) on the set of points of the classifying topos of $\mathbb{P}$. If, instead of taking the theory $\mathbb{P}$ of prime filters, we had considered the propositional theory of prime ideals (axiomatized over the same signature in the obvious way), we would have obtained a classifying topos inequivalent to the small Zariski topos of A, in spite of the fact that the two theories have the same models in $\mathbf{Set}$. In fact, in order to prove that their categories of set-based models are equivalent, one has to use the law of excluded middle, a principle which is not sound for general toposes, even for very simple ones such as the Sierpinski topos (i.e. the topos of sheaves on the Sierpinski space).

- **The classifying topos for integral domains**

The theory of *integral domains* is the theory obtained from the theory of commutative rings with unit by adding the axioms

$$(0 = 1 \vdash_{[]} \bot);$$

$$(x \cdot y = 0 \ \vdash_{x,y} \ x = 0 \vee y = 0)\ .$$

Proposition 7.31. The theory of *integral domains* is classified by the topos $\mathbf{Sh}(\mathbf{Rng}_{f.g.}^{op}, J)$ of sheaves on the opposite of the category $\mathbf{Rng}_{f.g.}$ of finitely generated rings with respect to the topology J on $\mathbf{Rng}_{f.g.}^{\mathrm{op}}$ defined by: given a cosieve S in $\mathbf{Rng}_{f.g.}$ on an object A, $S \in J_2(A)$ if and only if

- either A is the zero ring and S is the empty sieve on it or
- S contains a non-empty finite family $\{\pi_{a_i} : A \to A/(a_i) \mid 1 \leq i \leq n\}$ of canonical projections $\pi_{a_i} : A \to A/(a_i)$ in $\mathbf{Rng}_{f.g.}$ where $\{a_1, \ldots, a_n\}$ is any set of elements of A such that $a_1 \cdot \ldots \cdot a_n = 0$.

For a proof of this proposition, see [6].

- **The topos of simplicial sets**

Definition 7.32. The theory $\mathbb{I}$ of *intervals* is written over a one-sorted signature having a binary relation symbol $\leq$ and two constants b and t, and has as axioms the following sequents:

$$(\top \vdash_x x \leq x);$$

$$(x \leq y \wedge y \leq z \vdash_{x,y,z} x \leq z);$$

$$(x \leq y \wedge y \leq x \vdash_{x,y} x = y);$$

$$(\top \vdash_x x \leq t \wedge b \leq x);$$

$$(b = t \vdash \bot);$$

$$(\top \vdash_{x,y} x \leq y \vee y \leq x)\ .$$

Proposition 7.33. The theory $\mathbb{I}$ is classified by the topos $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets.

A detailed proof of this proposition may be found in section VIII.8 of [29].

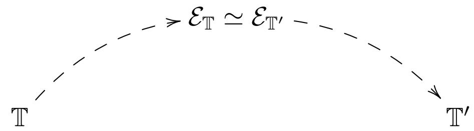
8 Toposes as 'bridges'

In this section we describe the unifying methodology 'toposes as bridges' introduced in [10]. This general technique will be systematically applied throughout the book, either implicitly or explicitly, in a variety of different contexts, so it is essential for the reader to acquire familiarity with it.

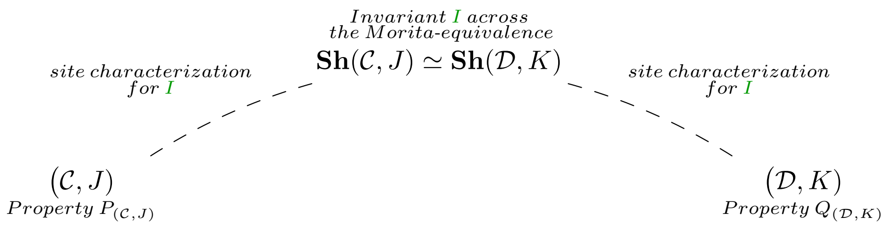
8.1 The 'bridge-building' technique

The 'bridge-building' technique allows to construct topos-theoretic 'bridges' connecting distinct mathematical theories with each other.

Specifically, if $\mathbb{T}$ and $\mathbb{T}'$ are two Morita-equivalent theories (that is, geometric theories classified by the same topos) then their common classifying topos can be used as a 'bridge' for transferring information between them:



The transfer of information between $\mathbb{T}$ and $\mathbb{T}'$ takes place by expressing topos-theoretic *invariants* (that is, properties or constructions on toposes which are stable under categorical equivalences) defined on their common classifying topos directly in terms of the theories $\mathbb{T}$ and $\mathbb{T}'$. This is done by associating to each of the two theories a site of definition for its classifying topos (for example, the geometric syntactic site) and then considering topos-theoretic invariants on the classifying topos from the points of view of the two sites of definition. More precisely, suppose that $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are two sites of definition for the same topos, and that I is a topos-theoretic invariant. Then one can seek site characterizations for I, that is, in the case I is a property (the case of I being a 'construction' admits an analogous treatment), a logical equivalence of the kind 'the topos $\mathcal{E}$ satisfies I if and only if $(\mathcal{C}, J)$ satisfies a property $P_{(\mathcal{C},J)}$ (written in the language of the site $(\mathcal{C}, J)$)' and, similarly for $(\mathcal{D}, K)$, a logical equivalence of the kind 'the topos $\mathbf{Sh}(\mathcal{D}, K)$ satisfies I if and only if $(\mathcal{D}, K)$ satisfies a property $Q_{(\mathcal{D},K)}$':



Clearly, such characterizations immediately lead to a logical equivalence between the properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, which can thus be seen as different manifestations of a unique property, namely I, in the context of the two different sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$.

