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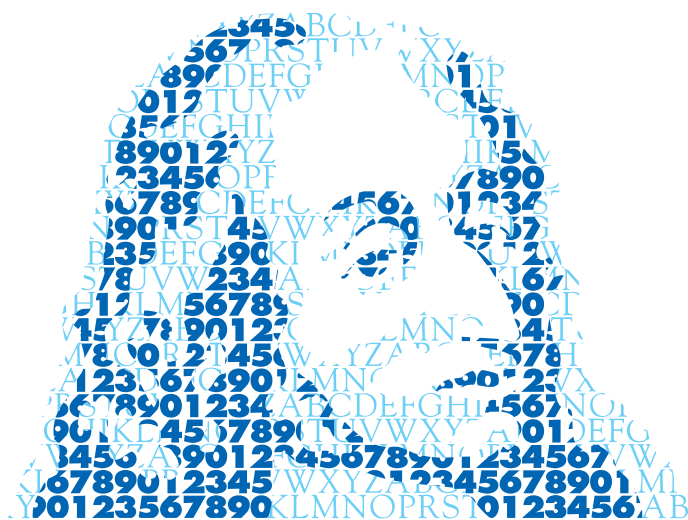

BLAISE PASCAL

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A generalization of Pascal's triangle using powers of base numbers

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Abstract

In this paper we generalize the Pascal triangle and examine the connections among the generalized triangles and powering integers respectively polynomials. We emphasize the relationship between the new triangles and the Pascal pyramids, moreover we present connections with the binomial and multinomial theorems.

1. Introduction

The interesting and really romantic Pascal triangle has been a favorite research field for mathematicians for a very long time. The table of binomial coefficients has been named after Blaise Pascal, a French scientist and writer. Although the triangle had been known as early as in ancient India, and later in Persia, China and Europe in the Middle Ages by a number of scientists before Pascal, he generalized known results and gave a number of new properties, which he formulated in nineteen theorems ([3]).

The Pascal triangle has a number of generalizations.

We can construct the generalized Pascal triangles of sth order (or kind s, sometimes referred to as the s-arithmetical triangles), from the generalized binomial coefficients of order s. This idea was first published in 1956 by John E. Freund ([5]). These triangles have been intensively investigated in the last decades, we cite some important properties in Sections 4 and 5.

Other interesting planar generalizations are the Lucas, Fibonacci, Catalan and other arithmetical triangles, however these constructions diverge from our topic.

The so-called Pascal pyramid is constructed from trinomial coefficients, which occur in the expansions $(x + y + z)^n$ (first mentioned by E. B. Rosenthal, 1960, [3]). Each of the outer faces of the pyramid are Pascal triangles. We can extend this idea to the multi-dimensional case (with $dim \geq 4$), so Pascal hyperpyramids can be constructed from multinomial coefficients ([3]).

In this paper we present another type of generalization. This idea is based on the well-known fact (see e.g. the paper of Robert L. Morton from 1964, [8]) that from the nth row of the Pascal triangle with *positional* addition we get the nth power of 11 (Figure 1), where n is a non-negative integer, and the indices in the rows and columns run from 0.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$\cdots$$
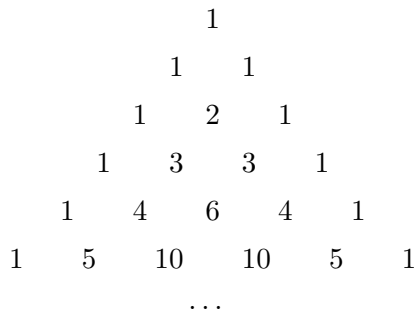
$$1 = 11^0, 11 = 11^1, 121 = 11^2, 1331 = 11^3, 14641 = 11^4, 161051 = 11^5, \ldots$$

FIGURE 1. Powers of 11 in the Pascal triangle

This comes directly from the binomial equality

$$\binom{n}{0} 10^n + \binom{n}{1} 10^{n-1} + \binom{n}{2} 10^{n-2} + \cdots + \binom{n}{n-1} 10^1 + \binom{n}{n} 10^0 = 11^n.$$

$$(1.1)$$

2. ab-based triangles

Let us construct triangles, in which the powers of other numbers appear. To achieve this, let us consider the Pascal triangle as the 11-based triangle, and take the following definition.