In fact, one does not necessarily need 'if-and-only-if' site characterizations in order to build 'bridges': in order to establish an implication between a property $P_{(\mathcal{C},J)}$ of a site $(\mathcal{C}, J)$ and a property $Q_{(\mathcal{D},K)}$ of another site of definition $(\mathcal{D}, K)$ of the same topos, it suffices to find an invariant I such that $P_{(\mathcal{C},J)}$ implies I on $\mathbf{Sh}(\mathcal{C}, J)$ and I on $\mathbf{Sh}(\mathcal{D}, K)$ implies $Q_{(\mathcal{D},K)}$.

The 'bridge' technique allows to interpret and study many dualities and equivalences arising in different fields of mathematics by means of the investigation of the characterization of topos-theoretic invariants in terms of sites. In other words, the representation theory of Grothendieck toposes becomes a sort of 'meta-theory of mathematical duality', which makes it possible to effectively compare distinct mathematical theories with each other and transfer knowledge between them. In the following sections we discuss more in detail the subject of *Morita-equivalences*, which play in our context the role of 'decks' of our 'bridges', and of *site characterizations* for topos-theoretic invariants, which constitute their 'arches'.

Incidentally, it should be noted that this method could be generalized to the case of 'bridges' whose deck is given by some kind of relationship between toposes which is not necessarily an equivalence, in the presence of properties or constructions of toposes which are invariant with respect to such a relation. Nonetheless, the advantage of focusing on Morita-equivalences is twofold; on one hand, it is convenient because, due to the fact that every property expressed in categorical language is automatically invariant with respect to categorical equivalence, we dispose of an unlimited number of invariants readily available to consider, whilst on the other, it realizes a unification of 'concrete' properties of different theories by interpreting them as different manifestations of a unique property lying at the topos-theoretic level.

8.2 *Decks of 'bridges'*: Morita-equivalences

Let us first recall from [26] the following classical definition.

Definition 8.1. Two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ are said to be *Morita-equivalent* if they have equivalent classifying toposes, equivalently if they have equivalent categories of models in every Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$, that is for each Grothendieck topos $\mathscr{E}$ there is an equivalence of categories

$$\tau_{\mathscr{E}} : \mathbb{T}\text{-mod}(\mathscr{E}) \to \mathbb{T}'\text{-mod}(\mathscr{E})$$

such that for any geometric morphism $f : \mathscr{F} \to \mathscr{E}$ the following diagram commutes (up to isomorphism):

$$\begin{array}{ccc} \mathbb{T}\text{-mod}(\mathscr{E}) & \xrightarrow{\tau_{\mathscr{E}}} & \mathbb{T}'\text{-mod}(\mathscr{E}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbb{T}\text{-mod}(\mathscr{F}) & \xrightarrow[\tau_{\mathscr{F}}]{} & \mathbb{T}'\text{-mod}(\mathscr{F}) \end{array}$$

Note that 'to be Morita-equivalent to each other' defines an *equivalence relation* of the collection of all geometric theories.

Given the level of technical sophistication of this definition, it is reasonable to wonder if Morita-equivalences naturally arise in Mathematics and, in case, if there are systematic ways for 'generating' them. The following remarks are meant to show that the answer to both questions is positive.

- If two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ have equivalent categories of models in the category $\mathbf{Set}$ then, provided that the given categorical equivalence is established by only using constructive logic (that is, by avoiding in particular the law of excluded middle and the axiom of choice) and geometric constructions (that is, by only using set-theoretic constructions which involve finite limits and small colimits, equivalently which admit a syntactic formulation involving only equalities, finite conjunctions, (possibly) infinitary disjunctions and existential quantifications), it is reasonable to expect the original equivalence to 'lift' to a Morita-equivalence between $\mathbb{T}$ and $\mathbb{T}'$. Indeed, as we saw in section 5.5, a Grothendieck topos behaves logically as a 'generalized universe of sets' in which one can perform most of the classical set-theoretic constructions with the only significant exception of arguments requiring non-constructive principles; hence we can naturally expect to be able to generalize the original equivalence between the categories of set-based models of the two theories to the case of models in arbitrary Grothendieck toposes; moreover, the fact that the constructions involved in the definition of the equivalence are geometric ensures that the above-mentioned naturality condition for Morita-equivalences is satisfied (since geometric constructions are preserved by inverse image functors of geometric morphisms). As examples of 'lifting' of naturally arising categorical equivalences to Morita-equivalences we mention the one between MV-algebras and abelian ℓ-groups with strong unit (cf. [16]) and that between abelian ℓ-groups and perfect MV-algebras (cf. [17]).

- Two cartesian (in particular, finitary algebraic) theories $\mathbb{T}$ and $\mathbb{T}'$ have equivalent categories of models in $\mathbf{Set}$ if and only if they are Morita-equivalent (or, equivalently, if and only if their cartesian syntactic categories are equivalent). Indeed, $\mathbb{T}\text{-mod}(\mathbf{Set}) \simeq \mathbb{T}'\text{-mod}(\mathbf{Set})$ if and only if f.p.$\mathbb{T}$-mod$(\mathbf{Set}) \simeq$ f.p.$\mathbb{T}'$-mod$(\mathbf{Set})$, if and only if $\mathcal{C}_{\mathbb{T}}^{\text{cart}} \simeq \mathcal{C}_{\mathbb{T}'}^{\text{cart}}$, if and only if $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$ (cf. Theorem 7.26).

- If two geometric (resp. regular, coherent) theories have equivalent geometric (resp. regular, coherent) syntactic categories (i.e., they are bi-interpretable in the sense of Definition 7.15) then they are Morita-equivalent. This follows at once from the fact that the classifying topos can be constructed as the category of sheaves on the syntactic site of the theory and the logical Grothendieck topologies on the syntactic categories are defined intrinsically in terms of the categorical structures of the syntactic categories. Anyway, as it can be naturally expected, the most interesting Morita-equivalences are not of this form.

  In particular, if two finitary first-order theories are bi-interpretable (in the sense of classical Model Theory) then their Morleyizations (in the sense of Proposition 7.1) are Morita-equivalent. In fact, Proposition 7.2(ii) immediately implies that the syntactic Boolean pretoposes of two classical first-order theories are equivalent if and only if their Morleyizations are Morita-equivalent (i.e. their classifying toposes are equivalent).

- Two associative rings with unit are Morita-equivalent (in the classical, ring-theoretic, sense) if and only if the algebraic theories axiomatizing the (left) modules over them are Morita-equivalent (in the topos-theoretic sense). In fact, two rings are Morita-equivalent if and only if the cartesian syntactic categories of these theories are equivalent. Indeed, by the first remark above, these theories are Morita-equivalent if and only if their categories of set-based models are equivalent, that is if and only if the categories of (left) modules over the two rings are equivalent. Specifically, for each ring R the theory axiomatizing its (left) R-modules can be defined as the theory obtained from the algebraic theory of abelian groups by adding one unary function symbol for each element of the ring and writing down the obvious equational axioms which express the conditions in the definition of R-module.

- Other notions of Morita-equivalence for various kinds of algebraic or geometric structures considered in the literature can be reformulated as equivalences between different representations of the same topos, and hence as Morita-equivalences between different geometric theories. For instance:
  - Two topological groups are Morita-equivalent (in the sense of [31]) if and only if the toposes of continuous actions over them are equivalent. A natural analogue of this notion for topological and localic groupoids has been studied by several authors and a summary of the main results in contained in section C5.3 of [26]).
  - Two small categories are Morita-equivalent (in the sense of [20]) if and only if the corresponding presheaf toposes are equivalent, that is if and only if their Cauchy-completions (also called Karoubian completions) are equivalent.
  - Two inverse semigroups are Morita-equivalent (in the sense of [32] or, equivalently, of [21]) if and only if their classifying toposes (as defined in [22]) are equivalent (cf. [21]).
- Categorical dualities or equivalences between 'concrete' categories can often be seen as arising from the process of 'functorializing' Morita-equivalences which express structural relationships between each pair of objects corresponding to each other under the given duality or equivalence (cf. for example [10], [13] and [14]). In fact, the theory of geometric morphisms of toposes provides various natural ways of 'functorializing' bunches of Morita-equivalences.
- Different sites of definition for a given topos can be interpreted logically as Morita-equivalences between different theories (cf. Theorem 7.11); in fact, the converse also holds, in the sense that any Morita-equivalence gives canonically rise to two different sites of definition of the common classifying topos. The representation theory of Grothendieck toposes in terms of sites and, more generally, any technique that one may employ for obtaining a different site of definition or representation for a given topos thus constitutes an effective tool for generating Morita-equivalences (see, for instance, Theorem 3.7).
- The usual notions of *spectra* for mathematical structures can be naturally interpreted in terms of classifying toposes, and the resulting sheaf representations as arising from Morita-equivalences between an 'algebraic' and a 'topological' representation of such toposes. More specifically, Cole's general theory of spectra [18] (cf. also section 6.5 of [25] for a succinct overview of this theory) essentially amounts to establishing, given two geometric theories $\mathbb{T}$ and $\mathbb{S}$ such that $\mathbb{S}$ is a quotient of $\mathbb{T}$, and a class $\mathbb{A}$ of 'admissible $\mathbb{T}$-model homomorphisms' (relative to $\mathbb{S}$), the existence of a Grothendieck topos classifying the '$\mathbb{T}$-quotients' of a given $\mathbb{S}$-model M in a Grothendieck topos $\mathcal{E}$ (relatively to $\mathbb{A}$). Coste introduced in [19] alternative representations of such classifying toposes, identifying in particular simple sets of conditions under which they can be represented as toposes of sheaves on a topological space. He then derived from the equivalence between two of these representations, one of essentially algebraic nature and the

other of topological nature, a criterion for the canonical homomorphism from the given structure to the global sections of the associated structure sheaf to be an isomorphism.

- The notion of Morita-equivalences materializes in many situations the intuitive feeling of 'looking at the same thing in different ways', meaning, for instance, describing the same structure(s) in different languages or constructing a given object in different ways. Concrete examples of this general remark can be found for instance in [10] and [14], where the different constructions of the Zariski spectrum of a ring, of the Gelfand spectrum of a C^*-algebra, and of the Stone-Cech compactification of a topological space are interpreted as Morita-equivalences between different theories.

- Different ways of looking at a given mathematical theory can often be formalized as Morita-equivalences. Indeed, different ways of describing the structures axiomatized by a given theory can often give rise to a theory written in a different language whose models (in any Grothendieck topos) can be identified, in a natural way, with those of the former theory and which is therefore Morita-equivalent to it.