Definition 2.1. Let a and b be integers, with $0 \leq a, b \leq 9$. We get the kth element in the nth row of the 'ab'-based triangle if we add b-times the $k-1^{\text{th}}$ element in the $n-1^{\text{th}}$ row to a-times the kth element in the $n-1^{\text{th}}$ row. If $k-1 < 0$ or $k > n$ (i.e., the element in the $n-1^{\text{th}}$ row does not exist according to the normal implementation) then we consider this element to be 0 (Figure 2). The indices in the rows and columns run from 0.
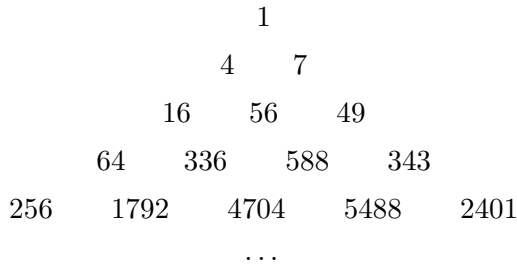
$$1$$
$$4 \qquad 7$$
$$16 \qquad 56 \qquad 49$$
$$64 \qquad 336 \qquad 588 \qquad 343$$
$$256 \qquad 1792 \qquad 4704 \qquad 5488 \qquad 2401$$
$$\cdots$$

FIGURE 2. The 47-based triangle

Remark 2.2. Although the construction of triangles of coefficients in expansions of $(a + bx)^n$ – i.e. our ab-based triangles – has already been mentioned by few other authors in the last years (see e.g. [4] and [9]), systematic analysis as here was published first only in [6] and [7]. The same is true for the more general case as triangles of coefficients in expansions of $(a + bx + cx^2)^n$, see Definition 3.1.

Let us introduce the notation $E_{n,k}^{ab}$ for the kth element in the nth row of the ab-based triangle. Thus, our definition takes the form $E_{n,k}^{ab} = bE_{n-1,k-1}^{ab} + aE_{n-1,k}^{ab}$ with $n \geq 1$ and $E_{0,0}^{ab} = 1$. Here $E_{n,k}^{11} = C_n^k$ is the element of the Pascal triangle.

The structure of the ab-based triangle is relatively simple and closely related to the Pascal triangle.

Proposition 2.3. *In the ab-based triangle we have $E_{n,k}^{ab} = a^{n-k}b^k C_n^k$.*

Proof. We prove by induction. In the first row the assertion is true. Let us now assume, that in the $n-1^{\text{th}}$ row $E_{n-1,k-1}^{ab} = a^{n-k}b^{k-1}C_{n-1}^{k-1}$ and

$E_{n-1,k}^{ab} = a^{n-k-1}b^k C_{n-1}^k$ hold. Then for $E_{n,k}^{ab}$ by the definition we have

$$E_{n,k}^{ab} = b \cdot a^{n-k}b^{k-1}C_{n-1}^{k-1} + a \cdot a^{n-k-1}b^k C_{n-1}^k = a^{n-k}b^k C_n^k.$$ □

As a consequence of this result, by positional addition from the nth row of the ab-based triangle we get the nth power of the base-number ab, since similarly, as in (1.1), from the expansion of ab^n we get

$$(10a + b)^n = \binom{n}{0}a^n 10^n + \binom{n}{1}a^{n-1}b10^{n-1} + \cdots$$
$$+ \binom{n}{n-1}ab^{n-1}10 + \binom{n}{n}b^n. \tag{2.1}$$

Example 2.4. In the 3rd row of the 47-based triangle $64 \cdot 10^3 + 336 \cdot 10^2 + 588 \cdot 10 + 343 = 103823 = 47^3$.