- A geometric theory *alone* generates an infinite number of Morita-equivalences, via its 'internal dynamics'. In fact, any way of looking at a geometric theory as an extension of a geometric theory written in its signature provides a different representation of its classifying topos, as a subtopos of the classifying topos of the latter theory (cf. the Duality Theorem of [6]).

- As we already remarked in 3.2, different separating sets for a given topos give rise to different sites of definition for it; indeed, for any separating set of objects $\mathcal{C}$ of a Grothendieck topos , we have an equivalence $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J^{\text{can}}|_{\mathcal{C}})$, where $J^{\text{can}}|_{\mathcal{C}}$ is the Grothendieck topology on $\mathcal{C}$ induced by the canonical topology on $\mathcal{E}$. In particular, for any topological space X and any basis $\mathcal{B}$ for it, we have an equivalence $\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathcal{B}, J^{\text{can}}|_{\mathcal{B}})$ (cf. [10] and [14] for examples of dualities arising from Morita-equivalences of this form when the topology can be characterized intrinsically in terms of an invariant on the topos).

8.3 *Arches of 'bridges'*: Site characterizations

As we remarked above, the 'arches' of topos-theoretic 'bridges' should be provided by site characterizations for topos-theoretic invariants, that is results connecting invariant properties (resp. constructions) on toposes and properties (resp. constructions) of their sites of definition (written in their respective languages).

It thus becomes crucial to investigate the behaviour of topos-theoretic invariants with respect to sites. As a matter of fact, such behaviour is often very natural, in the sense that topos-theoretic invariants generally admit natural site characterizations.

For instance, bijective characterizations for a wide class of geometric invariants of toposes, notably including the property of a topos to be atomic (resp. locally connected, localic, equivalent to a presheaf topos, compact, two-valued) were obtained in [11].

Moreover, it was shown in [15] that a wide class of logically-inspired invariants of topos, obtained by interpreting first-order formulae written in the language of Heyting algebras, admit elementary 'if and only if' site characterizations.

Also, we shall see in the book that several notable invariants of subtoposes admit natural site chacterizations as well as explicit logical descriptions in terms of the theories classified by the relevant toposes.

Topos-theoretic invariants relevant in Algebraic Geometry and Homotopy Theory, such as for example the cohomology and homotopy groups of toposes, also admit, at least in many important cases, natural characterizations in terms of sites.

It should be noted that, whilst it is often possible to obtain, by using topos-theoretic methods, site characterizations for topos-theoretic invariants holding for large classes of sites, such criteria can be highly non-trivial as far as their mathematical depth is concerned (since the representation theory of toposes is by all means a non-trivial subject). Therefore, when combined with specific Morita-equivalences to form 'bridges', they can lead to deep results on the relevant theories, especially when the given Morita-equivalence is a non-trivial one. These insights can actually be quite surprising, when observed from a concrete point of view (that is, from the point of view of the two theories related by the Morita-equivalence), since a given topos-theoretic invariant may manifest itself in very different ways in the context of different sites. For example, as shown in [4], for any site $(\mathcal{C}, J)$ the topos $\mathbf{Sh}(\mathcal{C}, J)$ satisfies the invariant property to be De Morgan if and only if for any object c of the category $\mathcal{C}$ and any J-closed sieve R on c the sieve

$$\{f : d \to c \mid (f^*(R) = R_d) \text{ or (for any } g : e \to d, g^*(f^*(R)) = R_e \Rightarrow g \in R_d\}$$

is J-covering, where $R_c := \{f : d \to c \mid \emptyset \in J(d)\}$ (for any c).

Such property specializes, on a presheaf topos, to the condition on the underlying category to satisfy the right Ore condition, while on a topos of sheaves on a topological space it specializes to the property of the space to be extremally disconnected. We shall see in the book many other examples enlightening the natural behaviour of topos-theoretic invariants with respect to sites.

The 'centrality' of topos-theoretic invariants in mathematics is well-illustrated by the fact that, in spite of their apparent remoteness from the more 'concrete' objects of study in mathematics, once translated at the level of sites or theories, they often specialize to construction of natural mathematical or logical interest. Besides homotopy and cohomology groups of toposes, whose 'concrete' instantiations in the context of topological spaces and schemes have been of central importance in Topology and Algebraic Geometry since decades ago, a great deal of other invariants, including those which might seem at first sight to be too abstract to be connected to any problem of natural mathematical interest, can be profitably used to shed light on classical theories. For example, even an abstract logically-inspired construction such as the *DeMorganization* of a topos, introduced in [4] as the largest dense subtopos of a given topos satisfying De Morgan's law, was shown in [5] to yield, when applied to a specific topos such as the classifying topos of the coherent theory of fields, the classifying topos of a very natural mathematical theory, namely the theory of fields of finite characteristic which are algebraic over their prime fields. The author's papers contain several other examples, some of which will be presented in the later chapters of the book.

It should be noted that the arches of our 'bridges' need not necessarily be 'symmetric', that is arising from the instantiation in the context of two given sites of a unique site characterization holding for both of them. As an example, take the property of a topos to be coherent: this property does not admit an 'elementary' 'if and only if' site characterization holding for all sites, but it admits such a site characterization holding for all trivial sites (i.e. sites in which the Grothendieck topology is trivial) and an implicative characterization of the kind 'if a theory is coherent, then its classifying topos is coherent'. These characterizations can for instance be combined together to obtain a 'bridge' yielding a result on coherent theories classified by a presheaf topos.