We have a nice connection with the binomial theorem.

Proposition 2.5. *The elements in the nth row of the ab-based triangle are the coefficients of the polynomial $(ax + by)^n$.*

Proof. This result follows from Proposition 2.3, if we substitute $10a$ with ax and b with by in (2.1). □

Example 2.6. From the 47-based triangle $(4x + 7y)^3 = 64x^3 + 336x^2y + 588xy^2 + 343y^3$.

It is interesting to check how the properties of the Pascal triangle are modified in the ab-based triangles. Some of them (e.g. symmetry, divisibility) are straightforward, some others will be investigated more generally in the next sections.

We focus here only on some properties, which rely on the fact that the base-number has only two digits.

An important property of the Pascal triangle is that the sum of elements in the ascending diagonals form the Fibonacci sequence. A direct consequence of the construction is that in the ab-based triangles we have Fibonacci-like recurrence sequences $G_i = bG_{i-2} + aG_{i-1}$, with $G_0 = 1$ and $G_1 = a$. (Figure 3).

An interesting connection between the Pascal pyramid and the ab-based triangles will be discussed in Section 4.
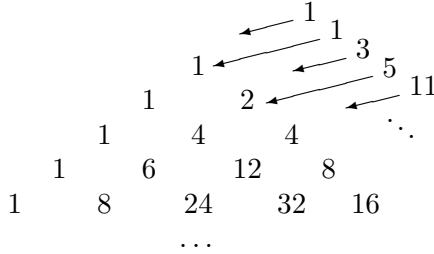
Figure 3. The recurrence sequence $G_i = 2G_{i-2} + G_{i-1}$ in the 12-based triangle

Finally, we mention that the ab-based triangles have already some practical applications. In 1996, Sven J. Cyvin, Jon Brunvoll and Bjørg N. Cyvin published several articles to solve the so-called isomer enumeration problem for polycyclic conjugated hydrocarbons and for the unbranched α-5 catapolyheptagons (see e.g. in [4]). To achieve this, they used certain triangular matrices (denoted by $A(x,y)$ with integer parameters x and y), which were defined by a recurrence relation essentially identical as in Definition 2.1. E.g., in [4], the authors used matrix $A(4,2)$ (our 42-based triangle).

3. Further generalization

The base-number of such triangles can consist of more than two digits, too.

Definition 3.1. Let $0 \leq a_0, a_1, \ldots, a_{m-1} \leq 9$ be integers. We get the kth element in the nth row of the $a_0 a_1 \ldots a_{m-1}$-based triangle if we multiply the $k - m^{\text{th}}$ element in the $n - 1^{\text{th}}$ row by a_{m-1}, the $k - m + 1^{\text{th}}$ element in the $n - 1^{\text{th}}$ row by a_{m-2}, $\ldots$, the kth element in the $n - 1^{\text{th}}$ row by a_0, and add the products. If for some index i we have $k - m + i < 0$ or $k - m + i > n(m-1)$ (i.e., an element in the $n - 1^{\text{th}}$ row does not exist) then we consider this element to be 0. The indices in the rows and columns run from 0 (Figure 4).

Using our notation introduced above, we have
$$E_{n,k}^{a_0 a_1 \ldots a_{m-1}} = a_{m-1} E_{n-1,k-m}^{a_0 \ldots a_{m-1}} + a_{m-2} E_{n-1,k-m+1}^{a_0 \ldots a_{m-1}} + \cdots$$
$$+ a_1 E_{n-1,k-1}^{a_0 \ldots a_{m-1}} + a_0 E_{n-1,k}^{a_0 \ldots a_{m-1}}.$$

5

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 2 & 5 & 7 & & \\
 & 4 & 20 & 53 & 70 & 49 & \\
8 & 60 & 234 & 545 & 819 & 735 & 343
\end{array}
$$

$\ldots$

Figure 4. The 257-based triangle

As we already mentioned, triangles with base-number $11\ldots 1$ (s pieces of 1 digits) have already been known for decades as generalized Pascal triangles of sth order ([3]). However, they were introduced using a combinatorial approach, by the generalization of the binomial coefficients ([5]). [1]

The most common such triangle is the generalized Pascal triangle of order three (our 111-based triangle), sometimes referred as trinomial triangle – because of the connection with the trinomial coefficients, see Section 4.

Similarly, as in the generalized Pascal triangles of mth order, in the $a_0 a_1 \ldots a_{m-1}$-based triangles the number of the elements in the nth row is $N_{n,m} = n(m-1) + 1$.

As by the ab-based triangles, we have the same result about the powers of the base-number.

Theorem 3.2. *From the nth row of the $a_0 a_1 \ldots a_{m-1}$-based triangle by positional addition we get the nth power of the number $a_0 a_1 \ldots a_{m-1}$.*

Proof. We prove by induction. In the first row of the triangle is obviously the first power of the number $a_0 a_1 \ldots a_{m-1}$.

Let us now assume, that in the $n-1^{\text{th}}$ row ($n > 1$) we have elements $b_0, b_1, \ldots, b_p$ with $p = (n-1)(m-1)$, and from these elements with positional addition we get the $n-1^{\text{th}}$ power of the base-number $a_0 a_1 \ldots a_{m-1}$. Let us write out $(a_0 10^{m-1} + a_1 10^{m-2} + \cdots + a_{m-2} 10 + a_{m-1})^n$ as

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[1]The generalized binomial coefficient $\binom{n}{m}_s$ is the number of different ways of distributing m objects among n cells where each cell may contain at most $s-1$ objects ([3]). This is the element in the mth column of the nth row in the generalized Pascal triangles of sth order. For $s = 2$ we get the "normal" binomial coefficients and the Pascal triangle.

$$(b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1} 10 + b_p) a_0 10^{m-1}$$
$$+ (b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1} 10 + b_p) a_1 10^{m-2}$$
$$+ \ldots$$
$$+ (b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1} 10 + b_p) a_{m-2} 10$$
$$+ (b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1} 10 + b_p) a_{m-1}.$$

By adding these expressions (using $p = mn - m - n + 1$) we get

$$a_0 b_0 10^{mn-n} + (a_0 b_1 + a_1 b_0) 10^{mn-n-1} + (a_0 b_2 + a_1 b_1 + a_2 b_0) 10^{mn-n-2} + \cdots$$
$$+ (a_{m-1} b_0 + a_{m-2} b_1 + \cdots + a_1 b_{m-2} + a_0 b_{m-1}) 10^{mn-m-n+1} + \cdots$$
$$+ (a_{m-2} b_p + a_{m-1} b_{p-1}) 10 + a_{m-1} b_p,$$

and this is exactly the number we get after positional addition from the nth row of the triangle. □

Example 3.3. In the 2nd row of the 257-based triangle $4 \cdot 10^4 + 20 \cdot 10^3 + 53 \cdot 10^2 + 70 \cdot 10 + 49 = 66049 = 257^2$.