The level of mathematical 'depth' of the results obtained by applying the 'bridge' technique can vary enormously, depending on the complexity of the site characterizations and of the given Morita-equivalence. Still, as we shall see at various points of the book, even very simple invariants applied to easily established Morita-equivalences can yield surprising insights which would be hardly attainable, or not even imaginable, otherwise.

Lastly, it is worth noting that sites are by no means the only mathematical objects that one can use for representing toposes. For instance, Grothendieck toposes can be represented by using groupoids (either topological or localic) or quantales. It clearly makes sense to apply the 'bridge' technique, described above for sites, also in the case of these representations, replacing the site characterizations for the given invariant with appropriate characterizations of it in terms of the mathematical objects used for representing the topos.

8.4 Some simple examples

In this section we shall derive, as applications of the technique 'toposes as bridges', some logical results which settle questions which we had left unanswered in the previous sections of the chapter.

Recall that for any fragment of geometric logic, there is a corresponding notion of provability for theories in that fragment; for example, we have a notion of provability of regular sequents in regular logic and a notion of provability of coherent sequents in coherent logic. A natural question is whether these notions of provability are compatible with each other, that is if the notion of provability in a given fragment of logic specializes to the notions of provability in a smaller fragment when applied to sequents lying the smaller fragment (note that any theory in a given fragment can always be considered as a theory in a larger fragment). Similar questions can be posed concerning the notion of bi-interpretability and of set-based completeness of a theory in a given fragment of logic.

There is a natural topos-theoretic way of dealing with all these questions, which exploits the fact that for theories in a given fragment of geometric logic one has multiple syntactic representations of their classifying topos, each corresponding to a particular larger fragment of logic in which the theory can be considered. For example, given a coherent theory $\mathbb{T}$ over a signature Σ, we have two syntactic representations of its classifying topos, as the category of sheaves $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ on the coherent syntactic site $(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ of $\mathbb{T}$ and as the category of sheaves $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$ on the geometric syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$ of $\mathbb{T}$. Transferring appropriate invariants across these different representations will yield the desired results:

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$$

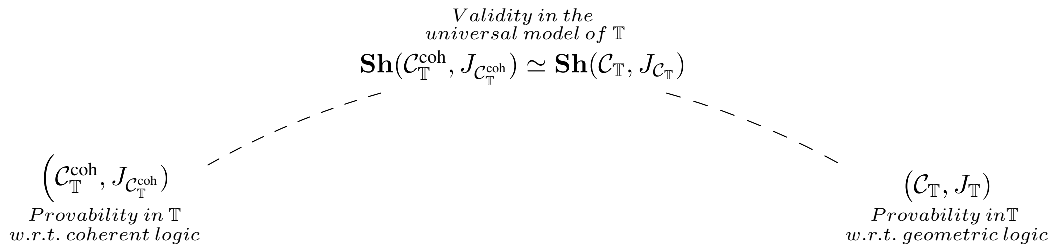
$$(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}}) \qquad\qquad\qquad (\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$$

We shall address this questions one by one, enlightening in each case the relevant topos-theoretic invariants and site characterizations.

Below, by a *fragment* L of geometric logic we mean either cartesian, regular, coherent or geometric logic. Clearly, each of this fragments is contained in the successive in the list. By a sequent in cartesian (resp. regular, coherent, geometric) logic we mean a cartesian (resp. regular, coherent, geometric) sequent.

Theorem 8.2. *Let $\mathbb{T}$ be a theory in a fragment L of geometric logic and σ a sequent in L over its signature. Then σ is provable in $\mathbb{T}$, regarded as a theory in L, if and only if it is provable in $\mathbb{T}$, regarded as a theory in any larger fragment of geometric logic containing L.*

Proof. We shall give the proof of the theorem in the case $\mathbb{T}$ is a coherent theory, regarded both in coherent logic and in geometric logic. The other cases can be proved in a completely analogous way.



From the fact that the two Yoneda embeddings $y^{\text{coh}} : \mathcal{C}_{\mathbb{T}}^{\text{coh}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ and $y : \mathcal{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$ are conservative, it follows that a coherent sequent over Σ is valid in the universal model of $\mathbb{T}$ lying in the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ if and only if it is valid in the universal model $M_{\mathbb{T}}^{\text{coh}}$ of $\mathbb{T}$ lying in its coherent syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$, i.e. if and only if it is provable in $\mathbb{T}$ by using coherent logic (cf. Theorem 6.5). Similarly, one obtains that a geometric sequent over Σ is provable in the universal model of $\mathbb{T}$ lying in the topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$ if and only if it is provable in $\mathbb{T}$ by using geometric logic. But the property of validity of a given sequent in the universal model of a geometric theory is a topos-theoretic invariant, whence a coherent sequent is provable in $\mathbb{T}$ by using coherent logic if and only if it is provable in $\mathbb{T}$ by using geometric logic, as required. □

Let us now turn to the topic of 'classical completeness' of theories in fragements of geometric logic. In section 6.5 we introduced the property, for a theory considered within a given fragment of geometric logic, of having enough models (in $\mathbf{Set}$). It is natural to wonder if this property depends on the fragment of logic in which the theory is considered. We shall now see, by applying again the 'bridge technique', that this is not the case.

Theorem 8.3. *Let $\mathbb{T}$ be a theory in a given fragment L of geometric logic. Then $\mathbb{T}$ has enough models (in $\mathbf{Set}$), regarded as a theory in L, if and only if it has enough models (in $\mathbf{Set}$), regarded as a theory in any larger fragment of geometric logic containing L.*

Proof. As for the previous result, we shall give the proof of the theorem in the case $\mathbb{T}$ is a coherent theory, regarded both in coherent logic and in geometric logic, the other cases being completely analogous.

The result naturally arises from the transfer of an invariant, namely the property of a topos to have enough points, across the two different representations $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$ of the classifying topos of $\mathbb{T}$. Indeed, in terms of the site $(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ the invariant rephrases as the condition that every coherent sequent over Σ which is satisfied in all the $\mathbf{Set}$-based models of $\mathbb{T}$ should be provable in $\mathbb{T}$ by using coherent logic, while in terms of the site $(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$ it rephrases as the condition that every geometric sequent over Σ which is satisfied in all the $\mathbf{Set}$-based models of $\mathbb{T}$ should be provable in $\mathbb{T}$ by using geometric logic:

Property of having enough points

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathcal{C}_{\mathbb{T}}})$$

$$\left(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}}\right) \qquad\qquad (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Classical completeness w.r.t. coherent logic — *Classical completeness w.r.t. geometric logic*

The above-mentioned site characterizations for the property of the classifying topos to have enough points can be proved as follows. By definition, $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ has enough points if and only if the inverse image functors f^* of the geometric morphisms $f : \mathbf{Set} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ are jointly conservative. Now, since the geometric morphism $f_M : \mathbf{Set} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ corresponding to a $\mathbb{T}$-model M in $\mathbf{Set}$ satisfies $f^*(y^{\text{coh}}(M_{\mathbb{T}}^{\text{coh}})) = M$ (where $M_{\mathbb{T}}^{\text{coh}}$ is the universal model of $\mathbb{T}$ lying in the coherent syntactic category $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ of $\mathbb{T}$ and $y^{\text{coh}} : \mathcal{C}_{\mathbb{T}}^{\text{coh}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathcal{C}_{\mathbb{T}}^{\text{coh}}})$ is the Yoneda embedding) then, by Theorem 6.5 and the fact that y^{coh} is conservative, any coherent sequent over Σ which is satisfied in every $\mathbb{T}$-model M in $\mathbf{Set}$ is satisfied in $M_{\mathbb{T}}^{\text{coh}}$, equivalently is provable in $\mathbb{T}$. A similar characterization can be obtained for the geometric syntactic site for $\mathbb{T}$. □

Finally, let us discuss the independence of the notion of bi-interpretability of theories from the fragment of logic in which they are considered.

Theorem 8.4. *Let $\mathbb{T}$ and $\mathbb{S}$ be two theories in a fragment L of geometric logic. Then $\mathbb{T}$ and $\mathbb{S}$ are bi-interpretable within the fragment L if and only if they are bi-interpretable within any fragment of geometric logic containing L.*

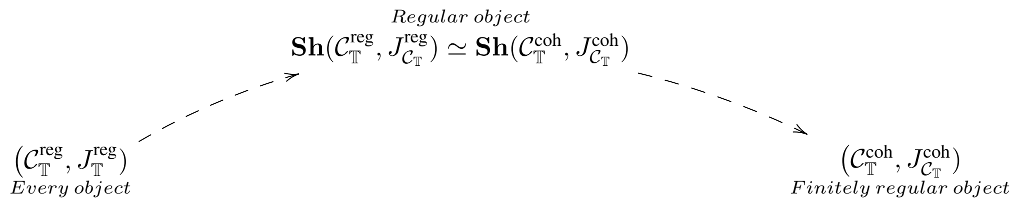
Proof. We shall first give the proof of the theorem in the case of two regular theories $\mathbb{T}$ and $\mathbb{S}$, considered both in regular logic and in coherent logic, and then discuss how to modify the proof in the other cases.

The 'only if' direction follows immediately from the universal properties of syntactic regular and coherent categories. Indeed, by Theorem 6.10 any equivalence $\mathcal{C}_{\mathbb{T}}^{\text{reg}} \simeq \mathcal{C}_{\mathbb{S}}^{\text{reg}}$ induces an equivalence between the categories of models of $\mathbb{T}$ and $\mathbb{S}$ in any regular, and hence in particular in any coherent, category $\mathcal{D}$, naturally in $\mathcal{D}$. The 2-dimensional Yoneda Lemma thus yields an equivalence $\mathcal{C}_{\mathbb{T}}^{\text{coh}} \simeq \mathcal{C}_{\mathbb{S}}^{\text{coh}}$, as desired.

The 'if' direction can be proved by using the 'bridge technique', as follows.

In order to conclude that $\mathcal{C}_{\mathbb{T}}^{\text{reg}} \simeq \mathcal{C}_{\mathbb{S}}^{\text{reg}}$ starting from $\mathcal{C}_{\mathbb{T}}^{\text{coh}} \simeq \mathcal{C}_{\mathbb{S}}^{\text{coh}}$ it will be sufficient to characterize $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$ as a full subcategory of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ in invariant terms not depending on $\mathbb{T}$. Since both these categories embed fully in the classifying topos of $\mathbb{T}$ (as the topologies $J_{\mathbb{T}}^{\text{reg}}$ and $J_{\mathcal{C}_{\mathbb{T}}}^{\text{coh}}$ are subcanonical) it would be enough to characterize in topos-theoretic invariants terms the objects of the classifying topos for $\mathbb{T}$ which come from the regular site of $\mathbb{T}$. As we remarked in section 7.3, we cannot expect to be able to distinguish through an invariant property of the classifying topos the objects which come from the regular site from those which lie in its effective completion; nonetheless, we can identify an invariant property enjoyed by the objects of the classifying topos coming from the regular site of the theory, namely regularity. Indeed, the property of the site $(\mathcal{C}_{\mathbb{T}}^{\text{reg}}, J_{\mathbb{T}}^{\text{reg}})$ to be regular easily implies that the image under the Yoneda embedding $y^{\text{reg}} : \mathcal{C}_{\mathbb{T}}^{\text{reg}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{reg}}, J_{\mathbb{T}}^{\text{reg}})$ of any object of $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$ is a regular object of $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{reg}}, J_{\mathbb{T}}^{\text{reg}})$.