Proposition 3.4. *The elements in the nth row of the general triangle are exactly the coefficients of the polynomial $(a_0 + a_1 x + \cdots + a_{m-1} x^{m-1})^n$, where the kth element is the coefficient of x^k.*

Proof. We prove by induction. In the first row the statement is true. Let us now assume that in the $n - 1^{\text{th}}$ row there are the coefficients of the polynomial $(a_0 + a_1 x + \cdots + a_{m-1} x^{m-1})^{n-1}$, and the kth element is the coefficient of x^k. If we multiply this polynomial by $a_0, a_1 x, \ldots, a_{m-1} x^{m-1}$, and sum the results (similarly as in the proof of Theorem 3.2), we get the nth power of the basic polynomial. But according to the forming rules of the triangle, the coefficients of this polynomial are exactly the elements of the nth row. □

Example 3.5. From the 3rd row of the 257-based triangle (Figure 4)
$$(2 + 5x + 7x^2)^3 = 8 + 60x + 234x^2 + 545x^3 + 819x^4 + 735x^5 + 343x^6.$$

These results show that we have the "right" to call the new triangles as generalized Pascal triangles, since their general properties are very similar to that of the Pascal triangle.

In Proposition 2.5 we saw a connection of the ab-based triangle with the binomial theorem. Thus, we expect the $a_0a_1 \ldots a_{m-1}$-based triangle to have a relation with the multinomial (sometimes referred as the polynomial) theorem. However the structure of the latter triangle is much more complicated. See for example the triangle with base-number abc (Figure 5). The elements in the nth row are some sums of the coefficients of the polynomials $(ax + by + cz)^n$.

$$1$$

$$a \quad\quad b \quad\quad c$$

$$a^2 \quad\quad 2ab \quad\quad 2ac + b^2 \quad\quad 2bc \quad\quad c^2$$

$$a^3 \quad 3a^2b \quad 3a^2c + 3ab^2 \quad 6abc + b^3 \quad 3ac^2 + 3b^2c \quad 3bc^2 \quad c^3$$

a^4	$4a^3b$	$4a^3c$	$12a^2bc$	$6a^2c^2$	$12abc^2$	$4ac^3$	$4bc^3$	c^4
		$+6a^2b^2$	$+4ab^3$	$+12ab^2c$	$+4b^3c$	$+6b^2c^2$		
				$+b^4$				

$$\ldots$$

Figure 5. The abc-based triangle

To discover the connection of the general triangle with the multinomial theorem we need deeper analysis.

Definition 3.6. For the digits of the base-number $a_0a_1 \ldots a_{m-1}$ let the *weight* of a digit be its distance from the centerline. So $w(a_0) = -w(a_{m-1})$, $w(a_1) = -w(a_{m-2})$, ... If the base number is odd, then $w(a_{(m-1)/2}) = 0$. Let the unit of the weights be the distance of two neighboring elements in the triangle, i.e., $w(a_i) = w(a_{i+1}) + 1$.

Example 3.7. In the abc-based triangle $w(a) = -1$, $w(b) = 0$ and $w(c) = 1$, in the $abcd$-based triangle $w(a) = -1.5$, $w(b) = -0.5$, $w(c) = 0.5$ and $w(d) = 1.5$.

We would like to extend this idea to the elements of the other rows. As the elements of the triangle are sums, consider first the parts of them. For such an expression let the weight of the part be the sum of the weights of its digits. If a digit is on the ith power, then we count its weight i-times.

Example 3.8. One part of the 2nd element in the 3rd row of the *abc*-based triangle is $3a^2c$ (Figure 5). For this expression we have $w(3a^2c) = 2w(a) + w(c) = -1$.

Lemma 3.9. *In an element of the general triangle the weights of the parts are identical, and this weight is the distance of the element from the centerline.*

Proof. This result follows by induction directly from the construction of the triangle. □

In the following let us call the identical weights of the parts the weight of the element.

Lemma 3.10. *Let us consider an expression $a_0^{i_0} a_1^{i_1} \ldots a_{m-1}^{i_{m-1}}$ for which $i_0 + i_1 + \cdots + i_{m-1} = n$. Then we can find this expression with some kind of coefficient as a part of the element with the same weight in the nth row of the general triangle.*

Proof. Let us assume indirectly that this expression does not exist in the nth row of the general triangle as a part of the element with a corresponding weight. We should get this expression from parts of elements of the previous row

$$(a_0^{i_0-1} a_1^{i_1} \ldots a_{m-1}^{i_{m-1}}, a_0^{i_0} a_1^{i_1-1} \ldots a_{m-1}^{i_{m-1}}, \ldots, a_0^{i_0} a_1^{i_1} \ldots a_{m-1}^{i_{m-1}-1})$$

with multiplication (by a_0, a_1, ..., a_{m-1}). Thus these parts of elements can't exist in the previous row. Proceeding backwards with this method we conclude that in the first line some digits of the base-number do not exist, and this is a contradiction. □

Lemma 3.11. *For the coefficient d of the expression $d a_0^{i_0} a_1^{i_1} \ldots a_{m-1}^{i_{m-1}}$ with $i_0 + i_1 + \cdots + i_{m-1} = n$ in the nth row of the general triangle we have $d = \frac{n!}{i_0! i_1! \ldots i_{m-1}!}$.*

Proof. This lemma can be proved by induction. This is left to the motivated reader. Now we omit the details. □

By these three Lemmas we proved the following result, which gives the desired connection with the multinomial theorem (4.3):

Theorem 3.12. *The elements in the nth row of the $a_0 a_1 \ldots a_{m-1}$-based triangle are exactly such sums of the coefficients of the polynomial $(a_0 x_0 + a_1 x_1 + \cdots + a_{m-1} x_{m-1})^n$ in which the weights of the parts are identical.*

Remark 3.13. With this result we presented a method to determine directly an element of the general triangle in a given position, too. However, because of the complexity of the construction, we cannot have such a nice formula for the elements as in the *ab*-based triangle.

4. **Relationship with the Pascal pyramid**

Richard C. Bollinger proved in [2] that the generalized binomial coefficients can be expressed by the means of multinomial coefficients as

$$\binom{n}{m}_s = \sum_{n_1, n_2, \ldots, n_s} \binom{n}{n_1, n_2, \ldots, n_s}, \tag{4.1}$$

where the summation is over all *s*-part compositions $n_1, n_2, \ldots, n_s$ of n such that a) $n_1 + n_2 + \cdots + n_s = n$, and b) $0n_1 + 1n_2 + \cdots + (s-1)n_s = m$, where the multinomial coefficients [2] are determined by

$$\binom{n}{n_1, n_2, \ldots, n_s} = \frac{n!}{n_1! n_2! \ldots n_s!}. \tag{4.2}$$

Equation (4.1) presents a direct connection of generalized Pascal triangles of sth order (for us now triangles with base-number $11 \ldots 1$) with the Pascal hyperpyramids and the multinomial theorem (see (4.3) with $n_1 + n_2 + \cdots + n_m = n$), similarly as our Theorem 3.12.

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1, n_2, \ldots, n_m} \frac{n!}{n_1! n_2! \ldots n_m!} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}. \tag{4.3}$$

For $s = 3$ rule (4.1) can be presented as follows: if we add the trinomial coefficients in the layers of the Pascal pyramid with identical weights, we get the elements of the 111 based triangle (see Figure 6 with $a = b = c = 1$).

   As it is already known, the Pascal pyramid can be constructed in the way that we choose base numbers in the first layer a, b, c, and so the nth layer consists of the coefficients in the expansion of $(a + b + c)^n$ (first published by G. Garcia, 1967, [3]). We will call this kind of pyramid in the following as the *general* Pascal pyramid.

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[2]The combinatorial sense of the multinomial coefficient is as follows: (4.2) gives the number of ways that n different objects may be distributed among s cells, where the number of objects in the kth cell is n_k, with $k = 1, 2, \ldots, s$ ([3]).