On the other hand, one can easily characterize in terms of the coherent syntactic site of $\mathbb{T}$ the property of an object $\{\vec{x} . \phi\}$ of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ to be sent by the Yoneda embedding $y^{\text{coh}} : \mathcal{C}_{\mathbb{T}}^{\text{coh}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathbb{T}}^{\text{coh}})$ to a regular object of $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathbb{T}}^{\text{coh}})$: $y^{\text{coh}}(\{\vec{x} . \phi\})$ is regular in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\text{coh}}, J_{\mathbb{T}}^{\text{coh}})$ if and only if for any finite family $\{\psi_i(\vec{x}) \mid i \in I\}$ of coherent formulae in the same context, $\phi \vdash_{\vec{x}} \bigvee_{i \in I} \psi_i$ provable in $\mathbb{T}$ implies $\phi \vdash_{\vec{x}} \psi_i$ for some $i \in I$. Let us refer to this property of objects $\{\vec{x} . \phi\}$ of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ as to 'finite regularity'.



Now, it can easily be proved by induction on the structure of coherent formulae that every coherent formula is provably equivalent in coherent logic to a finite disjunction of regular formulae in the same context. This clearly implies that every finitely regular coherent formula $\{\vec{x} . \phi\}$ is isomorphic in $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ to a regular formula. We can thus conclude that the full subcategory of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ on the finitely regular formulae is equivalent to the category $\mathcal{C}_{\mathbb{T}}^{\text{reg}}$, and similarly for the theory $\mathbb{S}$. It follows that the equivalence $\mathcal{C}_{\mathbb{T}}^{\text{coh}} \simeq \mathcal{C}_{\mathbb{S}}^{\text{coh}}$ restricts to an equivalence $\mathcal{C}_{\mathbb{T}}^{\text{reg}} \simeq \mathcal{C}_{\mathbb{S}}^{\text{reg}}$, as desired.

In the case of a cartesian theory considered in a larger fragment of geometric logic, the notion to use in place of that of regular object is that of irreducible object (the only difference in the proof being that in order to show that every formula which is sent by the Yoneda embedding to an irreducible object of the classifying topos is isomorphic, in the relevant syntactic category, to a cartesian formula, one simply uses Proposition 7.17).

In the case of coherent theories, the notion to use is instead that of compact object. □

8.5 Final remarks: a theory of 'structural translations'

The view underlying the methodology 'toposes as bridges' described above consists in regarding a topos as an object which, together with all its different representations, embodies a great amount of relationships existing between the different theories classified by it. Any topos-theoretic invariant behaves like a 'pair of glasses' which allows to discern certain information which is 'hidden' in a given Morita-equivalence. Toposes can thus act as 'universal translators' across different mathematical theories which share the same classifying topos.

From a technical point of view, the main reason for the effectiveness of the 'bridge' technique is two-fold: on one hand, as we have argued in section 8.3, topos-theoretic invariants usually manifest themselves in significantly different ways in the context of different sites; on the other, due to the very well-behaved nature of the representation theory of Grothendieck toposes in terms of sites, the site characterizations formally expressing such relationships are essentially canonical and can often be derived by means of rather mechanical 'calculations'.

Unlike the traditional, 'dictionary-oriented' method of translation based on a 'renaming', according to a given 'dictionary', of the primitive constituents of the information across the two different languages, the 'invariant-oriented' translations realized by topos-theoretic 'bridges' consist in 'structural unravelings' of appropriate invariants across different representations of the toposes involved, rather than through the use of an explicit description of the Morita-equivalence serving as 'dictionary'. In fact, for the transfer of 'global' properties of toposes, it is only the *existence* of a Morita-equivalence that really matters, and we can well ignore its explicit description, since, by its very definition, a topos-theoretic invariant is stable under any kind of categorical equivalence. If one wants to establish more 'specific' results, one can use invariant properties of objects of toposes rather than properties of the 'whole topos', in which case an explicit description of the Morita-equivalence is of course needed, but for investigating most of the 'global' properties of theories this is not at all necessary.

We have already hinted above to the fact that there is an strong element of *automatism* implicit in the 'bridge' technique. In fact, in order to obtain insights on the Morita-equivalence under consideration, in many cases one can just readily apply to it general characterizations connecting properties of sites and topos-theoretic invariants. Still, the results generated in this way are in general non-trivial; in some cases they can be rather 'weird' according to the usual mathematical standards (although they might still be quite deep) but, with a careful choice of Morita-equivalences and invariants, one can easily get interesting and natural mathematical results. In fact, a lot of information that is not visible with the usual 'glasses' is revealed by the application of this machinery.