Now from Theorem 3.12 directly follows:

Corollary 4.1. *Adding the elements in the layers of the general pyramid in the way that the weights of the parts are identical, we get the elements of the abc based triangle.*

Example 4.2. For layer 4 this connection is presented in Figure 6.
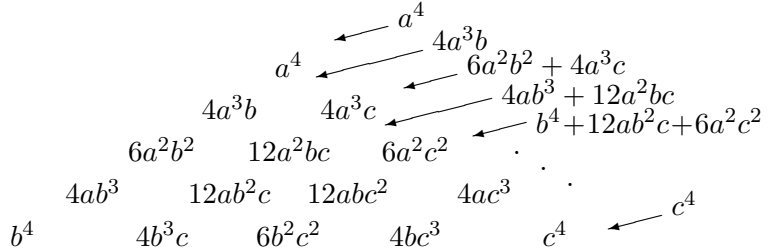

FIGURE 6. Connection between the *abc*-based triangle and the general Pascal pyramid

We can extend this result theoretically to the multi-dimensional case. Thus, from an m-dimensional Pascal hyperpiramid which consists of the coefficients in the expansions of $(a_0 + a_1 + \cdots + a_{m-1})^n$, with the same method we get the elements of the $a_0 a_1 \ldots a_{m-1}$-based triangle.

Another interesting connection can be observed between the Pascal pyramid and the *ab*-based triangles. Lior Manor mentioned in 2004 ([9]) that by adding the elements in the rows of the layers in the pyramid, we get the elements of the 12-based triangle. [3]

However, by the analysis of the change of this property to the general pyramid we can discover that something more general is true.

Theorem 4.3. *Adding the elements in the rows of the nth layer of the general pyramid, we get the elements in the nth row of the triangle with base numbers AB, where $A = a$ and $B = b + c$.*

Proof. Mary Basil presented in 1968 ([1]) a rule for constructing the elements of the nth layer of the Pascal pyramid as

$$\binom{n}{n - m_1, m_1 - m_2} = \binom{n}{m_1}\binom{m_1}{m_2}, \tag{4.4}$$

---

[3]Called by him as the triangle of coefficients in expansion of $(1 + 2x)^n$.

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with $m_1 = 0, 1, \ldots, n$ (rows) and $m_2 = 0, 1, \ldots, m_1$ (columns). Moreover ([1]), in the general pyramid the element in the nth layer with row and column index m_1 and m_2 is

$$\binom{n}{m_1}\binom{m_1}{m_2}a^{n-m_1}b^{m_1-m_2}c^{m_2} \tag{4.5}$$

We know that

$$(a+(b+c))^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}(b+c) + \binom{n}{2}a^{n-2}(b+c)^2 + \ldots + \binom{n}{n}(b+c)^n. \tag{4.6}$$

Now if we expand the powers of $(b+c)^i$ in (4.6), using (4.5) our statement follows. $\square$

Remark 4.4. Rule (4.4) says, in effect, that we get the elements of the nth layer by taking rows of the Pascal triangle until n, rotating the last row counterclockwise through the angle $90°$, and then multiplying the elements of this column by the elements of the triangle ([3]).

In Figure 7 we arranged the parts of the right hand side of (4.6) in rows and illustrated the proof of Theorem 4.3.

FIGURE 7. Illustration of the proof of Theorem 4.3

5. Some properties of the generalized triangles

As we already mentioned, a number of mathematicians examined how the properties of binomial coefficients can be extended to the generalized

binomial coefficients of order s. Similarly, it seems to be a very interesting and exciting research field, to determine more general connections among the elements in the general triangles.

One interesting property follows immediately from Theorem 3.12.

Corollary 5.1. *In the nth row of the $a_0 a_1 \ldots a_{m-1}$-based triangle the sum of the elements (with normal addition) is $(a_0 + a_1 + \cdots + a_{m-1})^n$.*

Proof. If we set in the polynomial $(a_0 x_0 + a_1 x_1 + \cdots + a_{m-1} x_{m-1})^n$ the values of x_i-s as $1 = x_0 = x_1 = \cdots = x_{m-1}$, then from Theorem 3.12 in the nth row of the triangle there are the coefficients of the "polynomial" $(a_0 + a_1 + \cdots + a_{m-1})^n$. $\qquad\square$

Remark 5.2. For the generalized Pascal triangle of order s from this corollary we get the known fact (see e.g. in [3]) that the sum in the nth row is s^n, while for $s = 2$ it follows the famous combinatorial equality

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

For an arbitrary triangle let us denote the sum of the elements in the nth row with even index by S_n^e and similarly, the sum of the elements with odd index by S_n^o.

Theorem 5.3. *In the nth row of the $a_0 a_1 \ldots a_{m-1}$-based triangle we have*

$$S_n^e - S_n^o = (a_0 - a_1 + a_2 - \cdots + (-1)^{m-1} a_{m-1})^n. \qquad (5.1)$$

For the proof of this theorem we will use the following lemma.

Lemma 5.4. *Let us assume that in the $b_0 b_1 \ldots b_{m-1}$-based triangle the digits of the base-number are of alternating sign e.g. b_0 is positive, b_1 is negative, b_2 is positive, b_3 is negative, and so on. Then this property remains true for all of the rows of the triangle, i.e. in every row is the 0th element is negative, the 1st element is positive, and so on. Moreover, the absolute value of the elements is the same, as the value of the elements in the triangle with base $|b_0||b_1| \ldots |b_{m-1}|$.*

Proof. We prove by induction. In the first row the statement is true. Let us now assume that in the $n - 1^{\text{th}}$ row the assumption is true. By the definition of the triangle, we calculate the elements of the nth row as follows: we multiply together the elements of alternating sign in the $n-1^{\text{th}}$

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row by the digits of the base number, which are of alternating sign, too. Thus, we multiply together elements with signs + and +, − and −, ..., or − and +, + and −, ..., alternately, so we have elements in the nth row with alternating sign. Not considering the signs we calculated the same way the elements in the triangle with base $|b_0||b_1|\ldots|b_{m-1}|$, and from the induction assumption the absolute value of the elements in the $n-1^{\text{th}}$ rows are the same. Thus, the absolute value of the elements in the nth rows are the same, too. □

Now we prove the theorem.

Proof. Let us apply now Corollary 5.1 for the $b_0b_1\ldots b_{m-1}$-based triangle in Lemma 5.4. So

$$(b_0+b_1+\cdots+b_{m-1})^n = E_{0,n}^{b_0b_1\ldots b_{m-1}} + E_{1,n}^{b_0b_1\ldots b_{m-1}} + \cdots + E_{n(m-1),n}^{b_0b_1\ldots b_{m-1}} \quad (5.2)$$

where on the right-hand side of (5.2) we have simply the sum of the elements in the nth row. If we choose the b_i-s such that $b_i = a_i$, if i is even, and $b_i = -a_i$, if i is odd, then (5.1) follows directly from (5.2). □

In an arbitrary triangle let $S_n^e + S_n^o = A$ and $S_n^e - S_n^o = B$. Then $S_n^e = \frac{A+B}{2}$ and $S_n^o = \frac{A-B}{2}$. Using Theorem 5.3 and Corollary 5.1 we are able to determine both of these sums without constructing the triangle.

Example 5.5. In the 257-based triangle (Figure 4) we have $S_1^e = 9 = (14^1 + 4^1)/2$ and $S_1^o = 5 = (14^1 - 4^1)/2$, $S_2^e = 4 + 53 + 49 = (14^2 + 4^2)/2$ and $S_2^o = 20 + 70 = (14^2 - 4^2)/2$, and so on.

As a special result, for the generalized Pascal triangle of sth order it follows the already known fact (see e.g. in [3]) that either $S_n^e - S_n^o = 1^n$, if s is odd, or $S_n^e - S_n^o = 0^n$, if s is even (this is the case for the Pascal triangle, too).

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