The range of applicability of the 'bridge' technique is very broad within mathematics, by the very generality of the notion of topos (and of that of geometric theory). Through this method, results are generated transversally to the various mathematical fields, in a 'uniform' way which is determined by the form of the toposes involved and the invariants considered on them. Notice that this way of doing mathematics is inherently 'upside-down': instead of starting with simple ingredients and combining them to build more complicated structures, one assumes as primitive ingredients rich and sophisticated (meta-)mathematical entities, namely Morita-equivalences and topos-theoretic invariants, and proceeds to extracting from them 'concrete' information relevant for classical mathematics.

Acknowledgements: I am grateful to Alain Connes, Vincent Lafforgue, Noson Yanofsky and Anna Carla Russo for useful comments on a preliminary version of this text.

References

[1] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Séminaire de Géométrie Algébrique du Bois-Marie, année 1963-64; second edition published as Lecture Notes in Math., vols. 269, 270 and 305, Springer-Verlag (1972).

[2] A. Blass and A. Ščedrov, Classifying topoi and finite forcing, *Journal of Pure and Applied Algebra* 28 (2), 111-140 (1983).

[3] C. Butz and I. Moerdijk, Representing topoi by topological groupoids, *Journal of Pure and Applied Algebra*, 130 (3), 223-235 (1998).

[4] O. Caramello, De Morgan classifying toposes, *Advances in Mathematics* 222 (6), 2117-2144 (2009).

[5] O. Caramello and P.T. Johnstone, De Morgan's law and the theory of fields, *Advances in Mathematics* 222 (6), 2145-2152 (2009).

[6] O. Caramello, Lattices of theories, *arXiv:math.CT/0905.0299* (2009), 84 pages

[7] O. Caramello, Universal models and definability, *Mathematical Proceedings of the Cambridge Philosophical Society* 152 (2) 279-302 (2012).

[8] O. Caramello, Syntactic characterizations of properties of classifying toposes, *Theory and Applications of Categories* 26 (6), 176-193 (2012).

[9] O. Caramello, The unification of Mathematics via Topos Theory, *arXiv:math.CT/1006.3930*.

[10] O. Caramello, A topos-theoretic approach to Stone-type dualities, *arXiv:math.CT/1006.3930* (2011), 158 pages.

[11] O. Caramello, Site characterizations for geometric invariants of toposes, *Theory and Applications of Categories* Vol. 26, No. 225, pp 710-728 (2012).

[12] O. Caramello, A general method for building reflections, arXiv:math.CT/1112.3603 (2011), 23 pages, to appear in *Applied Categorical Structures*.

[13] O. Caramello, Priestley-type dualities for partially ordered structures, *arXiv:math.CT/1203.2800* (2012), 44 pages.

[14] O. Caramello, Gelfand spectra and Wallman compactifications, *arxiv:math.CT/1204.3244* (2012), 50 pages.

[15] O. Caramello, Topologies for intermediate logics, *arxiv:math.CT/1205.2547* (2012), 21 pages, revised version to appear in *Mathematical Logic Quarterly*.

[16] O. Caramello and A. C. Russo, The Morita-equivalence between MV-algebras and abelian ℓ-groups with strong unit, *arxiv:math.CT/1312.1272* (2013), 27 pages, revised version to appear in the *Journal of Algebra*.

[17] O. Caramello and A. C. Russo, A topos-theoretic perspective on the equivalence between abelian ℓ-groups and perfect MV-algebras, to shortly appear on the Mathematics ArXiv.

[18] J. Cole, The bicategory of topoi, and spectra, unpublished text available at the address
`http://www.oliviacaramello.com/Unification/ColeBicategoryTopoiSpectra.pdf`

[19] M. Coste, Localisation, spectra and sheaf representation, in *Applications of Sheaves*, Lecture Notes in Mathematics 753, 212-238 (1979).

[20] B. Elkins and J.A. Zilber, Categories of actions and Morita equivalence, *Rocky Mountain J. Math.* 6 (2), 199-226 (1976).

[21] J. Funk, M. V. Lawson and B. Steinberg, Characterizations of Morita equivalent inverse semigroups, *Journal of Pure and Applied Algebra* 215 (9) 2262-2279 (2011).

[22] J. Funk and P. Hofstra, Topos theoretic aspects of semigroup actions, *Theory and Applications of Categories*, 24 (6), 117-147 (2010).

[23] R. Goldblatt, *Topoi, the categorial analysis of logic*, Elsevier Science Publishers B. V. (1984).

[24] M. Hakim, *Topos annelés et schémas relatifs*, Springer-Verlag (1972).

[25] P. T. Johnstone, *Topos theory*, Academic Press (1977).

[26] P. T. Johnstone, *Sketches of an Elephant: a topos theory compendium. Vols. 1 and 2*, Oxford University Press (2002).

[27] A. Joyal, M. Tierney, An extension of the Galois theory of Grothendieck, *Mem. Amer. Math. Soc.* 51, no. 309, vii + 71 pp. (1984).

[28] W. Lawvere, Functorial Semantics of Algebraic Theories, 1963, republished in *Reprints in Theory and Applications of Categories* 5, 1-121 (2004).

[29] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic: a first introduction to topos theory*, Springer-Verlag (1992).

[30] M. Makkai and G. Reyes, *First-order categorical logic*, Lecture Notes in Mathematics (Springer-Verlag) vol. 611 (1977).

[31] I. Moerdijk, Morita equivalence for continuous groups, *Math. Proc. Cam. Phil. Soc.*, 103 (1), 97-115 (1988).

[32] S. Talwar, Morita equivalence for semigroups, *J. Austral. Math. Soc. (Series A)* 59, 81-111 (1995).